ON COMBINATORIAL CALCULATIONS FOR THE JAMES–HOPF MAPS

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Abstract. We give some formulas of the James-Hopf maps by using combinatorial methods. An application is to give a product decomposition of the spaces $\Omega\Sigma^2(X)$.

1. Introduction

In this paper, we give some formulas of the James-Hopf maps by using combinatorial methods.

Let X be a pointed space. The James-Hopf map $H_k : J(X) \to J(X^{(k)})$ is defined by setting

$$H_k(x_1x_2\cdots x_n) = \prod_{1\le i_1<i_2<\cdots<i_k\le n} (x_{i_1}x_{i_2}\cdots x_{i_k})$$

with left lexicographical order in the product, where $X^{(k)}$ is the k-fold self smash product of X and $J(X)$ is the James construction of X. The first result is as follows.

Theorem 1.1. *Let $X = \Sigma X'$ be a suspension of a CW-complex X'. Then the composite*

$$J(\vee_{l\ge k}X^{(l)}) \overset{\widetilde{\vee_{l\ge k}S_l}}{\to} JX \overset{H_k}{\to} JX^{(k)}$$

is a loop map, where $S_l = [[E,\cdots,E] : X^{(l)} \to JX$ is the l-fold Samelson product and $\widetilde{\vee_{l\ge k}S_l} : J(\vee_{l\ge k}X^{(l)}) \to JX$ is the homomorphism of topological monoids induced by $\vee_{l\ge k}S_l$.

Corollary 1.2. *Let $X = \Sigma X'$ be a suspension of a CW-complex X'. Then*

$$JX^{(l)} \overset{\tilde{S}_l}{\to} JX \overset{H_k}{\to} JX^{(k)}$$

is a loop map for $l \ge k$.

The second calculation is to give a decomposition of the compositions of James-Hopf maps.

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Notation 1.3. *The map* $L_{k.l} : X^{(kl)} \to JX^{(kl)}$ *is defined by*

$$L_{k,l}(x_1 \wedge x_2 \wedge \cdots \wedge x_k l) =$$

$$\prod_{1 \le j_1 < \cdots < j_l = kl, 1 \le l_1^{j_s} < \cdots < l_k^{j_s} = j_s} (x_{l_1^{j_1}} \wedge \cdots \wedge x_{l_k^{j_1}} \wedge \cdots \wedge x_{l_1^{j_s}} \wedge \cdots \wedge x_{l_k^{j_l}})$$

where $(l_1^{j_1}, \cdots, l_k^{j_1}, \cdots, l_1^{j_l}, \cdots, l_k^{j_l})$ *runs over shuffles of* $(1, \cdots, kl)$ *with left lexicographical oder.*

Let $\tilde{L}_{k,l} : JX^{(kl)} \to J^{(kl)}$ *denote the homomorphism of topological monoids induced by* $L_{k,l}$.

Proposition 1.4. *Let* $X = \Sigma X'$ *be a suspension of a CW-complex* X'. *Then*

$$H_l \circ H_k \simeq \tilde{L}_{k,l} \circ H_{kl} : JX \to JX^{(kl)}$$

An application is to give a product decomposition of $\Omega\Sigma X$ if X is a suspension.

Notation 1.5. *Let* $X = \Sigma X'$. *The map* $\beta_n : X^{(n)} \to X^{(n)}$ *is defined by induction* $\beta_n = \beta_{n-1} \wedge 1 - (1, 2, \ldots, n) \circ (\beta_{n-1} \wedge 1)$ *and* $\beta_2 = id - (1, 2)$, *where* $(1, 2, \ldots, n) \in \Sigma_n$ *acts on* $X^{(n)}$*by permutating coordinates. Let* p *be a prime. Denote* $L_n(X) = hocolim_{\frac{1}{n}\beta_n} X^{(n)}$ *localized at* p *if* $n \not\equiv 0$ *(mod* p*).*

Notice that the mod p homology of the spaces $L_n(X)$ are represented by the Lie elements of weight n in the tensor algebra $T(\overline{H}_*(X, Z/pZ)$.

Theorem 1.6. *Let* $X = \Sigma X'$ *be a suspension of a CW-complex* X'. *and let* $1 < k_1 < k_2 < \cdots$ *be a sequence of integers so that*
(1) $k_j \not\equiv 0(p)$ *for each* $j \ge 1$
(2) k_j *is not a multiple of any* k_i *else for each* j.
Then

$$JX \simeq \prod_j J(L_{k_j}(X)) \times ?$$

localized at the prime p.

The article is organized as follows. In Section 2, we introduce the groups $K_n(k)(X)$. The combinatorial calculations and the proofs of Theorem 1.1 and Proposition 1.4 are given in Section 3. The proof of Theorem 1.6 is given in Section 4.

2. The groups $K_n(k)(X)$

In this section, we consider certain subgroup in the group $[X^n, J(X^{(k)}]$, where X^n is the n-fold self Cartesion product of X and $X^{(k)}$ is the K-fold self smash product of X. When $k = 1$, these groups have been studied by Fred Cohen [C3].

Notation 2.1. *Let $X = \Sigma X'$ be the suspension of a space. Denote $K_n(k)(X)$ the subgroup of $[X^n, JX^{(k)}]$ generated by the homotopy classes of the maps*

$$\pi_{i_1\cdots i_k} : X^n \to JX^{(k)}$$

with $\pi_{i_1\cdots i_k}(x_1, \cdots, x_n) = E(x_{i_1} \wedge \cdots \wedge x_{i_k})$, where X^n is the n-fold self Cartesian product of X, $X^{(k)}$ is the k-fold self smash product of X and $E : X^{(k)} \to JX^{(k)}$ is the suspension. If there is no confusion, we simply denote $K_n(k)$ for $K_n(k)(X)$ and denote K_n or $K_n(X)$ for $K_n(1)(X)$. Denote $\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\}$ for the homotopy class of $\pi_{i_1\cdots i_k}$ and denote $\{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_k}^{[n_k]}\}$ for the homotopy class of the map $f : X^n \to JX^{(k)}$ with $f(x_1, \cdots, x_n) = E([n_1](x_{i_1}) \wedge [n_2](x_{i_2}) \wedge \cdots \wedge [n_k](x_{i_k}))$, where $[n] : X \to X$ is the composite

$$X \xrightarrow{\psi} \bigvee^n X \xrightarrow{\bigtriangledown} X,$$

where ψ is the comultiplication and $\bigtriangledown$ is the fold map.

