ON THE HOMOLOGY OF CONFIGURATION SPACES $C((M,M_o)\times \mathbf{R}^n;X)$

JIE WU

Abstract. The homology with coefficients in a field of the configuration spaces $C(M\times\mathbf{R}^n, M_o\times\mathbf{R}^n;X)$ is determined in this paper.

1. Introduction.

The purpose of this paper is to determine the homology (with coefficients in a field $\mathbf{F}$) of configuration spaces $C((M,M_o)\times\mathbf{R}^n;X)$. The homology groups

$$H_*(C((M,M_o)\times\mathbf{R}^n;X);\mathbf{F})$$

are determined by dim M, $H_*(M,\, M_o;\,\mathbf{F})$ and $H_*(X;\mathbf{F})$. This answers a conjecture of F. Cohen and L. Taylor [10].

Let A be a topological space. The ordered configuration space $\widetilde{C}^k(A)$ is the subspace of A^k consisting of all k-tuples $(a_1,...,a_k)$ such that $a_i\neq a_j$ for $i\neq j$, where $A^k=A\times ...\times A$ with k copies. Define the configuration space

$$C(A,A_o;X)=\coprod_{k=1}^{\infty}\widetilde{C}^k(A)\times_{\Sigma_k}X^k/\approx$$

where A_o is a closed subspace of A, X is a space with non-degenerate base-point $*$, and $\approx$ is generated by

$$(a_1,...,a_k;x_1,...,x_k)\approx(a_1,...,a_{k-1};x_1,...,x_{k-1})$$

if $a_k\in A_o$ or $x_k=*$.

The spaces in the title are given by $C((A,A_o)\times\mathbf{R}^n;X)=C(A\times\mathbf{R}^n,\,A_o\times\mathbf{R}^n;\,X)$. The applications of configuration spaces can be found in [1, 3, 6, 7, 10, 11, 14, 17, 18]. The homology of configuration spaces for various cases can be found in [4, 5, 8, 9, 10, 11, 12, 13, 20]. We will consider the case that A is a manifold and A_0 is a submanifold and $n>0$.

In this paper, a space X will always mean a compactly generated weak Hausdorff space with nondegenerate base-point $*$ such that $(X,*)$ is an NDR-pair. Homology groups are always taken with coefficients in the field $\mathbf{F}$;$S_*(-)$ willalways mean the

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singular chain complex with coefficients in the field F. A manifold will always mean smooth triangulated manifold. For each finitedimensional graded $\mathbf{F}$-module V_* and a connected space X, define

$$\mathcal{C}^m(V_*;X)=\bigotimes_{q=0}^{m}\bigotimes^{\beta_q(V_*)} H_*(\Omega^{m-q}S^mX)$$

where $\beta_q(V_*)=\dim_{\mathbf{F}}(V_q)$ is the q-th Betti number of V_* and $\Omega^{m-q}S^mX=*$ if $q>m$.

If $V_q=0$ for all $q\geq m$, then each term $H_*(\Omega^{m-q}S^mX)$ in $\mathcal{C}^m(V_*;X)$ is an algebra with an algebra filtration induced by $\Omega^{m-q}S^mX\simeq C(\mathbf{R}^{m-q},S^qX)$. This filters $\mathcal{C}^m(V_*;X)$ by the tensor product filtration. The natural filtration of configuration spaces will be given in section 2. Now our main theorems are as follows

Theorem A. *Let M be a smooth triangulated compact manifold and let M_o be a smooth triangulated compact submanifold of M,then*

(1): *for each simply connected space X and $n\geq 1$, there is an $\mathbf{F}$-filtered **module** isomorphism*

$$\theta:\mathcal{C}^{\dim M+n}(H_*(M,M_o);X)\longrightarrow H_*C((M,M_o)\times\mathbf{R}^n,X)$$

(2): *for each simply connected space X and $n\geq 2$, there is an $\mathbf{F}$-filtered **algebra** isomorphism*

$$\theta:\mathcal{C}^{\dim M+n}(H_*(M,M_o);X)\longrightarrow H_*C((M,M_o)\times\mathbf{R}^n;X)$$

Theorem B. *Let M be a smooth triangulated compact manifold and let M_o be a smooth triangulated compact submanifold of M, then there exist isomorphisms*

$$\overline{H}*C((M,M_o)\times\mathbf{R}^n;X)\approx\bigoplus_{k=1}^{\infty}\sigma^{-2k}\mathcal{D}_k^{m+n}(H_*(M,M_o);S^2X)$$

as $\mathbf{F}$-modules for any space X and $n\geq 1$, where $m=$dim M, $\mathcal{D}_k^{m+n}(V_*;X)=$

$$F_k\mathcal{C}^{m+n}(V_*;X)/F_{k-1}\mathcal{C}^{m+n}(V_*;X)$$

and σ^{-t} denotes the t-th desuspension of graded modules.

