HARMONIC ANALYSIS ON THE TWISTED FINITE POINCARÉ UPPER HALF-PLANE

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ABSTRACT. We prove that the induced representation from a non trivial character of the Coxeter torus of $\mathrm{GL}(2, \mathrm{F})$, for a finite field F, is multiplicity-free; we give an explicit description of the corresponding (twisted) spherical functions and a version of the Heisenberg Uncertainty Principle.

1. Introduction

Let F be a finite field, with q elements, and E be its unique quadratic extension. Put $G = \mathrm{GL}(2, \mathrm{F})$ and denote by K the Coxeter torus of G, realized as the subgroup of all matrices $m_z (z \in E^\times)$ of the maps $w \mapsto zw (w \in E)$ with respect to a fixed F - basis of E. Recall that the finite homogeneous space $\mathcal{H} = G/K$ may be looked upon as the finite analogue of (the double cover of) the classical Poincaré Upper Half Plane (see [4]). Harmonic analysis on $\mathcal{H}$ amounts to decompose the induced representation $\mathrm{Ind}_K^G \mathbf{1}$ from the unit character $\mathbf{1}$ of K to G. We are interested here in the "twisted" version of this, i.e., the decomposition of the induced representation $\mathrm{Ind}_K^G \Phi$ from a non (necessarily) trivial character Φ of K to G. The real analogue of this case has been considered in [1]. We prove that this representation is multiplicity-free, taking advantage of the fact that this is so for $\mathrm{Ind}_K^G \mathbf{1}$ (see [4]) and reducing the computation of the multiplicities in $\mathrm{Ind}_K^G \Phi$ to the ones in $\mathrm{Ind}_K^G \mathbf{1}$. We also give an explicit description of the corresponding (twisted) spherical functions. Finally, we give a version of the Heisenberg Uncertainty Principle

2. The Multiplicity One Theorem for $\mathrm{Ind}_K^G \Phi$.

2.1. **The case $\Phi = \mathbf{1}$.** We consider first the special case $\Phi = \mathbf{1}$ in which the multiplicity one theorem follows from a geometric argument. In fact, we have

$$\mathrm{Ind}_K^G \mathbf{1} \simeq (\mathbf{L^2}(\mathcal{H}), \tau),$$

where $L^2(\mathcal{H}))$ stands for the space of all complex functions on $\mathcal{H}$ endowed with the usual canonical scalar product, and τ denotes the natural representation of G in

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$L^2(\mathcal{H})$, defined by $(\tau_g f)(z) = f(g^{-1}.z)$, where $z \mapsto g.z$ is the homographic action of G on $\mathcal{H}$, given by

$$g.z = \frac{az+b}{cz+d} \qquad\qquad \text{for} g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, z \in \mathcal{H}.$$

Definition 1. *For all $z, w \in \mathcal{H}$, we put*

$$D(z,w) = \frac{N(z-w)}{N(z-\bar{w})}$$

with the convention that $D(z,w) = \infty$ if $w = \bar{z}$.

Proposition 1. *D is an orbit classifying invariant function for the homographic action of G in $\mathcal{H} \times \mathcal{H}$.*

Corollary 1. *The commuting algebra of $(L^2(\mathcal{H}), \tau)$ is commutative.*

This follows from the fact that, the classifying invariant D being symmetric, the G-orbits in $\mathcal{H} \times \mathcal{H}$ are also symmetric. □

2.2. **The case of general Φ.** Let's denote by ϕ the restriction of Φ to $F^\times$. We will prove that every twisting of an irreducible representation π_θ^d of G (where the superscript d denotes the dimension of π and θ its character parameter) by the character $(\Phi + \Phi^q)$ is isomorphic to a representation of the form $\pi_{\theta'}^{d'} + \pi_{\theta''}^{d''}$, when restricted to K. In fact we will work with the characters χ_θ^d of the irreducible representations π_θ^d of G, for which we keep the notations of [5] or [2].

Lemma 1. *On K we have*
$(\Phi + \Phi^q)\chi_{\alpha,\alpha}^q = \chi_{\Phi(\alpha\circ N)}^{q-1} + \chi_{\phi\alpha,\alpha}^{q+1}$,
$(\Phi + \Phi^q)\chi_{\alpha,\alpha}^1 = \chi_{\phi\alpha,\alpha}^{q+1} - \chi_{\Phi(\alpha\circ N)}^{q-1}$,
$(\Phi + \Phi^q)\chi_{\alpha,\beta}^{q+1} = \chi_{\phi\alpha,\beta}^{q+1} + \chi_{\alpha,\phi\beta}^{q+1}$,
$(\Phi + \Phi^q)\chi_{\Lambda}^{q-1} = \chi_{\Phi\Lambda}^{q-1} + \chi_{\Phi^q\Lambda}^{q-1}$. □

Now for a character χ of G, we have $\chi \circ \mathrm{Frob} = \chi$ on K, as it follows from the character table. Therefore $\sum_K \bar{\Phi}(k^q)\chi(k) = \sum_K \bar{\Phi}(k)\chi(k)$ because Frob is an involutive automorphism.

