A PRODUCT DECOMPOSITION OF $\Omega_0^3\Sigma\mathbf{R}P^2$

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Abstract. We give a specific product decomposition of the base-point path connected component of the triple loop space of the suspension of the projective plane.

1. Introduction

In this paper, we give a specific product decompositions of the base-point path connected component of the triple loop space of the suspension of the projective plane, which is denoted by $\Omega_0^3\Sigma\mathbf{R}P^2$. Let $\mathbf{R}P_a^b = \mathbf{R}P^b/\mathbf{R}P^{a-1}$, let $X < n >$ be the n-connected covering of the space X and let $P^n(2) = \Sigma^{n-2}\mathbf{R}P^2$, the n-dimensional mod 2 Moore space. In particular, $P^3(2) = \Sigma\mathbf{R}P^2$. Our theorem is as follows.

Theorem 1.1. *There is a homotopy equivalence*

$$\Omega_0^3P^3(2) \simeq \Omega^2(S^3\langle 3\rangle) \times \Omega_0^3(P^6(2) \vee \Sigma\mathbf{R}P_2^4)$$

localized at 2.

This theorem was first given in my thesis where we only gave a product decomposition of $\Omega_0^4(\Sigma\mathbf{R}P^2)$. The author would like to thank his thesis advisor Fred Cohen for pointing out that our original decomposition for $\Omega_0^4(\Sigma\mathbf{R}P^2)$ can be delooping. He showed that the power map

$$4 : \Omega^2(S^3\langle 3\rangle) \to \Omega^2(S^3\langle 3\rangle)$$

is null localized at 2, his proof will be used in this paper. The application of this theorem to calculate the homotopy groups $\pi_r(P^3(2))$ for $r \leq 11$ was given in my thesis [Wu]. An independent calculation for $\pi_*(P^3(2))$ can be found out in [Mu]. The homotopy groups of $\Sigma\mathbf{R}P^2$ may be useful in low dimensional topology.

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2. Proof of the Theorem 1.1

In this section, all spaces are localized at 2 and $H_*(X)$ means the homology with coefficients in field $\mathbf{F}_2$, the field with two elements. Let $j : P^3(2) \to BSO(3)$ be the inclusion of the 3-skeleton of $BSO(3)$ and let X_2 be the pull back in the diagram

$$\begin{array}{ccc} BSO(2) & \to & BSO(3) \\ \uparrow & & \uparrow \\ X_2 & \to & P^3(2), \end{array}$$

where the map $BSO(2) \to BSO(3)$ is induced by the canonical injection $SO(2) \to SO(3)$. Thus there is a fibration

$$S^2 \to X_2 \to P^3(2).$$

Let $i : S^2 \to X_2$ be the inclusion of the fibre.

Lemma 2.1. *The map* $\Omega^2(i) : \Omega^2(S^2\langle 3\rangle) \to \Omega^2(X_2\langle 3\rangle)$ *is null.*

Proof: Consider the commutative diagram

$$\begin{array}{ccccccccc} \to & \pi_3(BSO(3)) & \xrightarrow{0} & \pi_2(S^2) & \hookrightarrow & \pi_2(BSO(2)) & \to & \pi_2(BSO(3)) & \to 0 \\ & \uparrow & & \| & & \uparrow & & \uparrow & \\ \to & \pi_3(P^3(2)) & \xrightarrow{0} & \pi_2(S^2) & \overset{i_*}{\hookrightarrow} & \pi_2(X_2) & \to & \pi_2(P^3(2)) & \to 0 \end{array}$$

where the top and bottom rows are long exact sequence of homotopy groups. Thus $\pi_1(X_2) = 0$, $\pi_2(X_2) \cong \pi_2(BSO(2)) = \pi_2(\mathbf{C}P^\infty) \cong \mathbf{Z}$ and $i_* : \pi_2(S^2) \to \pi_2(X_2)$ is of degree 2. Let $\phi : S^2 \to X_2$ be a representive of the generator in $\pi_2(X_2)$. Then there is a homotopy commutative diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{i} & X_2 \\ \| & & \uparrow \phi \\ S^2 & \xrightarrow{[2]} & S^2, \end{array}$$