The article is organized as follows. In section 2, some basic properties of configuration spaces are studied. some product decompositions of certain configuration spaces are given in section 3. The proofs of Therem A and Theorem B will be given in section 4. The author would like to thank Professors Xueguang Zhou and Fred Cohen for their many helpful discussions and encouragement during the writing of the manuscript.The author also would like to thank Miguel Xicotencatl for his help to type the manuscript.

2. Basic properties of configuration spaces.

In this section, some basic properties of $C(A, A_o; X)$ are recalled. Given an embedding $(A, A_o) \to (A', A'_o)$ of space-pairs and a pointed map $X \to X'$, there is an induced map $C(A, A_o; X) \longrightarrow C(A', A'_o; X')$. Hence the homotopy type of $C(A, A_o; X)$ is an invariant of the (relative) isotopy type of (A, A_o) and the homotopy type of $(X, *)$. The length of a configuration induces a natural filtration of $C(A, A_o; X)$ by the closed subspaces

$$F_k C = F_k C(A, A_o; X) = \coprod_{j=1}^{k} \widetilde{C}^j(A) \times_{\Sigma_j} X^j / \approx$$

$F_o C = *$, $F_1 C = A/A_o \wedge X$, see [3]. It is easy to see that $C(A \amalg A', A_o \amalg A'_o;\, X)$ is homeomorphic to the product

$$C(A, A_o; X) \times C(A', A'_o; X)$$

and the homeomorphism preserves the filtration. Hence $C(A, A_o; X)$ is a filtered H-space if there is an embedding $e : (A \amalg A, A_o \amalg A_o) \longrightarrow (A, A_o)$ such that $e|_{\text{first copy of } (A,A_o)}$ and $e|_{\text{second copy of } (A,A_o)}$ are (relatively) isotopic to the identity map of (A, A_o). In particular, $C((A, A_o) \times \mathbf{R}^n; X)$ is a filtered H-space for each $n \geq 1$.

Define

$$\widetilde{C}^k(A|A_o) = \{(a_1, ..., a_k) \in \widetilde{C}^k(A) |\text{ some } a_j \in A_o\}$$

and

$$(X|*)^k = \{(x_1, ..., x_k) \in X^k |\text{ some } x_j = *\} .$$

If (A, A_o) is a relative CW-complex, there are Σ_k-equivalent cofibrations $\widetilde{C}^k(A|A_o) \to \widetilde{C}^k(A)$ and $(X|*)^k \to X^k$ and therefore there are cofibrations

$$\widetilde{C}^k(A|A_o) \times_{\Sigma_k} X^k \cup \widetilde{C}^k(A) \times_{\Sigma_k} (X|*)^k \longrightarrow \widetilde{C}^k(A) \times_{\Sigma_k} X^k$$

and $F_{k-1}C \to F_k$ with the same cofibre $D_k(A, A_o; X) = F_k C / F_{k-1} C \cong$

$$\widetilde{C}^k(A) \times_{\Sigma_k} X^k / (\widetilde{C}^k(A|A_o) \times_{\Sigma_k} X^k \cup \widetilde{C}^k(A) \times_{\Sigma_k} (X|*)^k)$$

see [2, pp 231-239; 14 pp 162-172 and Thm. 7.1]. We call $D_k(A, A_o; X)$ the k-adic construction of $C(A, A_o; X)$.

Proposition 2.1. *If (A, A_o) is a relative CW-complex and $k \geq 1$, then there is an isomorphism of $\mathbf{F}$-modules*

$$\overline{H}_* \mathcal{D}_k(A, A_o; X) \longrightarrow H_* \left(S_*(\widetilde{C}^k(A), \widetilde{C}^k(A|A_o)) \otimes_{\Sigma_k} (\overline{H}_*(X))^{\otimes k} \right)$$

The following lemma is useful.

Lemma 2.2. *Let C be a free Σ_k-chain complex and let K and L be chain complexes. If the chain maps $f, g : K \to L$ are homotopic, then $1 \otimes f^{\otimes k}$ and $1 \otimes g^{\otimes k} : C \otimes L^{\otimes k} \longrightarrow C \otimes K^{\otimes k}$ are Σ_k-equivariently homotopic, where Σ_k acts diagonally on $C \otimes L^{\otimes k}$ and $C \otimes K^{\otimes k}$.*

Proof. Let I be the unit chain complex with 0-simplexes $\overline{0}$ and $\overline{1}$ and 1-simplex $\overline{I}$ and the differential $\partial(\overline{I}) = \overline{1} - \overline{0}$, and let $D : I \otimes K \to L$ the chain homotopy between f and g, the composite of Σ_k-equivariant chain maps

$$I \otimes C \otimes K^{\otimes k} \xrightarrow{\varphi \otimes 1} I^{\otimes k} \otimes C \otimes K^{\otimes k} \approx C \otimes (I \otimes K)^{\otimes k} \xrightarrow{1 \otimes D^{\otimes k}} C \otimes L^{\otimes k}$$

is the required Σ_k-equivariant chain homotopy, where the Σ_k-equivariant chain map $\varphi : I \otimes C \to I^{\otimes k} \otimes C$ is defined as follows.

