THE RENORMALIZATION METHOD AND QUADRATIC-LIKE MAPS

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ABSTRACT. The renormalization of a quadratic-like map is studied. The three-dimensional Yoccoz puzzle for an infinitely renormalizable quadratic-like map is discussed. For an unbranched quadratic-like map having the *a priori* complex bounds, the local connectivity of its Julia set is proved by using the three-dimensional Yoccoz puzzle. The generalized version of Sullivan's sector theorem is discussed and is used to prove his result that the Feigenbaum quadratic polynomial has the *a priori* complex bounds and is unbranched. A dense subset on the boundary of the Mandelbrot set is constructed so that for every point of the subset, the corresponding quadratic polynomial is unbranched and has the *a priori* complex bounds.

Contents

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1. Introduction

Let $P_c(z) = z^2 + c$ be a quadratic polynomial. A central problem in the study of the dynamics of P_c is to understand the topology and geometry of the Julia set J_c of P_c. The filled-in Julia set K_c of P_c is, by definition, the set of points not going to infinity under iterations of P_c. The Julia set J_c of P_c is, by definition, the boundary of K_c.

In order to have the more penetrating study of the dynamics of a quadratic polynomial P_c, Douady and Hubbard [DH3] introduced the concept of a quadratic-like map. In §1, we review the definition of a quadratic-like map and the work of Douady and Hubbard [DH3] which proves that a quadratic-like map with connected Julia set is hybrid equivalent to a unique quadratic polynomial. We also review the result due to Douady and Yoccoz (see [HUB, MI2]) about the landing of external rays at a repelling periodic point of a quadratic polynomial. In the same section, we also review some fundamental results about the Julia set of a quadratic-like map and some basic facts of hyperbolic geometry.

The Mandelbrot set $\mathcal{M}$ is the set of complex parameters c such that the Julia set J_c of P_c is connected. The Julia set J_c in the complement of $\mathcal{M}$ is a Cantor set. Douady and Hubbard [DH1] proved that $\mathcal{M}$ is connected. They further conjectured that $\mathcal{M}$ is locally connected. Many important research problems follow from this conjecture. For example, the hyperbolicity conjecture (explained below) would follow from this conjecture [DH2]. A quadratic polynomial is hyperbolic if it has an attractive or super-attractive periodic point in the complex plane. Let $\mathcal{H}P$ be the set of parameters c such that P_c is hyperbolic. The hyperbolicity conjecture says that $\mathcal{H}P$ is open and dense in $\mathcal{M}$. To study the local connectivity of the Mandelbrot set $\mathcal{M}$, it would be helpful to answer the question: for which c in $\mathcal{M}$ is the Julia set J_c locally connected ? Yoccoz made substantial progress in this direction. He proved that if P_c is non-renormalizable (or finitely renormalizable), then the Julia set J_c is locally connected. We discuss his result in §3. (Using this result, Yoccoz further proved that $\mathcal{M}$ is locally connected at a finitely renormalizable point c (see [HUB])). Also in §3, we discuss the two-dimensional Yoccoz puzzle of a quadratic-like map and its relation with the renormalizability of this quadratic-like map.

There remain many points in $\mathcal{M}$ which are infinitely renormalizable. In §2, we define infinitely renormalizable quadratic-like maps and define infinitely renormalizable folding mappings, and discuss the relation between the two definitions.

In §4, we prove one of our main results: for an unbranched infinitely renormalizable quadratic-like map having the *a priori* complex bounds, its Julia set is locally connected. We prove this result by using the three-dimensional Yoccoz puzzle of an infinitely renormalizable quadratic-like map. Also in §4, we prove that the filled-in Julia set of any renormalization of a renormalizable quadratic-like map, about the period of the two-dimensional Yoccoz puzzle, does not depend on the choices of renormalization domains. In particular, the renormalized filled-in Julia set is the limiting component in two-dimensional Yoccoz puzzle containing the critical point.

In §5, we discuss Sullivan's sector theorem. We prove a generalized version. The proof repeatedly applies the sharpest version of Koebe's distortion theorem (see [BIE]), and uses Sullivan's idea about using hyperbolic contraction to trap points in a hyperbolic neighborhood.

Using the generalized version of Sullivan's sector theorem, we prove in §6 his result that the Feigenbaum quadratic polynomial has the *a priori* complex bounds and is unbranched. By combining this with the result in §4, we complete the proof of the result which was first announced in [JIH] and which says that the Julia set of the Feigenbaum quadratic polynomial is locally connected.

In §7, we construct a subset $\tilde{\Upsilon}$ of the Mandelbrot set which is dense on the boundary $\partial\mathcal{M}$ of the Mandelbrot set $\mathcal{M}$ such that for every point c in this subset, the corresponding quadratic polynomial $P_c(z) = z^2 + c$ is unbranched and infinitely renormalizable and has the *a priori* complex bounds. Thus, for c in $\tilde{\Upsilon}$, the Julia set J_c of a quadratic polynomial $P_c(z) = z^2 + c$ is locally connected. A similar result concerning about the local connectivity of the Mandelbrot set at infinitely renormalizable points is proved in [JI4].

This article benefited from reading J. Hubbard's paper [HUB] and J. Milnor's paper [MI2], from talking with D. Sullivan, and from listening the talk given by J.-C. Yoccoz in Denmark. During the writing of this article, several e-mails from C. McMullen provided great help. Some statements in this article are more precise because of his suggestions. Also from his e-mails, I decided to include the papers [JI1,JIH,JI2,JI3] into this self-contained article. During my study of the work of Sullivan, F. Gardiner and J. Hu provided a lot help. I thank them all. This research was partially carried out [JI1] when the author was in the IMS at Stony Brook. This research is partially supported by an grant from the NSF and by awards from the PSC-CUNY. Research at MSRI is supported in part by NSF grant # DMS-9022140.

2. Quadratic Polynomials, Quadratic-Like Maps, and Hyperbolic Geometry

Let $P_c(z) = z^2 + c$ be a quadratic polynomial. Let $\mathbf{C}$ be the complex plane. Let $V = \{z \in \mathbf{C} \mid |z| < r\}$ be a disk in $\mathbf{C}$. For r large enough, $U = P_c^{-1}(V)$ is a simply connected domain, its closure is relatively compact in V, and $P_c : U \to V$ is a holomorphic, proper map of degree two (see Fig. 1). This is a model of an object defined by Douady and Hubbard [DH3].

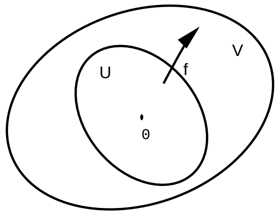

Fig. 1

Definition 1. *A quadratic-like map is a triple (U, V, f) where U and V are simply connected domains isomorphic to a disc with $\overline{U} \subset V$, and where $f : U \to V$ is a holomorphic, proper map of degree two (see Fig. 1).*

Remark 1. A proper map f means that $f^{-1}(K)$ is compact for every compact set K. Douady and Hubbard [DH3] also defined a polynomial-like map.

Suppose (U, V, f) is a quadratic-like map. We use

$$K_f = \cap_{n=0}^{\infty} f^{-n}(U)$$

to denote the set of z in U such that images of z under iterations of f are all in U. The set K_f is called the filled-in Julia set. It is a compact subset of U. The Julia set J_f of f is the boundary of K_f.

A quadratic-like map (U, V, f) has only one branched point b at which the derivative $f'(b)$ of f equals zero. We call b the critical point of f and, without loss of generality, we always assume that $b = 0$. The following theorem is well-known.

Theorem 1. *Suppose (U, V, f) is a quadratic-like map. If the critical point 0 is not in K_f, then $K_f = J_f$ is a Cantor set of zero Lebesgue measure in $\mathbf{C}$. And moreover, the set K_f (or J_f) is connected if and only if the critical point 0 is in K_f.*

Suppose that Ω is a domain in $\mathbf{C}$ and that f is a self-map of Ω. A point z in Ω is called a periodic point of period $k \geq 1$ if $f^{\circ k}(z) = z$ and $f^{\circ i}(z) \neq z$ for $1 \leq i < k$. The number $\lambda = (f^{\circ k})'(z)$ is called the multiplier (or eigenvalue) of f at z. A periodic point of period 1 is called a fixed point. A periodic point z of f is said to be repelling, attractive, or neutral if $|\lambda| > 1$, $0 < |\lambda| < 1$, or $|\lambda| = 1$. Moreover, a periodic point is called super-attractive if $\lambda = 0$, and is called parabolic if $\lambda = e^{2\pi i p/q}$ where p and q are integers. For an attractive or super-attractive or parabolic periodic point of period $k \geq 1$ of a quadratic-like map $f : U \to V$, let $O(p) = \{f^{\circ i}(p)\}_{i=0}^{k-1}$ be the periodic orbit of f. The set

$$B(p) = \{z \in V \mid f^{\circ n}(p) \to O(p) \text{ as } n \to \infty\}$$

is called the basin of $O(p)$. Let $CB(f^{\circ i}(p))$ be the connected component containing $f^{\circ i}(p)$ of $B(p)$. The set $IB(p) = \cup_{i=0}^{k-1} CB(f^{\circ i}(p))$ is called the immediate basin of

$O(p)$. The proofs of the following two theorems can be found in Blanchard's survey article [BLA].

Theorem 2. *Suppose z_0 is a super-attractive periodic point of period k of a holomorphic function f defined on Ω. Then there is a neighborhood U of z_0, a unique holomorphic diffeomorphism $h : U \to h(U)$ with $h(z_0) = 0$ and $h'(z_0) = 1$, and a unique integer $n > 1$ such that*

$$h \circ f^{\circ k} \circ h^{-1}(z) = z^n$$

on $h(U)$.

Theorem 3. *Let (U, V, f) be a quadratic-like map and let $J_f = \partial K_f$ be its Julia set. Let E_f be the set of all repelling periodic points of f. Then*

(1) J_f is completely invariant, i.e., $f(J_f) = J_f$ and $f^{-1}(J_f) = J_f$;

(2) J_f is perfect, i.e., $J'_f = J_f$, where J'_f means the set of limit points of J_f;

(3) E_f is dense in the Julia set J_f, i.e., $\overline{E_f} = J_f$;

(4) for any z in V, the limit set of $\{f^{-n}(z)\}_{n=0}^{\infty}$ is J_f;

(5) J_f has no interior point;

(6) If f has an attractive or super-attractive or parabolic periodic point p in V, then the immediate basin $IB(p)$ contains the critical point 0 and the critical orbit $O(0) = \{f^{\circ n}(0)\}_{n=0}^{\infty}$;

(7) If f has neither any attractive, nor any super-attractive, nor any neutral periodic points in V, then $K_f = J_f$.

Let (U, V, f) and (U', V', g) be two quadratic-like maps. They are topologically conjugate if there is a homeomorphism h from a neighborhood $K_f \subset X \subset U$ to a neighborhood $K_g \subset Y \subset U'$ such that $h \circ f = g \circ h$ on X where K_f and K_g are filled-in Julia sets. If h is quasiconformal (see [AH1]) (respectively, holomorphic), then they are quasiconformally (respectively, holomorphically) conjugate. If h can be chosen such that $h_{\overline{z}} = 0$ a.e. on K_f, then they are hybrid equivalent. Let

$$I(f) = \{g \mid g \text{ is hybrid equivalent to } f\}$$

be the inner class of f. The following theorem is due to Douady and Hubbard.

Theorem 4 [DH3]. *If (U, V, f) is a quadratic-like map such that K_f is connected, then there is a unique quadratic polynomial $P(z) = z^2 + c_f$ in $I(f)$.*

Let $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ be the extended complex plane. Then ∞ is a super-attractive fixed point of any quadratic polynomial $P_c(z) = z^2 + c$. The filled-in Julia set K_c of P_c is the set of all points not going to infinity under iterations of P_c. Let $D_r = \{z \in \mathbf{C} \mid |z| < r\}$. As we mentioned in the beginning of this section, for $r > 1$ large enough, $U = P_c^{-1}(D_r)$ is a simply connected domain and $\overline{U} \subset D_r$.

Thus (U, V, P_c) for $V = D_r$ is a quadratic-like map. In particular, (D_r, D_{r^2}, P_0) is a quadratic-like map for every $r > 1$; the filled-in Julia set K_0 of P_0 is the closed unit disk $\overline{D}_1$. Applying Theorem 2, there is a holomorphic diffeomorphism h_1 defined on a neighborhood $\overline{\mathbf{C}} \setminus \overline{D}_r$ (for $r > 1$ large) about ∞ such that $h_1(\infty) = \infty$, $h_1'(\infty) = 1$, and such that

$$h_1^{-1} \circ P_c \circ h_1(z) = z^2$$

on $\overline{\mathbf{C}} \setminus \overline{D}_r$. Let $B_1(\infty) = h_1(\overline{\mathbf{C}} \setminus \overline{D}_r)$ and let $B_n(\infty) = P_c^{-(n-1)}(B_1(\infty))$. If the filled-in Julia set K_c of P_c is connected, then all

$$P_c : B_n(\infty) \cap \mathbf{C} \to B_{n-1}(\infty) \cap \mathbf{C}$$

are unramified covering maps of degree two. We can inductively define holomorphic diffeomorphisms h_n on $\overline{\mathbf{C}} \setminus \overline{D}_{r^{\frac{1}{2^n}}}$ such that

$$h_n^{-1} \circ P_c \circ h_n(z) = z^2$$

for z in $\overline{\mathbf{C}} \setminus \overline{D}_{r^{\frac{1}{2^n}}}$ and for $n > 1$. As n tends to infinity, we get a holomorphic diffeomorphism h_∞ defined on $\overline{\mathbf{C}} \setminus \overline{D}_1$ such that

$$h_\infty^{-1} \circ P_c \circ h_\infty(z) = z^2 \tag{1}$$

for all z in $\overline{\mathbf{C}} \setminus \overline{D}_1$. Therefore, $B_c(\infty) = h_\infty(\overline{\mathbf{C}} \setminus \overline{D}_1)$ is the basin of ∞ for P_c and $K_c = \overline{\mathbf{C}} \setminus B_c(\infty)$. Furthermore, for every $r > 1$ and for $U_r = h_\infty(D_r)$, (U_r, U_{r^2}, P_c) is a quadratic-like map and its filled-in Julia set is always K_c. Let $S^R = \{z \in \mathbf{C} \mid |z| = R\}$ and let $s_R = h_\infty(S^R)$ for $R > 1$. Then

$$P_c(s_R) = s_{R^2}. \tag{2}$$

The topological circle s_R for every $R > 1$ is called an equipotential curve of P_c. A curve

$$e_\theta = h_\infty(\{z \in \overline{\mathbf{C}} \mid |z| > 1, \arg(z) = \theta\})$$

for $0 \leq \theta < 2\pi$ is called an external ray of P_c. Then

$$P_c(e_\theta) = e_{2\theta}. \tag{3}$$

Remark 2. Let

$$G(z) = \max\{0, \lim_{n\to\infty} \frac{1}{2^n} \log |P_c^{\circ n}(z)|\}$$

be the Green's function of K_c in $\overline{\mathbf{C}}$. Then $G(P_c(z)) = 2G(z)$. For any $R > 1$, the equipotential curve $s_R = G^{-1}(\log R)$ is a level curve of G.

If the mapping h_∞ in Equation (1) can be extended continuously to the unit circle S^1, then we have a unique continuous map $H : \overline{C} \setminus D_1 \to \overline{\mathbf{C}} \setminus \mathring{K}_f$ such that $H|(\overline{C} \setminus \overline{D}_1) = h_\infty$. Using H, we can define an equivalence relation on S^1: $z_1 \sim z_2$ if and only if $H(z_1) = H(z_2)$. Let $[z]$ be the equivalent class of z. Then $\tilde{P}_0([z]) = [P_0(z)]$ defines a map of the quotient space $X = S^1/\sim$, since $z_1^2 \sim z_2^2$ if $z_1 \sim z_2$. The dynamical system $(\tilde{P}_0, X)$ is topologically conjugate to (P_c, J_c) by $\tilde{H}([z]) = H(z)$. The question arises:

Question 1. *For which c can h_∞ be extended continuously to the unit circle S^1 ?*

A connected set X in $\mathbf{C}$ is locally connected if for any point p in X and any neighborhood V about p, there is another neighborhood $U \subset V$ about p such that $U \cap X$ is connected. The following classical theorem proved by Carathéodory in one complex variable gives a sufficient and necessary condition to extend h_∞ continuously to S^1. The proof of this theorem can be found in [MI1].

Theorem 5 [CAR]. *Let h be a Riemann mapping from D_1 onto a simply connected open domain Ω. Then h can be extended continuously to the unit circle S^1 if and only if the boundary $\partial\Omega$ (as well as Ω) is locally connected.*

Remark 3. If $\partial\Omega$ is a Jordan curve, then h can be extended to a homeomorphism from $\overline{D}_1$ onto $\overline{\Omega}$. Moreover, if $\partial\Omega$ is made of finite number of analytic curves, then the extension restricted to the unit circle S^1 has non-zero derivative at every point other than a corner (see [BIE]).

We have an equivalent question by just concerning the topology of a Julia set now:

Question 2. *For which c is J_c locally connected ?*

An external ray e_θ lands at J_c if e_θ has only one limit point at J_c. An external ray is periodic with period m if $e_\theta \cap P_c^{\circ i}(e_\theta) = \{\infty\}$ for $1 \leq i < m$ and if $P_c^{\circ m}(e_\theta) = e_\theta$. The following theorem is proved by Douady and Yoccoz (refer to [MI1,MI2,HUB]).

Theorem 6. *Let $P_c(z) = z^2 + c$ be a quadratic polynomial with connected Julia set J_c. Then every repelling periodic point of P_c is a landing point of finitely many periodic external rays with the same period.*

Let $\mathcal{S}$ be a Riemann surface and let $(\tilde{\mathcal{S}}, \pi)$ be the universal cover of $\mathcal{S}$, where $\tilde{\mathcal{S}}$ is a simple connected Riemann surface and $\pi : \tilde{\mathcal{S}} \to \mathcal{S}$ is the universal covering map. From the Uniformization Theorem (see [AH2]), we can identify $\tilde{\mathcal{S}}$ with one of the extended complex plane $\overline{\mathbf{C}}$, the complex plane $\mathbf{C}$, or the open unit disk D_1. A Riemann surface $\mathcal{S}$ is hyperbolic if $\tilde{\mathcal{S}} = D_1$.

Let $\mathcal{D}$ be the hyperbolic disk which is the open unit disk with the hyperbolic metric

$$d_H s = \frac{|dz|}{1 - |z|^2}.$$

Let d_H be the hyperbolic distance. Every holomorphic diffeomorphism $h : D_1 \to D_1$, which is not a linear fractional transformation, strictly decreases the hyperbolic distance d_H, i.e,

$$d_H(h(z_1), h(z_2)) < d_H(z_1, z_2)$$

for all z_1 and z_2 in $\mathcal{D}$. For a hyperbolic Riemann surface $\mathcal{S}$, one can define the hyperbolic distance $d_{H,S}$ from d_H and π. Any holomorphic map $h : \mathcal{S} \to \mathcal{S}$, which is not an isometry with respect to $d_{H,\mathcal{S}}$, strictly decreases this hyperbolic distance $d_{H,S}$, i.e., for any z_1 and z_2 in $\mathcal{S}$,

$$d_{H,\mathcal{S}}(h(z_1), h(z_2)) < d_{H,\mathcal{S}}(z_1, z_2).$$

A bounded domain Ω in $\mathbf{C}$ is a hyperbolic Riemann surface. An important family of hyperbolic Riemann surfaces is the family of bounded doubly connected domains in $\mathbf{C}$. A bounded doubly connected domain Ω is called an annulus. From complex analysis (see [BIE]), any annulus is holomorphically diffeomorphic to a unique round annulus $A_r = \{z \in \mathbf{C} \mid r < |z| < 1\}$ for $0 < r < 1$. The number

$$\mathrm{mod}(\Omega) = -\log r$$

is called the modulus of Ω. It is a conformal invariant, i.e., $\mathrm{mod}(h(\Omega)) = \mathrm{mod}(\Omega)$ whenever h is a conformal homeomorphism from Ω onto $h(\Omega)$.