Lemma 2.2. *Let $X = \Sigma X'$. Then, in the group $[X^n, JX^{(k)}]$, the following identities hold:*
(1)

$$\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\} = 1$$

if $i_s = i_t$ for some $1 \le s < t \le k$
(2)

$$\{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_k}^{[n_k]}\} = \{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\}^{n_1 n_2\cdots n_k}$$

(3)

$$[[\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\}, \{x_{i_{k+1}}|x_{i_{k+2}}|\cdots|x_{i_{2k}}\}, \cdots, \{x_{i_{(l-1)k+1}}|x_{i_{(l-1)k+2}}|\cdots|x_{i_{lk}}\}] = 1$$

if $i_s = i_t$ for some $1 \le s < t \le kl$, where $[[a_1, a_2, \cdots, a_l] = [\cdots[a_1, a_2], \cdots,], a_l]$ with $[x, y] = x^{-1}y^{-1}xy$.
(4)

$$[[\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\}^{n_1}, \{x_{i_{k+1}}|x_{i_{k+2}}|\cdots|x_{i_{2k}}\}^{n_2}, \cdots, \{x_{i_{(l-1)k+1}}|x_{i_{(l-1)k+2}}|\cdots|x_{i_{lk}}\}^{n_l}] =$$

$$[[\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\}, \{x_{i_{k+1}}|x_{i_{k+2}}|\cdots|x_{i_{2k}}\}, \cdots, \{x_{i_{(l-1)k+1}}|x_{i_{(l-1)k+2}}|\cdots|x_{i_{lk}}\}]^{n_1 n_2\cdots n_l}$$

Proof: (1) By a shuffle map, we may assume that $i_1 = i_2$. Notice that $\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\}$ is represented by the composite

$$X^n \stackrel{p_{i_1\cdots i_k}}{\to} X^{(k)} \stackrel{E}{\to} JX^{(k)}$$

where $p_{i_1\cdots i_k}(x_1,\cdots,x_n) = x_{i_1}\wedge x_{i_2}\wedge\cdots\wedge x_{i_k}$. Since $i_1 = i_2$, there is a commutative diagram

$$\begin{array}{ccccc} X^n & \stackrel{p_{i_1 i_2\cdots i_k}}{\to} & X^{(k)} & \stackrel{E}{\to} & JX^{(k)} \\ \| & & \bar\Delta\uparrow & & \\ X^n & \stackrel{p_{i_1 i_3\cdots i_k}}{\to} & X^{(k-1)} & & \end{array}$$

where $\bar\Delta : X \to X\wedge X$ is the reduced diagonal map. Since $X = \Sigma X'$ is a coH-space, $\bar\Delta$ is null. Thus (1) follows.

(2) The element $\{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_k}^{[n_k]}\}$ is represented by the composite

$$X^n \stackrel{p_{i_1\cdots i_k}}{\to} X^{(k)} \stackrel{[n_1]\wedge\cdots\wedge[n_k]}{\to} X^{(k)} \stackrel{E}{\to} JX^{(k)}$$

The assertion follows from the following homotopy commutative diagram

$$\begin{array}{ccc} X^{(k)} & \stackrel{E}{\to} & JX^{(k)} \\ \phi_1\uparrow & & \uparrow\phi_2 \\ X^{(k)} & \stackrel{E}{\to} & JX^{(k)}, \end{array}$$

where $m : JX^{(k)} \to JX^{(k)}$ is the power map of degree m, $\phi_1 = [n_1]\wedge\cdots\wedge[n_k]$ and $\phi_2 = n_1\cdots n_k$.

(3) The element $[[\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\},\cdots,\{x_{i_{(l-1)k+1}}|x_{i_{(l-1)k+2}}|\cdots|x_{i_{lk}}\}]$ is represented by the composite

$$X^n \stackrel{q}{\to} (X^{(k)})^{(l)} \stackrel{\phi}{\to} JX^{(k)}$$

where $\phi = [[E,E],\cdots,E]$ and

$q(x_1,x_2,\cdots,x_n) = (x_{i_1}\wedge\cdots\wedge x_{i_k})\wedge(x_{i_{k+1}}\wedge\cdots\wedge x_{i_{2k}})\wedge\cdots\wedge(x_{i_{(l-1)k+1}}\wedge\cdots\wedge x_{i_{lk}})$.

Since $i_s = i_t$ for some $1\le s<t\le kl$, there is a commutative diagram

$$\begin{array}{ccc} X^n & \stackrel{q}{\to} & X^{(kl)} \\ \| & & \uparrow\bar\Delta \\ X^n & \stackrel{q'}{\to} & X^{(kl-1)} \end{array}$$

for a choice of reduced diagonal map $\bar\Delta$. Thus (3) follows.

(4) The element $[[\{x_{i_1}|x_{i_2}|\cdots|x_{i_k}\}^{[n_1]},\cdots,\{x_{i_{(l-1)k+1}}|x_{i_{(l-1)k+2}}|\cdots|x_{i_{lk}}\}^{[n_l]}]$ is represented by the composite

$$X^n \xrightarrow{q} (X^{(k)})^{(l)} \xrightarrow{E^{(l)}} (JX^{(k)})^{(l)} \xrightarrow{n_1\wedge\cdots\wedge n_l} (JX^{(k)})^{(l)} \xrightarrow{[[id,\cdots,id]} JX^{(k)}$$

where $q(x_1,\cdots,x_n) = x_{i_1}\wedge\cdots\wedge x_{i_{kl}}$. the assertion follows from the homotopy commutative diagram

$$\begin{array}{ccccc}
(JX^{(k)})^{(l)} & \xrightarrow{n_1\wedge\cdots\wedge n_l} & (JX^{(k)})^{(l)} & \xrightarrow{[[id,\cdots,id]} & JX^{(k)} \\
\psi\uparrow & & \uparrow\psi & & \| \\
(X^{(k)})^{(l)} & \xrightarrow{[n_1]\wedge\cdots\wedge[n_l]} & (X^{(k)})^{(l)} & \xrightarrow{\phi} & JX^{(k)} \\
\phi\downarrow & & & & \| \\
JX^{(k)} & \xrightarrow{=} & JX^{(k)} & \xrightarrow{n_1n_3\cdots n_l} & JX^{(k)},
\end{array}$$

where $\phi = [[E,E],\cdots,E]$ and $\psi = (E)^{(l)}$.