Let $\mathcal{B}$ be a Σ_k-basis of C, define $\varphi(\overline{0} \otimes c) = \overline{0}^k \otimes c$, $\varphi(\overline{1} \otimes c) = \overline{1}^k \otimes c$ for each$c \in \mathcal{B}$ and define

$$\varphi(\overline{I} \otimes \sigma \cdot c) = \sigma \cdot (\overline{0}^{k-1} \otimes \overline{I} + \sum_{j=1}^{k-2} \overline{0}^{k-1-j} \otimes \overline{I} \otimes \overline{1}^j + \overline{I} \otimes \overline{1}^{k-1}) \otimes \sigma \cdot c$$

for each $\sigma \in \Sigma_k$.

Proof of Proposition 2.1. By Eilenberg-Zilber Theorem, there are isomorphisms

$$\overline{H}_* D^k(A, A_o; X) \approx H_*(S_*(\tilde{C}^k(A), \tilde{C}^k(A|A_o)) \otimes_{\Sigma_k} S_*(X^k, (X|*)^k))$$

$$\approx H_*(S_*(\tilde{C}^k(A), \tilde{C}^k(A|A_o)) \otimes_{\Sigma_k} (S_*(X, *))^{\otimes k}).$$

Since $S_*(\tilde{C}^k(A), \tilde{C}^k(A|A_o)$ is a free Σ_k-chain complex over $\mathbf{F}$ and $S_*(X, *) \simeq \overline{H}_*(X)$ as chain complexes over $\mathbf{F}$, where $\overline{H}_*(X)$ with trivial differential (see[19, Lemma VIII.3.1]). The assertion follows by the above Lemma.

Proposition 2.3. *If (A, A_o) is a relative CW-complex, then*

$$\overline{H}_* D_k(A, A_o; X) \approx \sigma^{-2k} \overline{H}_* D_k(A, A_o; S^2 X)$$

where σ^{-t} is the t-th desuspension.

Proof. By Proposition 2.1, there are isomorphisms

$$\overline{H}_* D^k(A, A_o; X) \approx H_*(S_*(\tilde{C}^k(A), \tilde{C}^k(A|A_o)) \otimes_{\Sigma_k} (\overline{H}_*(X))^{\otimes k})$$

and

$$\overline{H}_* D^k(A, A_o; S^2 X) \approx H_*(S_*(\tilde{C}^k(A), \tilde{C}^k(A|A_o)) \otimes_{\Sigma_k} (\overline{H}_*(S^2 X))^{\otimes k})$$

$$\approx \sigma^{2k} H_*(S_*(\tilde{C}^k(A), \tilde{C}^k(A|A_o) \otimes_{\Sigma_k} \overline{H}_*(X)^{\otimes k})$$

Proposition 2.4. *If (A, A_o) is a relative CW-complex with an embedding*

$$\coprod_{k=1}^{\infty} \tilde{C}^k(A)/\Sigma_k \to R^{\infty},$$

then there is a stable equivalence

$$\sigma : C(A, A_o; X) \longrightarrow \bigvee_{k=1}^{\infty} D_k(A, A_o; X)$$

via stable equivalences

$$\sigma_k : F_k C(A, A_o; X) \longrightarrow \bigvee_{j=1}^{k} D_j(A, A_o; X)$$

Proof. If $(A, A_o) = (M, M_o)$ a manifold-pair, this is proved in [3, Prop. 3]. For the general cases, one proceeds in the same way.

Proposition 2.5. *Let X be a path connected space and let (A, A_o) be a relative CW-complex with an embedding $\coprod_{k=1}^{\infty} \tilde{C}^k(A)/\Sigma_k \longrightarrow R^{\infty}$. There exists an isomorphism of $\mathbf{F}$-modules*

$$\varphi : H_* C(A, A_o; \bigvee_{\alpha \in I} S^{n_\alpha}) \longrightarrow H_* C(A, A_o; X)$$

where $\{n_\alpha | \alpha \in I\}$ is determinined by $\overline{H}_*(X)$.

Proof. Let $\{x_\alpha | \alpha \in I\}$ be a basis of $\overline{H}_*(X)$ and let $n_\alpha = |x_\alpha|$ for $\alpha \in I$, we have an $\mathbf{F}$-isomorphism

$$\varphi_1 : \overline{H}_*(\vee_\alpha S^{n_\alpha}) \to \overline{H}_*(X)$$

Now the assertion follows by Proposition 2.1 and 2.3.

Proposition 2.6. *Let M be a smooth compact manifold and let M_o and N be the smooth compact submanifolds of M with codim$N = 0$. If $N/M_o \cap N$ or X is path connected, then*

$$C(N, N \cap M_o; X) \to C(M, M_o; X) \to C(M, N \cup M_o; X)$$

is a quasifibration.

[5, pp. 113].