Let Ω be a bounded doubly connected domain in $\mathbf{C}$. Then the complement $\mathbf{C} \setminus \Omega$ of Ω in $\mathbf{C}$ has two components. One is a connected and simply connected compact set E and the other is an unbounded set F. From the Grötzsch argument (see [AH1]), the bounded component E is a single point if and only if $\mathrm{mod}(\Omega) = \infty$ (see [BRH]).

Let Ω be an annulus. If $\Omega_1 \subseteq \Omega$ is a subannulus, then

$$\mathrm{mod}(\Omega_1) \leq \mathrm{mod}(\Omega). \tag{4}$$

If Ω_1, $\Omega_2 \subseteq \Omega$ are two disjoint subannuli, then

$$\mathrm{mod}(\Omega_1) + \mathrm{mod}(\Omega_2) \leq \mathrm{mod}(\Omega). \tag{5}$$

The proofs of these two inequalities can be found in the book of Ahlfors [AH1].

Let E be a connected and simply connected compact subset of the open unit disk D_1. Let

$$\mathrm{mod}(D_1, E) = \sup_{\Omega}\{\mathrm{mod}(\Omega),\ \text{where } \Omega \subseteq D_1 \setminus E \text{ is a round subannulus}\}.$$

Note that $\mathrm{mod}(D_1, E) = \infty$ if E is a single point.

Let $\mathrm{diam}_H(E) = \sup_{z_1,z_2\in E} d_H(z_1, z_2)$ be the hyperbolic diameter of E in D_1. The following theorem can be found in the book of McMullen [MC1].

Theorem 7. *The hyperbolic diameter $d_H(E)$ and the $mod(D_1, E)$ are inversely related:*

$$diam_H(E) \to 0 \qquad \Longleftrightarrow \qquad mod(D_1, E) \to \infty$$

and

$$diam_H(E) \to \infty \qquad \Longleftrightarrow \qquad mod(D_1, E) \to 0.$$

More precisely, there is a constant $C > 0$ such that

$$C^{-1} diam_H(E) \leq \exp(-mod(D_1, E)) \leq C diam_H(E)$$

when $diam_H(E)$ is small, while

$$\frac{C}{diam_H(E)} \geq mod(D_1, E) \geq C^{-1}\exp(-diam_H(E))$$

when $dima_H(E)$ is large.

3. Renormalizable Quadratic-Like Maps

Let (U_0, V_0, f_0) be a quadratic-like map and suppose its filled-in Julia set K_{f_0} is connected. The map $f_0 : U_0 \to V_0$ is renormalizable if there are an integer $n \geq 2$ and an open subdomain U_1 of U_0 such that $0 \in U_1$ and such that $(U_1, V_1, f_0^{\circ n})$ is a quadratic-like map with connected filled-in Julia set, where $V_1 = f^{\circ n}(U_1)$. Let $f_1 = f_0^{\circ n}|U_1$. The filled-in Julia set $K_{f_1} = K_{f_1}(n, U_1)$ is *a priorily* dependent on the choice of U_1 (refer to Theorem 13). The domain U_1 is called a renormalization and (U_0, V_0, f_0) is called renormalizable about n. Otherwise, (U_0, V_0, f_0) is called non-renormalizable.

The quadratic-like map (U_0, V_0, f_0) is infinitely renormalizable if there is a strictly increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that (U_0, V_0, f_0) is renormalizable about n_k for $k = 1, 2, \ldots$. Otherwise, (U_0, V_0, f_0) is called finitely renormalizable.

A quadratic-like map (U_0, V_0, f_0) is called real if $f_0(U_0 \cap \mathbf{R}^1) \subset V_0 \cap \mathbf{R}^1$ and $g = f_0|U_0 \cap \mathbf{R}^1$ is a real folding map. Let (U_0, V_0, f_0) be a real quadratic-like map and suppose its filled-in Julia set K_{f_0} is connected. Suppose $g = f_0|U_0 \cap \mathbf{R}^1$ has a fixed point $p \in \mathbf{R}^1$ with positive multiplier $f_0'(p)$. Let $p' \neq p$ be another inverse image of p under g, that is, $g(p') = p$. Conjugating by a linear fractional transformation, we may assume that $p = -1$ and $p' = 1$ and that

$$[-1, 1] = \cap_{n=0}^{\infty} g^{-n}(V_0 \cap \mathbf{R}^1) = K_f \cap R^1.$$

Hence g is a folding map with unique quadratic critical point 0.

Let g be a folding map of $[-1, 1]$ such that $g(-1) = g(1) = -1$ and such that 0 is a unique quadratic critical point. We say g is renormalizable about $n > 1$ if there is a subinterval I of $[-1, 1]$ such that $0 \in \mathring{I}$, such that $g^{\circ i}(I) \cap \mathring{I} = \emptyset$ for all $0 < i < n$, and such that $g^{\circ n}(I) \subseteq I$. Otherwise, g is non-renormalizable. We say g is infinitely renormalizable if there is a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that g is renormalizable about n_k for all $k > 0$. Otherwise, g is called finitely renormalizable. The next theorem shows that for a real quadratic-like map, both definitions of renormalization are essentially equivalent.

Theorem 8. *Let (U_0, V_0, f_0) be a real quadratic-like map and suppose its filled-in Julia set K_{f_0} is connected. Suppose f_0 has neither neutral, nor attractive, nor super-attractive periodic points. Then f_0 is renormalizable if and only if the folding map $g = f_0|[-1, 1]$ is renormalizable.*

Proof. Suppose the folding map $g = f_0|[-1,1]$ is renormalizable. This means that there is a maximal closed subinterval I of $[-1,1]$ and an integer $n>1$ such that 0 is in $\mathring{I}$, such that $g^{\circ i}(I)\cap \mathring{I}=\emptyset$ for $0<i<n$, and such that $g^{\circ n}(I)\subseteq I$. One of the endpoints of I is fixed by $g^{\circ n}$. It is a repelling fixed point. Take a neighborhood T' of I such that $f^{\circ n}|(L\cup R)$ is expanding, where $L\cup R = T'\setminus I$. Let $I_i = g^{\circ i}(I)$ for $0\leq i\leq n$. The inverse h_n of $g^{\circ (n-1)}: I_1\to I_n$ is a diffeomorphism and can be extended to a diffeomorphism on an open interval $T\supset I$. Take $T\subset T'$. Let $M' = h_n(T)$ and $M = g^{-1}(M')$. Then $\overline{M}$ is a subset of T. Because the critical orbit $CO=\{f_0^{\circ k}(0)\}_{k=0}^{\infty}$ of f_0 is in the real line $\mathbf{R}^1$, h_n can be extended to $V_1 = (V_0\setminus \mathbf{R}^1)\cup T$ analytically. The image V_1' of V_1 under this extension is contained in $U' = (U_0\setminus\mathbf{R}^1)\cup M'$. Let $U_1 = f_0^{-1}(V_1')$. Then $\overline{U}_1\subset V_1$ and (U_1,V_1,f_1) for $f_1 = f_0^{\circ n}|U_1$ is a quadratic-like map. Since $g^{\circ n}(I)\subseteq I$, the filled-in Julia set K_{f_1} is connected. Therefore, (U_0,V_0,f_0) is renormalizable about n and U_1 is a renormalization.

Suppose (U_0,V_0,f_0) is renormalizable about $n>1$. Let U_1 be a renormalization and set $V_1 = f^{\circ n}(U_1)$. Then (U_1,V_1,f_1) for $f_1 = f_0^{\circ n}|U_1$ is a quadratic-like map with the connected filled-in Julia set K_{f_1}. Let $I = K_{f_1}\cap\mathbf{R}^1$. Then $I = \cap_{i=0}^{\infty} f_1^{-i}(V_1\cap\mathbf{R}^1)$. For every $1\leq i<n$, $f^{\circ i}(I)\cap\mathring{I}=\emptyset$ else $f^{\circ n}$ would have at least three fixed points in $I\cup f^{\circ i}(I)$ with one of them either attractive or parabolic. Since $f_1(0)$ is in I, $f_0^{\circ n}(I)\subseteq I$. Therefore, $g = f_0|[-1,1]$ is a renormalizable folding map. ∎

4. Two-Dimensional Yoccoz Puzzles and Renormalizability

In this section, we discuss a technique in the study of non-renormalizable quadratic polynomials, due to Yoccoz, and some of its applications to renormalization.

Let $P_c(z) = z^2+c$ be a quadratic polynomial with connected filled-in Julia set K_c. The external ray e_0 of P_c is the only one fixed by P_c (see Fig. 2). It lands either at a repelling or at a parabolic fixed point β of P_c (see [MI1]). Suppose β is repelling. Applying Theorem 6, we see that e_0 is the only external ray landing at β. Thus $K_c\setminus\{\beta\}$ is connected. We call β the non-separate fixed point of P_c. Let $\alpha\neq\beta$ be the other fixed point of P_c. If α is either an attractive or a super-attractive fixed point, then $J_c = K_c\setminus (\cup_{n=0}^{\infty} P_c^{-n}\big(D(\alpha)\big)$ for a small disk centered at α. The Julia set J_c is a Jordon curve; every external ray lands at a unique point in J_c (see Remark 3). If α is a repelling fixed point, there are at least two periodic external rays landing at α. We use $R_0(\alpha)$ to denote the union of a cycle of periodic external rays of period q landing at α (see Fig. 2). The set $R_0(\alpha)$ cuts $\mathbf{C}$ into finitely many simply connected domains Ω_0, Ω_1, $\ldots$, Ω_{q-1}. Each domain contains points in the Julia set J_c. Thus $K_c\setminus\{\alpha\}$ is disconnected. We call α the separate fixed point of P_c.

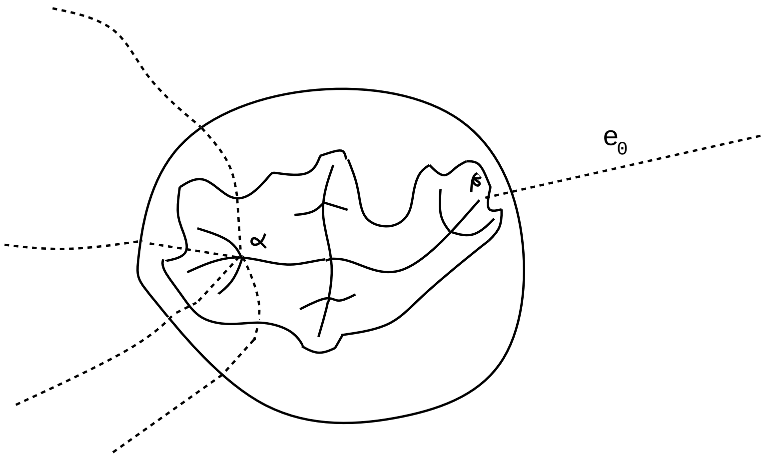

Fig. 2

Henceforth, we assume that the fixed points β and α are both repelling. Let s_r be a fixed equipotential curve of P_c and let U_r be the open domain bounded by s_r. Then $(U_{\sqrt{r}}, U_r, P_c)$ is a quadratic-like map. The set $R_0(\alpha)$ cuts U_r into finitely many simply connected domains. Let C_0 be the closure of the domain containing 0, and let $B_{0,i}$ be the closure of the domain containing $P_c^{\circ i}(0)$ for $1 \leq i < q$. Since $R_0(\alpha)$ is forward invariant under P_c, the image under P_c of $C_0 \cap K_c$ or $B_{0,i} \cap K_c$, for every $1 \leq i < q$, is the union of some of $C_0 \cap K_c$, $B_{0,1} \cap K_c$, $\ldots$, $B_{0,q-1} \cap K_c$. The set $\eta_0 = \{C_0, B_{0,1}, \ldots, B_{0,q-1}\}$ is called the original partition. We note that it is not a Markov partition because $P_c|C_0$ is a proper, holomorphic map of degree two. (But $P_c|B_{0,i}$ is a holomorphic diffeomorphism for every $1 \leq i < q$.)

Let $\Gamma_n = P_c^{-n}(\alpha)$ and let $R_n(\alpha) = P_c^{-n}\Big(R_0(\alpha)\Big)$ for $n \geq 0$. The set $R_n(\alpha)$ is the union of some external rays landing at points in Γ_n; it cuts the domain $U_{r^{\frac{1}{2^n}}}$ into a finite number of simply connected domains. Let C_n be the closure of the domain containing 0 and let $B_{n,1}$, $\ldots$, B_{n,k_n} be the closures of others. Since $P_c(\Gamma_n) = \Gamma_{n-1}$, the image of C_n or $B_{n,i}$ under P_c is one of C_{n-1}, $B_{n-1,1}$, $\ldots$, $B_{n-1,k_{n-1}}$, for $1 \leq i \leq k_n$ and $n \geq 1$. Then $P_c|C_n$ is holomorphic, proper branch covering map of degree two; all $P_c|B_{n,i}$ are holomorphic diffeomorphisms. The set $\eta_n = \{C_n, B_{n,1}, \ldots, B_{n,k_n}\}$ is called the n^{th}-partition. The sequence $\eta = \{\eta_n\}_{n=0}^{\infty}$ is called the two-dimensional Yoccoz puzzle for P_c. A similar puzzle for certain cubic polynomial is constructed by Branner and Hubbard [BRH]. Yoccoz used this puzzle while studying the local connectivity of a non-renormalizable quadratic polynomial as follows.

Let $\Gamma_\infty = \cup_{n=0}^{\infty}\Gamma_n$. For any x in $K_c \setminus \Gamma_\infty$, there is one and only one sequence $\{D_n(x)\}_{n=0}^{\infty}$ such that $x \in D_n(x) \in \eta_n$. For any x in Γ_∞, there are q such sequences. We call such a sequence

$$x \in \cdots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \cdots \subseteq D_1(x) \subseteq D_0(x)$$

an x-end. In particular,

$$0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_2 \subseteq C_1 \subseteq C_0$$

is called the critical end.

Suppose $D_{n+1} \subseteq D_n$ are domains in η_{n+1} and η_n for $n \geq 0$. Define $A_n = D_n \setminus \mathring{D}_{n+1}$. If $D_{n+1} \subset \mathring{D}_n$, then $\mathring{A}_n$ is a non-degenerate annulus and its modulus (denoted $\mathrm{mod}(A_n)$) is greater than 0. Otherwise, A_n is a degenerate annulus and its modulus $\mathrm{mod}(A_n)$ is zero. The domain D_n is critical if $D_n = C_n$ and $D_{n+1} = C_{n+1}$; it is semi-critical if $D_n = C_n$ but $D_{n+1} \neq C_{n+1}$; it is non-critical if $D_n \neq C_n$. If D_n is non-critical, then

$$\mathrm{mod}(A_n) = \mathrm{mod}(P_c(A_n))$$

since $P_c : A_n \to P_c(A_n)$ is a conformal homeomorphism. If D_n is critical, then

$$\mathrm{mod}(A_n) = \frac{\mathrm{mod}\big(P_c\big(A_n\big)\big)}{2}$$

since $P_c : A_n \to P_c(A_n)$ is a proper, holomorphic, unramified covering map of degree two. If D_n is semi-critical, then

$$\frac{\mathrm{mod}\big(P_c\big(A_n\big)\big)}{2} \leq \mathrm{mod}(A_n).$$

For a point x in K_c and an x-end

$$x \in \cdots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \cdots \subseteq D_1(x) \subseteq D_0(x),$$

we define

$$D_{nm}(x) = P_c^{\circ m}(D_{n+m}(x))(= D_n(P_c^{\circ m}(x)))$$

and

$$A_{nm}(x) = D_{nm}(x) \setminus \mathring{D}_{(n+1)m}(x)$$

for $n, m \geq 0$. The tableau $T(x) = (a_{nm})_{n\geq 0, m\geq 0}$ is an $\infty \times \infty$-matrix defined as follows: $a_{nm} = 1$ if $D_{nm}(x) = C_n$, and $a_{nm} = 0$ if $D_{nm}(x) \neq C_n$. The tableau $T(0) = \Big(a^0_{nm}\Big)_{n\geq 0, m\geq 0}$ is called the critical tableau. Note that

$$P_c^{\circ m}(x) \in \cdots \subseteq D_{nm}(x) \subseteq D_{(n-1)m}(x) \subseteq \cdots \subseteq D_{1m}(x) \subseteq D_{0m}(x)$$

is a $P_c^{\circ m}(x)$-end.

Lemma 1. *The tableau $T(x)$ satisfies the following rules:*

(T1) *if $a_{nm} = 1$ for $n, m \geq 0$, then $a_{im} = 1$ for all $0 \leq i \leq n$,*

(T2) *if $a_{nm} = 1$ for $n, m \geq 0$, then $a_{(n-i)(m+j)} = a^0_{(n-i)j}$ for all $0 \leq i + j \leq n$,*

(T3) *if*

i) *$a_{nm} = 1$ and $a_{(n+1)m} = 0$ for $n, m \geq 0$ and if*

ii) *$a_{(n-i)(m+i)} = 1$ for $1 \leq i \leq n$ and $a_{(n-j)(m+j)} = 0$ for $0 < j < i$,*

then $a^0_{(n-i+1)i} = 1$ implies $a_{(n-i+1)(m+i)} = 0$.

Proof. Rule **(T1)** is valid because

$$0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_1 \subseteq C_0.$$

Rule **(T2)** is valid because if $a_{nm} = 1$, then $D_{im}(x) = C_i$ for all $0 \leq i \leq n$. Thus $D_{(n-i)(m+j)}(x) = f^{\circ j}(C_{n-i})$ for all $0 \leq i + j \leq n$.

We prove Rule **(T3)**. Conditions **i)** and **ii)** imply that $P_c^{\circ i} : C_n \to C_{n-i}$ is a degree two holomorphic proper branch covering map. Condition **i)** and Rule **(T2)** imply $a_{k(m+j)} = a^0_{kj}$ for $0 \leq k + j \leq n$. Now $a^0_{(n-i+1)i} = 1$ and $a^0_{j0} = 1$ for all $0 \leq j < \infty$ imply that $P_c^{\circ i} : C_{n+1} \to C_{n-i+1}$ is a degree two holomorphic proper branch covering map. Thus $C_{n+1} = P_c^{-i}(C_{n-i+1}) \cap C_n$. Assume $a_{(n-i+1)(m+i)} = 1$. Then $D_{(n+1)m}(x) = P_c^{-i}(C_{n-i+1}) \cap C_n = C_{n+1}$. This contradicts $a_{(n+1)m} = 0$. ∎

Lemma 2. *For any domain D in η_n, for $n \geq 0$, $D \cap K_c$ is connected.*

Proof. Since the domain D is bounded by finitely many external rays $\Pi = \{e_{\theta_i}\}_{i=1}^m$ and by some equipotential curve, then $\partial D \cap K_c$ consists of a finite number of points $\{p_i\}_{i=1}^n$. Every p_i is a landing point of two external rays in Π. Suppose $D \cap K_c$ is not connected for some D in η_n. Then there are two disjoint open sets U and V such that $D \cap K_c = (D \cap K_c \cap U) \cup (D \cap K_c \cap V)$. Suppose that $p_1, \ldots, p_k$ are in U and that $p_{k+1}, \ldots, p_n$ are in V. The two external rays in Π landing at p_i cut $\mathbf{C}$ into two open domains. Let W_i be the one which is disjoint from D. Then $U' = U \cup \cup_{i=1}^k W_i$ and $V' = V \cup \cup_{i=k+1}^n W_i$ are two disjoint open sets and $K_c = (U' \cap K_c) \cup (V' \cap K_c)$. This contradicts the fact that K_c is connected. ∎

Lemma 2 implies that for any x-end

$$x \in \cdots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \cdots \subseteq D_1(x) \subseteq D_0(x),$$

the intersection $L_x = \cap_{n=0}^\infty D_n(x)$ is a compact connected non-empty set containing x. Let $T(x) = (a_{nm})_{n\geq 0, m\geq 0}$ be the tableau of the x-end. It is non-recurrent if there is an integer $N \geq 0$ such that $a_{nm} = 0$ for all $n \geq N$ and all $m \geq 1$. Otherwise, $T(x)$ is recurrent.