Lemma 2.3. *Let $X = \Sigma X'$ and let $H_k : JX \to JX^{(k)}$ be the k-th James-Hopf map. Let $(H_k)_* : [X^n, JX] \to [X^n, JX^{(k)}]$ be the function induced by H_k. Then, for $y = x_{i_1}^{n_1}x_{i_2}^{n_2}\cdots x_{i_l}^{n_l} \in K_n$,*

$$\begin{aligned}
(H_k)_*(y) &= \prod_{1\le j_1<\cdots<j_k\le l} \{x_{i_{j_1}}|x_{i_{j_2}}|\cdots|x_{i_{j_k}}\}^{n_{j_1}n_{j_2}\cdots n_{j_k}} \\
&= \prod_{1\le j_1<\cdots<j_k\le l} \{x_{i_{j_1}}^{[n_{j_1}]}|x_{i_{j_2}}^{[n_{j_2}]}|\cdots|x_{i_{j_k}}^{[n_{j_k}]}\}
\end{aligned}$$

in $K_n(k)$ with left lexicographical order.

Proof: The element y is represented by the composite

$$X^n \xrightarrow{q} X^l \xrightarrow{\prod_{j=1}^l [n_j]} X^l \xrightarrow{p} JX,$$

where p is the projection and $q(x_1,\cdots,x_n) = (x_{i_1},\cdots,x_{i_l})$. The assertion follows from the definition of H_k and the above lemma.

3. Combinatorial Calculations

In this section, we give some combinatorial calculations and give proofs of Theorem 1.1 and Proposition 1.4, where Theorem 1.1 is Theorem 3.10 and Proposition 1.4 is Proposition 3.14.

Definition 3.1. *The tensor product $\otimes : [Z, JX]\times[Z, JY] \to [Z, J(X\wedge Y)]$ is defined as follows:*
$[f]\otimes[g]$ is represented by the composite

$$Z \xrightarrow{\bar{\Delta}} Z\wedge Z \xrightarrow{f\wedge g} JX\wedge JY \xrightarrow{c} J(X\wedge Y)$$

for $f : Z \to JX$ and $g : Z \to JY$, where $c((x_1 \cdots x_n) \wedge (y_1 \cdots y_m)) = (x_1 \wedge y_1) \cdot (x_2 \wedge y_1) \cdots (x_n \wedge y_1) \cdot (x_1 \wedge y_2) \cdot (x_2 \wedge y_2) \cdots (x_n \wedge y_2) \cdots (x_1 \wedge y_m) \cdot (x_2 \wedge y_m) \cdots (x_n \wedge y_m)$

Remark 3.2. *The tensor product was introduced by F.Cohen[C2]*

Lemma 3.3. *Let $X = \Sigma X'$ and let $\otimes : [X^n, JX^{(k)}] \times [X^n, JX^{(l)}] \to [X^n, JX^{(k+l)}]$ be the tensor product. Then*

$$- \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{[n_l]}\} : K_n(k) \to K_n(k+l)$$

is a group homomorphism. Furthermore,

$$ab \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{[n_l]}\} = ba \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{[n_l]}\}$$

i.e. $- \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{n_l]}\}$ factors though the abelianization $K_n(k)/[K_n(k), K_n(k)]$.

Proof: Let $a = \{x_{j_1}|\cdots|x_{j_k}\}$ be a generator in $K_n(k)$. Then

$$\begin{aligned} a^{-1} \otimes \{x_{i_1}^{[n_1]}\cdots x_{i_l}^{[n_l]}\} &= \{x_{j_1}^{[-1]}|x_{j_2}|\cdots|x_{j_k}\} \otimes \{x_{i_1}^{[n_1]}\cdots x_{i_l}^{[n_l]}\} \\ &= \{x_{j_1}^{[-1]}|x_{j_2}|\cdots|x_{j_k}|x_{i_1}^{[n_1]}|\cdots|x_{i_l}^{[n_l]}\} \\ &= (\{x_{j_1}|x_{j_2}|\cdots|x_{j_k}\} \otimes \{x_{i_1}^{[n_1]}|\cdots x_{i_l}^{[n_l]}\})^{-1} \end{aligned}$$

Let $a_1, a_2, \cdots, a_s$ be sequence of generators in $K_n(k)$. Then

$$\begin{aligned} (a_1 \cdots a_s) \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{[n_l]}\} &= (a_1 \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{[n_l]}\}) \cdots \\ &\quad (a_s \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{[n_l]}\}). \end{aligned}$$

Thus $- \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{[n_l]}\}$ is a group homomorphism. By the Lemma 2.2,

$$ab \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{[n_l]}\} = ba \otimes \{x_{i_1}^{[n_1]}|x_{i_2}^{[n_2]}|\cdots|x_{i_l}^{[n_l]}\}$$

The assertion follows.

Lemma 3.4. *Let $X = \Sigma X'$ and let $(H_k)_* : [X^n, JX] \to [X^n, JX^{(k)}]$ be the function induced by $H_k : JX \to JX^{(k)}$. Then, for $a = x_{i_1}^{n_1} \cdots x_{i_p}^{n_p}$ and $y = x_{i_{p+1}}^{n_{p+1}} \cdots x_{i_{p+q}}^{n_{p+q}}$ in K_n,*

$$(H_k)_*(a \cdot y) = (H_k)_*(a) \prod_{j=1}^{q} (\prod_{1 \le l_1 < \cdots < l_{s-1} < l_s = j, 1 \le s \le k} (H_{k-s})_*(a) \otimes \{x_{i_{p+l_1}}^{[n_{p+l_1}]}|\cdots|x_{i_{p+l_s}}^{[n_{p+l_s}]}\}),$$

where the order of $\prod_{1 \le l_1 < \cdots < l_{s-1} < l_s = j, 1 \le s \le k}$ can be chosen to any order.