Now let M be an m-manifold and let W be a m-manifold without boundary which contains M, e.g. $W = M$ if M is closed, or $W = M \cup \partial M \times [0, 1)$ if M has boundary. Let ξ be the principal $O(m)$-bundle of the tangent bundle of W. Let $\Gamma_{\xi[S^m X]}(B, B_o)$ be the space of cross sections of $\xi[S^m X]$ which are defined on B and take values at $\infty \wedge X$ on B_o for each subspace pair (B, B_o) in W, where $\xi[S^m X]$ is the associated bundle and $O(m)$ acts diagonally on $S^m X = S^m \wedge X$, trivially on X and canonically on $S^m \cong R^m \cup \{\infty\}$.

Proposition 2.7. *Let M be a smooth compact manifold and let M_o be a smooth compact submanifold of M. If M/M_o or X is path connected, then there is a (weak) homotopy equivalence*

$$C(M, M_o; X) \to \Gamma_{\xi[S^m X]}(W - M_o, W - M)$$

[5, Proposition 3.1 and 3, pp. 178]

Remark 2.8. *By Proposition 2.6,there is a homotopy equivalence*

$$C((M, M_o) \times \mathbf{R}^n; X) \simeq \Omega^n C(M, M_o; S^n X)$$

if M/M_o or X is path connected.

3. Decomposition Theorems

In this section, assume that (M, M_o) is a smooth triangulated compact manifold-pair with $m =$dim$M$ and W is a smooth m-manifold without boundary which contains M and ξ_W is the principal $O(m)$-bundle of the tangent bundle of W (see Proposition 2.6). Let $\overline{W} = M$ if M is closed, or $\overline{W} = M \times [0, \frac{1}{2}]$ if M has boundary.

Lemma 3.1. *If X is path connected, then there is a (weak) homotopy equivalence*

$$C((M, M_o) \times \mathbf{R}; X) \to \Gamma_{\xi_W[\Omega S^{m+1}X]}(W - M_o, W - M)$$

where $O(m)$ acts on $\Omega S^{m+1}X$ via the functor $\Omega S(-)$,i.e. by taking the homeomorphism $\Omega S(\sigma \wedge id|_X) : \Omega S^{m+1}X \longrightarrow \Omega S^{m+1}X$ for each $\sigma \in O(m)$.

Proof. Notice that $(M, M_o) \times \mathbf{R}$ is isotopic to $(M, M_o) \times I$. By Proposition 2.6, it is easy to see that

$$C((M, M_o) \times \mathbf{R}; X) \simeq \Gamma_{\xi_{W\times\mathbf{R}}[S^{m+1}X]}((W - M_o, W - M) \times (I, \partial I))$$

Since $\xi_{W\times\mathbf{R}}[S^{m+1}X] = \pi^*\xi_W[S^{m+1}X]$, where $\pi : W \times \mathbf{R} \to W$ is the projection, there is

$$\begin{aligned} &\Gamma_{\xi_{W\times\mathbf{R}}[S^{m+1}X]}((W - M_o, W - M) \times (I, \partial I)) \\ &\simeq \Gamma_{\xi_W[\Omega S^{m+1}X]}(W - M_o, W - M) \end{aligned}$$

Lemma 3.2. *Let (N, N_o) be a compact submanifold-pair in $\overline{W}$ such that $N \subseteq \overline{W} - \partial\overline{W}$, then there is a (weak) homotopy equivalence*

$$C((\overline{W} - \nu(N_o), \overline{W} - \nu(N)) \times \mathbf{R}; X) \to \Gamma_{\xi_W[\Omega S^{m+1}X]}(N, N_o)$$

if X is path connected, where$(\nu(N), \nu(N_o))$ is an open tubular neighborhood of (N, N_o) in $\overline{W} - \partial\overline{W}$.

Proof. By Lemma 3.1, there is a homotopy equivalence

$$\begin{aligned} C((\overline{W}-\nu(N_o),\overline{W}-\nu(N))\times\mathbf{R};X) &\simeq \Gamma_{\xi_W[\Omega S^{m+1}X]}(W-(\overline{W}-\nu(N)),W-(\overline{W}-\nu(N_o))) \\ &\cong \Gamma_{\xi_W[\Omega S^{m+1}X]}(\nu(N),\nu(N_o)) \\ &\simeq \Gamma_{\xi_W[\Omega S^{m+1}X]}(N,N_o) \end{aligned}$$

Lemma 3.3. *Let (N,N_o) be a submanifold-pair in W. There is a homeomorphism*

$$\Gamma_{\xi_W[\Omega S^{mn+1}X]}(N,N_o)\cong\Gamma_{\xi_{W^n}[\Omega S^{mn+1}X]}(\Delta_n(N),\Delta_n(N_o))$$

for $n\geq 1$, where $\Delta_n : W\to W^n = W\times ...\times W$ the diagonal map, $O(mn)$ acts on $\Omega S^{mn+1}X$ via the functor $\Omega S(-)$,see Lemma 3.1 as above, and $O(m)$ acts on $\Omega S^{mn+1}X$ via the diagonal inclusion $O(m)\to O(mn), \sigma\to diag(\sigma,\sigma,...,\sigma)$.