Lemma 3. *If $T(x)$ is non-recurrent, then $L_x = \{x\}$.*

Proof. Suppose $N \geq 0$ is an integer such that $a_{nm} = 0$ for all $n \geq N$ and all $m \geq 1$. Then, for $n > N$, every

$$P_c^{\circ(n-N-1)} : D_{n1}(x) \to D_{N(n-N)}(x)$$

is a holomorphic diffeomorphism. Thus $\mathrm{mod}(A_{n0}(x))$ is greater than or equal to $\mathrm{mod}(A_{N(n-N)}(x))/2$ for every $n > N$. There are only finitely many different annuli in $\{A_{Nm}(x)\}_{m=0}^{\infty}$ because η_N has only finitely many domains. If there were infinitely many non-degenerate annuli in $\{A_{Nm}(x)\}_{m=0}^{\infty}$, then there would be infinitely many non-degenerate annuli in $\{A_{n0}(x)\}_{n=0}^{\infty}$ whose moduli are the same. This would imply that

$$\mathrm{mod}(D_0(x) \setminus L_x) \geq \sum_{n=0}^{\infty} \mathrm{mod}(A_{n0}(x)) = \infty.$$

Therefore $L_x = \{x\}$.

If there are only finitely many non-degenerate annuli in $\{A_{Nm}(x)\}_{m=0}^{\infty}$, The proof uses results from hyperbolic geometry. Let B_{N1} be the domain in η_N containing the critical value $P_c(0)$. Since $P_c^{\circ i}(x)$ do not enter C_{N+1} for all $0 < i < \infty$, then $P_c^{\circ i}(x)$ does not enter B_{N1} for all $2 \leq i < \infty$. Let us thicken B_{Ni} to an open simply connected domain $\tilde{B}_{Ni}$ such that $B_{Ni} \subset \tilde{B}_{Ni}$ and such that $P_c(0)$ is not in $\tilde{B}_{Ni}$ for $1 < i \leq k_N$. The map P_c has two inverse branches g_{i1} and g_{i2} defined on $\tilde{B}_{Ni}$ for every $1 < i \leq k_N$. we consider $\tilde{B}_{Ni}$ to be a hyperbolic Riemann surface with hyperbolic distance $d_{H,i}$ for every $1 < i \leq k_N$, where k_N is the number of elements in η_N. Then if g_{ik}, for $1 < i \leq k_N$ and $k = 1$ or 2, sends B_{Ni} into B_{Nj} for some $1 < j \leq k_N$, then it strictly contracts these hyperbolic distances; more precisely, there is a constant $0 < \lambda < 1$ such that $d_{H,j}(g_{ik}(x), g_{ik}(y)) < \lambda d_{H,i}(x,y)$ for x and y in B_{Ni} and for $k = 0$ and 1. Therefore, there is a constant $C > 0$ such that for any $D_{n1}(x)$ and for any $n > N$,

$$d(D_{n1}(x)) = \max_{y,z \in D_{n0}(x)} |y - z| \leq C\lambda^{n-N-1}$$

since $D_{(n-i)i}(x)$ is in one of $B_{N2}, \ldots, B_{Nk_N}$ for every $2 \leq i \leq n-N$. Thus $d(D_{n0}(x))$ tends to zero as n goes to infinity and $L_x = \{x\}$. ∎

The critical end

$$0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_2 \subseteq C_1 \subseteq C_0$$

is important. Let $A_{n0}(0) = C_n \setminus \mathring{C}_{n+1}$.

Lemma 4. *If $\sum_{n=0}^{\infty} mod(A_{n0}(0)) = \infty$, then for any x in K_c and any x-end*

$$x \in \cdots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \cdots \subseteq D_1(x) \subseteq D_0(x),$$

$L_x = \{x\}$.

Proof. Consider the tableau $T(x) = (a_{nm})_{n\geq 0, m\geq 0}$. If $T(x)$ is non-recurrent, the lemma follows from Lemma 3.

Suppose $T(x)$ is recurrent. If there is a column which is entirely 1's, then there are integers $M \geq 0$ and $N \geq 0$ such that $a_{iM} = 1$ for all $i \geq 0$ and $a_{nm} = 0$ for all $n \geq N$ and $0 \leq m < M$. Thus $P_c^{\circ M} : D_{n0}(x) \to D_{(n-M)M} = C_{n-M}$ is a holomorphic diffeomorphism for every $n \geq N$. This implies

$$m(D_0(x) \setminus L_x) \geq \sum_{n=0}^{\infty} \mathrm{mod}(A_{n0}(x)) \geq \sum_{n=N}^{\infty} \mathrm{mod}(A_{n0}(x)) = \sum_{n=N-M}^{\infty} \mathrm{mod}(A_{n0}(0)) = \infty.$$

So $L_x = \{x\}$.

Suppose that there is no column which is entirely 1's. Let $N > 0$ be an integer such that $a_{n0} = 0$ for $n \geq N$. For any $n \geq N$, let $m_n > 0$ be the integer such that $a_{nm_n} = 1$ and $a_{ni} = 0$ for $0 \leq i < m_n$. Then $P_c^{\circ m_n} : D_{(n+m_n-1)0}(x) \to D_{(n-1)m_n}(x) = C_{n-1}$ is a holomorphic diffeomorphism. Remember that $A_{(n-1)0}(x) = D_{(n-1)0}(x) \setminus \mathring{D}_{n0}(x)$ and $A_{(n-1)0}(0) = C_{n-1} \setminus \mathring{C}_n$. We have

$$\mathrm{mod}\Big(A_{(n+m_n-1)0}(x)\Big) = \mathrm{mod}\Big(A_{(n-1)0}(0)\Big).$$

Let $q_n = n + m_n - 1$. Then $q_N < q_{N+1} < \cdots < q_n < q_{n+1} < \cdots$. Thus

$$\begin{aligned}\mathrm{mod}(D_0(x) \setminus L_x) &\geq \sum_{n=N}^{\infty} \mathrm{mod}\Big(A_{n0}(x)\Big) \geq \sum_{n=N}^{\infty} \mathrm{mod}\Big(A_{q_n 0}(x)\Big) \\ &= \sum_{n=N}^{\infty} \mathrm{mod}\Big(A_{(n-1)0}(0)\Big) = \infty.\end{aligned}$$

This implies that $L_x = \{x\}$. ∎

The first column of the critical tableau $T(0) = (a^0_{nm})_{n\geq 0, m\geq 0}$ is entirely 1's. If $T(0)$ has another column which is entirely 1's, that is, if there is an integer $m > 0$ such that $a_{im} = 1$ for all $i \geq 0$, then we call $T(0)$ a periodic critical tableau.

Theorem 9 [YOC]. *The critical tableau $T(0)$ is periodic if and only if P_c is renormalizable.*

Proof. Suppose $T(0)$ is periodic. Let $n_1 > 0$ be the smallest integer such that $a_{in_1} = 1$ for all $i \geq 0$. Let $N \geq 0$ be the smallest integer such that $a_{ij} = 0$ for all $i \geq N$ and $0 < j < n_1$. For any $n \geq n_1 + N$, $P_c^{\circ n_1} : C_n \to C_{n-n_1}$ is a degree two proper holomorphic branch covering map. Thus $\{P^{\circ kn_1}(0)\}_{k=0}^{\infty}$ is contained in C_{n_1+N}. If $C_{n_1+N} \subset \mathring{C}_N$, then $P_c^{\circ n_1} : \mathring{C}_{n_1+N} \to \mathring{C}_N$ is a quadratic-like map with connected filled-in Julia set and $\mathring{C}_{n_1+N}$ is a renormalization about n_1. Thus P_c is renormalizable.

In general, let us consider a small open disk $D(\alpha)$ centered at the separate fixed point α of P_c such that

$$\overline{D}(\alpha) \subset D'(\alpha) = P_c^{\circ n_1}\Big(D(\alpha)\Big)$$

and such that

$$D'(\alpha) \cap \{P_c^{\circ i}(0)\}_{i=0}^{n_1+N} = \emptyset.$$

Thicken C_0 and B_{0i} for $1 \leq i \leq q-1$ as follows. Suppose C_0 (respectively, B_{0i}) is bounded by two external rays R_{θ_1} and R_{θ_2} of angles θ_1 and θ_2. Let $\epsilon > 0$ be a small number such that the domains

$$U_1 = \cup_{\theta_1-\epsilon<\theta<\theta_1+\epsilon}(R_\theta \setminus D_\alpha) \qquad \text{and} \qquad U_2 = \cup_{\theta_2-\epsilon<\theta<\theta_2+\epsilon}(R_\theta \setminus D_\alpha)$$

are disjoint from $\{P_c^{\circ i}(0)\}_{i=0}^{n_1+N}$. Let

$$\tilde{C}_0 = (U_1 \cup C_0 \cup U_2 \cup D(\alpha)) \cap U_r \qquad \text{(respectively,} \qquad \tilde{B}_{0i} = (U_1 \cup B_{0i} \cup U_2 \cup D(\alpha)) \cap U_r)$$

where U_r is the domain bounded by the equipotential curve s_r. Let

$$\tilde{\eta}_0 = \{\tilde{C}_0, \tilde{B}_{01}, \ldots, \tilde{B}_{0(q-1)}\}$$

and let

$$\tilde{\eta}_n = P^{-n}(\tilde{\eta}_n) = \{\tilde{C}_n, \tilde{B}_{n1}, \ldots, \tilde{B}_{nk_n}\}$$

for $1 \leq n \leq n_1 + N$. The diffeomorphism $g = P_c^{-(n_1-1)} : C_N \to P_c(C_{n_1+N})$ can be extended to $\tilde{C}_N$. Let B' be the image of $\tilde{C}_N$ under g. Then $\tilde{C}_{n_1+N} = P_c^{-1}(B')$. Let $\tilde{\mathring{C}}_n$ denote the interior of $\tilde{C}_n$ for $0 \leq n \leq n_1 + N$. Then $\tilde{C}_{n_1+N} \subset \tilde{\mathring{C}}_N$. Thus

$$P_c^{\circ n_1} : \tilde{\mathring{C}}_{n_1+N} \to \tilde{\mathring{C}}_N$$

is a quadratic-like map and $\tilde{\mathring{C}}_{n_1+N}$ is a renormalization about n_1. This proves the "only if" part.

Now suppose P_c is renormalizable. Let U_1 be a renormalization about n_1, that is, (U_1, V_1, f_1) is a quadratic-like map with connected filled-in Julia set K_{f_1} where $f_1 = P_c^{\circ n_1}|U_1$ and $V_1 = f_1(U_1)$. The map f_1 has two fixed points β_1 and α_1 in U_1. Let

$$\alpha_1 \in \cdots \subseteq D_n(\alpha_1) \subseteq D_{n-1}(\alpha_1) \subseteq \cdots \subseteq D_1(\alpha_1) \subseteq D_0(\alpha_1)$$

be an α_1-end. There is a $D_k(\alpha_1)$ such that $K_{f_1} \subset D_k(\alpha_1)$ and $D_k(\alpha_1) = C_k$. Since

$$K_{f_1} \subseteq f_1^{-1}(U_1 \cap C_k) \subseteq U_1 \cap C_k,$$

then $P_c^{\circ n_1}$ sends C_{k+in_1} to $C_{k+(i-1)n_1}$ for all $i > 0$. Thus $T(0)$ is periodic. It is the "if" part. ∎

We define a function τ on the set $\mathbf{N}$ of natural numbers by using the critical tableau $T(0) = (a_{nm}^0)_{n\geq 0, m\geq 0}$ as follows: $\tau(n) = m$ if $a_{(n-i)i}^0 = 0$ for $0 < i < n - m$ and if $a_{m(n-m)}^0 = 1$; if there is no such integer $m > 0$, then $\tau(n) = -1$.

If the critical tableau $T(0)$ is periodic, then there are integers $n_1 > 0$ and $N \geq 0$ such that $a_{in_1} = 1$ for all $i \geq 0$ and such that $a_{ij} = 0$ for $i \geq N$ and $0 < j < n_1$. Thus $\tau(n) = n - n_1$ for $n \geq N + n_1$.

If the critical tableau $T(0)$ is non-recurrent, then there is the smallest integer $N \geq 0$ such that $a_{nm} = 0$ for all $n \geq N$ and $m > 0$. Thus the image $\tau(\mathbf{N})$ is contained in the finite set $\{-1, 0, 1, \ldots, N-1\}$.

If the critical tableau $T(0)$ is not periodic and is recurrent, then every row of $T(0)$ has infinitely many 1's and every column (except for the 0^{th}-column) has a 0. An integer $n \geq 0$ is noble if for every entry a^0_{nk} such that $a^0_{nk} = 1$, we have $a^0_{(n+1)k} = 1$.

Lemma 5. *If the critical tableau $T(0)$ is not periodic and is recurrent, then the function τ satisfies the following properties:*

(i) For any integer $m \geq 0$, $\tau^{-1}(m)$ is not empty.

(ii) If $m \geq 0$ is noble, then $\tau^{-1}(m)$ contains at least two different integers.

(iii) If $\tau(n) = m$ and if m is noble, then n is also noble.

(iv) If $\tau^{-1}(m)$ contains only one integer n, then n is noble.

Proof. We prove **(i)** first. Consider any m-row in $T(0)$ for $m \geq 0$. Let $k > 0$ be the integer such that $a^0_{mi} = 0$ for $0 < i < k$ and such that $a^0_{mk} = 1$. From **(T1)**, $a^0_{(m+k-i)i} = 0$ for $0 < i < k$. Thus $\tau(m+k) = m$.

To prove **(ii)**, suppose $m \geq 0$ is noble. Let k be the same integer as that in the proof of **(i)**. Let m_1 be the integer such that $a^0_{m_1 k} = 1$ and such that $a^0_{ik} = 0$ for all $i > m_1$. Consider $a^0_{(m_1-k)(2k)}$, $a^0_{(m_1-2k)(3k)}$, $\ldots$, and $a^0_{(m_1-(i-1)k)(ik)}$ where $m_1 - ik \leq m < m_1 - (i-1)k$. From the tableau rules **(T1)** and **(T3)**, $a^0_{(m_1-k+1)(2k)} = 0$, $a^0_{(m_1-2k+1)(3k)} = 0$, $\ldots$, $a^0_{(m_1-(i-1)k+1)(ik)} = 0$. If $m = m_1 - ik$, from the tableau rules **(T1)** and **(T3)**, $a^0_{m(ik)} = 1$. Since m is noble, $a^0_{(m+1)(ik)} = 1$. But from the tableau rules **(T1)** and **(T3)**, $a^0_{(m+1)(ik)} = 0$. The contradiction implies that $m > m_1 - ik$.

Now from the tableau rule **(T2)**, $a^0_{m(k+m_1-m)} = 0$. Let $k_1 > k + m_1 - m$ be the integer such that $a^0_{mi} = 0$ for $k + m_1 - m < i < k_1$ and such that $a^0_{mk_1} = 1$. Then $a^0_{(m+k_1-i)i} = 0$ for $0 < i < k_1$ and $a^0_{mk_1} = 1$. This says that $\tau(m + k_1) = m$.

To prove **(iii)**, suppose $\tau(n) = m$ where m is noble. For any $a^0_{nk} = 1$, since $a^0_{(n-i)i} = 0$ for $0 < i < n - m$ and $a^0_{m(n-m)} = 1$ and since the tableau rule **(T1)**, we have $a^0_{(n-i)(k+i)} = 0$ for $0 < i < n - m$ and $a^0_{m(k+n-m)} = 1$. Since m is noble, $a^0_{(m+1)(n-m)} = 1$. Assume $a^0_{(n+1)k} = 0$. From the tableau rule **(T3)**, $a^0_{(m+1)(k+n-m)} = 0$. This contradicts to that m is noble. Thus $a^0_{(n+1)k} = 1$. This means that n is noble.

Now we prove **(iv)**. Suppose $n > 0$ is the only integer such that $\tau(n) = m$. We first consider $a^0_{(m+1)(n-m)}$. If $a^0_{(m+1)(n-m)} = 0$, then we would have an integer $k > n - m$ such that $a^0_{mi} = 0$ for $n - m < i < k$ and such that $a^0_{mk} = 1$. From the tableau rule **(T1)**, $a^0_{(m+k-i)i} = 0$ for $0 < i < k$. This would imply that $\tau(m+k) = m$, which contradicts the assumption. Thus, $a^0_{(m+1)(n-m)} = 1$. If there is an entry $a^0_{nk_1} = 1$ with $a^0_{(n+1)k_1} = 0$, then $k_1 > n - m$ and, from the tableau rule **(T3)**, $a^0_{m(n-m+k_1)} = 1$ and $a^0_{(m+1)(n-m+k_1)} = 0$. Consider the smallest integer $k_2 > n - m + k_1$ such that $a^0_{mi} = 0$ for $k_1 + n - m < i < k_2$ and $a^0_{mk_2} = 1$. From the tableau rule **(T1)**, $a^0_{(m+k_2-i)i} = 0$ for $k_1 + n - m < i < k_2$. So we can find another integer $n_0 \geq k_2 - k_1 + m > n$ such that $\tau(n_0) = m$. This would contradict the assumption. ∎

Theorem 10 [YOC]. *Suppose $P_c(z) = z^2 + c$ is a recurrent quadratic polynomial. The critical tableau $T(0)$ is periodic if and only if L_0 contains more than one point.*

Proof. We use the same notation as in the proof of Theorem 9 and the proof of Lemma 5. Suppose $T(0)$ is periodic. Then for $n > N + n_1$

$$P_c^{\circ n_1} : C_{n+1} \to C_{n-n_1+1}$$

is a degree two proper holomorphic branch covering map. Replacing C_{n+1} by $\tilde{C}_{n+1}$ if it is necessary, we may assume that this map is a quadratic-like map. Since L_0 is the filled-in Julia set of this map, it contains more than one point. This is the "only if" part.

To prove the "if" part, suppose $T(0)$ is not periodic. We will prove that L_0 contains only one point. Since P_c is recurrent, there are infinitely many 1's in every row of $T(0)$, that is, $T(0)$ is recurrent. Consider the first partition

$$\eta_1 = \{C_1, B_{11}, \ldots, B_{1(q-1)}, B_{01}, \ldots B_{0(q-1)}\},$$

where $B_{0i} = B_{0,i}$ for $1 \leq i < q$ and where $B_{1i} \subseteq C_0$ and $P_c(B_{1i}) = B_{0,i}$ for $1 \leq i < q$. (Remember that $\eta_0 = \{C_0, B_{0,1}, \ldots, B_{o,q-1}\}$ is the original partition.) Let $c(n) = P_c^{\circ n}(0)$. If the critical orbit $CO = \{c(n)\}_{n=0}^{\infty}$ is contained in the union

$$C_1 \cup B_{01} \cup \ldots \cup B_{0(q-1)},$$

then $T(0)$ is periodic of period q. Hence there must be one critical value $c(n)$ in $B_{11} \cup \cdots \cup B_{1(q-1)}$. Let $c(n)$ be in B_{1i}. The annulus $A_{0n}(0) = C_0 \setminus \mathring{B}_{1i}$ is non-degenerate. Pull back $A_{0n}(0)$ by P_c along $A_{i(n-i)}(0)$ for $0 \leq i \leq n$; we get a non-degenerate annulus $A_{n0}(0)$.

Now consider $\tau^{-k}(n)$. For each m in $\tau^{-k}(n)$,

$$\mathrm{mod}\Big(A_{m0}(0)\Big) \geq \frac{\mathrm{mod}\Big(A_{n0}(0)\Big)}{2^k}.$$

If the number of $\tau^{-k}(n)$ is greater than or equal to 2^k for every $k > 0$, then

$$\operatorname{mod}(C_0 \setminus L_0) \geq \sum_{m=1}^{\infty} \operatorname{mod}\Big(A_{m0}(0)\Big) \geq \sum_{k=1}^{\infty} \sum_{m\in\tau^{-k}(n)} \operatorname{mod}\Big(A_{m0}(0)\Big) \geq \sum_{k=1}^{\infty} \operatorname{mod}\Big(A_{n0}(0)\Big)$$
$$= \Big(\operatorname{mod}\Big(A_{n0}(0)\Big)\Big) \cdot \sum_{k=1}^{\infty} 1 = \infty.$$

So $L_0 = \{0\}$.