Proof: By induction on q. If $q = 0$, the assertion is trivial. Suppose that the assertion holds for $q - 1$ and $y = x_{i_{p+1}}^{n_{p+1}} \cdots x_{i_{p+q}}^{n_{p+q}}$. Let b denote $a \cdot x_{i_{p+1}}^{n_{p+1}} \cdots x_{i_{p+q-1}}^{n_{p+q-1}}$. By lemma 2.3,

$$(H_k)_*(b \cdot x_{i_{p+q}}^{n_{p+q}}) = (H_k)_*(b) \cdot \prod_{1 \le l_1 < \cdots < l_{k-1} \le p+q-1} \{x_{i_{l_1}}^{[n_{l_1}]}|\cdots|x_{i_{l_{k-1}}}^{[n_{l_{k-1}}]}|x_{i_{p+q}}^{[n_{p+q}]}\}).$$

Let

$$w=\prod_{1\le l_1<\cdots<l_{k-1}\le p+q-1}\{x_{i_{l_1}}^{[n_{l_1}]}|\cdots|x_{i_{l_{k-1}}}^{[n_{l_{k-1}}]}|x_{i_{p+q}}^{[n_{p+q}]}\}).$$

By Lemma 2.2, the elements in the product w commute each other. Thus

$$\begin{aligned} w &= (H_{k-1})_*(b)\otimes\{x_{i_{p+q}}^{[n_{p+q}]}\}\\ &= (H_{k-1})_*(a)\otimes\{x_{i_{p+q}}^{[n_{p+q}]}\}+\\ \sum_{j=1}^{q-1}\sum_{1\le l_1<\cdots<l_{s-1}<l_s=j,1\le s\le k-1}&(H_{k-s-1})_*(a)\otimes\{x_{i_{p+l_1}}^{[n_{p+l_1}]}|\cdots|x_{i_{p+l_s}}^{[n_{p+l_s}]}\}\otimes\{x_{i_{p+q}}^{[n_{p+q}]}\}\\ &= \sum_{1\le l_1<\cdots<l_s=q,1\le s\le k}(H_{k-s})_*(a)\otimes\{x_{i_{p+l_1}}^{[n_{p+l_1}]}|\cdots|x_{i_{p+l_s}}^{[n_{p+l_s}]}\}. \end{aligned}$$

The assertion follows.

Lemma 3.5. *Let $X=\Sigma X'$ and let $(H_k)_*:[X^n,JX]\to[X^n,JX^{(k)}]$ be induced by $H_k:JX\to JX^{(k)}$. Then*

$$(H_k)_*([[x_{i_1}^{n_1},\cdots,x_{i_m}^{n_m}])\otimes x_j=1$$

in $[X^n,JX^{(k+1)}]$ for $m>k\ge1$.

Proof: By induction on k. If $k=1$, H_1 is the identity and the assertion follows from Lemma 3.3. Suppose that the assertion holds for $<k$ and consider $(H_k)_*([[x_{i_1}^{n_1},\cdots,x_{i_m}^{n_m}])\otimes x_j$ with $m>k$. Let $<\tilde{x_j}>$ denote the subgroup of $K_n(k+1)$ generated by all of the elements $\{x_{a_1}|\cdots|x_{a_{k+1}}\}$ so that $a_{k+1}=j$. Then, by Lemma 2.2, $<\tilde{x_j}>$ is abelian. Thus the product is the sum in $<\tilde{x_j}>$. Let $z_j=x_{i_j}^{n_j}$. Then

$$\begin{aligned} (H_k)_*([[z_1,\cdots,z_m])\otimes x_j &= (H_k)_*([[z_1,\cdots,z_{m-1}]^{-1}\cdot z_m^{-1}\cdot[[z_1,\cdots,z_{m-1}]\cdot z_m)\otimes x_j\\ &= \sum_{s+t=k}(H_s)_*([[z_1,\cdots,z_{m-1}]^{-1})\otimes(H_t)_*([[z_1,\cdots,z_{m-1}])\otimes x_j\\ &+\sum_{s+t=k-1}(H_s)_*([[z_1,\cdots,z_{m-1}]^{-1})\otimes z_m^{-1}\otimes(H_t)_*([[z_1,\cdots,z_{m-1}])\otimes x_j\\ &+\sum_{s+t=k-1}(H_s)_*([[z_1,\cdots,z_{m-1}]^{-1})\otimes(H_t)_*([[z_1,\cdots,z_{m-1}])\otimes z_m\otimes x_j\\ &+\sum_{s+t=k-2}(H_s)_*([[z_1,\cdots z_{m-1}]^{-1})\otimes z_m^{-1}\otimes(H_t)_*([[z_1,\cdots,z_{m-1}])\otimes z_m\otimes x_j. \end{aligned}$$

By Lemma 2.2,

$$(H_s)_*([[z_1,\cdots z_{m-1}]^{-1})\otimes z_m^{-1}\otimes(H_t)_*([[z_1,\cdots,z_{m-1}])\otimes z_m\otimes x_j=1.$$

By induction, for $1\le t\le k-1$,

$$(H_s)_*([[z_1,\cdots,z_{m-1}]^{-1})\otimes(H_t)_*([[z_1,\cdots,z_{m-1}])\otimes x_j=1,$$

$$(H_s)_*([[z_1,\cdots,z_{m-1}]^{-1})\otimes z_m^{-1}\otimes (H_t)_*([[z_1,\cdots,z_{m-1}])\otimes x_j=1,$$

and

$$(H_s)_*([[z_1,\cdots,z_{m-1}]^{-1})\otimes (H_t)_*([[z_1,\cdots,z_{m-1}])\otimes z_m\otimes x_j=1$$