Proof. The diagonal map $\Delta_n : W\to W^n$ induces a bundle map $\overline{\Delta}_n : \xi_W\to\xi_{W^n}$. Consider the induced map on the total spaces of bundles $\overline{\Delta}_n\times 1 : E\xi_W\times\Omega S^{mn+1}X\longrightarrow E\xi_{W^n}\times\Omega S^{mn+1}X$.

For each (x,y)$\in E\xi_W\times\Omega S^{mn+1}X$ and $\sigma\in O(m)$, notice that

$$\begin{aligned} \overline{\Delta}_n\times 1(x\sigma,\sigma^{-1}y) &= ((x,\dots,x)\tilde{\sigma},\Omega S(\sigma^{-1}\wedge\dots\wedge\sigma^{-1}\wedge 1_X)y) \\ &= ((x,\dots,x)\tilde{\sigma},\tilde{\sigma}^{-1}y) \end{aligned}$$

where $\tilde{\sigma}=\begin{pmatrix}\sigma & & \\ & \ddots & \\ & & \sigma\end{pmatrix}\in O(mn)$.

Hence $\overline{\Delta}_n\times 1$ induces a map

$$\begin{aligned} \Delta : E(\xi_W[\Omega S^{mn+1}X]) = E\xi_W\times_{O(m)}\Omega S^{mn+1}X &\longrightarrow E\xi_{W^n}\times_{O(mn)}\Omega S^{mn+1}X \\ &= E(\xi_{W^n}[\Omega S^{mn+1}X]). \end{aligned}$$

Furthermore Δ induces a homeomorphism

$$\Gamma_{\xi_W[\Omega S^{mn+1}X]}(N,N_o)\cong\Gamma_{\xi_{W^n}[\Omega S^{mn+1}X]}(\Delta_n(N),\Delta_n(N_o)),$$

since the induced map is 1-1,onto and open.

The following lemma follows from the naturality of the Samelson products.

Lemma 3.4. *There exists $O(m)$-map $\Phi_n : (S^mX)^{(n)}\longrightarrow\Omega S^{m+1}X$ which represents the Samelson products $[[E,E],\dots,E]$ for each $n\geq 2$, where $X^{(n)}$ is the reduced join of n copies of X and $E : S^mX\longrightarrow\Omega S^{m+1}X$ is the suspension.*

Now we give some decomposition theorems.

Theorem C. *Let (M, M_o) be a smooth triangulated compact manifold-pair, $n+m+1$ even and $n \geq 1$. There exist a manifold-pair $(\tilde{M}, \tilde{M}_o)$ so that $\dim \tilde{M} = 2m$ and*

$$C((M, M_o) \times \mathbf{R}; S^n)_{(p)} \overset{w}{\simeq} C((\tilde{M}, \tilde{M}_o) \times \mathbf{R}, S^{2n})_{(p)} \times C(M, M_o; S^n)_{(p)}$$

where $m = \dim M$ and p is an odd prime or zero and $\overset{w}{\simeq}$ means the (weak) homotopy equivalence

Proof. By Lemma 3.1, there is a homotopy equivalence

$$C((M, M_o) \times \mathbf{R}; S^n) \simeq \Gamma_{\xi_W[\Omega S^{m+1+n}]}(W - M_o, W - M)$$

The inclusion $j : S^{m+n} \to \Omega S^{m+1+n}$ is an $O(m)$-map. By Lemma 3.4, there is an $O(m)$-Samelson product $[j, j] : S^{2m+2n} \to \Omega S^{m+1+n}$, which induces an $O(m)$-H-map $\overline{[j, j]} : \Omega S^{2m+1+2n} \to \Omega S^{m+1+n}$.

Now consider the $O(m)$-map

$$(\overline{[j, j]}, j) : \Omega S^{2m+1+2n} \times S^{m+n} \longrightarrow \Omega S^{m+1+n},$$

which induces a bundle map

$$\xi_W[\Omega S^{2m+1+2n} \times S^{m+n}] \longrightarrow \xi_W[\Omega S^{m+1+n}]$$

and therefore a map

$$\Gamma_{\xi_W[\Omega S^{2m+1+2n} \times S^{m+n}]}(W - M_o, W - M) \longrightarrow \Gamma_{\xi_W[\Omega S^{m+1+n}]}(W - M_o, W - M)$$

Notice that

$$\begin{gathered}\Gamma_{\xi_W[\Omega S^{2m+1+2n} \times S^{m+n}]}(W - M_o, W - M) \simeq \\ \simeq \Gamma_{\xi_W[\Omega S^{2m+1+2n}]}(W - M_o, W - M) \times \Gamma_{\xi_W[S^{m+n}]}(W - M_o, W - M)\end{gathered}$$

By Lemma 3.3

$$\begin{aligned}\Gamma_{\xi_W[\Omega S^{2m+1+2n}]}(W - M_o, W - M) &\simeq \Gamma_{\xi_{W^2}[\Omega S^{2m+1+2n}]}(\Delta(W - M_o), \Delta(W - M)) \\ &\simeq C((\tilde{M}, \tilde{M}_o) \times \mathbf{R}; S^{2n}) \text{ (Lemma 3.2)}\end{aligned}$$

where $\dim \tilde{M} = 2m$. Since $(\overline{[j, j]}, j)$ is homotpy equivalent after p-localization, the assertion follows by induction on the handle decomposition of M.