Hence, the multiplicity of π in $\mathrm{Ind}_K^G \Phi$ equals $\frac{1}{2}\sum_K (\bar{\Phi} + \bar{\Phi}^q)(k)\pi(k)$ and so it is just the average of the multiplicities in $\mathrm{Ind}_K^G \mathbf{1}$ of two representations of G (one of which may be virtual!)

Remark 1. *Put $\pi_{\alpha,\alpha}^{q+1} = \pi_\alpha^q + \pi_\alpha^1$ and $\pi_{\alpha\circ N}^{q-1} = \pi_\alpha^q - \pi_\alpha^1$ for every $\alpha \in (F^\times)^\wedge$. It is easy to check than in the degenerate cases $\alpha = \beta$ (for $\pi = \pi_{\alpha,\beta}^{q+1}$) and $\Lambda = \Lambda^q$ (for $\pi = \pi_\Lambda^{q-1}$) we find for the multiplicities $m_1(\pi)$*

$$m_1(\pi_{\alpha,\alpha}^{q+1}) = 1 \qquad\qquad (\alpha \in (F^\times)^\wedge) \tag{1}$$

and

$$m_1(\pi^{q-1}_{\alpha\circ N}) = -\delta_{\alpha,1} \qquad (\alpha \in (F^\times)^\wedge) \tag{2}$$

Using the fact that the multiplicities of the irreducible representations of G in $\mathrm{Ind}_K^G \mathbf{1}$ are at most one and also equations (1) and (2), we get that the multiplicities are also at most one in the more general case of $\mathrm{Ind}_K^G \Phi$. □

2.3. **The multiplicities $m_{\Theta,d}(\Phi)$ of π^d_Θ in $\mathrm{Ind}_K^G \Phi$ for general $\Phi \in (E^\times)^\wedge$.** In Table 1 below, π^d_Θ denotes an irreducible representation of G, of dimension d and parameter Θ. Then $d \in \{1, q, q+1, q-1\}$ and Θ is of the form $\{\alpha, \beta\}$, with $\alpha, \beta \in (F^\times)^\wedge$ or $\{\Lambda, \Lambda^q\}$ with $\Lambda \in (E^\times)^\wedge$

TABLE 1. The multiplicities $m_{\Theta,d}(\Phi)$

π^d_Θ	$m_{\Theta,d}(\Phi)$
$\pi^1_{\alpha,\alpha}$	$\delta_{\alpha^2,\phi}$
$\pi^q_{\alpha,\alpha}$	$\delta_{\alpha^2,\phi} - \delta_{\alpha\circ N,\Phi}$
$\pi^{q+1}_{\alpha,\beta}$	$\delta_{\alpha\beta,\phi}$
$\pi^{q-1}_{\Lambda,\Lambda^q}$	$\delta_{\lambda,\phi} - \delta_{\Lambda,\Phi} - \delta_{\Lambda^q,\Phi}$

NOTATIONS. Here $\alpha, \beta \in (F^\times)^\wedge$ with $\alpha \neq \beta$ and $\Phi, \Lambda \in (E^\times)^\wedge$ with $\Lambda \neq \Lambda^q$, and λ (resp. ϕ) denotes the restriction of the character Λ (resp. Φ) to $(F^\times)^\wedge$.

3. The twisted spherical functions

3.1. **The averaging construction.** In this section G denotes an arbitrary finite group, K a subgroup of $G$and Φ a one dimensional representation of K. We notice that the spherical functions for the representation $\mathrm{Ind}_K^G \Phi$ are obtained as weighted averages of the characters of G. More precisely:

Definition 2. *Let $L^1(G)$ be the group algebra of G, realized as the convolution algebra of all complex functions of G and let $L^1_\Phi(G,K)$ be the convolution algebra of all complex functions f on G such that*

$$f(kgk') = \Phi(k) f(g) \Phi(k')$$

for all $g \in G, k, k' \in K$. For any $f \in L^1(G)$ put

$$(P_\Phi f)(g) = \frac{1}{|K|} \sum_{k \in K} \Phi^{-1}(k) f(kg)$$

for all $g \in G$.