If there is an integer $k > 0$ such that the number of $\tau^{-k}(n)$ is less than 2^k, then there are pre-images $m > q$ of n under iterates of τ such that m is the only pre-image of q under τ. From **(iii)**, m is noble. Hence $\tau^{-k}(m)$ are all noble and contain at least 2^k different integers. Moreover

$$\operatorname{mod}(A_{p0}(0)) = \frac{\operatorname{mod}(A_{m0}(0))}{2^k}$$

for every p in $\tau^{-k}(m)$. Therefore,

$$\operatorname{mod}(C_0 \setminus L_0) \geq \sum_{k=1}^{\infty} \operatorname{mod}\Big(A_{k0}(0)\Big) \geq \sum_{k=1}^{\infty} \sum_{p\in\tau^{-k}(m)} \operatorname{mod}\Big(A_{p0}(0)\Big) \geq \sum_{k=1}^{\infty} \operatorname{mod}\Big(A_{m0}(0)\Big)$$
$$= \Big(\operatorname{mod}\Big(A_{m0}(0)\Big)\Big) \cdot \sum_{k=1}^{\infty} 1 = \infty.$$

Again we have $L_0 = \{0\}$. This completes the “if” part. ■

Theorem 11 [YOC]. *If $P_c(z) = z^2 + c$ is a non-recurrent or recurrent non-renormalizable quadratic polynomial, then its filled-in Julia set K_c is locally connected.*

Proof. Let α be the separate fixed point of P_c. Construct the two-dimensional Yoccoz puzzle for P_c. For any x in K_c, let

$$x \in \cdots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \cdots \subseteq D_1(x) \subseteq D_0(x)$$

be an x-end. If P_c is non-recurrent, then the critical tableau is non-recurrent. Lemma 3 and Lemma 4 imply that the diameter $d(D_n(x))$ tends to zero as n goes to infinity. If P_c is recurrent and non-renormalizable, then $T(0)$ is recurrent and is not periodic. Lemma 4 and Theorem 10 imply that the diameter $d(D_n(x))$ tends to zero as n goes to infinity.

If x is not a preimage of α under any iterate of P_c, then x is an interior point of $D_n(x)$ for all $n \geq 0$. From Lemma 2, $\{D_n(x)\}$ is a basis of connected neighborhoods at x. If x is a preimage of α under some iterate of P_c, then there are q different x-ends,

$$x \in \cdots \subseteq D_{i,n}(x) \subseteq D_{i,(n-1)}(x) \subseteq \cdots \subseteq D_{i,1}(x) \subseteq D_{i,0}(x)$$

where q is the period of the external rays landing at α. Let $\tilde{D}_n(x) = \cup_{i=1}^{q} D_{i,n}(x)$. Then x is an interior point of $\tilde{D}_n$. Since $K_c \cap D_{1,n}(x), \ldots, K_c \cap D_{q,n}(x)$ have a common point x, from Lemma 2, $K_c \cap \tilde{D}_n(x)$ is connected. So $\{\tilde{D}_n(x)\}_{n=0}^{\infty}$ is a basis of connected neighborhoods at x. ∎

From Theorem 4, all arguments in this section apply to a quadratic-like map. Suppose that (U, V, f) is a quadratic-like map and that its filled-in Julia set K_f is connected. Suppose two fixed points β and α of f are repelling. Let β be the non-separate fixed point of f, that is, $K_f \setminus \{\beta\}$ is connected, and let α be the separate fixed point of f, that is, $K_f \setminus \{\alpha\}$ is disconnected. Since (U, V, f) is hybrid equivalent to a quadratic polynomial P_c, there is a quasiconformal homeomorphism H defined on V such that

$$H \circ f = P_c \circ H$$

on U. We call $e_{\theta,f} = H^{-1}(e_\theta \cap H(U))$ the external ray of angle θ of f where e_θ is the external ray of P_c of angle θ.

Two points $H(\beta)$ and $H(\alpha)$ are non-separate and separate fixed points of P_c, respectively. Suppose Γ is the union of a cycle of periodic external rays landing at $H(\alpha)$. Let $\Gamma' = H^{-1}(\Gamma \cap H(U))$. The set Γ' cuts the domain U into q domains. Each of them contains points in the filled-in Julia set K_c. Let C_0 be the domain containing 0 and let $B_{0,i}$ be the domain containing $f^{\circ i}(0)$ for $1 \leq i < q$. The partition

$$\eta_0 = \{C_0, B_{0,1}, \ldots, B_{0,q-1}\}$$

is called the original partition for f. Let $\Gamma'_n = f^{-n}(\Gamma')$ and $U_n = f^{-n}(U)$. Then Γ'_n cuts U_n into finitely many domains. Let C_n be the domain containing 0 and $B_{n,i}$ for $1 \leq i \leq k_n$ be others. Then

$$\eta_n = \{C_n, B_{n,1}, \ldots, B_{n,k_n}\}$$

is called the n^{th}-partition for f. We use $f^{-n}(\eta_0)$ to denote η_n, i.e., $\eta_n = f^{-n}(\eta_0)$, for $1 \leq n < \infty$. We have that $f(C_n)$ and $f(B_{n,i})$ for $1 \leq i \leq k_n$ are in η_{n-1} for $n > 0$ (set $k_0 = q - 1$). We call $\eta = \{\eta_n\}_{n=0}^{\infty}$ the two-dimensional Yoccoz puzzle of f. Let $\Lambda = \cup_{n=0}^{\infty} f^{-n}(\alpha)$. Let $L_0 = \cap_{n=0}^{\infty} C_n$ be the connected component of $K_f \setminus \Lambda$ containing 0. We state Theorems 10 and 11 in the following form.

Theorem 12 [YOC]. *Suppose (U, V, f) is a recurrent quadratic-like map. Then (U, V, f) is renormalizable if and only if L_0 contains more than one point. Moreover, if (U, V, f) is non-renormalizable, then any connected component of $K_f \setminus \Lambda$ consists of only one point and K_f is locally connected.* ∎

5. Infinitely Renormalizable Quadratic Julia Sets and Three-Dimensional Yoccoz Puzzles

Suppose (U, V, f) is a renormalizable quadratic-like map with connected filled-in Julia set K_f. Let $\eta = \{\eta_n\}_{n=0}^{\infty}$ be the two-dimensional Yoccoz puzzle for f. From the previous section, the critical tableau $T(0) = (a^0_{nm})_{n\geq 0, m\geq 0}$ is periodic of period n_1. Let

$$0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_1 \subseteq C_0$$

be the critical end. There is an integer $N > 0$ such that $a^0_{ij} = 0$ for all $i \geq N$ and for all $0 < j < n_1$. Let $f_0 = f^{\circ n_1}|\mathring{C}_{N+n_1}$. Then $f_0 : \mathring{C}_{N+n_1} \to \mathring{C}_N$ is a proper, holomorphic branch cover of degree two. Assume $C_{N+n_1} \subset \mathring{C}_N$. (Otherwise, we can replace C_n with $\tilde{C}_n$ (see Theorem 9)). Then $(\mathring{C}_{N+n_1}, \mathring{C}_N, f_0)$ is a quadratic-like map and its filled-in Julia set is $L_0 = \cap_{n=0}^{\infty} C_n$.

Theorem 13. *Suppose (U, V, f) is a renormalizable quadratic-like map with connected filled-in Julia set K_f. For any renormalization U_1 about n_1, let $f_1 = f^{\circ n_1}|U_1$ and let $V_1 = f_1(U_1)$. Then the filled-in Julia set K_{f_1} (or the Julia set J_{f_1}) of (U_1, V_1, f_1) is always L_0 (or ∂L_0).*

Proof. Let $U' = \mathring{C}_{N+n_1} \cap U_1$ and let U'' be the connected component of U' containing 0. Let $f_2 = f^{\circ n_1}|U''$ and $V'' = f_2(U'')$. Then $f_2 : U'' \to V''$ is a degree two branch covering. It is also proper because

$$f_2^{-1}(K) = f_0^{-1}(K) \cap f_1^{-1}(K)$$

for any compact set K of V''. Since $C_{N+n_1} \subset \mathring{C}_N$ and $\overline{U}_1 \subset V_1$, then

$$C_{N+n_1} \cap \overline{U}_1 \subset \mathring{C}_N \cap V_1.$$

Thus $\overline{U''} \subset V''$ because they are the connected components of U' and $\mathring{C}_N \cap V_1$ containing 0. Both U'' and V'' are simply connected and isomorphic to a disc because they are intersections of simply connected domains each of which is isomorphic to a disc. Therefore, (U'', V'', f_2) is a quadratic-like map. Let K_{f_2} be the filled-in Julia set of (U'', V'', f_2). Since a filled-in Julia set is completely invariant and since 0 is in U'', the two inverse images of 0 under f_2 are in K_{f_2}. But these two points are also inverse images of 0 under f_0 and under f_1. Therefore, they are both in L_0 and in K_{f_1}. Using this argument, the set Ξ of all inverse images of 0 under iterates of f_2 is contained in K_{f_2} and is also contained in L_0 and in K_{f_1}. Therefore,

$$K_{f_1} = K_{f_2} = L_0 \text{ (or } J_{f_1} = J_{f_2} = \partial L_0)$$

because each of ∂L_0, $J_{f_1} = \partial K_{f_1}$, and $J_{f_2} = \partial K_{f_2}$ is the closure of Ξ (see Theorem 3). ∎

As we saw in §2, the definition of the filled-in Julia set of a renormalization about n_1 depends on choices of domains in renormalization. But the renormalized filled-in Julia set is actually canonical; it is independent of choices of domains in renormalization and is the limiting piece containing 0 in the two-dimensional Yoccoz puzzle from Theorem 12.

Suppose (U_1, V_1, f_1) is a recurrent renormalizable quadratic-like map with connected filled-in Julia set K_1. We call K_1 (or J_1) a quadratic filled-in Julia set (or quadratic Julia set). It is renormalizable if the corresponding quadratic-like map is renormalizable. Let β_1 and α_1 be the non-separate and separate fixed points of f_1, i.e., $K_1 \setminus \{\beta_1\}$ is still connected and $K_1 \setminus \{\alpha_1\}$ is disconnected. Let $\Lambda_1 = \cup_{n=0}^{\infty} f_1^{-n}(\alpha_1)$. Let $K_2 = L_0$ be the connected component of $K_1 \setminus \Lambda_1$ containing 0. From Theorem 12, K_1 is renormalizable if and only if K_2 contains more than one point. The quadratic filled-in Julia set K_2 is called the renormalization of K_1.

Inductively, let K_i be the renormalization of K_{i-1}. Let $f_i = f_{i-1}^{\circ n_{i-1}}$ for $i \geq 2$, where n_{i-1} is the period of the critical tableau $T^{i-1}(0) = (a_{nm}^0(i-1))_{n\geq 0, m\geq 0}$ of the two-dimensional Yoccoz puzzle for $(U_{i-1}, V_{i-1}, f_{i-1})$. Let β_i and α_i be the non-separate and the separate fixed points of f_i, i.e., $K_i \setminus \{\beta_i\}$ is still connected and $K_i \setminus \{\alpha_i\}$ is disconnected. Let $\Lambda_i = \cup_{n=0}^{\infty} f_i^{-n}(\alpha_i)$ and let K_{i+1} be the connected component of $K_i \setminus \Lambda_i$ containing 0. Then K_i is renormalizable if and only if K_{i+1} contains more than one point. Here K_i, for $i > 1$, is called the i^{th}-renormalization of K_1. Theorem 12 can be generalized as follows.

Theorem 14 [YOC]. *Suppose that (U_1, V_1, f_1) is a recurrent quadratic-like map and that K_1 is its filled-in Julia set. The quadratic Julia set K_1 is finitely renormalizable if and only if there is an integer $m \geq 1$ such that $K_1, \ldots, K_m$ contains more than one point and such that K_{m+1} contains only the point 0. Moreover, if K_1 is finitely renormalizable, then K_1 is locally connected.*

Proof. The first part of the theorem follows directly from Theorem 12. We prove the second part. Let α_m be the separate fixed point of f_m. Let Γ_m be a cycle of periodic external rays of f_1 landing at α_m (refer to the end of §3). Using Γ_m, we can construct the two-dimensional Yoccoz puzzle: let η_0^m be the set consisting of the closures of the connected components of $V_1 \setminus \Gamma_m$. Let $\eta_n^m = f_1^{-n}(\eta_0^m)$ for $n \geq 1$. Let C_n^m be the member of η_n^m containing 0. Since f_m is non-renormalizable, we use a proof similar to that of Theorem 10 to show that $\sum_{n=0}^{\infty} \mathrm{mod}(A_{n0}^m(0)) = \infty$, where $A_{n0}^m(0) = C_n^m \setminus \mathring{C}_{n+1}^m$. Applying Lemma 4, for every x-end

$$x \in \cdots \subseteq D_n^m(x) \subseteq D_{n-1}^m(x) \subseteq \cdots \subseteq D_1^m(x) \subseteq D_0^m(x),$$

$L_x = \cap_{n=0}^{\infty} D_n^m(x)$ contains only x. By using a similar argument to the proof of Theorem 11, we can now show that K_1 is locally connected. ∎

Now let us consider an infinitely renormalizable quadratic-like map (U_1, V_1, f_1). Let K_1 be the filled-in Julia set of f_1. Let K_i be the i^{th}-renormalization of K_1. Then $\mathcal{K} = \{K_i\}_{i=1}^{\infty}$ is a sequence of renormalizations of K_1. Let $\{(U_i, V_i, f_i)\}_{i=1}^{\infty}$ be a sequence of renormalizations with filled-in Julia set K_i where $f_i = f_{i-1}^{\circ n_{i-1}}$ and where n_{i-1} is the period of the critical tableau $T^{i-1}(0)$ of the two-dimensional Yoccoz puzzle for $(U_{i-1}, V_{i-1}, f_{i-1})$, for $i \geq 2$. Suppose U_i is a renormalization of $(U_{i-1}, V_{i-1}, f_{i-1})$.

We describe (U_1, V_1, f_1) as $(n_1, n_2, \dots)$-infinitely renormalizable. The grid $\{T^i(0)\}_{i=1}^{\infty}$ is called the three-dimensional critical tableau for (U_1, V_1, f_1). Let $c(n) = f_1^{\circ n}(0)$. The critical orbit of f_1 is $CO = \{c(n)\}_{n=0}^{\infty}$. Let $GCO = \cup_{k=0}^{\infty} \cup_{n=0}^{\infty} f_1^{-k}(c(n))$ be the grand critical orbit of f_1.

Definition 2. *An infinitely renormalizable quadratic-like map (U_1, V_1, f_1) has a priori complex bounds if there are a constant $\lambda > 0$ and a sequence of renormalizations $\{(U_{i_k}, V_{i_k}, f_{i_k})\}_{k=1}^{\infty}$ of f_1 such that*

$$mod(V_{i_k} \setminus U_{i_k}) \geq \lambda$$

for all $k \geq 1$.

Theorem 15. *Suppose (U_1, V_1, f_1) is an infinitely renormalizable quadratic-like map having the a priori complex bounds. Its filled-in Julia set K_1 is locally connected at every point in GCO.*

Proof. Suppose, without loss of generality, that $\{U_i, V_i, f_i\}_{i=1}^{\infty}$ is the sequence of renormalizations in Definition 2. Let $\lambda > 0$ be the constant in Definition 2. Then $\{U_i\}_{i=1}^{\infty}$ is a sequence of nested domains containing 0. Consider the annulus $A_i = \overline{U}_i \setminus U_{i+1}$ for $i \geq 1$. For each $i \geq 1$, let $cv_i = f_i(0)$ and let γ_i be a curve in $V_i \setminus f_i^{\circ 2}(U_{i+1})$ connecting cv_i and a point on the boundary of V_i. Let $0 \in U_i' \subset U_i$ be the connected component of the pre-image of $V_i \setminus \gamma_i$ under $f_i^{\circ 2}$. Then $f_i^{\circ 2} : U_i' \to V_i \setminus \gamma_i$ is a degree two branch covering. Moreover, $f_i : U_i' \to f_i(U_i') \subset U_i$ is also a degree two branch covering map. Thus $f_i : U_i \setminus U_i' \to V_i \setminus f(U_i')$ is a degree two branch covering map. This implies that

$$\mathrm{mod}(U_i \setminus U_i') = \frac{1}{2} \cdot \mathrm{mod}(V_i \setminus f_i(U_i')).$$

But $V_i \setminus U_i$ is a sub-annulus of $V_i \setminus f_i(U_i')$. So

$$\mathrm{mod}(U_i \setminus U_i') \geq \frac{1}{2} \cdot \mathrm{mod}(V_i \setminus U_i) > \frac{\lambda}{2}.$$

Remember that U_{i+1} is the domain of the renormalization $f_{i+1} = f_i^{n_i}$ where $n_i \geq 2$. We have $U_{i+1} \subset U_i'$. Hence $U_i \setminus U_i'$ is a sub-annulus A_i. So

$$\mathrm{mod}(A_i) > \frac{\lambda}{2}.$$

Let $A_\infty = \cap_{i=1}^{\infty} U_i$. Since $U_1 \setminus A_\infty = \cup_{i=1}^{\infty} A_i$,

$$\mathrm{mod}(U_1 \setminus A_\infty) \geq \sum_{i=1}^{\infty} \mathrm{mod}(A_i) = \infty.$$

Thus, $A_\infty = \{0\}$. This implies that the diameter $d(U_i)$ tends to 0 as i goes to infinity.

Let α_i be the separate fixed point of f_i. Let Γ_i be a cycle of periodic external rays of f_1 landing at α_i (refer to the end of §3). Let η_0^i be the set consisting of the closures of the connected components of $U_1 \setminus \Gamma_i$ and let $\eta_n^i = f_1^{-n}(\eta_1^i)$ for $n \geq 1$. Then $\{\eta_n^i\}_{n=0}^\infty$ is a puzzle for f_1. Let C_n^i be the member of η_n^i containing 0. Consider the critical end

$$0 \in \cdots \subseteq C_n^i \subseteq C_{n-1}^i \subseteq \cdots \subseteq C_1^i \subseteq C_0^i$$

and the critical tableau $T^i(0) = (a_{nm}^0(i))_{n\geq 0, m\geq 0}$. Since (U_1, V_1, f_1) is $(n_1, n_2, \dots)$-renormalizable, $T^i(0)$ is periodic of period $m_i = \prod_{j=1}^i n_j$. Thus there is an integer $N > 0$ such that $a_{nm}^0(i) = 0$ for $n \geq N$ and $0 < i < m_i$. Thus

$$f_1^{\circ m_i} = f_i^{\circ n_i} : \mathring{C}_{m_i+N}^i \to \mathring{C}_N^i$$

is a degree two proper holomorphic map. We may assume that $C_{m_i+N}^i \subset \mathring{C}_N^i$ (otherwise, we can modify C_n^i as the proof of Theorem 9). Therefore,

$$f_{i+1} = f_1^{\circ m_i} = f_i^{\circ n_i} : \mathring{C}_{m_i+N}^i \to \mathring{C}_N^i$$

is a quadratic-like map. Since

$$K_{i+1} = \cap_{j=0}^\infty f_1^{-jm_i}(C_N^i) = \cap_{j=0}^\infty C_{jm_i+N}^i,$$

there is a $C_{k(i)}^i$ contained in U_i. The diameter $d(C_{k(i)}^i)$ of $C_{k(i)}^i$ tends to zero as i goes to infinity. From Lemma 2, $C_{k(i)}^i \cap K_1$ is connected. So $\{C_{k(i)}^i\}_{i=1}^\infty$ is a basis of connected neighborhoods at 0.

For any $x = f_1^{\circ n}(0)$, consider $\{D_{i,n}(x) = f_1^{\circ n}(C_{k(i)}^i)\}_{i=1}^\infty$. It is a basis of connected neighborhoods at x. For any y in $f_1^{-m}(x)$ (where 0 is not in $f^{-n}(x)$ for $0 < n \leq m$), there is an open neighborhood W of y such that $f_1^{\circ m} : W \to f^{\circ m}(W)$ is a homeomorphism. Let g be its inverse. Then $\{g(D_{i,n}(x))\}_{i=1}^\infty$ is a basis of connected neighborhoods at y. ∎

Suppose (U_1, V_1, f_1) is an $(n_1, n_2, \dots)$-infinitely renormalizable quadratic-like map. We call the puzzle $\{\{\eta_n^i\}_{n=0}^\infty\}_{i=1}^\infty$ constructed in Theorem 15, the three-dimensional Yoccoz puzzle for (U_1, V_1, f_1). Let $m_i = \prod_{j=1}^i n_j$ and $\{K_i\}_{i=1}^\infty$ be the sequence of renormalizations of K_1. Let $c(j) = f^{\circ j}(0)$ for $j \geq 0$, and let $CO = \{c(j)\}_{j=0}^\infty$ be the critical orbit of f_1.