Theorem D. *Let (M, M_o) be a smooth triangulated compact manifold-pair. If X_1, $\ldots, X_k$ are connected spaces, there exist manifold pairs*

$$(M_\omega, M_{o\omega})$$

such that $C((M, M_o) \times \mathbf{R}; X_1 \vee \ldots \vee X_k)$ is (weak) homotopy equivalent to the (weak) product

$$\prod_\omega C((M_\omega, M_{o\omega}) \times \mathbf{R}; \omega(X_1, \ldots, X_k))$$

where ω runs over all addmissible words in $x_1, \ldots, x_k$ (the addmissible words see [21, pp. 511-514]) and $\omega(X_1, \ldots, X_k) = X_1^{(a_1)} \wedge \ldots \wedge X_k^{(a_k)}$ and $X^{(n)}$ is the reduced join of n copies of X and a_i is the number of occurences of x_i in the word ω.

Proof. Denote $X = X_1 \vee \ldots \vee X_k$. By Lemma 3.1, there is a homotopy equivalence

$$C((M, M_o) \times \mathbf{R}; X) \simeq \Gamma_{\xi_W[\Omega S^{m+1}X]}(W - M_o, W - M)$$

Consider the inclusions

$$i_t : S^m X_t \to \Omega S^{m+1} X \qquad \text{for } 1 \leq t \leq k$$

By Lemma 3.4, there are $O(m)$-Samelson products

$$\omega(i_1, \ldots, i_k) : S^{l(\omega)\cdot m}\omega(X_1, \ldots, X_k) \longrightarrow \Omega S^{m+1} X$$

for each word ω, where $l(\omega)$, is the length of ω, which induces an $O(m)$-H-map

$$\tilde{\omega}(i_1, \ldots, i_k) : \Omega S^{l(\omega)\cdot m}\omega(X_1, \ldots, X_k) \longrightarrow \Omega S^{m+1} X$$

Consider the $O(m)$-map

$$f = \prod_\omega \tilde{\omega}(i_1, \ldots, i_k) : \prod_\omega \Omega S^{l(\omega)\cdot m}\omega(X_1, \ldots, X_k) \longrightarrow \Omega S^{m+1} X$$

which induces a bundle map

$$\xi_W[\prod_\omega \Omega S^{l(\omega)\cdot m}\omega(X_1, \ldots, X_k)] \longrightarrow \xi_W[\Omega S^{m+1} X]$$

and therefore a map

$$\begin{aligned} \tilde{f} : \Gamma_{\xi_W[\prod_\omega \Omega S^{l(\omega)\cdot m}\omega(X_1, \ldots, X_k)]}&(W - M_o, W - M) \\ &\longrightarrow \Gamma_{\xi W[\Omega S^{m+1}X]}(W - M_o, W - M). \end{aligned}$$

By Hilton-Milnor Theorem, f is a (weak) homotopy equivalence. By induction on the handle decomposition of M,$\tilde{f}$ is a (weak) homotopy quivalence.

Notice that

$$\begin{aligned} &\Gamma_{\xi_W[\prod_\omega \Omega S^{l(\omega)\cdot m}\omega(X_1, \ldots, X_k)]}(W - M_o, W - M) \\ &\simeq \Gamma_{\xi_W[\prod_\omega \Omega S^{l(\omega)+1}\omega(X_1, \ldots, X_k)]}(\overline{W} - v(M_o), \overline{W} - v(M)) \\ &\cong \prod_\omega \Gamma_{\xi_W[\Omega S^{l(\omega)+1}\omega(X_1, \ldots, X_k)]}(\overline{W} - v(M_o), \overline{W} - v(M)) \\ &\cong \prod_\omega \Gamma_{\xi_{W^{l(\omega)}}[\Omega S^{l(\omega)+1}\omega(X_1, \ldots, X_k)]}(\Delta_{l(\omega)}(\overline{W} - v(M_o)), \Delta_{l(\omega)}(\overline{W} - v(M))) \\ &\simeq \prod_\omega C((M_\omega, M_{o\omega}) \times \mathbf{R}; \omega(X_1, \ldots, X_k)) \qquad \text{(Lemma 3.2)} \end{aligned}$$

where $\dim M_\omega = \dim W^{l(\omega)} = l(\omega) \cdot m$.

The assertion follows.