where $[2] : S^2 \to S^2$ is of degree 2. Now consider the homotopy commutative diagram

$$\begin{array}{ccc} \Omega S^2 & \xrightarrow{H_2} & \Omega S^3 \\ \Omega([2]) \downarrow & & \downarrow \Omega([4]) \\ \Omega S^2 & \xrightarrow{H_2} & \Omega S^3, \end{array}$$

where H_2 is the second James-Hopf map. Notice that S^3 is an H-space. We have

$$\Omega([4]) \simeq 4 : \Omega S^3 \to \Omega S^3,$$

where $4 : \Omega S^3 \to \Omega S^3$ is the power map of degree 4. Consider the EHP sequence $S^3 \to \Omega(S^3\langle 3\rangle) \xrightarrow{H_2} \Omega S^5$, the map $S^3 \to \Omega(S^3\langle 3\rangle)$ is a representative of the nontrivial element η in $\pi_4(S^3) \cong \mathbf{Z}/2$. Notice that the composite

$$S^3 \xrightarrow{\eta} \Omega(S^3\langle 3\rangle) \xrightarrow{2} \Omega(S^3\langle 3\rangle)$$

is null. Thus the map $2 \simeq \Omega(2) : \Omega^2(S^3\langle 3\rangle) \to \Omega^2(S^3\langle 3\rangle)$ factors through $\Omega(H_2) : \Omega^2(S^3\langle 3\rangle) \to \Omega^2 S^5$ up to homotopy. By a result of Cohen [C1, or C2], the map

$$2 \cdot \Omega(H_2) : \Omega^2(S^3\langle 3\rangle) \to \Omega^2 S^5$$

is null. Thus the power map

$$4 : \Omega^2(S^3\langle 3\rangle) \to \Omega^2(S^3\langle 3\rangle)$$

is null. Notice that the map $\Omega(H_2) : \Omega^2(S^2\langle 3\rangle) \to \Omega^2(S^3\langle 3\rangle)$ is a homotopy equivalence. Thus $\Omega^2([2]) : \Omega^2(S^2\langle 3\rangle) \to \Omega^2(S^2\langle 3\rangle)$ is null and so the boundary map $\Omega^2(i) : \Omega^2(S^2\langle 3\rangle) \to \Omega^2(X_2\langle 3\rangle)$ is null, which is the assertion.

Corollary 2.2. *There is a homotopy equivalence*

$$\Omega_0^3 P^3(2) \simeq \Omega^2(S^3\langle 3\rangle) \times \Omega_0^3(X_2).$$

Lemma 2.3. *The (mod 2) homology of $X_2\langle 2\rangle$ is as follows:*

$$\bar{H}_r(X_2\langle 2\rangle) = \begin{cases} \mathbf{Z}/2 & r=3,4,6 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & r=5 \\ 0 & otherwise. \end{cases}$$

Proof: Consider the homotopy commutative of fibre sequences

$$\begin{array}{ccccc} S^2 & \to & BSO(2) = \mathbf{C}P^\infty & \to & BSO(3) \\ \| & & \uparrow & & \uparrow \\ S^2 & \to & X_2 & \to & P^3(2) \\ \uparrow & & \uparrow & & \uparrow \\ * & \to & X_2\langle 2\rangle & \xrightarrow{=} & X_2\langle 2\rangle \end{array}$$

Thus the homotopy fibre of $X_2\langle 2\rangle \to P^3(2)$ is $SO(3)$. The assertion follows by inspecting the Serre spectral sequence for the principle fibre sequence

$$SO(3) \to X_2\langle 2\rangle \to P^3(2).$$

Recall that$H_*(\Omega P^3(2)) \cong T(u,v)$ as algebras with $dim(v) = 2$ and $Sq^1_* v = u$, where $T(u,v)$ is the tensor algebra generated by u and v.

Lemma 2.4. *$H_*(\Omega(X_2\langle 2\rangle)) \cong T(V)$ as subalgebras of $H_*(\Omega P^3(2))$, where V has a basis*

$$\{u^2, [u,v], v^2 + u[u,v], [u^2,v], [v^2,u] + u[u^2,v]\}$$

in $T(u,v) \cong H_*(\Omega P^3(2))$ and $[x,y]$ is the commutator of x and y.