Also, by induction, we have

$$(H_{k-1})_*([[z_1,\cdots,z_{m-1}]^{-1})\otimes z_m^{-1}\otimes x_j=1$$

and

$$(H_{k-1})_*([[z_1,\cdots,z_{m-1}]^{-1})\otimes z_m\otimes x_j=1.$$

Thus

$$\begin{aligned}(H_k)_*([[z_1,\cdots,z_m])\otimes x_j &= (H_k)_*([[z_1,\cdots,z_{m-1}]^{-1})\otimes x_j+(H_k)_*([[z_1,\cdots,z_{m-1}])\otimes x_j\\ &= (H_k)_*([[z_1^{-1},\cdots,z_{m-1}])\otimes x_j+(H_k)_*([[z_1,\cdots,z_{m-1}])\otimes x_j\end{aligned}$$

with $m-1\geq k$. Now $(H_k)_*[[z_1,\cdots,z_{m-1}]$ is represented by

$$X^n \xrightarrow{q} X^{(m-1)} \xrightarrow{[n_1]\wedge\cdots\wedge[n_{m-1}]} X^{(m-1)} \xrightarrow{[[E,\cdots,E]} JX \xrightarrow{H_k} JX^{(k)}$$

and $(H_k)_*[[z_1^{-1},\cdots,z_{m-1}]$ is represented by

$$X^n \xrightarrow{q} X^{(m-1)} \xrightarrow{[-n_1]\wedge[n_2]\wedge\cdots\wedge[n_{m-1}]} X^{(m-1)} \xrightarrow{[[E,\cdots,E]} JX \xrightarrow{H_k} JX^{(k)}$$

where $q(x_1,x_2,\cdots,x_n)=x_{i_1}\wedge\cdots\wedge x_{i_{m-1}}$. Thus

$$(H_k)_*[[z_1^{-1},\cdots,z_{m-1}]=-(H_k)_*[[z_1,\cdots,z_{m-1}]$$

and the assertion follows.

Remark 3.6. *This lemma does not hold for $m\leq k$, e.g. $(H_1)_*(x_1)\otimes x_2=\{x_1|x_2\}$ and $(H_2)_*([x_1,x_2])\otimes x_3=\{x_1|x_2|x_3\}\cdot\{x_2|x_1|x_3\}^{-1}$.*

Definition 3.7. *Let G be a group. The **lower central series** Γ^kG is defined by induction*

$$\Gamma^1G=G$$

and

$$\Gamma^kG=[\Gamma^{k-1}G,G]$$

for $k\geq 2$.

Theorem 3.8. *Let $X=\Sigma X'$ and let $(H_k)_*\colon[X^n,JX]\to[X^n,JX^{(k)}]$ be induced by $H_k\colon JX\to JX^{(k)}$. Then*

$$(H_k)_*(a\cdot y)=(H_k)_*(a)\cdot(H_k)_*(y)$$

for $a\in\Gamma^kK_n$ and $y\in K_n$, i.e. $(H_k)_*$ restricted to K_n is a Γ^kK_n-map.

Proof: It suffices to show that

$$(H_k)_*([[x_{i_1}^{n_1},\cdots,x_{i_m}^{n_m}]\cdot y) = (H_k)_*[[x_{i_1}^{n_1},\cdots,x_{i_m}^{n_m}]\cdot (H_k)_*(y)$$

for $m\geq k$. Let $y=x_{j_1}^{l_1}\cdots x_{j_t}^{n_t}$. Let $z_j=x_{i_j}^{n_j}$ and $y_s=x_{j_s}^{l_s}$. By Lemma 3.4,

$$(H_k)_*([[z_1,\cdots,z_m]\cdot y_1\cdots y_t) =$$

$$(H_k)_*([[z_1,\cdots,z_m)\prod_{j=1}^{t}(\prod_{1\leq i_1<\cdots<i_{s-1}<i_s=j,1\leq s\leq k}(H_{k-s})_*([[z_1,\cdots,z_m])\otimes\{y_{i_1}|\cdots|y_{i_s}\}).$$

By Lemma 3.5,

$$(H_{k-s}([[z_1,\cdots,z_m])\otimes\{y_{i_1}|\cdots|y_{i_s}\}=1$$

for $1\leq s\leq k-1$. Thus

$$(H_k)_*([[z_1,\cdots,z_m]\cdot y_1\cdots y_t) = (H_k)_*([[z_1,\cdots,z_m)\prod_{j=1}^{t}(\prod_{1\leq i_1<\cdots<i_{k-1}<i_k=j}\{y_{i_1}|\cdots|y_{i_k}\}).$$

$$= (H_k)_*([[z_1,\cdots,z_m])(H_k)_*(y_1\cdots y_t).$$

The assertion follows.

Lemma 3.9. *(see also [B1]) Let X be a path-connected CW-complex and let $f,g: JX\to\Omega Y$ so that $f|_{J_nX}\simeq g|_{J_nX}$ for each n. Then $f\simeq g$.*

Proof: There is a homotopy equivalence $\Phi:\Sigma\vee_{n=1}^{\infty}X^{(n)}\to\Sigma JX$ so that $\Phi_n=\Phi|_{\Sigma\vee_{j=1}^{n}X^{(j)}}:\Sigma\vee_{j=1}^{n}X^{(j)}\to\Sigma J_nX$ are homotopy equivalences. Denote $f',g':\Sigma JX\to Y$ the adjoints of f and g, respectively. Let $i_j:\Sigma X^{(j)}\to\Sigma\vee_{n=1}^{\infty}X^{(n)}$ be the canonical inclusion. Since $f|_{J_nX}\simeq g|_{J_nX}$ for each n, $f'|_{\Sigma J_n}\circ\Phi_n\simeq g'|_{\Sigma J_n}\circ\Phi_n$ for each n and therefore $f'\circ\Phi\circ i_j\simeq g'\circ\Phi\circ i_j$ for each j. Let $F_j:\Sigma X^{(j)}\wedge I^+\to Y$ be a homotopy between $f'\circ\Phi\circ i_j$ and $g'\circ\Phi\circ i_j$. Then $F=\vee_{j=1}^{\infty}F_j:(\Sigma\vee_{j=1}^{\infty}X^{(j)})\wedge I^+\to Y$ is a homotopy between $f'\circ\Phi$ and $g'\circ\Phi$. The assertion follows.

Theorem 3.10. *Let $X=\Sigma X'$ be a suspension of a CW-complex X'. Then the composite*

$$J(\vee_{l\geq k}X^{(l)})\overset{\tilde{\vee_{l\geq k}S_l}}{\to}JX\overset{H_k}{\to}JX^{(k)}$$

is a loop map, where $S_l=[[E,\cdots,E]:X^{(l)}\to JX$ is the l-fold Samelson product and $\tilde{\vee_{l\geq k}S_l}:J(\vee_{l\geq k}X^{(l)})\to JX$ is the homomorphism of topological monoids induced by $\vee_{l\geq k}S_l$.