4. Proofs of Theorems A and B.

Lemma 4.1. *Let $(N, N_o) \subseteq (M, M_o)$ be the smooth compact submanifold pair of (M, M_o) with $\dim N = \dim M$ and let X be a simply connected space. If $H_*(N, N_o) \to H_*(M, M_o)$ is onto, then*

$$H_*C((N, N_o) \times \mathbf{R}^n; X) \longrightarrow H_*C((M, M_o) \times \mathbf{R}^n; X)$$

is onto for $n \geq 1$.

Proof. Since $(M, M_o) \times \mathbf{R}^n$ is isotopic to $(M \times I^{n-1}, M_o \times I^{n-1}) \times \mathbf{R}$, it is sufficient to show that

$$H_*C((N, N_o) \times \mathbf{R}; X) \longrightarrow H_*C((M, M_o) \times \mathbf{R}; X)$$

is onto.

Step1: Assume that $X = S^n$ with $n > 1$.
If $m + 1 + n$ is odd or $\mathbf{F}$ is of characteristic 2, this was proved in [5, Th. A], where $m = \dim M$.
Now assume that $m+1+n$ is even and $\mathbf{F}$ is of characteristic $\neq 2$. By Theorem C, there is a p-homotopy equivalence

$$C((M, M_o) \times \mathbf{R}; S^n) \simeq C((\tilde{M}, \tilde{M}_o) \times \mathbf{R}; S^{2n}) \times C(M, M_o; S^n)$$

and

$$C((N, N_o) \times \mathbf{R}; S^n) \simeq C((\tilde{N}, \tilde{N}_o) \times \mathbf{R}; S^{2n}) \times C(N, N_o; S^n)$$

where $\dim \tilde{M} + 1 + 2n = \dim \tilde{N} + 1 + 2n = 2m + 2n + 1$
and $\dim M + n = \dim N + n = m + n$.
The assertion follows by [5, Th. A].

Step 2: Assume that $X = S^{d_1} \vee \ldots \vee S^{d_k}$ with $d_j > 1$.
By Theorem D, there is a (weak) homotopy equivalence

$$C((M, M_o) \times \mathbf{R}; X) \simeq \prod_{\omega} C((M_\omega, M_{o\omega}) \times \mathbf{R}; \omega(X))$$

where $\omega(X) = S^{d_1 a_1} \wedge \ldots \wedge S^{d_k a_k} = S^{d_1 a_1 + \cdots + d_k a_k}$.
The assertion follows from Step 1.

Step 3: Assume $H_*(X)$ is a finite dimensional $\mathbf{F}$-vector space. The assertion follows from Step 2 and Proposition 2.4.

Step 4: General case.
There exist X_α such that $H_*(X) = \lim_\alpha H_*(X_\alpha)$ and $H_*(X_\alpha)$ is finite dimensional $\mathbf{F}$-vector space for each α. The assertion follows from Step 3.

Proof of Theorem A. First we will prove the absolute case $M_o = \phi$ by induction on on a handle decomposition of M. If M is a disjoint union $M_1 \amalg M_2$, then

$$C(M \times \mathbf{R}^n; X) \cong C(M_1 \times \mathbf{R}^n; X) \times C(M_2 \times R^n; X)$$

Hence we can restrict to connected manifolds and start with $M = I^m$, the assertion is obvious for $M = I^m$. Assume that the assertion holds for M and $\overline{M} = M \cup D$ with $D \cong I^m$ a handle of index q i.e. $D \cap M \cong I^{m-q} \times \partial I^q$, $q \geq 1$ since M is connected. There is a cofibration

$$M \xrightarrow{i} \overline{M} \xrightarrow{j} (\overline{M}, M) \simeq (I^m, I^{m-q} \times \partial I^q) \simeq (S^q, *)$$

and alternative

I. $H_q(\overline{M}) \xrightarrow{j_*} H_q(S^q)$ is onto.

II. $H_q(\overline{M}) \longrightarrow H_q(S^q)$ is zero.

Case I. Consider the quasifibration

$$C(M \times \mathbf{R}^n; X) \longrightarrow C(\overline{M} \times \mathbf{R}^n; X) \xrightarrow{C(j)} C((\overline{M}, M) \times \mathbf{R}^n; X)$$
$$\simeq \Omega^{m+n-q} S^{m+n} X$$

Since j_* is onto, $C(j)_*$ is onto and the Serre spectral sequence for the quasifibration above collapses. Hence there is a short exact sequence of Hopf algebras

$$H_* C(M \times \mathbf{R}^n; X) \rightarrowtail H_* C(\overline{M} \times \mathbf{R}^n; X) \twoheadrightarrow H_* \Omega^{m+n-q} S^{m+n} X$$

By Proposition 2.3, there is a commutative diagram

$$\begin{array}{ccc} \Sigma^\infty C(\overline{M} \times \mathbf{R}^n; X) & \xrightarrow{p} & \Sigma^\infty D_k \overline{M} \times \mathbf{R}^n; X) \\ \downarrow & & \downarrow \\ \Sigma^\infty C((\overline{M}, M) \times \mathbf{R}^n; X) & \xrightarrow{p} & \Sigma^\infty D_k((\overline{M}, M) \times \mathbf{R}^n; X) \end{array}$$

for each $k \geq 1$, where p is the projection.