Proof: Notice that $\Omega j_* : H_*(\Omega P^3(2)) \to H_*(SO(3))$ is an epimorphism and

$$\pi_1(\Omega P^3(2)) \cong \pi_1(SO(3)).$$

Thus the Serre spectral sequence for the fibre sequence

$$\Omega(X_2\langle 2\rangle) \overset{\Omega i}{\to} \Omega P^3(2) \overset{\Omega j}{\to} SO(3)$$

collapses and therefore there is a short exact sequence of Hopf algebras

$$1 \to H_*(\Omega(X_2\langle 2\rangle)) \to H_*(\Omega P^3(2)) \to H_*(SO(3)) \to 1.$$

Thus $H_*(\Omega(X_2\langle 2\rangle))$ is the cotensor product of $\mathbf{F}_2$ and $H_*(\Omega P^3(2))$ over $H_*(SO(3))$ (see [MM]).

We first show that $V \subseteq H_*(\Omega(X_2\langle 2\rangle))$. Let ψ be the comultiplication in the Hopf algebra $H_*(\Omega P^3(2)) \cong T(u,v)$. Then ψ is determined by the generators.

$$\psi(u) = u \otimes 1 + 1 \otimes u$$

and

$$\psi(v) = v \otimes 1 + 1 \otimes v + u \otimes u.$$

The comultiplication ψ on the basis for V is as follows:

$$\psi(u^2) = u^2 \otimes 1 + 1 \otimes u^2,$$

$$\psi([u,v]) = [u,v] \otimes 1 + 1 \otimes [u,v],$$

$$\psi(v^2 + u[u,v]) = (v^2 + u[u,v]) \otimes 1 + 1 \otimes (v^2 + u[u,v]) + u^2 \otimes u^2,$$

$$\psi([u^2,v]) = [u^2,v] \otimes 1 + 1 \otimes [u^2,v],$$

and

$$\psi([u,v^2] + u[u^2,v]) = ([u,v^2] + u[u^2,v]) \otimes 1 + 1 \otimes ([u,v^2] + u[u^2,v]).$$

Notice that $H_*(SO(3)) \cong E(u,v)$, the exterior algebra generated by u and v. Thus $V \subseteq H_*(\Omega(X_2\langle 2\rangle))$. Let $\phi : T(V) \to H_*(\Omega(X_2\langle 2\rangle))$ be the algebra map induced by the inclusion $V \subseteq H_*(\Omega(X_2\langle 2\rangle))$ and let $A = \Omega i_* \circ \phi(T(V))$. Then A is the subalgebra of $H_*(\Omega P^3(2))$ generated by V, which is also a tensor algebra. Notice that $V \to Q(A)$ is an isomorphism. Thus $A \cong T(V)$ (see [CMN, Proposition 3.9]) and so $\phi : T(V) \to H_*(\Omega(X_2\langle 2\rangle))$ is a monomorphism. Notice that Poincare series

$$\chi(H_*(\Omega(X_2\langle 2\rangle))) = \chi(H_*(\Omega P^3(2)))/\chi(H_*(SO(3)))$$

$$= (1 - t - t^2)^{-1}(1 + t + t^2 + t^3)^{-1} = (1 - t^2 - t^3 - 2t^4 - t^5)^{-1} = \chi(T(V)).$$

Thus $\phi : T(V) \to H_*(\Omega(X_2\langle 2\rangle))$ is an isomorphism, which is the assertion.