Notice that the operator P_Φ is just convolution with the idempotent function $\varepsilon_K^\Phi \in L^1G$ which coincides with $|K|^{-1}\Phi$ on K and vanishes elsewhere. Moreover $L^1_\Phi(G,K)$ may be writen as $\varepsilon_K^\Phi * L^1G * \varepsilon_K^\Phi$ and its elements f are characterized by the properties

$$\varepsilon_K^\Phi * f = f = f * \varepsilon_K^\Phi$$

Lemma 2. *Let χ be the character of an irreducible representation π of G. Then $P_\Phi(\chi)(e) \neq 0$ iff π appears in $\operatorname{Ind}_K^G \Phi$.* □

Lemma 3. *$P_\Phi(\chi)$ is a non-zero function iff it doesn't vanish for $g = e$.* □

Proposition 2. *The mapping P_Φ is an algebra epimorphism from the center $Z(L^1G))$ of the convolution algebra L^1G onto the center $Z(L^1_\Phi(G,K))$ of the convolution algebra $L^1_\Phi(G,K)$.*

Proof: We have

$$\begin{array}{rclcl} P_\Phi(f_1 * f_2) & = & \varepsilon_K^\Phi * (f_1 * f_2) & = & (f_1 * \varepsilon_K^\Phi) * f_2 \\ & = & (f_1 * \varepsilon_K^\Phi * \varepsilon_K^\Phi) * f_2 & = & (\varepsilon_K^\Phi * f_1) * (\varepsilon_K^\Phi * f_2) \\ & = & P_\Phi f_1 * P_\Phi f_2. & & \end{array}$$

since f_1 is central and ε_K^Φ is idempotent. Moreover the dimension d of the image of $Z(L^1G))$ under P_Φ is the number of irreducible characters χ of G such that $P_\Phi(\chi) \neq 0$; but $P_\Phi(\chi) \neq 0$ iff $(P_\Phi\chi)(e) \neq 0$ and, the number $(P_\Phi(\chi))(e)$ being the multiplicity in $\operatorname{Ind}_K^G \Phi$ of the representation π of G whose character is χ, we see that d is just the number of irreducible representations π of G appearing in $\operatorname{Ind}_K^G \Phi$, i. e. the dimension of the center of $L^1_\Phi(G,K)$. □

Corollary 2. *The nonzero functions that satisfy the functional equation*

$$h(x)h(y) = \int_K \bar{\Phi}(k)h(xky)\,dk$$

linearly span the center of the algebra $L^1_\Phi(G,K)$.

Proof: The functions h that satisfy the above functional equation are exactly the complex multiples of the functions $P_\Phi(\chi)$; for a proof (see [6]). Therefore the corollary follows. □

3.2. **Explicit formulae for the twisted spherical functions.** Define

$$S_\Lambda^\Phi(a) = -(q^2-1)^{-1} \sum_{(z,w)\in\Gamma_a} \Phi^{-1}(z)\Lambda(w)$$

for $\Lambda \in (E^\times)^\wedge$ and $a \in F^\times$, where Γ_a denotes the set of all $(z,w) \in E^\times \times E^\times$ such that $N(w) = aN(z)$ and $Tr(w) = 2(a+1)^{-1}Tr(z)$.

Then the spherical function ζ_Λ^Φ of G associated to the cuspidal character χ_Λ^{q-1} of G is given on the representatives $d(a,1) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $\quad (a \in F^\times)$ for the K - double cosets in G, by

$$\zeta_\Lambda^\Phi(d(a,1)) = S_\Lambda^\Phi(a) + q(q+1)^{-1}\delta_{a,1}\delta_{\lambda,\phi},$$

where λ (resp. ϕ) denotes the restriction of the character Λ (resp. Φ) of $E^\times$ to $F^\times$.

Notice that $a = 1$ corresponds to the origin in $\mathcal{H}$ and $a = -1$ corresponds to the antipode of the origin in $\mathcal{H}$. It is not difficult to check that these formulae for the spherical functions are equivalent to the ones given in [4] for the case $\Phi = 1$.

3.3. A new form for the cuspidal spherical functions for $\Phi = \mathbf{1}$ (char $F \neq 2$).

For $a \neq 1$, one has the following new expression for the spherical functions estimated in [3]

$$\zeta_\Lambda^\Phi(a) = (q+1)^{-1} \sum_{u \in U} \varepsilon(Tr(u) - (a + a^{-1}))(\varepsilon\omega)(u),$$

for $a \neq 1$, where ε denotes the sign character of $F^\times$.