Proof: Let $\tilde{\beta}:J(\vee_{l\geq k}X^{(l)})\to JX^{(k)}$ be the homomorphism of topological monoids induced by the composite

$$\vee_{l\geq k}X^{(l)}\overset{\vee_{l\geq k}S_l}{\to}JX\overset{H_k}{\to}JX^{(k)}.$$

By Lemma 3.9, it suffices to show that

$$\tilde{\beta}|_{J_n(\vee_{l\geq k}X^{(l)})}\simeq H_k\circ \tilde{\vee_{l\geq k}S_l}|_{J_n(\vee_{l\geq k}X^{(l)})}$$

for each n. Let $j_n : J_n(\vee_{l\geq k}X^{(l)}) \to J(\vee_{l\geq k}X^{(l)})$ denote the inclusion and $p_n : (\vee_{l\geq k}X^{(l)})^n \to J_n(\vee_{l\geq k}X^{(l)})$ the projection. Notice that

$$p_n^* : [J_n(\vee_{l\geq k}X^{(l)}), \Omega Y] \to [(\vee_{l\geq k}X^{(l)})^n, \Omega Y]$$

is a monomorphism for any space Y. It suffice to show that

$$\tilde{\beta}\circ j_n\circ p_n \simeq H_k\circ \tilde{\vee_{l\geq k}S_l}\circ j_n\circ p_n.$$

Now, by the splitting theorem for $\Sigma(\vee_{l\geq k}X^{(l)})^n$, it suffices to show that

$$\tilde{\beta}\circ j_n\circ p_n\circ\Phi_{l_1\cdots l_n} \simeq H_k\circ \tilde{\vee_{l\geq k}S_l}\circ j_n\circ p_n\circ\Phi_{l_1\cdots l_n}$$

for all $l_1, l_2, \cdots, l_n \geq k$, where $\Phi_{l_1\cdots l_n} : X^{(l_1)}\times\cdots\times X^{(l_n)} \to (\vee_{l\geq k}X^{(l)})^n$ is the canonical inclusion. Let $\tilde{K}_n$ be the subgroup of $[X^{l_1+l_2+\cdots+l_n}, J(\vee_{l\geq k}X^{(l)})]$ generated by $\tilde{x}_1, \cdots, \tilde{x}_n$, where $\tilde{x}_i$ is represented by the composite

$$X^{l_1+\cdots+l_n} \xrightarrow{q} \prod_{j=1}^n X^{(l_j)} \xrightarrow{q} X^{(l_i)} \overset{i}{\hookrightarrow} (\vee_{l\geq k}X^{(l)})^n \xrightarrow{j_n\circ p_n} J(\vee_{l\geq k}X^{(l)}),$$

where q is the projection and i is the injection. Consider the monomorphism

$$q* : [X^{(l_1)}\times\cdots\times X^{(l_n)}, J(\vee_{l\geq k}X^{(l)})] \to [X^{l_1+\cdots+l_n}, J(\vee_{l\geq k}X^{(l)})].$$

Then

$$q*([j_n\circ p_n\circ\Phi_{l_1\cdots l_n}]) = \tilde{x}_1\cdot\tilde{x}_2\cdots\tilde{x}_n,$$

where $[f]$ is the homotopy class of f. Now consider the homomorphism

$$\tilde{\vee_{l\geq k}S_{l}}_* : [X^{l_1+\cdots+l_n}, J(\vee_{l\geq k}X^{(l)})] \to [X^{l_1+\cdots+l_n}, JX].$$

Then

$$\tilde{\vee_{l\geq k}S_{l}}_*(\tilde{x}_i) = [[x_{l_1+\cdots+l_{i-1}+1}, \cdots, x_{l_1+l_2+\cdots+l_i}]$$

By Theorem 3.8,

$$(H_k)_*\circ\tilde{\vee_{l\geq k}S_{l}}_*(\tilde{x}_1\cdots\tilde{x}_n) = \prod_{i=1}^n (H_k)_*\circ\tilde{\vee_{l\geq k}S_{l}}_*(\tilde{x}_i) = \tilde{\beta}_*(\tilde{x}_1\tilde{x}_2\cdots\tilde{x}_n).$$

Thus

$$H_k\circ\tilde{\vee_{l\geq k}S_l}\circ j_n\circ p_n\circ\Phi_{l_1\cdots l_n}\circ q \simeq \tilde{\beta}\circ j_n\circ p_n\circ\Phi_{l_1\cdots l_n}\circ q$$

and therefore

$$H_k\circ\tilde{\vee_{l\geq k}S_l}\circ j_n\circ p_n\circ\Phi_{l_1\cdots l_n} \simeq \tilde{\beta}\circ j_n\circ p_n\circ\Phi_{l_1\cdots l_n}.$$

The assertion follows.

Remark 3.11. *When $k = 2$, this is Lemma 2.5 in [C2]. The homology view of this statement was given in [CT].*

Proposition 3.12. *Let $X=\Sigma X'$ be a suspension of a CW-complex X'. Then the diagram*

$$\begin{array}{ccc} JX^{(l)}\times JX & \stackrel{\tilde{S}_l\cdot 1_{JX}}{\to} & JX \\ \phi\downarrow & & \downarrow H_k \\ JX^{(k)}\times JX^{(k)} & \stackrel{m}{\to} & JX^{(k)} \end{array}$$

homotopy commutes for $l\geq k$, where $\phi=H_k\circ\tilde{S}_l\times H_k$ and m is the multiplication.

Proof: Let $\bar{K}_{n,m}$ be the subgroup of $[X^{nl+m}, JX^{(l)}\times JX]$ generated by $\bar{x}_1,\cdots,\bar{x}_n$ and $\bar{y}_1,\cdots,\bar{y}_m$, where $\bar{x}_i$ and $\bar{y}_j$ are represented by $q_i, r_j\colon X^{nl+m}=(X^l)^n\times X^m\to JX^{(l)}\times JX$ with

$$q_i(a_1,\cdots,a_n;b_1,\cdots,b_m)=p_l(a_i)$$

and

$$r_j(a_1,\cdots,a_n;b_1,\cdots,b_m)=E(b_j)$$

for $a_s\in X^l$ and $b_t\in X$, where p_l is the composite $X^l\to X^{(l)}\to JX^{(l)}$. Let $\mu\colon JX\times JX\to JX$ be the mutiplication. Consider

$$\mu\circ(\tilde{S}_l\times id)_*\colon[X^{nl+m}, JX^{(l)}\times JX]\to[X^{nl+m}, JX].$$

