FINITE TYPE CONDITIONS ON REINHARDT DOMAINS

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Abstract. In this paper we prove that, if p is a boundary point of a smoothly bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$, then the variety type at p is identical to the regular type.

In this paper we study the finite type conditions on pseudoconvex Reinhardt domain. We prove that, if p is a boundary point of a smoothly bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$, then the variety type at p is identical to the regular type. In a forthcoming paper, we will study the biholomorphically invariant objects (*e.g.*, the Bergman kernel and metric, the Kobayashi and Carathéodory metrics) on a pseudoconvex Reinhardt domain of finite type.

We first recall some definitions. A domain $\Omega \subset \mathbb{C}^n$ is Reinhardt if

$$(e^{i\theta_1} z_1, \dots , e^{i\theta_n} z_n) \in \Omega$$

whenever $(z_1, \dots , z_n) \in \Omega$ and $0 \leq \theta_j \leq 2\pi, 1 \leq j \leq n$. Denote

$$\mathcal{Z}_j = \{(z_1, \dots , z_n) \in \mathbb{C}^n; \quad z_j = 0\},$$

for $j = 1, \dots , n$. Let $\mathcal{Z} = \bigcup_{j=1}^n \mathcal{Z}_j$. Define $L\colon \mathbb{C}^n \setminus \mathcal{Z} \to \mathbb{R}^n$ by

$$L(z_1, \dots , z_n) = (\log |z_1|, \dots , \log |z_n|)$$

and

$$L^*(z_1, \dots , z_n) = (\log z_1, \dots , \log z_n).$$

where, in the second case, the logarithm takes the principle branch, and L^* is defined locally near every point of $\mathbb{C}^n \setminus \mathcal{Z}$.

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Let $\Omega = \{z \in \mathbb{C}^n; \quad \rho(z) < 0\}$ be a bounded domain with smooth boundary. A boundary point $p \in b\Omega$ is of finite *variety type* m if

$$m = \sup \left\{ \frac{v(\rho \circ f)}{v(f)} \right\} < \infty \tag{1}$$

where the supremum is taken over all analytics disc $f \colon \Delta \to \mathbb{C}^n$ such that $f(0) = p$. We use $v(f)$ to denote the order of vanishing of f at 0 (see [K] for details on this notion). When the supremum in (1) is taken over all regular analytic discs f (*i.e.*, $f'(0) \neq 0$), then m is called the *regular type* of p; when the supremum in (1) is taken over all complex lines through p, then m is called the *line type* of p.

It has been proved by McNeal [M] that, for a boundary point of a smoothly bounded convex domain, the variety type is identical to the line type. This result was generalized by Boas and Straube [B-S] to a star-like boundary point (see Theorem 4 below for the exact statement). Meanwhile, J. Yu [Y] showed that the Catlin multi-type is identical to the D'Angelo q-type for a boundary point of a convex domain. In this paper, we shall prove that, for a boundary point of a pseudoconvex Reinhardt domain, the regular type is identical to the variety type.

Lemma 1. *(Reinhardt) If Ω is a pseudoconvex Reinhardt domain, then $L(\Omega \setminus \mathcal{Z})$ is convex. Furthermore, if $\Omega \cap \mathcal{Z}_j \neq \varnothing$, then $(z_1, \dots, z_{j-1}, \lambda z_j, z_{j+1}, z_n) \in \Omega$ whenever $(z_1, z_2, \dots, z_n) \in \Omega$ and $|\lambda| \leq 1$.*

Lemma 2. *Let Ω be a smoothly bounded pseudoconvex Reinhardt domain and let $p \in b\Omega$. Then there is a neighborhood U of p such that $b\Omega \cap U$ has a plurisubharmonic defining function $\rho(z)$.*

Proof. If $p \in \Omega \setminus \mathcal{Z}$, then $L(z)$ is well-defined in a neighborhood of p. Since $L(\Omega \setminus \mathcal{Z})$ is convex, there is a local convex defining function $\tilde{\rho}(u_1, \dots, u_n)$ of $L(\Omega \setminus \mathcal{Z})$ near $L(p)$. Let $\rho = \tilde{\rho}(L(z))$. Then ρ is a plurisubharmonic defining function of $b\Omega$ near p.

If $p \in \Omega \cap \mathcal{Z}$, then, without lose of generality, we may assume that

$$p = (p_1, \dots, p_k, 0, \dots, 0),$$

with $p_j \neq 0$, $0 \leq j \leq k$. Let $\sigma(z)$ be the Euclidean distance from z to $b\Omega$. Then $\sigma(z)$ is smooth in a neighborhood of $b\Omega$ and satisfy $\sigma(z_1, \dots, z_n) = \sigma(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$. It follows that $\frac{\partial\sigma}{\partial z_j}(z_1, \dots, z_n) = 0$ whenever $z_j = 0$. Since $\frac{\partial\sigma}{\partial z_j}(p) = 0$, $k+1 \leq j \leq n$, one of $\frac{\partial\sigma}{\partial z_j}(p)$, $1 \leq j \leq k$, must not be zero.