Definition 3. *An infinitely renormalizable quadratic-like map (U_1, V_1, f_1) is unbranched if there are a constant $\lambda > 0$ and a sequence of domains $\{W_k\}_{k=1}^\infty$ such that $W_k \supset K_{i_k}$, such that*

$$mod(W_k \setminus K_{i_k}) \geq \lambda,$$

and such that $W_k \cap CO = \{c(jm_{i_k})\}_{j=0}^\infty$ for all $k \geq 1$

Theorem 16. *Suppose (U_1, V_1, f_1) is an infinitely renormalizable unbranched quadratic-like map having the a priori complex bounds. Then its filled-in Julia set K_1 is locally connected.*

Proof. Suppose, without loss of generality, that $k = i$ and $i_k = i$ in Definition 3, and that $\{U_i, V_i, f_i\}_{i=1}^{\infty}$ is a sequence of renormalizations in Definition 2. Let $\lambda > 0$ be a constant satisfying Definitions 2 and 3.

Let $\{\{\eta_n^i\}_{n=0}^{\infty}\}_{i=1}^{\infty}$ be the three-dimensional Yoccoz puzzle for (U_1, V_1, f_1). Let $\{C_{k(i)}^i\}_{i=1}^{\infty}$ be the basis of connected neighborhoods constructed in Theorem 15. By choosing $k(i)$ large enough, we can have

$$\mathrm{mod}(W_i \setminus C_{k(i)}^i) \geq \frac{\lambda}{2}$$

for all $i \geq 1$.

If $x = 0$, Theorem 15 says that K_1 is locally connected at x. For each $x \neq 0$ in K_1, there are two cases: either (1) x is non-recurrent, which means that there is an integer $i \geq 1$ such that $\{f_1^{\circ n}(x)\}_{n=0}^{\infty} \cap C_{k(i)}^i = \emptyset$; or else (2) x is recurrent.

In case (1), we prove it by applying the results in hyperbolic geometry. Let x be a non-recurrent point. Then there is a $C_{k(i)}^i$ such that the orbit $O(x) = \{f_1^{\circ n}(x)\}_{n=0}^{\infty}$ is disjoint from the interior of $C_{k(i)}^i$. Let r be an external ray of f_1 cutting $U_1 \setminus f_1(C_{k(i)}^i)$ into a simply connected domain Ω. Consider two inverse branches Q_1 and Q_2 of $f_1|\Omega$.

Let $\tilde{\Omega} = V_1 \setminus (f_1(C_{k(j)}^j) \cup r)$ for some $j > i$ such that $\overline{\Omega} \subset \tilde{\Omega}$. Let $\tilde{Q}_1$ and $\tilde{Q}_2$ be the inverses of $f_1|\tilde{\Omega}$ and let $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ be the images of $\tilde{\Omega}$ under $\tilde{Q}_1$ and $\tilde{Q}_2$, respectively. Consider $\tilde{\Omega}$, $\tilde{\Omega}_1$, and $\tilde{\Omega}_2$ as hyperbolic Riemann surfaces. Then $Q_1 : \Omega \to \Omega_1$ and $Q_2 : \Omega \to \Omega_2$ strictly contract the hyperbolic distances of $\tilde{\Omega}$, $\tilde{\Omega}_1$, and $\tilde{\Omega}_2$. Since Ω contains finite number of connected components of $K_1 \setminus f_1(C_{k(i)}^i)$, we can cut Ω into finite number of simply connected domains Ω_1', ..., Ω_l' such that $\Omega_k' \cap K_1$ is connected for each $1 \leq k \leq l$. Since the orbit $O(x)$ is contained in Ω, from the images of Ω_1', ..., Ω_l' under the semi-group generated by Q_1 and Q_2, we can get a basis of connected neighborhoods at x. Therefore, K_1 is locally connected at x.

In case (2), $f_1^{\circ n}(x)$ enters $C_{k(i)}^i$ infinitely many times. For each $i \geq 1$, consider the puzzle $\eta^i = \{\eta_n^i\}_{n=0}^{\infty}$. For the x-end,

$$x \in \cdots \subseteq D_n^i(x) \subseteq D_{n-1}^i(x) \subseteq \cdots \subseteq D_1^i(x) \subseteq D_0^i(x),$$

let $T^i(x) = (a_{nm}(i))_{n\geq 0, m\geq 0}$ be the corresponding tableau. Let q_i be the integer such that $a_{k(i)q} = 0$ for $0 \leq q < q_i$ and such that $a_{k(i)q_i} = 1$. Let $p_i = k(i) + q_i$. Then $g_{i,x} = f_1^{\circ q_i} : D_{p_i}^i(x) \to C_{k(i)}^i$ is a proper holomorphic diffeomorphism. Since there is no critical point in $W_i \setminus C_{k(i)}^i$, this diffeomorphism can be extended to a proper holomorphic diffeomorphism on W_i. Let $x_i = f_1^{\circ q_i}(x)$. We can modify W_j such that

$$\frac{\lambda}{2} \leq \mathrm{mod}(W_j \setminus C^j_{k(j)}) \leq 2\lambda.$$

From the previous theorem, the diameter $d(C^j_{k(j)})$ tends to zero as j goes to infinity. We can find an integer $j > i$ such that $W_j \subset C^i_{k(i)}$. Let

$$x_j = f_1^{\circ q_j}(x) = f_1^{\circ (q_j - q_i)}(x_i).$$

Consider the puzzle $\eta^j = \{\eta^j_n\}_{j=1}^{\infty}$. Let

$$x_i \in \cdots \subseteq D^j_n(x_i) \subseteq D^j_{n-1}(x_i) \subseteq \cdots \subseteq D^j_1(x_i) \subseteq D^j_0(x_i)$$

be the x_i-end and let $T^j(x_i) = (b_{nm}(j))_{n\geq 0, m\geq 0}$ be the corresponding tableau. Then one can check that $f_1^{\circ(q_j - q_i)}$ is a proper holomorphic diffeomorphism from

$$D^j_{k(j)+q_j-q_i}(x_i) \to C^j_{k(j)}.$$

In other words, $b_{k(j)m} = 0$ for $0 \leq m < q_j - q_i$, but $b_{k(j)(q_j - q_i)} = 1$. Let g_{ij} be the inverse of $f_1^{\circ(q_j-q_i)} : D^j_{k(j)+q_j-q_i}(x_i) \to C^j_{k(j)}$. Then g_{ij} can be extended to W_j. Since $C^i_{k(i)}$ is bounded by external rays landing at some pre-iamges of α_i under iterations of P_c and is a part of an invariant net under f_1, $W_{ij} = g_{ij}(W_j)$ is contained in $C^i_{k(i)}$. Thus

$$\mathrm{mod}(W_i \setminus W_{ij}) \geq \frac{\lambda}{2}.$$

Consider $X_i = g_{i,x}(W_i)$ and $X_j = g_{j,x}(W_j) = g_{i,x}(W_{ij})$. Then

$$\mathrm{mod}(X_i \setminus X_j) \geq \frac{\lambda}{2}.$$

Therefore, inductively, we can find a sequence of domains $\{X_{i_s}\}_{s=1}^{\infty}$ such that

$$\mathrm{mod}(X_{i_s} \setminus X_{i_{s+1}}) \geq \frac{\lambda}{2}$$

for $s \geq 1$. Thus the diameter of X_{i_s} tends to zero as s goes to infinity. For each puzzle $\eta^{i_s} = \{\eta^{i_s}_n\}_{n=0}^{\infty}$, consider both the x-end,

$$x \in \cdots \subseteq D^{i_s}_n(x) \subseteq D^{i_s}_{n-1}(x) \subseteq \cdots \subseteq D^{i_s}_1(x) \subseteq D^{i_s}_0(x)$$

and the corresponding tableau $T^{i_s}(x) = (a_{nm}(i_s))_{n\geq 0, m\geq 0}$. For each $k(i_s)$, there is an integer q_{i_s} such that $a^i_{k(i_s)q} = 0$ for $0 \leq q < q_{i_s}$ and such that $a^i_{k(i_s)q_{i_s}} = 1$. Then for $p_{i_s} = k(i_s) + q_{i_s}$, $f_1^{\circ q_{i_s}} : D_{p_{i_s}}(x) \to C_{k(i_s)}$ is a proper holomorphic diffeomorphism. This implies that

$$D^{i_s}_{p_{i_s}}(x) \subseteq X_{i_s}.$$

So the diameter $d(D^{i_s}_{p_{i_s}}(x))$ tends to zero as s goes to infinity. From Lemma 2, $\{D^{i_s}_{p_{i_s}}(x)\}_{s=1}^{\infty}$ forms a basis of connected neighborhoods at x. ∎

6. A Generalized Sullivan's Sector Theorem

Let $I = [0, 1]$ be the closed unit interval. Let $\mathcal{E}_0$ be the set of all functions G such that

(1) $G : I_G \supseteq I \to G(I_G)$ is a homeomorphism and $G(0) = 0$, $G(1) = 1$, and

(2) G can be extended to be a schlicht function g on $\mathbf{C}_G = (\mathbf{C} \setminus \mathbf{R}^1) \cup \mathring{I}_G$ preserving upper- and lower-half planes.

We assume I_G is the maximum interval satisfies **(1)** and **(2)** for each G in $\mathcal{E}_0$ and call it the definition interval of G. We will not distinguish g and G anymore. Take

$$S_\gamma(z) = r^{\frac{1}{\gamma}} e^{\frac{\theta}{\gamma} i} : \mathbf{C} \setminus \{x < 0\} \to \mathbf{C}$$

as the standard γ-root where $z = re^{\theta i}$ for $r > 0$, $-\pi < \theta < \pi$, and $\gamma > 1$. For every $a \le 0$, we call $L_a(z) = ES_\gamma(z - a) + F$ a γ-root at a where $E = 1/\Big((1-a)^{\frac{1}{\gamma}} - (-a)^{\frac{1}{\gamma}}\Big)$ and $F = -(-a)^{\frac{1}{\gamma}} E$ are determined by $L_a(0) = 0$ and $L_a(1) = 1$. Then L_a is an element in $\mathcal{E}_0$ whose definition interval is $[a, \infty)$.

Suppose that L_a is a γ-root at a and that G is an element in $\mathcal{E}_0$ whose definition interval is I_G. We say that L_a and G are compatible if $[a, 1] \subset G(I_G)$. For a compatible pair L_a and G, let $a' = G^{-1}(a)$, $J = [a', 1]$, $L \cup R = I_G \setminus J$, and $b = \min\{|L|, |R|\}$.

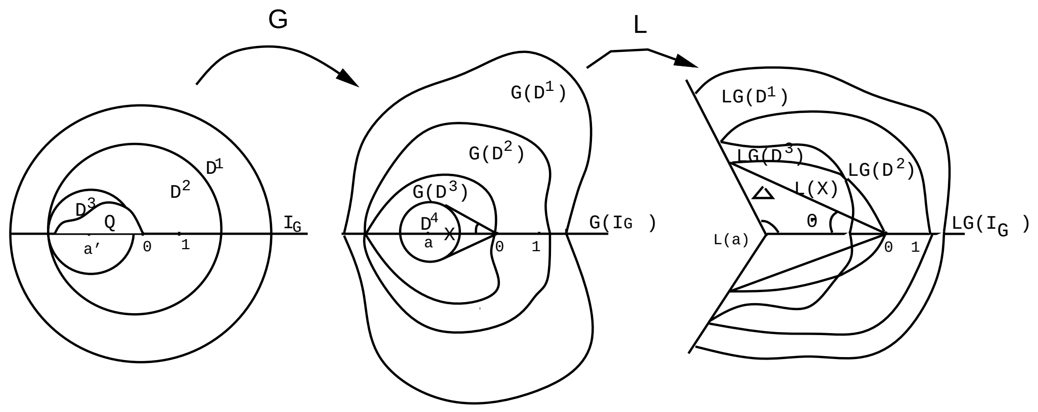

Fig. 3

We consider several disks related to a compatible pair L_a and G (see Fig. 3). Set D^1 to be the closed disk centered at the middle point $(1 + a')/2$ of $[a', 1]$ with diameter $1 + |a'| + 2b$. Set D^2 to be the closed disk centered at the middle point $(1 + a')/2$ of $[a', 1]$ with diameter $k = \min\{1 + |a'| + b, 2(1 + |a'|)\}$. Set D^3 to be the maximum closed disk centered at a' and contained in D^2. Then the radius of D^3 is $d = \min\{b/2, (1+|a'|)/2\}$. We note that

$$D^3 \subset D^2 \subset D^1.$$

The map G is a schlicht function on D^1. Let $\mu = (1+|a'|)/b$ and $\nu = (2\mu+3)^4$. From Koebe's distortion theorem (see [BIE]) for any ξ and η in D^2,

$$\nu^{-1} \leq \frac{|G'(\xi)|}{|G'(\eta)|} \leq \nu.$$

Using the fact that $G(I) = I$, there is at least one η in D^2 such that $|G'(\eta)| = 1$. Hence

$$\nu^{-1} \leq |G'(\xi)| \leq \nu$$

for all ξ in D^2. This implies that

$$\frac{|a'|}{\nu} \leq |a| \leq |a'| \cdot \nu.$$

Let $UH = \{z = x + yi \in \mathbf{C} \mid y > 0\}$ be the upper-half plane. For any z in UH, let $\theta(z) = \arg(z)$.

Theorem 17. *Suppose that $L = L_a$ and G are a pair of compatible elements in $\mathcal{E}_0$. There is a constant $0 < \theta < \pi$ depending only on μ such that the image $L(G(UH))$ of the upper-half plane under $L \circ G$ contains an open triangle Δ based on $[L(a), 0]$ whose angle at $L(a)$ is π/γ and whose angle at 0 is θ.*

Proof. The image $G(D^3)$ of D^3 under G contains the closed disk D^4 centered at a with radius d/ν. Similarly, for any $a' \leq x \leq 1$, consider the closed disk $D(x)$ centered at x with radius d. Then $D(x) \subset D^2$ and $G(D(x))$ contains the closed disk centered at $G(x)$ with radius d/ν. This implies that the convex-hull X of $\{0\} \cup D^4$ is contained in $G(D^2)$. Either $X = D^4$ or $X \cap UH$ has an angle at 0. In the later case, the angle φ of $X \cap UH$ at 0 has $\sin\varphi = d/(|a|\nu)$.

Since L is a γ-root for $\gamma > 1$, the convex set $L(X)$ contains a triangle Λ based on $[L(a), 0]$ whose angle at $L(a)$ is π/γ and whose angle ω at 0 can be calculated from

$$\frac{\sin\omega}{\sin(\frac{\pi}{\gamma}+\omega)} = \Big(\frac{d}{|a|\nu}\Big)^{\frac{1}{\gamma}}$$

through the law of sines. Because $d = \min\{b/2, (1+|a'|)/2\}$,

$$\frac{d}{|a|} \geq \frac{d}{|a'|\nu} \geq \min\{\frac{1}{2\mu\nu}, \frac{1}{2\nu}\}.$$

Hence Λ contains a triangle Δ based on $[L(a), 0]$ whose angle at $L(a)$ is still π/γ and whose angle θ at 0 is calculated from

$$\frac{\sin\theta}{\sin(\frac{\pi}{\gamma}+\theta)} = \Big(\min\{\frac{1}{2\mu\nu}, \frac{1}{2\nu}\}\Big)^{\frac{1}{\gamma}}. \qquad \blacksquare$$

Suppose $Q = (L\circ G)^{-1}(\overline{\Delta})$, where Δ is the triangle obtained in Theorem 17. Then

$$Q \subset D^2 \subset D^1.$$

Suppose $\{(L_i, G_i)\}_{i=0}^{n}$ is a sequences of compatible pairs in $\mathcal{E}_0$ where L_i is a γ-root at a_i. Let a_i', b_i, k_i, d_i, μ_i, ν_i, and θ_i be the numbers, and D_i^1, D_i^2, D_i^3, Δ_i, and Q_i be the sets corresponding to each compatible pair L_i and G_i. Let

$$\mathcal{L} = L_n\circ G_n\circ\cdots\circ L_i\circ G_i\circ\cdots\circ L_0\circ G_0.$$

Then $\mathcal{L}$ is a schlicht function defined on $\mathbf{C}_I = (\mathbf{C}\setminus\mathbf{R}^1)\cup\mathring{I}$.

Definition 4. *We call $\mathcal{L}$ a root-like map if there are constants $C > 0$ and $\lambda > 1$ such that*

- **(i)** $a_0 = 0$ and $a_1 \geq 1/C$,
- **(ii)** $|a_j| \geq \max\{(\lambda^{j-i}/C)\cdot|a_i|, \Big(1+(\lambda-1)/C\Big)\cdot|a_i|\}$ for all $1\leq i<j\leq n$, and
- **(iii)** $\mu_i < C$ for all $1\leq i\leq n$.

Theorem 18. *Suppose $\mathcal{L}$ is a root-like map. There is a constant $\theta > 0$ depending only on λ and C such that the image of the upper-half plane under $\mathcal{L}$ is contained in the sector*

$$Sec_\theta = \{z\in\mathbf{C} \mid 0\leq\arg(z)\leq\pi-\theta\}.$$

Before we prove this theorem, we introduce some basic results in hyperbolic geometry. Let $\mathbf{C}_I = (\mathbf{C}\setminus\mathbf{R}^1)\cup\mathring{I}$ be a plane domain. Then $q(z) = -z^2/(1-z^2)$ is a diffeomorphism from the upper-half plane UH onto $\mathbf{C}_I$. Consider UH to be a hyperbolic plane with Poincaré metric $d_{H,UH}s = |dz|/y$ for $z = x+yi$. This metric induces a hyperbolic metric $d_{H,\mathbf{C}_I}s = q_*(|dz|/y)$ on C_I. The plane domain $\mathbf{C}_I$ under this metric is a hyperbolic Riemann surface. Let $d = d_{H,\mathbf{C}_I}$ be the induced hyperbolic distance. We note that q maps the positive imaginary line in UH onto the interval I and maps the real line, which is the boundary of UH, onto the set $R^1\setminus\mathring{I}$.

Lemma 6. *A hyperbolic neighborhood $\Phi(r) = \{z\in\mathbf{C}_I \mid d(z,I) < r\}$ is the union of two Euclidean disks D^+ and D^-, symmetric to each other with respect to I, centered at c^+ and at $c^- = -c^+$, with the same radius $R^+ = R^-$. Moreover,*

$$R^+ = \frac{1}{2\sin\beta} \qquad \textit{and} \qquad c^+ = \frac{1}{2}+\frac{\cot\beta}{2}i$$

where

$$\beta = 4\cot^{-1}(e^r)$$

is the angle at 0 *between* ∂D^+ *and the negative real line (and the angle at* 1 *between* ∂D^+ *and the ray* $[1,\infty)$*).*

Proof. Consider the pre-image $\Phi' = q^{-1}(\Phi(r))$. It is a hyperbolic neighborhood in UH and consists of all points in UH whose hyperbolic distances to the half-line $l_+ = \{z = yi \mid y > 0\}$ are less than r. The boundary $\partial\Phi'$ consists of two rays starting from 0. Thus Φ' is a sector, symmetric with respect to the half-line l_+. Suppose $\beta/2$ is the outer angle of this sector (with respect to the real line). Since a geodesic in UH is a semi-circle or half-line perpendicular to the real line, it is easy to check that

$$\log\Big(\cot\frac{\beta}{4}\Big) = r.$$

Therefore, $\Phi(r)$ is the union of two disks D^+ and D^- symmetric with respect to I. The angle between ∂D^+ (or ∂D^-) and the negative real line is β. Moreover, every point z in ∂D^+ (or ∂D^-) views I under the same angle β, that is, every triangle $\triangle(0z1)$ has the angle β at z. Now consider the point u such that the segment $\overline{1u}$ is a diameter of D^+. The triangle $\triangle(u01)$ is a right triangle. We can calculate the length $2R^+$ of the segment $1u$ and length $|u|$ of the segment $\overline{0u}$ as follows:

$$2R^+ = \frac{1}{\sin\beta}, \qquad |u| = \cot\beta.$$

Therefore,

$$R^+ = \frac{1}{2\sin\beta} \qquad \text{and} \qquad c^+ = \frac{1}{2} + \frac{\cot\beta}{2} i. \qquad \blacksquare$$

Lemma 7. *Let* z *be a point in* $\mathbf{C}_I$ *and let* $\Phi(r) = D^+ \cup D^-$ *be the smallest hyperbolic neighborhood containing* z*. The Euclidean radius* R^+ *of* D^+ *(and* D^-*) is*

$$\frac{|z-1|}{2\sin(\arg(z))}.$$

Proof. Let u be the point in ∂D^+ such that the segment $\overline{1u}$ is a diameter of D^+. The angle of the triangle $\triangle(0z1)$ at z and the angle of the triangle $\triangle(0u1)$ at u are both β (see the previous lemma). Now applying the law of sines,

$$\frac{\sin(\arg(z))}{|1-z|} = \sin\beta = \frac{1}{|1-u|}.$$

Therefore,

$$2R^+ = \frac{|z-1|}{\sin(\arg(z))}. \qquad \blacksquare$$

Suppose $\mathcal{L}$ is a root-like map. Then there is a constant angle σ and a constant $C_0 > 0$ depending only on C such that $\theta_i \geq \sigma$ and $\nu_i \leq C_0$ for all $0 \leq i \leq n$.