Lemma 2.5. *The homology suspension* $\sigma_* : Q\bar{H}_*(\Omega(X_2\langle 2\rangle)) \to \bar{H}_{*+1}(X_2\langle 2\rangle)$ *is an isomorphism.*

Proof: Let x_2, x_3, x_4, y_4 and y_5 be in $H_*(\Omega(X_2\langle 2\rangle))$ corresponding to $u^2, [u,v], v^2 + u[u,v], [u^2,v]$ and $[v^2,u] + u[u^2,v]$ in $T(u,v)$, respectively. Consider mod 2 Serre spectral sequence $E^r_{*,*}$ for the principle fibre sequence

$$\Omega(X_2\langle 2\rangle) \to * \to X_2\langle 2\rangle.$$

Notice that, for $* \leq 5$, $\bar{H}_*(\Omega(X_2\langle 2\rangle))$ has a basis

$$x_2, x_3, x_2^2, y_4, x_4, [x_2, x_3], x_2x_3, y_5.$$

Let $a_{j+1} = \sigma_*(x_j)$ and $b_{j+1} = \sigma_*(y_j)$. Then

$$d_3(a_3) = x_2,$$

$$d_3(a_3x_2) = x_2^2,$$

$$d_3(a_3x_3) = x_2x_3,$$

$$d_4(a_4) = x_3,$$

$$d_4(a_4x_2) = x_3x_2,$$

and

$$E^r_{s,t} = 0$$

for $r \geq 6$,$s > 0$,$t > 0$ and $s + t \leq 6$. Thus there exists $a_5, b_5, b_6 \in E^2_{*,0}$ such that $d_5(a_5) = x_4$,$d_5(b_5) = y_4$ and $d_6(b_6) = y_5$. The assertion follows.

Lemma 2.6. *The element*

$$x_3 = [u, v]$$

in $H_3(\Omega(X_2\langle 2\rangle))$ *is spherical.*

Proof: Consider the exact sequence of homotopy groups

$$\pi_4(P^3(2)) \to \pi_3(SO(3)) \to \pi_3(X_2\langle 2\rangle) \to \pi_3(P^3(2)) \to \pi_2(SO(3)) = 0.$$

Notice that $\pi_3(SO(3)) = \mathbf{Z}$ and $\pi_4(P^3(2))$ is torsion. Thus $\pi_3(X_2\langle 2\rangle) = \mathbf{Z}$ or $\mathbf{Z}\oplus G$ for some 2-torsion abelian group G. Now $X_2\langle 2\rangle$ is 2-connected and $\mathbf{F}_2 = H_3(X_2\langle 2\rangle; \mathbf{F}_2) \cong \pi_3(X_2\langle 2\rangle) \otimes \mathbf{F}_2$. Thus $\pi_3(X_2\langle 2\rangle) = \mathbf{Z}$ and so $\pi_2(\Omega(X_2\langle 2\rangle)) = \mathbf{Z}$. Now the 3-skeleton $(\Omega(X_2\langle 2\rangle))_{(3)}$ of $\Omega(X_2\langle 2\rangle)$ is the homotopy cofibre of an attaching map $\phi : S^2 \to S^2$. Notice that

$$H_2((\Omega(X_2\langle 2\rangle))_{(3)}; \mathbf{Z}) \cong H_2(\Omega(X_2\langle 2\rangle); \mathbf{Z}) = \mathbf{Z}.$$

Thus ϕ is of degree 0 and so the 3-skeleton $(\Omega(X_2\langle 2\rangle))_{(3)} \simeq S^2 \vee S^3$. The assertion follows.

Proof of Theorem 1.1: By Corollary 2.2, it suffices to show that

$$X_2\langle 2\rangle \simeq P^6(2) \vee \Sigma\mathbf{R}P^4_2.$$

We use the notation in Lemmas 2.4 and 2.5. By Lemma 2.4, Steenrod operations on $\bar{H}_*(X_2\langle 2\rangle)$ are given by

$$Sq^2_* a_5 = a_3, Sq^1_* a_5 = a_4, Sq^1_* b_6 = b_5.$$

Thus

$$\bar{H}_*(X_2\langle 2\rangle) \cong M \oplus N$$

as modules over Steenrod algebra, where M is generated by a_3, a_4, a_5 and N is generated by b_5, b_6. By Lemma 2.6, the element x_3 is spherical in $H_3(\Omega(X_2\langle 2\rangle))$. Let $f : S^3 \to \Omega(X_2\langle 2\rangle)$ so that $f_*(\iota_3) = x_3$, where ι_3 is a generator for $H_3(S^3)$. Consider the relative Samelson product for the multiplicative fibre sequence