For simplicity of notation, we may assume that $\frac{\partial\sigma}{\partial z_1}(p) \neq 0$. Therefore $b\Omega$ is locally defined by an equation of the form $|z_1| + f(|z_2|, \dots, |z_n|) = 0$ near p. This in turn implies that $L(b\Omega \setminus \mathcal{Z})$ is defined by

$$u_1 + g(u_2, \dots, u_n) = 0$$

where

$$g(u_2, \ldots, u_n) = -\log(-f(e^{u_2}, e^{u_3}, \ldots, e^{u_n}))$$

for $(u_2, \ldots u_n) \in I_2 \times \cdots I_k \times I^{n-k}$. Here I_j is a neighborhood of $\log|p_j|$, $2 \leq j \leq k$, and $I = (-\infty, -C)$ ($C > 0$ is a large constant).

The convexity of $L(b\Omega \setminus \mathcal{Z})$ implies that $g(u_2, \ldots, u_n)$ is a convex function. Therefore

$$\begin{aligned}\rho(z_1, \ldots, z_n) &= \log|z_1| + g(\log|z_2|, \ldots, \log|z_n|)\\ &= \log|z_1| - \log(-f(|z_2|, \ldots, |z_n|))\end{aligned}$$

is a smooth plurisubharmonic defining function for $b\Omega$ near p. □

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formulate

For $z = (z_1, \ldots, z_n)$, let $\hat{z} = (z_2, \ldots, z_n)$.

Theorem 3([B-S]). *Suppose that a smooth hypersurface is defined by* $\operatorname{Re} z_1 + h(\hat{z}, \operatorname{Im} z_1) = 0$ *near the origin. If* $h(\hat{z}, 0) \geq 0$*, then the variety type of the hypersurface at the origin is obtained by analytic discs of form* $\phi(\zeta) = (0, \phi_2(\zeta), \ldots, \phi_n(\zeta))$*. Furthermore, if the hypersurface is star-like at the origin, i.e., there exists a* $\delta > 0$ *such that* $t \to h(a_2 t, \ldots, a_n t, 0)$ *is an increasing function on* $[0, \delta]$ *for all unit vectors* $(a_2, \ldots, a_n)$*, then the variety type at the origin is obtained by a complex line.*

Now we can prove the main theorem of this paper:

Theorem 4. *If* Ω *is a smoothly bounded pseudoconvex Reinhardt domain and if* $p \in b\Omega$*, then the variety type at* p *is identical to the regular type.*

Proof. It follows from the smoothness and the Reinhardt property of $b\Omega$ that p cannot be the origin (this can be easily seen from the proof of Lemma 2 in this paper or from Lemma 1.3 in [FIK]). We divide the proof into three cases.

If $p \in b\Omega \setminus \mathcal{Z}$, then by Lemma 1, $L^*(b\Omega \setminus \mathcal{Z})$ is convex near p. Therefore, it follows from the results of McNeal [Mc] and Boas/Straube [B-S] that the variety type of $L^*(b\Omega \setminus Z)$ at $L^*(p)$ is obtained by a complex line of the form $L^*(p) + a\zeta$. Thus, the variety type of $b\Omega$ at p is obtained by

$$\phi(\zeta) = (p_1 e^{a_1 \zeta}, \ldots, p_n e^{a_n \zeta}).$$

Therefore the variety type is identical to the regular type at p.

If $p \in b\Omega \cap \mathcal{Z}$, then the mapping L^* is not well-defined near p. Thus the above method fails. We first consider the case that only one coordinate of p is non-zero. We shall show that, in this case, the domain Ω is star-like at p. After a simple dilation and rotation, we may assume that $p = (1, 0, \ldots, 0)$ and $\operatorname{Re} z_1$ is the outward normal direction at p. It follows from the proof of Lemma 2 that $b\Omega$ is defined near p by

$$\log|z_1| + h(\hat{z}) = 0$$

where h is a smooth function defined near $\hat{p}$ such that

(a) $h(\hat{p}) = 0$ and $dh(\hat{p}) = 0$;
(b) $h(e^{i\theta_2} z_2, \ldots, e^{i\theta_n} z_n) = h(z_2, \ldots, z_n)$;
(c) The function h is plurisubharmonic. Furthermore,

$$h(z_2, \ldots, z_n) = g(\log|z_2|, \ldots, \log|z_n|)$$

for some convex function g and $z_i \neq 0$, $2 \leq i \leq n$.