For each $1 \leq i \leq n-1$, let $A_{i+1} = Q_{i+1} \setminus \Delta_i$. Let $A_1 = Q_1 \setminus \{z| \in \mathbf{C}, 0 \leq \arg(z) < \pi - \pi/\gamma\}$. Suppose $\Phi(r_i) = D_i^+ \cup D_i^-$ is the smallest hyperbolic neighborhood in $\mathbf{C}_I$ containing $A_i \neq \emptyset$ and let D_i^0 be the smallest disk centered at $1/2$ containing $\Phi(r_i)$. Let R_i^+ be the Euclidean radius of D_i^+ and let R_i be the Euclidean radius of D_i^0. Since z and 1 are in D_i^2, from Lemmas 6 and 7,

$$R_i \leq 2 \cdot R_i^+ \leq \frac{k_i}{\sin\theta_i} \leq \frac{2\cdot(1+|a_i'|)}{\sin\theta_i} \leq \frac{2\cdot(1+C)\cdot|a_i'|}{\sin\sigma} = C_1 \cdot |a_i'|$$

where $k_i = \min\{1+|a_i'|+b_i, 2(1+|a_i'|)\}$ is the diameter of D_i^2 and where $C_1 = 2\cdot(1+C)/\sin\sigma$ is a positive constant depending only on C (see Fig. 4).

For every $1 \leq i \leq n$, the definition interval of $\phi_i = L_i \circ G_i$ is $[a_i', a_i''] = [a_i', \infty) \cap I_{G_i}$. According to (iii) of Definition 4, the right-endpoint a_i'' satisfies

$$a_i'' \geq \frac{1+|a_i'|}{C} + 1 \geq 1 + \frac{|a_i'|}{C}.$$

For every $1 \leq i < j \leq n$,

$$|a_j'| \geq \frac{\lambda^{j-i}}{CC_0^2}|a_i'|.$$

Therefore, if $\tau = \min\{1, 1/(C\cdot C_0)^2\}$, we have

$$[a_j', a_j''] \supset I_{i,\tau} = [\lambda^{j-i}\tau a_i', \lambda^{j-i}\tau|a_i'|+1].$$

In other words, $\phi_j = L_j \circ G_j$ are schlicht functions on $\mathbf{C}_{I_{i,\tau}} = (\mathbf{C}\setminus\mathbf{R}^1)\cup I_{i,\tau}$ for all $1 \leq i < j \leq n$.

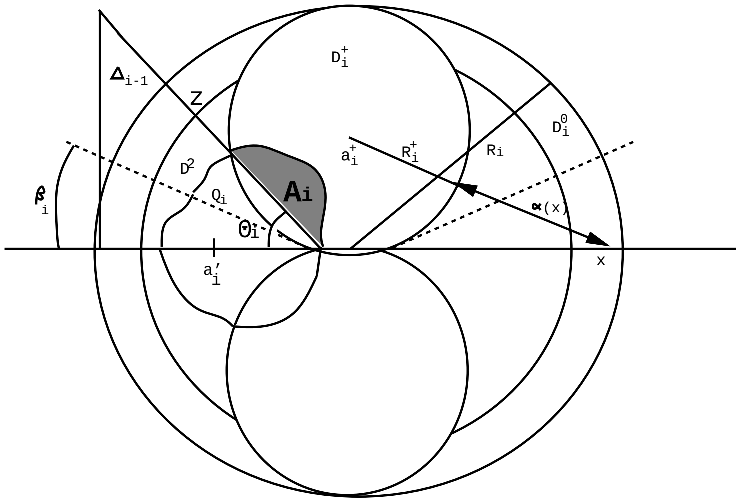

Fig. 4

Lemma 8. *There is a fixed integer $n_0 > 0$ depending only on C and on λ such that for any $0 \leq i < n$ and any $j \geq i + n_0$,*

$$|a'_j| + \frac{1}{2} > \lambda^{j-i-n_0} R_i \qquad \text{and} \qquad a''_j - \frac{1}{2} > \lambda^{j-i-n_0} R_i.$$

Proof. Take m_0 as the biggest integer such that

$$m \leq \frac{\log(\frac{C_1}{\tau})}{\log \lambda}.$$

From the above estimates, one can see that $n_0 = \max\{m_0 + 1, 0\}$ is the integer satisfying the lemma. ∎

Lemma 9. *There is a constant $C_2 > 0$ depending only on λ such that for any $0 \leq i < n - n_0$, let $\Pi_i = L_n \circ G_n \circ \cdots \circ L_j \circ G_j \circ \cdots \circ L_{i+n_0+1} \circ G_{i+n_0+1}$, the distortion of Π_i on $\Phi(r_i)$ is bounded by C_2, more precisely,*

$$\frac{|\Pi'_i(\xi)|}{|\Pi'_i(\eta)|} \leq C_2$$

for all ξ and η in $\Phi(r_i)$.

Proof. Consider the disk D^5_j centered at $1/2$ with radius $t_j = \min\{|a'_j| + 1/2, a''_j - 1/2\}$. The composition $\phi_j = L_j \circ G_j$ is a schlicht function on D^5_j. Let $r_{ij} = R_i/t_j$ be the ratio of the radii of D^0_i and D^5_j. Then

$$r_{ij} \leq \lambda^{i+n_0-j}$$

for $i + n_0 + 1 \leq j \leq n$. From Koebe's distortion theorem (see [BIE]),

$$\frac{|\phi'_j(\xi)|}{|\phi'_j(\eta)|} \leq \left(\frac{1 + r_{ij}}{1 - r_{ij}}\right)^4 \leq \left(\frac{1 + \lambda^{-k}}{1 - \lambda^{-k}}\right)^4$$

for $0 < k = j - i - n_0 \leq n - i - n_0$ and for all ξ and η in D^0_i. Since ϕ_j is a schlicht function on $\mathbf{C}_I$, it contracts the hyperbolic distance d on $\mathbf{C}_I$. Thus

$$\phi_j(\Phi(r_i)) \subset \Phi(r_i) \subset D^0_i.$$

Therefore, by the chain rule,

$$\frac{|\Pi'_i(\xi)|}{|\Pi'_i(\eta)|} = \prod_{j=n_0+1}^{n} \frac{|\phi'_j(\xi)|}{|\phi'_j(\eta)|} \leq C_2 = \left(\prod_{k=1}^{\infty} \frac{1 + \lambda^{-k}}{1 - \lambda^{-k}}\right)^4$$

for all ξ and η in $\Phi(r_i)$. ∎

Lemma 10. *There is a constant $C_3 > 0$ depending only on C such that for $0 \leq i \leq n$, for $i \leq j \leq i + n_0$, for $\phi_j = L_j \circ G_j$, and for all ξ and η in D_i^+ (or D_i^-),*

$$\frac{|\phi_j'(\xi)|}{|\phi_j'(\eta)|} \leq C_3.$$

Proof. Suppose x is a real number. Let $\alpha = \alpha(x) = \min\{|x - z| \mid z \in D_i^+ \cap UH\}$. Suppose that $c = c_i^+$ and that $R = R_i^+$ are the center and the radius of D_i^+. Suppose $h = h_i^+$ is the length of the segment $\overline{c\frac{1}{2}}$ (the straight line connecting c and $1/2$). Since the two triangles $\triangle(x\frac{1}{2}c)$ and $\triangle(0\frac{1}{2}c)$ are both right triangles, then

$$(\alpha + R)^2 = (\frac{1}{2} - x)^2 + h^2$$

and

$$R^2 = (\frac{1}{2})^2 + h^2.$$

Therefore,

$$\frac{\alpha + R}{R} = \sqrt{1 + \frac{x^2 - x}{R^2}}.$$

This implies that there is a constant $0 < C_4 < 1$ depending only on C such that for $x = a_j'$ or $x = a_j''$,

$$\frac{R}{\alpha + R} \leq C_4.$$

Now consider the largest disk D_j^6 centered at c such that ϕ_j is a schlicht function on it. Then the radius of D_j^6 is greater than or equal to $\min\{R + \alpha(a_j'), R + \alpha(a_j'')\}$. From Koebe's distortion theorem (see [BIE]),

$$\frac{|\phi_j'(\xi)|}{|\phi_j'(\eta)|} \leq C_3 = \left(\frac{1 + C_4}{1 - C_4}\right)^4$$

for any ξ and η in D_i^+. ∎

Combining Lemmas 9 and 10, we obtained the following estimate:

Lemma 11. *There is a constant $C_5 > 0$ depending on λ and on C such that for any $0 \leq i < n$, let $\Sigma_i = L_n \circ G_n \circ \cdots \circ L_j \circ G_j \circ \cdots \circ L_i \circ G_i$, the distortion of Σ_i on D_i^+ (or D_i^-) is bounded by C_5, more precisely,*

$$\frac{|\Sigma_i'(\xi)|}{|\Sigma_i'(\eta)|} \leq C_5$$

for all ξ and η in D_i^+ (or D_i^-).

Proof. Since each ϕ_j is a schlicht function on $\mathbf{C}_I$, it contracts the hyperbolic distance d on C_I. So

$$\phi_j(D_i^+) \subset D_i^+$$

for all $i \leq j \leq n$. If $n - i \leq n_0$, then from Lemma 10 and the chain rule,

$$\frac{|\Sigma_i'(\xi)|}{|\Sigma_i'(\eta)|} \leq C_3^{n_0}$$

for all ξ and η in D_i^+.

Now we consider $n - i > n_0$ and write $\Sigma_i = \Pi_i \circ \Theta_i$ where $\Theta_i = L_{i+n_0} \circ G_{i+n_0} \circ \cdots \circ L_i \circ G_i$ and where $\Pi_i = L_n \circ G_n \circ \cdots \circ L_{i+n_0+1} \circ G_{i+n_0+1}$. From Lemma 10,

$$\frac{|\Theta_i'(\xi)|}{|\Theta_i'(\eta)|} \leq C_3^{n_0}$$

for all ξ and η in D_i^+, and from Lemma 9,

$$\frac{|\Pi_i'(\xi)|}{|\Pi_i'(\eta)|} \leq C_2$$

for all ξ and η in D_i^+. Again, because Θ_i is a schlicht function on $\mathbf{C}_I$ and contracts the hyperbolic distance d on $\mathbf{C}_I$, we have $\Theta_i(D_i^+) \subset D_i^+$. Therefore, from the chain rule,

$$\frac{|\Sigma_i'(\xi)|}{|\Sigma_i'(\eta)|} \leq C_5 = C_2 \cdot C_3^{n_0}$$

for all ξ and η in D_i^+. ∎

Lemma 12. *Suppose G is in $\mathcal{E}_0$ and D is a closed simply connected convex domain with $I \subset D \subset C_G$. Then for all $z = x + yi$ with $y > 0$ in D*

$$\sin\Big(\arg(G(z))\Big) \geq \frac{\sin(\arg(z))}{N_0}$$

where $N_0 = \sup_{\xi,\eta \in D} |G'(\xi)/G'(\eta)|$ measures the distortion of G on D.

Proof. Since G maps UH into itself, it contracts the hyperbolic metric dz/y on UH. Suppose $z = x + yi$ with $y > 0$ and $G(z) = X + Yi$. Then $|G'(z)|y \leq Y$. Therefore,

$$\sin(\arg(z)) = \frac{y}{|z|} \leq \frac{Y}{|z| \cdot |G'(z)|} = \sin\Big(\arg(G(z))\Big) \cdot \frac{|G(z)|}{|z| \cdot |G'(z)|}.$$

So

$$\frac{\sin(\arg(z))}{N_0} \leq \sin\Big(\arg(G(z))\Big)$$

for all $z = x + yi$ with $y > 0$ in D. ■

Now we complete the proof of Theorem 18 as follows.

Proof of Theorem 18. For any z_0 in UH, let $z_{i+1} = L_i(G_i(z_i))$ for $0 \leq i \leq n$. Since $0 \leq \arg(z_1) \leq \pi/\gamma$, the smallest positive integer i such that z_i lies in Δ_i must be either bigger than zero or not exist. If such a positive integer does not exist, then $0 \leq \arg(z_{n+1}) \leq \pi - \theta_n \leq \pi - \sigma$. Now let us suppose that this smallest number exists and is $i_0 + 1$. Then z_{i_0} is in $A_{i_0} \neq \emptyset$ which is a subset of $D^+_{i_0}$. Since $0 < \arg(z_{i_0}) \leq \pi - \theta_i$, Lemmas 11 and 12 assure us there is a constant angle $0 < \theta \leq \sigma$ such that $0 < \arg(z_{n+1}) \leq \pi - \theta$. ■

7. Feigenbaum-Like Quadratic-Like Maps

In this section, we discuss Feigenbaum-like quadratic-like maps and prove the result of Sullivan which says that a Feigenbaum-like quadratic-like map has the *a priori* complex bounds and is unbranched.

Let us first recall some facts about infinitely renormalizable folding mappings. Let (U, V, f) be a real quadratic-like map, that is, $f(U \cap \mathbf{R}^1) \subseteq V \cap \mathbf{R}^1$. Conjugating by a linear fraction transformation, we may assume that $f(-1) = f(1) = -1$ and that $f|[-1, 1]$ is a folding map of $[-1, 1]$ with a unique quadratic critical point 0. For example, $P_t(z) = t - (1+t)z^2$ for $0 \leq t \leq 1$ is a real quadratic-like map whenever it is restricted to any domain bounded by an equipotential curve. Furthermore, suppose (U, V, f) is infinitely renormalizable. Then the filled-in Julia set K_f is connected. Let $\beta_0 = -1$ and let α_0 be the fixed point of f in $(-1, 1)$. Then β_0 is the non-separate fixed point, α_0 is the separate fixed point, and $K_f \cap \mathbf{R}^1 = [-1, 1]$. The mapping $f_0 = f|[-1, 1]$ is an infinitely $(n_1, n_2, \ldots, n_k, \ldots)$-renormalizable folding mapping where $\{n_k\}_{k=1}^{\infty}$ is the maximum sequence of integers such that f_0 is renormalizable about m_k for $m_k = \prod_{i=1}^{k} n_i$. Let $I_k = [-a_k, a_k]$ be the maximal interval containing 0 (set $m_0 = 0$ and $I_0 = [-1, 1]$) such that

(a) $f_0^{\circ m_k}$ is monotone when restricted to $[-a_k, 0]$ and to $[0, a_k]$,

(b) $f_0^{\circ m_k}(I_k) \subset I_k$,

(c) I_k, $f_0(I_k)$, $\ldots$, $f_0^{\circ(m_k-1)}(I_k)$ have pairwise disjoint interiors, and

(d) $f_0^{\circ m_k}$ has exactly two fixed points β_k and α_k in I_k where $a_k = |\beta_k|$.

There is a domain $U_k \supseteq I_k$ such that $(U_k, V_k, f^{\circ m_k})$ is a quadratic-like map with connected filled-in Julia set K_k, where K_k is the k^{th}-renormalization of K_1. Then β_k is the non-separate fixed point and α_k is the separate fixed point of this quadratic-like map and then $K_k \cap R^1 = I_k$. An infinitely renormalizable real quadratic-like map is called Feigenbaum-like if $n_k = 2$ for all k.

Suppose that $c(i) = f^{\circ i}(0)$ is the i^{th} critical value of f and that $J_k(i)$ is the interval bounded by $c(i)$ and $c(m_k + i)$ for $k \geq 0$ and $0 \leq i < m_k$. We note that $J_k(0) =$

$J_k(m_k)$. Then $f_0 : J_k(0) \to J_k(1)$ is folding for all $k \geq 0$ and $f_0 : J_k(i) \to J_k(i+1)$ is a homeomorphism for every $k \geq 1$ and $1 \leq i < m_k$. Let $\zeta_k = \{J_k(i)\}_{0 \leq i < m_k}$ for $k \geq 0$. Let $I_k(i) = f_0^{\circ i}(I_k)$, $0 \leq i < m_k$, and let $\xi_k = \{I_k(i)\}_{0 \leq i < m_k}$. Note that $J_k(i) \subseteq I_k(i)$. We use $LI_k(i)$ and $RI_k(i)$ to denote the intervals in ξ_k adjacent to $I_k(i)$ and on the left and right sides of $I_k(i)$, respectively (there is only $LI_k(1)$ or $RI_k(2)$ in ξ_k). Let $LI_k^+(i)$ be the smallest interval containing $LI_k(i)$ and the left end-point of $I_k(i)$ and let $RI_k^+(i)$ be the smallest interval containing $RI_k(i)$ and the right end-point of $I_k(i)$, for $i = 0$ or $3 \leq i < m_k$. Let $LI_k^+(2) = [-1, c(2)]$ and $RI_k^+(1) = [c(1), 1]$. Similarly, we can define $LJ_k(i)$ and $RJ_k(i)$ and $LJ_k^+(i)$ and $RJ_k^+(i)$ for $0 \leq i < m_k$. The following theorem is due to Sullivan.

Theorem 19 [SU2]. *There is a constant $C > 0$ such that*

$$\min\{|LI_k^+(i)|, |RI_k^+(i)|\} \geq C \cdot |I_k(i)|,$$

and such that

$$\min\{|LJ_k^+(i)|, |RJ_k^+(i)|\} \geq C \cdot |J_k(i)|.$$

for all $k \geq 0$ and $0 \leq i < m_k$.

Consider the slit domain $V_0 = V \setminus [c(1), \infty)$. The map $f|V_0$ has two inverse branches (see Fig. 5)

$$g_0 : V_0 \to U_{0,0} = U \cap \{z = x + yi \ \in C \mid x < 0\}$$

and

$$g_1 : V_0 \to U_{0,1} = U \cap \{z = x + yi \ \in C \mid x > 0\}.$$

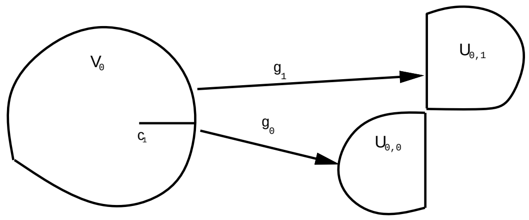

Fig. 5

For each $k \geq 1$, let $g_{s(i)} : J_k(i+1) \to J_k(i)$ for $1 \leq i < m_k$ where $s(i) = 0$ if $J_k(i)$ is contained in the negative half of the real line, and otherwise $s(1) = 1$. Then

$$\mathcal{A}_k = g_{s(1)} \circ g_{s(2)} \circ \cdots \circ g_{s(m_k-1)} : J_k(m_k) = J_k(0) \to J_k(1)$$

is a homeomorphism and can be extended homeomorphically to the maximum closed interval

$$T_k(m_k) \supseteq LJ_k^+(m_k) \cup J_k(m_k) \cup RJ_k^+(m_k).$$

Furthermore, $\mathcal{A}_k$ can be extended analytically to $V_k = (V \setminus \mathbf{R}^1) \cup \mathring{T}_k(m_k)$. Let us continue to use $\mathcal{A}_k$ to denote this extension. Let $U'_k = \mathcal{A}_k(V_k)$, and let $U_k = f^{-1}(U'_k)$ be the pre-image of U'_k under f. Since f has no attractive and no parabolic periodic point, then $g_0\Big(\mathcal{A}_k(T_k(m_k))\Big)$ and $g_1\Big(\mathcal{A}_k(T_k(m_k))\Big)$ are contained strictly in $T_k(m_k)$. Thus, $\overline{U}_k \subset V_k$ and

$$f^{\circ m_k} : U_k \to V_k$$

is a quadratic-like map (see Fig. 6).