4. Heisenberg Uncertainty Principle

For this section, G denotes an arbitrary finite group, K any subgroup of G and Φ any linear character of K.

Let $\hat{G}^\Phi$ be set of allthe equivalence classes of irreducible representations of G that contain the character Φ when restricted to K. For each equivalence class we choose, once and for all, a representative (π, V_π). As usual, for each f in $L^1(G)$, the Fourier Transform $\mathcal{F}(f)$, valued in the class (π, V_π), is the linear operator $\mathcal{F}$ in V_π defined by

$$\mathcal{F}(f)(\pi) := \pi(f) := \frac{1}{|G|}\int_G f(g)\pi(g^{-1})dg := \frac{1}{|G|}\sum_{g \in G} f(g)\pi(g^{-1}).$$

We recall the statement of the Plancherel theorem for a function $f \in L^1_\Phi(G, K)$

$$f(g) = \frac{1}{G} \sum_{\pi \in \hat{G}^\Phi} d_\pi \operatorname{trace}(\pi(f)\pi(g));$$

here $g \in G$ arbitrary and $d_\pi := \dim V_\pi$.

For any complex valued function f on G, let $|\operatorname{supp}(f)|$ denote the number of elements of the support of f. That is, the number of points of G where f takes nonzero values.

Proposition 3 (**Heisenberg Uncertainty Principle**). *For any nonzero function $f \in L^1_\Phi(G, K)$ we have*

$$|\operatorname{supp}(f)|\,(\sum_{\pi \in \operatorname{supp}(\mathcal{F}(f))} d_\pi) \geq |G|.$$

Here $\operatorname{supp}(\mathcal{F}(f))$ *is the subset of* $\hat{G}^\Phi$ *where* $\mathcal{F}(f)$ *does not vanish.*

Proof: For any function f on G we recall that

$$\|f\|_2^2 = \sum_{x\in G} |f(x)|^2; \ \ \|f\|_\infty = \max_{x\in G} |f(x)|; \ \ \|f\|_2^2 \leq \|f\|_\infty^2 |\operatorname{supp}(f)| \qquad (*)$$

From now on, we fix a $G-$invariant inner product on V_π. Then T^* denotes the adjoint of a linear operator T on V_π with respect to this inner product. Also $\|T\|$ denotes the Hilbert-Schmidt norm on $\operatorname{End} V_\pi$ defined by $\operatorname{trace}(TS^*)$, for $S, T \in \operatorname{End} V_\pi$

Since $f \in L^1_\Phi(G, K)$, as we pointed out before, the Plancherel Theorem says that we have that $\operatorname{supp}(f)$ is contained in $\hat{G}^\Phi$ and that

$$f(x) = \frac{1}{G} \sum_{\pi\in\hat{G}^\Phi} d_\pi \operatorname{trace}(\pi(f)\pi(x)).$$

The Cauchy–Schwarz inequality applied to the Hilbert-Schmidt inner product says that the first of the two following inequallities is true,

$$\operatorname{trace}(\pi(f)\pi(x)) \leq \|\pi(f)\|\|\pi(x)\| \leq \|\pi(f)\|,$$

the second inequality follows from the fact that $\|T\| = 1$ for a unitary operator.

Putting together the last two statements we get

$$\|f\|_\infty \leq \frac{1}{G} \sum_{\pi\in\hat{G}^\Phi} d_\pi \|\mathcal{F}(f)(\pi)\|$$

The classical Cauchy–Schwarz inequality and the fact that $d_\pi = d_\pi^{\frac{1}{2}} d_\pi^{\frac{1}{2}}$ imply that

$$\|f\|_\infty^2 \leq \frac{1}{|G|^2} \sum_{\pi\in\hat{G}^\Phi} d_\pi \|\mathcal{F}(f)(\pi)\|^2 \sum_{\pi\in\operatorname{supp}(\mathcal{F}(f))} d_\pi.$$

Now the L^2-version of Plancherel Theorem says that

$$\|f\|_2^2 = \frac{1}{|G|} \sum_{\pi\in\hat{G}^\Phi} d_\pi \|\mathcal{F}(f)(\pi)\|^2.$$

Therefore,

$$\|f\|_\infty^2 \leq \frac{1}{|G|} \|f\|^2 \sum_{\pi\in\hat{G}^\Phi} d_\pi.$$

Since f is nonzero, we apply (*) to the above inequality and get the desired result. □

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