Then $\mu\circ(\tilde{S}_l\times id)_*(\bar{x}_i)=[[x_{il+1},\cdots,x_{(i+1)l}]$ and $\mu\circ(\tilde{S}_l\times id)_*(\bar{y}_j)=x_{nl+j}$. Thus

$$\mu\circ(\tilde{S}_l\times id)_*(\bar{x}_1\cdots\bar{x}_n\cdot\bar{y}_1\cdots\bar{y}_m)=(\prod_{i=1}^n[[x_{il+1},\cdots,x_{(i+1)l}])\cdot x_{nl+1}\cdots x_{nl+m}.$$

By Theorem 3.8,

$$\begin{aligned} & H_k\circ\mu\circ(\tilde{S}_l\times id)_*(\bar{x}_1\cdots\bar{x}_n\cdot\bar{y}_1\cdots\bar{y}_m) \\ &\quad = (\prod_{i=1}^n(H_k)_*[[x_{il+1},\cdots,x_{(i+1)l}])\cdot(H_k)_*(x_{nl+1}\cdots x_{nl+m}) \\ &\quad = \mu\circ((H_k\circ\tilde{S}_l)\times H_k)_*(\bar{x}_1\cdots\bar{x}_n\cdot\bar{y}_1\cdots\bar{y}_m). \end{aligned}$$

Thus

$$H_k\circ\mu(\tilde{S}_l\times id)\circ q_{n,m}\simeq\mu\circ((H_k\circ\tilde{S}_l)\times H_k)\circ q_{n,m}$$

where $q_{n,m}$ is the composite $X^{nl+m}=(X^l)^n\times X^m\to J_n(X^{(l)})\times J_mX\hookrightarrow JX^{(l)}\times JX$. Since $q^*_{n,m}\colon[J_n(X^{(l)})\times J_mX,\Omega Y]\to[(X^l)^n\times X^m,\Omega Y]$ is a monomorphism,

$$H_k\circ\mu(\tilde{S}_l\times id)|_{J_nX^{(l)}\times J_mX}\simeq\mu\circ((H_k\circ\tilde{S}_l)\times H_k)|_{J_nX^{(l)}\times J_mX}.$$

The assertion follows from Lemma 3.9.

Notation 3.13. *The map $L_{k.l}: X^{(kl)} \to JX^{(kl)}$ is defined by*

$$L_{k,l}(x_1 \wedge x_2 \wedge \cdots \wedge x_k l) =$$

$$\prod_{1\leq j_1<\cdots<j_l=kl, 1\leq l_1^{j_s}<\cdots<l_k^{j_s}=j_s} (x_{l_1^{j_1}} \wedge \cdots \wedge x_{l_k^{j_1}} \wedge \cdots \wedge x_{l_1^{j_s}} \wedge \cdots \wedge x_{l_k^{j_l}}$$

where $(l_1^{j_1}, \cdots, l_k^{j_1}, \cdots, l_1^{j_l}, \cdots, l_k^{j_l})$ runs over shuffles of $(1, \cdots, kl)$ with left lexicographical oder.

Let $\tilde{L}_{k,l}: JX^{(kl)} \to J^{(kl)}$ denote the homomorphism of topological monoids induced by $L_{k,l}$.

Proposition 3.14. *Let $X = \Sigma X'$ be a suspension of a CW-complex X'. Then*

$$H_l \circ H_k \simeq \tilde{L}_{k,l} \circ H_{kl} : JX \to JX^{(kl)}$$

Proof: Consider $K_n \subseteq [X^n, JX]$. By Lemma 3.4,

$$(H_k)_*(x_1 \cdots x_n) = \prod_{j=1}^{n} \sum_{1\leq l_1^j<\cdots<l_k^j=j} \{x_{l_1^j}|\cdots|x_{l_k^j}\}.$$

By Lemma 2.2,

$$\{x_{l_1^{j_1}}|\cdots|x_{l_k^{j_1}}|\cdots|x_{l_1^{j_l}}|\cdots|x_{l_k^{j_l}}\} = 1$$

if some $l_t^{j_s}$ repeats. Thus

$$(H_l)_* \circ (H_k)_*(x_1 \cdots x_n) = \prod_{j=1}^{n} \sum_{1\leq j_1<\cdots<j_l=j, 1\leq l_1^{j_s}<\cdots<l_k^{j_s}=j_s} \{x_{l_1^{j_1}}|\cdots|x_{l_k^{j_1}}|\cdots|x_{l_1^{j_l}}|\cdots|x_{l_k^{j_l}}\}$$

so that no elements in $\{l_1^{j_1}, \cdots, l_k^{j_1}, \cdots, l_1^{j_l}, \cdots, l_k^{j_l}\}$ repeat. Thus

$$(H_l)_* \circ (H_k)_*(x_1 \cdots x_n) = (\tilde{L}_{k,l})_* \circ (H_{kl})_*(x_1 \cdots x_n)$$

and

$$H_l \circ H_k|_{J_n X} \circ p_n \simeq \tilde{L}_{k,l} \circ H_{kl}|_{J_n X} \circ p_n,$$

where $p_n: X^n \to J_n X$ is the projection. Hence

$$H_l \circ H_k|_{J_n X} \simeq \tilde{L}_{k,l} \circ H_{kl}|_{J_n X}$$

for each n. The assertion follows.

Remark 3.15. *If $X = \Sigma X'$, the order of the product in the definition of $L_{k,l}$ can be chosen in any way.*

Example 3.16. *Let $X = S^n$. Then $L_{2,2} : (S^n)^{(4)} = S^{4n} \to JS^{4n}$ is given by*

$$L_{2,2}(x_1 \wedge x_2 \wedge x_3 \wedge x_4) = (x_2 \wedge x_3 \wedge x_1 \wedge x_4) \cdot (x_1 \wedge x_3 \wedge x_2 \wedge x_4) \cdot (x_1 \wedge x_2 \wedge x_3 \wedge x_4)$$

Thus $L_{2,2}$ is of degree $2 + (-1)^n$ and the diagram

$$\begin{array}{ccc} JS^n & \xrightarrow{H_2} & JS^{2n} \\ H_4 \downarrow & & \downarrow H_2 \\ JS^{4n} & \xrightarrow{J(2+(-1)^n)} & JS^{4n} \end{array}$$

homotopy commutes. This coincides with the cohomology calculation.