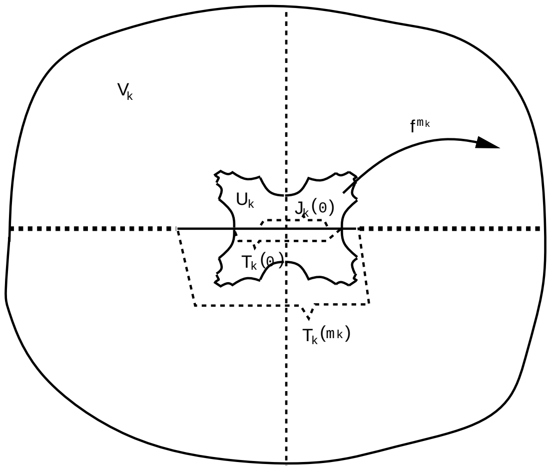

Fig. 6

Let $T_k(i) = g_{s(i)}(T_k(i+1))$ for $i = m_k - 1,\, m_k - 2,\, \ldots,\, 1$. Let $T_k(0) = f_0^{-1}(T_k(1))$. Then

$$J_k(0) = J_k(m_k) \subseteq I_k \subseteq T_k(0) \subset T_k(m_k).$$

The interval $T_k(m_k)$ is bounded by two critical values $c(q(k))$ and $c(r(k))$ of f; one of them is a maximum value and the other is a minimum value of $f^{\circ m_k}$. Suppose $T_k(1) = [d_1, e_1]$ where $d_1 < c(1) < e_1$. Let $d_i = f^{\circ i}(d_1)$ and let $e_i = f^{\circ i}(e_1)$ for $1 \leq i \leq m_k$. Note that d_{m_k} and e_{m_k} are $c(r(k))$ and $c(q(k))$. We normalize $T_k(i)$ into the unit interval $[0,1]$: let $l_i : T_k(i) \to [0,1]$ be the linear map such that $l_i(d_i) = 0$ and $l_i(e_i) = 1$ and let

$$\tilde{g}_{s(i)} = l_i \circ g_{s(i)} \circ l_{i+1}^{-1}$$

for $i = m_k - 1,\, \ldots,\, 2,\, 1$. The map $\tilde{g}_{s(i)}$ fixes 0 and 1 and is a univalent function defined on

$$\mathbf{C}_{[0,1],V} = \Big(\Big(\mathbf{C} \setminus \mathbf{R}^1\Big) \cup (0,1)\Big) \cap V.$$

The restriction $\tilde{g}_{s(i)}|[0,1]$ is a homeomorphism of $[0,1]$. Let

$$\mathcal{L}_k = \tilde{g}_{s(1)} \circ \tilde{g}_{s(2)} \circ \cdots \circ \tilde{g}_{s(m_k-1)}$$

for $k \geq 1$.

Lemma 13. *Suppose (U, V, f) is a Feigenbaum-like quadratic-like map. Then*

$$\{\mathcal{L}_k\}_{k=3}^{\infty}$$

are uniform root-like maps.

Proof. Since f is a Feigenbaum-like map, we have $m_k = 2^k$ and $T_k(m_k) = J_{k-2}(0)$ for $3 \leq k < \infty$. For any $k \geq 3$, consider the homeomorphism $f^{\circ(2^{k-1}-1)} : J_k(2^{k-1} + 1) \to J_k(2^k) = J_k(0)$. Its inverse $g_{s(2^{k-1}+1)} \circ \cdots \circ g_{s(2^k-1)} : J_k(2^k) \to J_k(2^{k-1}+1)$ can be extended to $T_{k-1}(2^{k-1})$, that is, we can consider

$$g_{s(2^{k-1}+1)} \circ \cdots \circ g_{s(2^k-1)} : T_{k-1}(2^{k-1}) \to T_{k-1}(1).$$

Let

$$G_0 = \tilde{g}_{s(2^{k-1}+1)} \circ \cdots \circ \tilde{g}_{s(2^k-1)}$$

and let

$$L_{a_0} = \tilde{g}_{s(2^{k-1})}.$$

Then G_0 is a univalent map (or holomorphic embedding) of $C_{[0,1],V}$ such that $G_0(0) = 0$ and $G_0(1) = 1$ and such that $G_0|[-1, 1]$ is a homeomorphism of $[0, 1]$. The map L_{a_0} is a square root at 0 (see Fig. 7).

Consider the homeomorphism $f^{\circ(2^{k-2}-1)} : J_k(2^{k-2} + 1) \to J_k(2^{k-1})$. Its inverse $g_{s(2^{k-2}+1)} \circ \cdots \circ g_{s(2^{k-1}-1)} : J_k(2^{k-1}) \to J_k(2^{k-2} + 1)$ can be extended to $T_{k-2}(2^{k-2})$, that is, we can consider

$$G_1' = g_{s(2^{k-2}+1)} \circ \cdots \circ g_{s(2^{k-1}-1)} : T_{k-2}(2^{k-2}) \to T_{k-2}(1).$$

Let

$$G_1 = \tilde{g}_{s(2^{k-2}+1)} \circ \cdots \circ \tilde{g}_{s(2^{k-1}-1)}$$

and let

$$L_{a_1} = \tilde{g}_{s(2^{k-2})}.$$

The map G_1 is a univalent map of $\mathbf{C}_{[0,1],V}$ and $G_1|[-1, 1]$ is a homeomorphism of $[0, 1]$. The map L_{a_1} is a square root at a_1. The pre-image of $c(1)$ under G_1' is $c(2^{k-2})$. One of the end-points of $T_k(2^{k-1})$ is 0; the other one is between $J_{k-1}(0)$ and one of $LJ_{k-1}(0)$ or $RJ_{k-1}(0)$. From Theorem 19, two components of $T_{k-2}(2^{k-2}) \setminus T_k(2^{k-1})$ have lengths greater than a constant C (obtained from Theorem 19) times the length of $T_k(2^{k-1})$. Thus (L_{a_1}, G_1) is a compatible pair and satisfies **(i)** and **(iii)** of Definition 4 (see Fig. 7).

Next we consider the homeomorphism $f^{\circ(2^{k-3}-1)} : J_k(2^{k-3} + 1) \to J_k(2^{k-2})$. Its inverse $g_{s(2^{k-3}+1)} \circ \cdots \circ g_{s(2^{k-2}-1)} : J_k(2^{k-2}) \to J_k(2^{k-3} + 1)$ can be extended to $T_{k-3}(2^{k-3})$; that is, we can consider

$$G_2' = g_{s(2^{k-3}+1)} \circ \cdots \circ g_{s(2^{k-2}-1)} : T_{k-3}(2^{k-3}) \to T_{k-3}(1).$$

Let

$$G_2 = \tilde{g}_{s(2^{k-3}+1)} \circ \cdots \circ \tilde{g}_{s(2^{k-2}-1)}$$

and let

$$L_{a_2} = \tilde{g}_{s(2^{k-3})}.$$

The map G_2 is a univalent map of $\mathbf{C}_{[0,1],V}$ and $G_2|[-1,1]$ is a homeomorphism of $[0,1]$. The map L_{a_2} is a square root at a_2. The preimage of $c(1)$ under G_2' is $c(2^{k-3})$. The interval $T_k(2^{k-2})$ is contained in $T_{k-1}(2^{k-1})$. From Theorem 19, two components of $T_{k-3}(2^{k-3})\setminus T_k(2^{k-2})$ have lengths greater than a constant C (obtained from Theorem 19) times the length of $T_k(2^{k-2})$. Thus (L_{a_2}, G_2) is a compatible pair and satisfies **(iii)** of Definition 4 (see Fig. 7).

In general, for $3 < i \leq k-1$, consider the homeomorphism $f^{\circ(2^{k-i}-1)} : J_k(2^{k-i} + 1) \to J_k(2^{k-i+1})$. Its inverse $g_{s(2^{k-i}+1)} \circ \cdots \circ g_{s(2^{k-i+1}-1)} : J_k(2^{k-i+1}) \to J_k(2^{k-i}+1)$ can be extended to $T_{k-i}(2^{k-i})$; that is, we can consider

$$G'_{i+1} = g_{s(2^{k-i}+1)} \circ \cdots \circ g_{s(2^{k-i+1}-1)} : T_{k-i}(2^{k-i}) \to T_{k-i}(1).$$

Let

$$G_{i+1} = \tilde{g}_{s(2^{k-i}+1)} \circ \cdots \circ \tilde{g}_{s(2^{k-i+1}-1)}$$

and let

$$L_{a_{i+1}} = \tilde{g}_{s(2^{k-i})}.$$

The map G_{i+1} is a univalent map of $\mathbf{C}_{[0,1],V}$ and $G_{i+1}|[0,1]$ is a homeomorphism. The map $L_{a_{i+1}}$ is a square root at a_{i+1}. The preimage of $c(1)$ under G'_{i+1} is $c(2^{k-i})$. The interval $T_k(2^{k-i+1})$ is contained in $T_{k-i+2}(2^{k-i+2})$. From Theorem 19, two components $T_{k-i}(2^{k-i}) \setminus T_k(2^{k-i+1})$ have lengths greater than a constant C times the length of $T_k(2^{k-i+1})$. Thus $(L_{a_{i+1}}, G_{i+1})$ is a compatible pair and satisfies **(iii)** of Definition 4.

One of the endpoints of $T_k(2^{k-i}+1)$ is to the left of $c(2^{k-i}+1)$; the other is in the interval in ζ_k which is adjacent to $J_k(2^{k-i}+1)$. The branch point of $g_{s(2^{k-i})}$ is always $c(1)$. From Theorem 19, we have a constant $\lambda > 1$ such that $\{a_i\}_{i=1}^{k}$ satisfies **(ii)** of Definition 4 for $C = 1$.

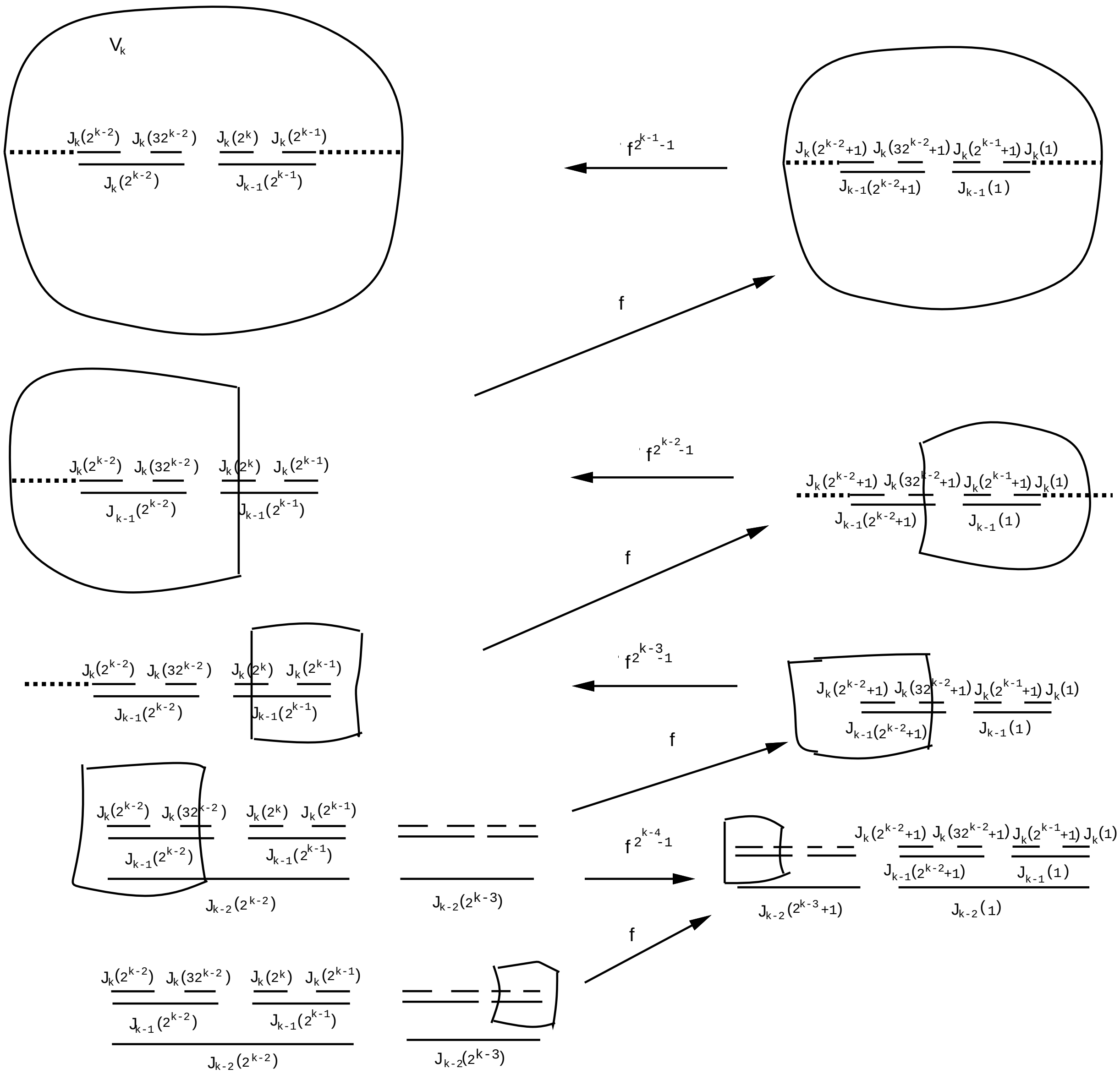

Fig. 7

For all $k \geq 3$, we therefore decompose

$$\mathcal{L}_k = L_{a_k} \circ G_k \circ L_{a_{k-1}} \circ G_{k-1} \circ \cdots \circ L_{a_1} \circ G_1 \circ L_{a_0} \circ G_0.$$

From the construction above, $\mathcal{L}_k$, $3 \leq k < \infty$, is uniform root-like map. ∎

Let (U, V, f) be a Feigenbaum-like quadratic-like map. Let $m_k = 2^k$. Consider the renormalizations

$$f^{\circ m_k} : U_k \to V_k$$

for $1 \leq k < \infty$ where $V_k = (V \setminus \mathbf{R}^1) \cup \mathring{T}k(m_k)$. Let $U_k \cap \mathbf{R}^1 = [-o_k, o_k]$. Let $w_{k,\theta}$ be the the ray starting at o_k with slop $\tan\theta$ for $0 < \theta < \pi/2$. Let $0 \in R_{k,\theta}$ be the domain bounded by $w_{k,\theta}$, $-w_{k,\theta}$, $\overline{w}_{k,\theta}$, and $-\overline{w}_{k,\theta}$.

Lemma 14. *Suppose (U, V, f) is a Feigenbaum-like quadratic-like map. Then there is a constant angle $\theta_0 > 0$ such that*

$$U_k \subseteq R_{k,\theta_0}$$

for all $k \geq 0$ (see Fig. 8).

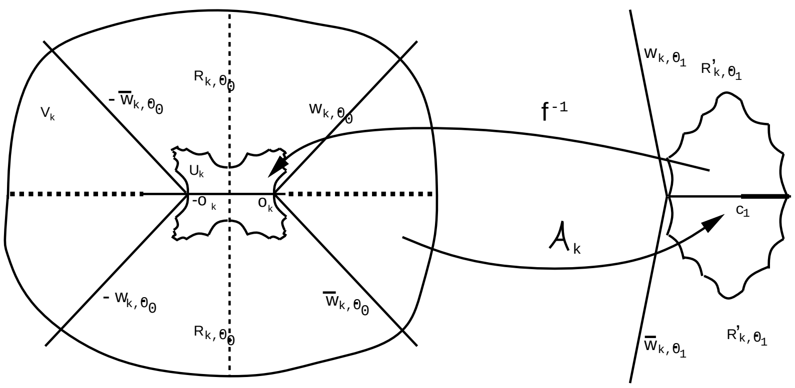

Fig. 8

Proof. Consider $f^{\circ(m_k-1)} : J_k(1) \to J_k(0)$. Its inverse has the maximum extension

$$\mathcal{A}_k = g_{s(1)} \circ g_{s(2)} \circ \cdots \circ g_{s(m_k-1)} : V_k = (V \setminus \mathbf{R}^1) \cup \mathring{T}_k(m_k) \to U'_k.$$

Remember that

$$\mathcal{L}_k = l_1 \circ \mathcal{A}_k \circ l_{m_k}^{-1}$$

where l_{m_k} and l_1 are the linear maps normalizing T_{m_k} and $T_k(1)$, respectively, to $[0, 1]$. Let

$$w'_{k,\theta} = \{z \in \mathbf{C} \mid \arg(z - d_1) = \theta, \Im(z) > 0\}$$

be the ray starting at d_1 with angle $0 \leq \theta \leq \pi$ where $U'_k \cap \mathbf{R}^1 = T_k(1) = [d_1, e_1]$ with $d_1 < c(1) < e_1$. Let $R'_{k,\theta}$ be the sector containing $c(1)$ bounded by $w'_{k,\theta}$ and $\overline{w'}_{k,\theta}$. Applying Theorem 18, there is a constant angle $0 < \theta_1 \leq \pi$ such that U'_k is contained in a sector domain R'_{k,θ_1} (see Fig. 8). Since $f : U \to V$ is a quadratic-like map, it is comparable with $z \mapsto z^2$ near 0. So there is a constant angle $0 < \theta_0 < \pi/2$ depending on θ_1 such that $U_k = f^{-1}(U'_k)$ is contained in R_{k,θ_0} (see Fig. 8). ∎

Take $I_0 = (-1,1)$. Let $\mathbf{C}_{I_0} = (\mathbf{C} \setminus R^1) \cup I_0$. Let $d = d_{H,\mathbf{C}_{I_0}}$ be the hyperbolic distance on $\mathbf{C}_{I_0}$. Let $\Omega_r = \{z \in \mathbf{C}_{I_0} \mid d(z, I_0) < r\}$ be a hyperbolic neighborhood. From Lemma 6, Ω_r is the union of two disks D^+_β and D^-_β centered at $c^+_\beta = i\cot\beta$ and $c^-_\beta = -i\cot\beta$ with radii $R^+_\beta = R^-_\beta = 1/\sin\beta$ where β is the angle between ∂D^+_β and the line $[1,\infty)$ at 1 (see Fig. 9). Using the law of cosines for the triangle $\Delta(c0z)$, for any $z = re^{i\phi}$ in ∂D^+_β,

$$r = \cot\beta\sin\phi + \sqrt{\csc^2\beta - \cot^2\beta\cos^2\phi}.$$

Let $q(z) = \sqrt{z}$ be the square root from $\mathbf{C} \setminus \{x < 0\}$ to the right half-plane RH. Let $\Pi^+_\beta = q(D^+_\beta)$. For $0 < \beta < \pi/4$, consider $\Omega_{r'} = D^+_{2\beta} \cup D^-_{2\beta}$. Let $z_0 = re^{i\tau} \neq 1$ be the intersection point of $\partial D^+_{2\beta}$ and Π^+_β. Then τ is the unique non-zero solution of the equation

$$\sqrt{\cot\beta\sin(2\phi) + \sqrt{\csc^2\beta - \cot^2\beta\cos^2(2\phi)}}$$
$$= \cot(2\beta)\sin\phi + \sqrt{\csc^2(2\beta) - \cot^2(2\beta)\cos^2\phi}.$$

Thus $\tau = \tau(\beta)$ is a strictly increasing function and $\tau \to 0$ as $\beta \to 0$. Let $z_0 - 1 = \tilde{r}e^{i\theta}$. Then $\theta = \theta(\tau)$ is a strictly increasing function and $\theta \to 0$ as $\tau \to 0$. Let $\theta = \theta \circ \tau(\beta)$. It is a strictly decreasing function and $\theta \to 0$ as $\beta \to 0$. Let $\beta = \beta(\theta)$ be its inverse function.