$$\Omega(X_2\langle 2\rangle) \to \Omega P^3(2) \to SO(3)$$

(see [N1]). There is a relative Samelson product

$$[f, E] : S^3 \wedge \mathbf{R}P^2 = P^5(2) \to \Omega(X_2\langle 2\rangle),$$

where $E : \mathbf{R}P^2 \to \Omega P^3(2)$ is the suspension. Notice that both ι_3 and u are primitive. Then

$$[f, E]_*(\iota_3 \otimes u) = [f_*(\iota_3), u] = [[u, v], u] = [u^2, v]$$

in $T(u, v)$. Let $\theta = \sigma \circ \Sigma[f, E]$ be the composite $P^6(2) \to \Sigma\Omega(X_2\langle 2\rangle) \xrightarrow{\sigma} X_2\langle 2\rangle$ and let u_5 and v_6 be the bases for $H_5(P^6(2))$ and $H_6(P^6(2))$, repectively. Then $\theta_*(u_5) = \sigma_*([u^2, v]) = b_5$ and $Sq^1_*\theta_*(v_6) = b_5$. Thus $\theta_*(v_6) = b_6$ which is the only nonzero element in $H_6(X_2\langle 2\rangle)$ and so

$$\theta_* : H_*(P^6(2)) \to H_*(X_2\langle 2\rangle)$$

is one to one. Let B^5 be the homotopy cofibre of $\theta : P^6(2) \to X_2\langle 2\rangle$. Then $\bar{H}_*(B^5)$ has a basis $\{a_3, a_4, a_5\}$. Notice that $a_4 = \sigma_*(x_3)$ is spherical. Thus B^5 is the homotopy cofibre of an attaching map $\phi : S^4 \to S^3 \vee S^4 = (X_2\langle 2\rangle)_{(4)}$. Notice that the homotopy fibre of the pinch map $X_2\langle 2\rangle \to B^5$ is 4-connected and $\pi_4(X_2\langle 2\rangle) \cong \pi_4(B^5)$. Thus the composite

$$S^4 \xrightarrow{\phi} S^4 \vee S^3 \to X_2\langle 2\rangle$$

is null, or the injection $S^4 \vee S^3 \to X_2\langle 2\rangle$ extends to a map $s : B^5 \to X_2\langle 2\rangle$. Now the composite

$$P^6(2) \vee B^5 \xrightarrow{\theta \vee s} X_2\langle 2\rangle \vee X_2\langle 2\rangle \xrightarrow{\bigtriangledown} X_2\langle 2\rangle$$

is a homotopy equivalence, where $\bigtriangledown$ is the fold map.

Consider homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccccc}
S^4 & \xrightarrow{\phi} & S^3 \vee S^4 & \to & B^5 & \to & S^5 \\
\| & & p\downarrow & & \downarrow & & \| \\
S^4 & \xrightarrow{[k]} & S^4 & \to & P^5(k) & \to & S^5,
\end{array}$$

where p is the projection. Notice that $Sq^1_* a_5 = a_4$. Thus $P^5(k) \simeq P^5(2)$ localized at 2 and $k = 2k'$ with $k' \equiv 1(2)$. Notice that $Sq^2_* a_5 = a_3$. Thus B^5 is not homotopy equivalent to $P^5(2) \vee S^3$ and therefore the composite

$$S^4 \xrightarrow{\phi} S^3 \vee S^4 \xrightarrow{p} S^3$$

is essential which is equal to η in $\pi_4(S^3) = \mathbf{Z}/2$, where p is the projection. Thus the homotopy class $[\phi] = \eta + 2k'[\iota]$ in $\pi_4(S^3 \vee S^4) \cong \mathbf{Z}/2 \oplus \mathbf{Z}$ with generators η and $[\iota]$, where $k' \equiv 1(2)$. Notice that $\Sigma\mathbf{R}P^4_2$ is the homotopy cofibre of $(\eta, [2]) : S^4 \to S^3 \vee S^4$. Thus $B^5 \simeq \Sigma\mathbf{R}P^4_2$ and the assertion follows.

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