Thus

$$\overline{H}_* D_k(\overline{M} \times \mathbf{R}^n) \longrightarrow \overline{H}_* D_k((\overline{M}, M) \times \mathbf{R}^n; X)$$

is an epimorphism and $F_r H_* C(\overline{M} \times \mathbf{R}^n; X) \longrightarrow F_r H_* \Omega^{m+n-q} S^{m+n} X$ is an epimorphism by Proposition 2.3.

Hence there exists an $\mathbf{F}$-map

$$\varphi : H_* \Omega^{m+n-q} S^{m+n} X \longrightarrow H_* C(\overline{M} \times \mathbf{R}^n; X)$$

such that

(1) φ preserves the filtration.

(2) $C(j)_* \circ \varphi = id$

Now the composite

$$H_* C(M \times \mathbf{R}^n; X) \otimes H_* \Omega^{m+n-q} S^{m+n} X \longrightarrow H_* C(\overline{M} \times \mathbf{R}^n; X)^{\otimes 2}$$
$$\longrightarrow H_* C(\overline{M} \times \mathbf{R}^n; X)$$

is an isomorphism of filtered modules and (1) follows.

If $n > 1$, let $\varphi : QH_*(\Omega^{m+n-q} S^{m+n} X) \longrightarrow QH_* C(\overline{M} \times \mathbf{R}^n; X)$ so that $QC(j)_* \circ \varphi = id$ and φ preserves the filtration.

The homomorphism

$$QH_*\Omega^{m+n-q}S^{m+n}X \longrightarrow QH_*C(\overline{M}\times R^n;X) \longrightarrow H_*C(\overline{M}\times \mathbf{R}^n;X)$$

induces an algebra map

$$H_*\Omega^{m+n-q}S^{m+n}X \longrightarrow H_*C(\overline{M}\times \mathbf{R}^n;X)$$

since $H_*C(\overline{M}\times \mathbf{R}^n;X)$ is a commutative algebra and $H_*\Omega^{m+n-q}S^{m+n}X$ is a free commutative algebra. Thus (2) follows.

Case II. Consider the quasifibration

$$\Omega C((\overline{M},M)\times \mathbf{R}^n;X) \longrightarrow C(M\times \mathbf{R}^n;X) \xrightarrow{C(i)} C(\overline{M}\times \mathbf{R}^n;X)$$

Since i_* is onto, $C(i)_*$ is onto and the Serre spectral sequence for the quasifibration above collapses. Hence

$$H_*C(\overline{M}\times \mathbf{R}^n;X) \approx H_*C(M\times \mathbf{R}^n;X)//H_*\Omega^{m+n-q+1}S^{m+n}X$$

Both (1) and (2) follow.

To treat the relative case, we can assume that M_o is part of a closed collar (see [5]). We will prove by induction om a handle decomposition of M, and start with $M=M_o$, the assertion is obvious for $M=M_o$ since $F(Mo,M_o)\simeq *$. Assume that the assertion holds for (M,M_o). If $\overline{M}=M\cup D$ with $D\cong I^m$ a hand of index q in M. Clearly we can assume that $q\geq 1$. There is a cofibration

$$(M,M_o)\to(\overline{M},M_o)\to(\overline{M},M)\simeq(I^m,I^{m-q}\times\partial I^q)$$

and again an alternative:

III. $H_q(\overline{M},M_o)\to H_q(\overline{M},M)$ is onto.

IV. $H_q(\overline{M},M_o)\to H_q(\overline{M},M)$ is zero.

For Case III and IV, the assertion follows from Lemma 4.1 similar to Case I and II respectively.

Proof of Theorem B. By Theorem A, there is an isomorphism of filtered $\mathbf{F}$-modules

$$H_*C((M,M_o)\times \mathbf{R}^n;S^2X)\approx \mathcal{C}^{m+n}(H_*(M,M_o);S^2X).$$

Hence

$$\overline{H}_*D_k((M,M_o)\times \mathbf{R}^n;S^2X)\approx \mathcal{D}^{m+n}_k(H_*(M,M_o);S^2X).$$

By Proposition 2.2 and Proposition 2.3, there are isomorphisms of $\mathbf{F}$-modules

$$\overline{H}_*D_k((M,M_o)\times \mathbf{R}^n;X)\approx \sigma^{-2k}\overline{H}_*D_k((M,M_o)\times \mathbf{R}^n;S^2X)$$

and

$$\overline{H}_*C((M,M_o)\times \mathbf{R}^n;X)\approx \bigoplus_{k=1}^{\infty}\overline{H}_*D_k((M,M_o)\times \mathbf{R}^n;X)$$

The assertion follows.

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MSRI, 1000 Centennial Drive, Berkeley, CA 94720-5070