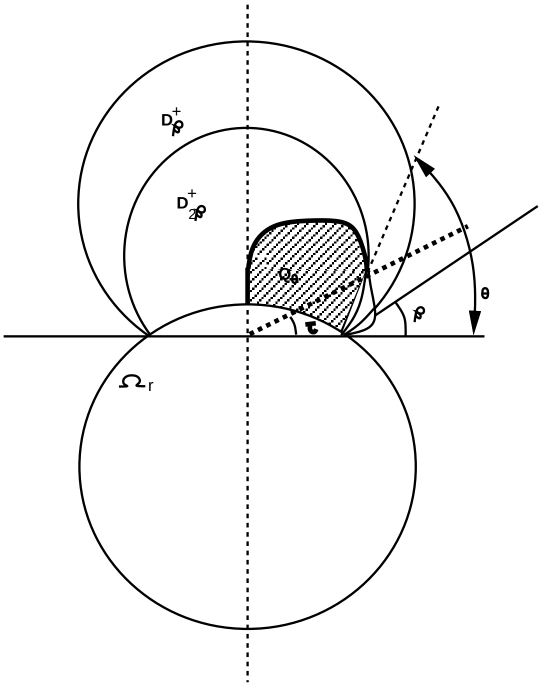

Fig. 9

For any $0 < \theta_0 < \pi/2$, let $0 < \beta_0 = \beta(\theta_0) < \pi/2$ and let

$$\mathcal{Q}_{\theta_0} = \{z \in \Pi_{\beta_0} \mid \theta_0 \leq \arg(z-1) \leq \frac{\pi}{2}\}.$$

Then $\mathcal{Q}_{\theta_0} \subseteq D^+_{2\beta_0}$. Let $\overline{\mathcal{Q}}_{\theta_0} = \{\overline{z} \mid z \in \mathcal{Q}_{\theta_0}\}$ and let $\mathcal{S}_{\theta_0} = \mathcal{Q}_{\theta_0} \cup (-\mathcal{Q}_{\theta_0}) \cup \overline{\mathcal{Q}}_{\theta_0} \cup (-\overline{\mathcal{Q}}_{\theta_0})$. For a number $0 < \nu_0 < 1$, let

$$\nu_0 \cdot \mathcal{S}_{\theta_0} = \{w = \nu_0 \cdot z \mid z \in \mathcal{S}_{\theta_0}\}.$$

Let $A_{\theta_0} = \Omega_r \setminus (\nu_0 \mathcal{S}_{\theta_0})$. From the above calculation, we have

Lemma 15. *There is a constant $C = C(\theta_0, \nu_0) > 0$ depending on θ_0 and on ν_0 such that the modulus $mod(A_{\theta_0})$ of the annulus A_{θ_0} is greater than C.*

Proof. Let $a = \mathrm{diam}(\nu_0 \mathcal{S}_{\theta_0})$ be the diameter of $\nu_0 \mathcal{S}_{\theta_0}$ and let $b = d(\partial(\nu_0 \mathcal{S}_{\theta_0}), \partial \Omega_r)$ be the distance between $\partial(\nu_0 \mathcal{S}_{\theta_0})$ and $\partial \Omega_r$. Then a/b is bounded from above by a constant depending only on θ_0 and on ν_0. This implies the lemma (by using Grötzsch argument (refer to [AH1])). ■

Suppose $0 < C_0 < 1$ is a constant. If we use $q_a(z) = \sqrt{z-a}/\sqrt{1-a}$ for $|a| < C_0$ to replace $q(z)$ in the above calculation, then Lemma 15 has a generalized version.

Lemma 15′. *There is a constant $C = C(\theta_0, \nu_0, C_0) > 0$ depending on θ_0, ν_0, and C_0 such that the modulus $\mathrm{mod}(A_{\theta_0})$ of the annulus A_{θ_0} is greater than C.*

Theorem 20 [SU2]. *A Feigenbaum-like quadratic-like map (U, V, f) has the a priori complex bounds and is unbranched.*

Proof. We use the same notation as in the previous lemmas. Suppose (U, V, f) is a Feigenbaum-like quadratic-like map. Let $m_k = 2^k$. Consider the renormalizations

$$f^{\circ m_k} : U_k \to V_k$$

for $1 \leq k < \infty$ where $V_k = (V \setminus \mathbf{R}^1) \cup \mathring{T}_k(m_k)$ and where $U_k \cap \mathbf{R}^1 = [-o_k, o_k]$. From Lemma 14, there is a constant angle $0 < \theta_0 < \pi/2$ such that

$$U_k \subseteq R_{k,\theta_0}$$

for $k > 0$. Let $\beta_0 = \beta(\theta_0)$.

Consider $f^{\circ(m_k-1)} : J_k(1) \to J_k(0)$. Its inverse has the maximum extension

$$\mathcal{A}_k = g_{s(1)} \circ g_{s(2)} \circ \cdots \circ g_{s(m_k-1)} : V_k \to U_k'$$

where $U_k' \cap \mathbf{R}^1 = T_k(1) = [d_1, e_1]$ with $d_1 < c(1) < e_1$. From Theorem 19 (and the bounded geometry property of the attractive Cantor set), there is a constant $C_0 > 0$ such that $C_0^{-1} \leq |c(1) - d_1|/|e_1 - c(1)| < C_0$ for all $k > 0$. We normalize $\mathring{T}_k(m_k) = (d_{m_k}, e_{m_k})$ to $(-1, 1)$ by the linear map s_1 such that $s_1(d_{m_k}) = 1$ and $s_1(e_{m_k}) = -1$. We normalize $(0, o_k)$ to $(0, 1)$ by the linear map s_2 such that $s_2(0) = 0$ and $s_2(o_k) = 1$. We normalize $\mathring{T}_k(1) = (d_1, e_1)$ to $(-1, 1)$ by the linear map s_3 such that $s_3(e_1) = -1$ and $s_3(d_1) = 1$. Let $a = s_3(c(1))$. Then there is a constant we still denote it as $0 < C_0 < 1$ such that $|a| < C_0$ for all $k > 0$. There is an integer $n_0 > 0$ such that for any $k > n_0$, $\Omega_r = D^+_{\beta_0} \cup D^-_{\beta_0}$ is contained in $s_1(V)$. Let $\mathcal{B}_k = s_3 \circ \mathcal{A}_k \circ s_1^{-1}$ and let $q_a(z) = s_2 \circ g_{s_{m_k}} \circ s_3^{-1}$. Then q_a is comparable with $\sqrt{z-a}/\sqrt{1-a}$. Since $\mathcal{B}_k$ contracts the hyperbolic distance $d_{H,C_{I_0}}$, then $\mathcal{B}_k(\Omega_r) \subseteq \Omega_r$. From Lemma 14,

$$X_k' = q_a(\mathcal{B}_k(\Omega_r)) \subseteq \mathcal{S}_{\theta_0} = \mathcal{Q}_{\theta_0} \cup (-\mathcal{Q}_{\theta_0}) \cup \overline{\mathcal{Q}}_{\theta_0} \cup (-\overline{\mathcal{Q}}_{\theta_0}).$$

Let $X_k'' = s_1 \circ s_3^{-1}(X_k')$. Then $X_k'' \subseteq \nu_0 S_{\theta_0}$ for all $k > 0$ where $\nu_0 > 0$ is a constant obtained from Lemma 4.2. Let $Y_k'' = \Omega_r$. Then from Lemma 15′,

$$\mathrm{mod}(Y_k'' \setminus X_k'') > C$$

for all $k > n_0$ where $C > 0$ is a constant.

Now let $X_k = s_1^{-1}(X_k'')$ and let $Y_k = s_1^{-1}(Y_k'')$. Then

$$f^{\circ m_k} : X_k \to Y_k$$

is quadratic-like map and $\mathrm{mod}(Y_k \setminus X_k) > C$ for all $k > n_0$. This means that (U, V, f) has the *a priori* complex bounds.

Let $W_k = Y_k \setminus (LJ_k(0) \cup RJ_k(0))$. Applying Theorem 19 and the above argument, there is a constant $C' > 0$ such that $\mathrm{mod}(W_k \setminus K_k) > C'$ for all $k > n_0$ where K_k is the filled-in Julia set of $f^{\circ m_k} : X_k \to Y_k$. But $W_k \cap CO = \{c(jm_k)\}_{j=0}^{\infty}$. So (U, V, f) is unbranched. This completes the proof. ∎

Theorem 20 and Theorem 16 now give us that

Corollary 1. *The filled-in Julia set K_f of a Feigenbaum-like quadratic-like map (U, V, f) is locally connected.*

Sullivan [SU2] (see also [MEV]) also proved that any real infinitely renormalizable quadratic-like map $f : U \to V$ of bounded type has the *a priori* complex bounds. Thus it is unbranched and its filled-in Julia set is locally connected from Theorem 16.

8. The Local Connectivity of Certain Infinitely Renormalizable Quadratic Julia Sets

We prove in this section the following result.

Theorem 21. *There is a subset $\tilde{\Upsilon}$ in $\mathcal{M}$ such that* **(1)** *$\tilde{\Upsilon}$ is dense in the boundary $\partial\mathcal{M}$ of the Mandelbrot set $\mathcal{M}$,* (2) *for every c in $\tilde{\Upsilon}$, P_c is unbranched, infinitely renormalizable and has the a priori complex bounds.*

From Theorem 16, we have

Corollary 2. *The filled-in Julia set K_c of P_c is locally connected for every c in $\tilde{\Upsilon}$.*

Proof of Theorem 21. Suppose c_0 is a Misiurewicz point in $\mathcal{M}$. Then there is an integer $m > 1$ such that $p = P_{c_0}^{\circ m}(0)$ is a repelling periodic point of period $k \geq 1$. Let α be the separate fixed point of P_{c_0}. Without loss of generality, we assume that P_{c_0} is non-renormalizable. (If P_{c_0} is renormalizable, it must be finitely renormalizable. We would then take α as the separate fixed point of the last renormalization of P_{c_0} (see §4)). Let Γ be the union of a cycle of external rays landing at α. Let γ be a fixed equipotential curve of P_{c_0}. Using Γ and γ, we construct the two-dimensional Yoccoz puzzle as follows (see §3). Let C_{-1} be the domain bounded by γ. The set Γ cuts C_{-1} into a finite number of closed domains. Let η_0 denote the set of these domains. Let $\eta_n = P_{c_0}^{-n}(\eta_0)$. Let C_n be the member of η_n containing 0.

Let

$$p \in \cdots \subseteq D_n(p) \subseteq D_{n-1}(p) \subseteq \cdots \subseteq D_1(p) \subseteq D_0(p)$$

be a p-end, where $D_n(p) \in \eta_n$. Let

$$c_0 \in \cdots \subseteq E_n(c_0) \subseteq E_{n-1}(c_0) \subseteq \cdots \subseteq E_1(c_0) \subseteq E_0(c_0)$$

be a c_0-end, where $E_n(c_0) \in \eta_n$. We have $P_{c_0}^{\circ(m-1)}(E_{n+m-1}(c_0)) = D_n(p)$.

Since the diameter $\mathrm{diam}(D_n(p))$ tends to zero as $n \to \infty$ and since p is a repelling periodic point, we can find an integer $l \geq m$ such that $|(P_{c_0}^{\circ k})'(x)| \geq \lambda > 1$ for all $x \in D_l(p)$ and such that

$$P_{c_0}^{\circ(m-1)} : E_{l+m-1}(c_0) \to D_l(p)$$

is a homeomorphism. Let $q > 0$ be the smallest integer such that $P_{c_0}^{\circ q}(D_l(p))$ contains 0, i.e., it is C_{r_0} in η_{r_0} where $r_0 \geq 0$. Then

$$f = P_{c_0}^{\circ q} : D_l(p) \to C_{r_0}$$

is a homeomorphism. Let $r > r_0$ be an integer such that $B_0 = f^{-1}(C_r) \subset D_l(p)$ does not contain p where C_r is the member of η_r containing 0. Then

$$P_{c_0}^{\circ q} : B_0 \to C_r$$

is a homeomorphism. The domain B_0 is a member of η_{r+q}. Let $B_n \subseteq D_{l+nk}$ be the pre-image of B_0 under $P_{c_0}^{\circ nk}|D_{l+nk}(p)$ for $n \geq 1$. The domain B_n is a member of η_{r+q+nk} and

$$P_{c_0}^{\circ(q+nk)} : B_n \to C_r$$

is a homeomorphism.

From the structural stability theorem (see [PRZ,SHU]), the points α and p and the sets Γ, C_r, D_n and B_n for $n \geq 1$ are all preserved by a small perturbation c of c_0 (refer to [JI4]). Therefore they can be constructed for P_c as long as c near c_0. Let U_0 be a neighborhood about c_0 such that the corresponding points $\alpha(c)$ and $p(c)$ and the corresponding sets $\Gamma(c)$, $C_r(c)$, $D_n(c)$, and $B_n(c)$ for $n \geq 1$ are all preserved for $c \in U_0$. Moreover, as n goes to infinity, the diameter $\mathrm{diam}(B_n(c))$ tends to zero and the set $B_n(c)$ approaches to $p(c)$ uniformly on U_0. Let

$$W_n = W_n(c_0) = \{c \in \mathbf{C} \mid P_c^m(0) \in B_n(c)\}.$$

Then from the result in [JI4], $W_n \subseteq U_0$ for n large enough.

For any $c \in W_n$, Let $R_n(c)$ be the preimage of $B_n(c)$ under the map $P_{c_0}^{\circ(m-1)}$: $E_{l+m-1}(c) \to D_l(p,c)$ and let $C_{r+q+nk+m}(c) = P_c^{-1}(R_n(c))$. The domain $C_{r+q+nk+m}(c)$ is the member containing 0 in $\eta_{r+q+m+nk}$ and

$$F_{n,c} = P_c^{\circ(q+nk+m)} : X_n(c) = \mathring{C}_{r+q+nk+m}(c) \to Y_n(c) = \mathring{C}_r(c)$$

is a quadratic-like map. Let $A_n(c) = \mathring{C}_r(c) \setminus C_{r+q+nk+m}(c)$. Since the diameter $\mathrm{diam}(C_{r+q+nk+m})(c))$ tends to zero as n goes to infinity uniformly in U_0, there is an integer $N_0 > 0$ such that

$$\mathrm{mod}(A_n(c)) \geq 1$$

for all $n \geq N_0$ and $c \in W_n$. Since

$$\{F_{n,c} : X_n \to Y_n \mid c \in W_n\}$$

is a full family of quadratic-like maps, W_n contains a copy $\mathcal{M}_n = \mathcal{M}_n(c_0)$ of the Mandelbrot set $\mathcal{M}$ (see [DH3]). For any $c \in \mathcal{M}_n$, P_c is once renormalizable and

$$CO(c) \cap C_{r+q+nk+m}(c) = \{c(j(q+nk+m))\}_{j=0}^{\infty}$$

where $CO(c) = \{c(i) = P_c^{\circ i}(0)\}_{i=0}^{\infty}$. Let

$$\tilde{\Upsilon}_1(c_0) = \cup_{n=N_0}^{\infty} \mathcal{M}_n.$$

We use the induction to complete the construction of the subset $\tilde{\Upsilon}(c_0)$ around c_0. Suppose we have constructed W_w where $w = i_0 i_1 \ldots i_{k-1}$ and $i_0 \geq N_0$, $i_1 \geq N_{i_1}$, ..., $i_{k-1} \geq N_{i_0 i_1 \ldots i_{k-2}}$. There is a parameter $c_w \in \mathcal{M}_w$ such that

$$F_w = F_{w,c_w} : X_w = X_w(c_w) \to Y_w = Y_w(c_w)$$

is hybrid equivalent (see §1) to $P(z) = z^2 - 2$. For F_w, let β_w and α_w be its non-separate and separate fixed points. Let $\tilde{\beta}_w$ be another preimage of β_w under F_w. Let Γ_w be the external rays of P_{c_w} landing at α_w. Let Y_{w0} be the domain containing 0 and bounded by ∂X_w and $F_w^{-1}(\Gamma_w)$. Let $\tilde{\beta}_w \in E_{w0}$ and $\beta_w \in E_{w1}$ be the components of the closure of $X_w \setminus Y_{w0}$. Let G_{w0} and G_{w1} be the inverses of $F_w|E_{w0}$ and $F_w|E_{w1}$. Let

$$D_{wn} = G_{w1}^{\circ n}(D_{w0})$$

and

$$B_{wn} = G_{w0}(D_{w(n-1)})$$

for $n \geq 1$. From the structural stability theorem (see [PRZ,SHU]), the points β_w and α_w and the sets Γ_w are all preserved by a small perturbation c of c_w. Therefore we can find a small neighborhood U_w about c_w such that the corresponding domains $D_{wn}(c)$ and $B_{wn}(c)$ can be constructed for P_c, $c \in U_w$ (refer to [JI4]). Let

$$W_{wn} = \{c \in \mathbf{C} \mid F_{w,c}(0) \in B_{wn}(c)\}.$$

The diameter $\mathrm{diam}(B_{wn}(c)) \to 0$ as $n \to \infty$ uniformly on U_w. From the result in [JI4], $W_{wn} \subseteq U_w$ for n large.

For each c in W_{wn}, $n \geq N_w$, let $X_{wn}(c) = F_{w,c}^{-1}(\mathring{B}_{wn}(c))$ and $Y_{wn}(c) = \mathring{Y}_{w0}(c)$. Then

$$F_{wn,c} = F_{w,c}^{\circ(n+1)} : X_{wn}(c) \to Y_{wn}(c)$$

is a quadratic-like map. Let

$$A_{wn}(c) = X_{wn}(c) \setminus \overline{Y}_{wn}(c).$$

Since the diameter $\mathrm{diam}(Y_{wn}(c))$ tends to zero as n goes to infinity uniformly in U_w. There is an integer $N_w > 0$ such that

$$\mathrm{mod}(A_{wn}(c)) \geq 1$$

for all $n \geq N_w$ and $c \in W_{wn}$. Since

$$\{F_{wn,c} : X_{wn} \to Y_{wn} \mid c \in W_{wn}\}$$

is a full family of quadratic-like maps, W_{wn} contains a copy $\mathcal{M}_{wn} = \mathcal{M}_{wn}(c_0)$ of the Mandelbrot set $\mathcal{M}$ (see [DH3]). For any $c \in \mathcal{M}_{wn}$, P_c is k-*times* renormalizable and $A_{wn}(c)$ contains no critical values of P_c. Let

$$\tilde{\Upsilon}_k(c_0) = \cup_w \cup_{n=N_w}^{\infty} \mathcal{M}_{wn}$$

where w runs over all sequences of integers of length k in the induction.

We have thus constructed a subset $\tilde{\Upsilon}(c_0) = \cap_{k=1}^{\infty} \tilde{\Upsilon}_k(c_0)$ such that every $c \in \tilde{\Upsilon}(c_0)$ is infinitely renormalizable and such that c_0 is a limit point of $\tilde{\Upsilon}(c_0)$. From the above construction, for every $c \in \tilde{\Upsilon}(c_0)$, P_c is unbranced and has the *a priori* complex bounds.

Let $\tilde{\Upsilon} = \cup_{c_0} \tilde{\Upsilon}(c_0)$ where c_0 runs over all Misiurewicz points in $\mathcal{M}$. Then for every $c \in \tilde{\Upsilon}$, P_c is unbranched infinitely renormalizable and has the *a priori* complex bounds. Since the set of Misiurewicz points is dense in $\partial\mathcal{M}$ (see [CAG]), the set $\tilde{\Upsilon}$ is dense in $\partial\mathcal{M}$. It completes the proof of the theorem. ∎

Remark 3. Douady (see [MI2]) constructed an example of an infinitely renormalizable quadratic polynomial whose filled-in Julia set is not locally connected.

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