For $a = (a_2, \ldots, a_n)$ with $\|a\| = 1$ and $a_i \neq 0$, $2 \leq i \leq n$, define

$$\begin{aligned} k_a(t) &= h(ta_2, \ldots, ta_n) \\ &= g(\log t + \log|a_2|, \ldots, \log t + \log|a_n|). \end{aligned}$$

It follows from the convexity of g that $\tilde{g}(s) = g(s + \log|a_2|, \ldots, s + \log|a_n|)$ is a convex function of s. By (a), we have

$$\lim_{s \to -\infty} \tilde{g}'(s) = 0.$$

Therefore, $\tilde{g}'(s) \geq 0$ for $s << -1$. It then follows that $k_a(t)$ is an increasing function of t. Thus, by the (proof of the) theorem of Boas/Straube, the variety type at p is obtained by a complex line of the form $\phi(\zeta) = (1, b_2\zeta, \ldots, b_n\zeta)$.

Now we treat the general situation for the case when $p \in b\Omega \cap \mathcal{Z}$. Without lost of generality, we may assume that the first k coordinates of p are 1's and the rest of the coordinates are 0's. Let $\sigma(z)$ be the Euclidean distance from z to $b\Omega$. It follows from the proof of Lemma 2 that one of $\frac{\partial\sigma}{\partial z_j}(p)$, $1 \leq j \leq k$ is not zero. For simplicity of notation, we assume that $\partial\sigma(p)/\partial z_1 \neq 0$. Let $z' = \Phi(z)$ be defined by

$$\begin{aligned} z_1' &= z_1^{\alpha_1} \cdots z_k^{\alpha_k}; \\ z_j' &= z_j, \quad 2 \leq j \leq n \end{aligned}$$

where $\alpha_j = \partial\sigma(p)/\partial z_j, 1 \leq j \leq k$.

It is obvious that $z' = \Phi(z)$ is biholomorphic in a neigborhood of p and $\operatorname{Re} z_1'$ is the outward normal direction to $b\Omega$ at p. This change of coordinates preserves the Reinhardt property of the domain near p. Changing coordinates if necessary, we may assume at the outset that $\operatorname{Re} z_1$ is the outward normal direction to $b\Omega$ at p. It follows from the proof of Lemma 2 that $b\Omega$ is defined near p by

$$\log|z_1| + h(z_2, \ldots, z_n) = 0,$$

where h is a function satisfying properties (a)-(c). By a similar argument, we can prove that, for fixed z_j, $2 \leq j \leq k$, near 1 and $a = (a_{k+1}, \ldots, a_n)$ with $\|a\| = 1$,

$$t \mapsto h(z_2, \ldots, z_k, ta)$$

is an increasing function of t for small $t > 0$. Thus we have

$$h(z_2, \ldots, z_n) \geq h(z_2, \ldots, z_k, 0, \ldots, 0). \tag{2}$$

Let $\rho(w_2, \dots, w_n) = h(e^{w_2}, \dots, e^{w_k}, w_{k+1}, \dots, w_n)$. Then $\rho(\hat{w})$ is a plurisubharmonic function. Furthermore, for fixed $(w_{k+1}, \dots, w_n)$, $\rho(\hat{w})$ is a convex function. Let $k(t) = \rho(tw_2, \dots, tw_k, 0, \dots, 0)$. Then $k(t)$ is a convex function with $k(0) = k'(0) = 0$. Therefore $k(1) = \rho(w_2, \dots, w_k, 0, \dots, 0) \geq 0$. Hence $h(\hat{z}) \geq 0$.

Claim. *The variety type of $b\Omega$ at p is obtained by analytic discs of the form $\phi(\zeta) = (1, \hat{\phi}(\zeta))$.*

The proof of the claim comes directly from the proof of the theorem of Boas and Straube [B-S]. We provide details here for the reader's convenience.

Let $r(z) = \log|z_1| + h(z_2, \dots, z_n)$. Let $\phi(\zeta) = (\phi_1(\zeta), \hat{\phi}(\zeta)) \colon \Delta \to \mathbb{C}^n$ be an analytic disc such that $\phi(0) = p$. We may assume that $v(r \circ \phi) > v(\phi)$. If $\phi_1(\zeta)$ is not identically zero, then there exists a sequence $\{\zeta_j\} \subset \Delta$, $\zeta_j \to 0$ such that $\{\phi_1(\zeta_j)\}$ is a sequence of real numbers and $\phi_1(\zeta_j) \to 1^+$. Therefore

$$\log|\phi_1(\zeta_j)| = |\log \phi_1(\zeta_j)| \leq r \circ \phi(\zeta_j).$$

Let $\psi(\zeta) = (1, \hat{\phi}(\zeta))$. Since the order of vanishing of a holomorphic function is determined by its order of vanishing along a sequence, we obtain that

$$v(\phi_1(\zeta) - 1) = v(\log|\phi_1(\zeta)|) \geq v(r \circ \phi).$$

This in turn implies that $v(\psi) = v(\phi)$. It then follows from the equality

$$|r \circ \psi(\zeta) - r \circ \phi(\zeta)| = |\log|\phi_1(\zeta)||$$

that $v(r \circ \psi) \geq v(r \circ \phi)$. This finishes the proof of the claim.