4. Proof of Theorem 1.6

Let $X = \Sigma X'$ be the suspension of a space X'. Recall that The map $\beta_n : X^{(n)} \to X^{(n)}$ is defined by induction

$$\beta_n = \beta_{n-1} \wedge id - (1, 2, \cdots, n) \circ (\beta_{n-1} \wedge id)$$

and $\beta_2 = id - (12)$, where $(1, 2, \cdots, n) \in \Sigma_n$ acts on $X^{(n)}$ by permutation of coordinates. Then $\beta_n \circ \beta_n = n\beta_n$ (see $[CW]$). If $n \not\equiv 0(p)$, we denote $L_n(x) = hocolim_{n^{-1}\beta_n} X^{(n)}$ localized at p for a prime p.

Proof of Theorem 1.6: Let $S_n = [[E, \cdots, E] : X^{(n)} \to JX$ be the n-fold Samelson product. Then S_n factors through $L_n(X)$ if $n \not\equiv 0(p)$, i.e. there a homotopy commutative diagram

$$\begin{array}{ccc} X^{(n)} & \xrightarrow{S_n} & JX \\ \downarrow & & \| \\ L_n(X) & \xrightarrow{S_n} & JX. \end{array}$$

Denote $\tilde{S}_n : J(L_n(X)) \to JX$ the homomorphism of topological monoids induced by $S_n : L_n(X) \to JX$. Let ϕ be the composite

$$\prod_{j=1}^{\infty} J(L_{k_j}(X)) \xrightarrow{\prod_j \tilde{S}_{k_j}} \prod_{j=1}^{\infty} JX \xrightarrow{multi.} JX,$$

where $\prod_{j=1}^{\infty}$ is the weak infinite product. Notice that $JX^{(n)}$ is $(n-1)$-connected if X is connected. Define $\psi : JX \to \prod_{j=1}^{\infty} J(L_{k_j}(X))$ to be the composite

$$JX \xrightarrow{\prod_j H_{k_j}} \prod_{j=1}^{\infty} JX^{(k_j)} \to \prod_{j=1}^{\infty} J(L_{k_j}(X)).$$

It suffices to show that

$$\psi \circ \phi : \prod_{j=1}^{\infty} J(L_{k_j}(X)) \to \prod_{j=1}^{\infty} J(L_{k_j}(X))$$

is a homotopy equivalence. Consider

$$\psi\circ\phi_*\colon PH_*\prod_{j=1}^{\infty}J(L_{k_j}(X))\to PH_*\prod_{j=1}^{\infty}J(L_{k_j}(X)),$$

where $H_*(-)=H_*(-;F_p)$ the homology with coefficients in the field F_p, the prime field with p elements. Notice that $PH_*\prod_{j=1}^{\infty}J(L_{k_j}(X))\cong\bigoplus_{j=1}^{\infty}PH_*JL_{k_j}(X)$. By Corollary 3.11, the composite

$$\phi_{i,j}\colon JX^{(k_j)}\hookrightarrow\prod_{j=1}^{\infty}JX^{(k_j)}\to JX\to\prod_{j=1}^{\infty}JX^{(k_j)}\to JX^{(k_i)}$$

is a loop map for $k_i\leq k_j$,i.e. $i\leq j$. Since k_j is not a multiple of k_i,

$$\phi_{i,j}|_{H_*X^{(k_j)}}\colon\bar{H}_*X^{(k_j)}\to\bar{H}_*JX^{(k_i)}$$

is zero for $k_i<k_j$ [CT, Proposition 5.3]. Now the composite

$$\phi'_{i,i}\colon JL_{k_i}(X)\hookrightarrow\prod_{j=1}^{\infty}JL_{k_j}(X)\to JX\to\prod_{j=1}^{\infty}JL_{k_j}(X)\to JL_{k_i}(X)$$

is a homotopy eqivalence [CW]. Thus the composite

$$PH_*JL_{k_j}(X)\ \stackrel{l_j}{\hookrightarrow}\ \bigoplus_{j=1}^{\infty}PH_*JL_{k_j}(X)\ \stackrel{\psi\circ\phi_*}{\to}\ \bigoplus_{j=1}^{\infty}PH_*JL_{k_j}(X)\ \stackrel{\pi_i}{\to}\ PH_*J(L_{k_i}(X))$$

is zero for $i<j$ and an isomorphism for $i=j$, where π_i is the projection. To check that

$$\psi\circ\phi_*\colon\bigoplus_{j=1}^{\infty}PH_*J(L_{k_j}(X))\to\bigoplus_{j=1}^{\infty}PH_*J(L_{k_j}(X))$$

is a monomorphism, suppose that

$$(\psi\circ\phi)_*(a_k+a_{k+1}+\cdots)=0$$

with $a_k\neq 0$, where $a_j\in PH_*JL_{k_j}(X)$. Then

$$\pi_k\circ(\psi\circ\phi)_*(a_k+a_{k+1}+\cdots)=\pi_k\circ(\psi\circ\phi)_*(a_k)+\sum_{j>k}\pi_k\circ(\psi\circ\phi)_*(a_j)$$

$$=\pi_k\circ(\psi\circ\phi)_*(a_k)=\pi_k\circ(\psi\circ\phi)_*\circ l_k(a_k)=0.$$

Thus $a_k=0$ which is a contradiction. Thus $\psi\circ\phi_*$ is a monomorphism and therefore an isomorphism for any finite CW-complex. Similarly, $\psi\circ\phi$ is a rational isomorphism. Thus $\psi\circ\phi$ is a homotopy equivalence for any finite CW-complex $X=\Sigma X'$. Notice that any CW-complex is a homotopy colimit of finite CW-complexes. The assertion follows.

Corollary 4.1. *Let $X = \Sigma X'$ be a suspension of a CW-complex X' and let $q_1 < q_2 < \cdots$ be all of the primes which are different from the prime p. Then*

$$JX \simeq \prod_{j=1}^{\infty} JL_{q_j}(X) \times ?$$

localized at p.

Corollary 4.2. *There exists a space Z_{n+1} so that*

$$\Omega P^{n+1}(2) \simeq (\prod_{j=1}^{\infty} \Omega\Sigma L_{p_j}(P^n(2))) \times Z_{n+1}$$

for $n \geq 3$, where $\{p_j\}$ is the set of odd primes.

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