We now return to the proof of the main theorem. Let $\tilde{\phi}(\zeta) = (\phi_2(\zeta), \dots, \phi_k(\zeta))$ and let $m = v(h \circ \hat{\phi})$. It follows from (2) that

$$v(h(\tilde{\phi}(\zeta), 0, \dots, 0)) \geq m. \tag{3}$$

Let $\beta_j = v(\phi_j)$, $k+1 \leq j \leq n$ and $\psi_j(\zeta) = \phi_j(\zeta)/\zeta^{\beta_j}$. For $\delta > 0$, we have

$$\begin{aligned}
\delta^m &\gtrsim \frac{1}{2\pi\delta} \int_{|\zeta|=\delta} h \circ \hat{\phi}(\zeta) |d\zeta| \\
&= \frac{1}{2\pi\delta} \int_{|\zeta|=\delta} h(\tilde{\phi}(\zeta), \delta^{\beta_{k+1}} \psi_{k+1}(\zeta), \dots \delta^{\beta_n} \psi_n(\zeta)) |d\zeta| \\
&\geq h(1, \dots, 1, \delta^{\beta_{k+1}} \psi_{k+1}(0), \dots \delta^{\beta_n} \psi_n(0)).
\end{aligned}$$

Therefore,

$$v(h(1, \dots, 1, \zeta^{\beta_{k+1}} \psi_{k+1}(0), \dots \zeta^{\beta_n} \psi_n(0))) \geq m. \tag{4}$$

It follows from (3) and (4) that the variety type at p can be realized by analytic discs of the form

$$\phi(\zeta) = (1, \phi_2(\zeta), \dots, \phi_k(\zeta), 0, \dots, 0) \quad \text{or} \quad \phi(\zeta) = (1, \dots, 1, \phi_{k+1}(\zeta), \dots, \phi_n(\zeta)).$$

However, it follows from the first and second cases of this proof that the variety type can actually be realized by analytic discs of form

$$\phi(\zeta) = (1, e^{a_2\zeta}, \dots, e^{a_k\zeta}, 0, \dots, 0), \tag{5}$$

or

$$\phi(\zeta) = (1, \dots, 1, a_{k+1}\zeta, \dots, a_n\zeta). \tag{6}$$

Indeed, to obtain the special form (5), consider the section of Ω by $\mathcal{Z}_{k+1,\dots,n} = \bigcap_{j=k+1}^n \mathcal{Z}_j$. Then, by Lemma 1.3 of [FIK], this section is a nonempty smoothly bounded set which is a finite collection of Reinhardt domains in $\mathcal{Z}_{k+1,\dots,n}$. The point p then becomes a point with all nonzero components, and therefore the first case of the proof applies.

To obtain the special form (6), one needs to consider the section of Ω by a complex affine subspace of the form $\{z_{j_1} = \cdots = z_{j_{k-1}} = 1\}$, where $1 \le j_i \le k$ for all i. Then we can use the argument from the second case of the proof. $\square$

Remark 5. (1) One certainly cannot expect that the line type at p is equal to the variety type. For example, let Ω be a pseudoconvex Reinhardt domain in $\mathbb{C}^2$ such that $b\Omega$ is locally defined by

$$\rho(z_1, z_2) = \log|z_1| + \log|z_2| + (\log|z_1| - \log|z_2|)^4$$

for $z = (z_1, z_2)$ near $p = (1, 1)$. Then the line type at p is 2 while the regular type at p is 4.

(2) The result of Yu [Y] cannot be generalized to Reinhardt domains. Let Ω be a pseudoconvex Reinhardt domain defined by

$$\rho(z_1, z_2, z_3) = |z_1|^2 + |z_2|^6 + |z_3|^6 + |z_2 z_3|^2 - 1.$$

Then the 3-tuple of the D'Angelo q-types at $p = (1, 0, 0)$ is $(\Delta_3, \Delta_2, \Delta_1) = (1, 4, 6)$ while the Catlin multi-type is $(1, 4, 4)$. We thank Professor Yu for pointing out this fact.

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