DOUGLAS ALGEBRAS WITH NO MAXIMAL SUBALGEBRA AND NO MINIMAL SUPERALGEBRA

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Abstract. We give several examples of Douglas Algebras that do not have any maximal subalgebra. We find a condition on these algebras that guarantees that some do not have any minimal superalgebra. We also show that if A is the only maximal subalgebra of a Douglas algebra B, then the algebra A does not have any maximal subalgebra.

1. Introduction

Let $\mathbf{D}$ denote the open unit disk in the complex plane and $\mathbf{T}$ the unit circle. By L^∞ we mean the space of essentially bounded measurable functions on $\mathbf{T}$ with respect to the normalized Lebesgue measure. We denote by H^∞ the space of all bounded analytic functions in $\mathbf{D}$. Via identification with boundary functions, H^∞ can be considered as a uniformly closed subalgebra of L^∞. Any uniformly closed subalgebra B strictly between L^∞ and H^∞ is called a Douglas algebra. $M(B)$ will denote the maximal ideal space of a Douglas algebra B. If we let $X = M(L^\infty)$, we can identify L^∞ with $C(X)$, the algebra of continuous functions on X. If C is the set of all continuous functions on $\mathbf{T}$, we set

$$H^\infty + C = \{h + g : g \in C,\, h \in H^\infty\}$$

$H^\infty + C$ then becomes the smallest Douglas algebra containing H^∞ properly. The function

$$q(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z - z_n}{1 - \bar{z}_n z}$$

is called a Blaschke product if $\sum_{n=1}^{\infty} (1 - |z_n|)$ converges. The set $\{z_n\}$ is called the zero set of q in $\mathbf{D}$. Here $\frac{|z_n|}{z_n} = 1$ is understood whenever $z_n = 0$. We call q an interpolating Blaschke product if

$$\inf_n \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_n z_m} \right| > 0\cdot$$

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An interpolating Blaschke product q is called sparse (or thin) if

$$\lim_{n\to\infty}\prod_{m\neq n}\left|\frac{z_m - z_n}{1-\bar{z}_n z_m}\right| = 1\cdot$$

The set

$$Z(q) = \{x \in M(H^\infty)\setminus \mathbf{D} : \mathbf{q(x) = 0}\}$$

is called the zero set of b in $M(H^\infty + C)$. Any function h in H^∞ with $|h| = 1$ almost everywhere on $\mathbf{T}$ is called an inner function. Since $|q| = 1$ for any Blaschke product, Blaschke products are inner functions. Let

$$QC = (H^\infty + C)\cap \overline{(H^\infty + C)}$$

and, for $x \in M(H^\infty + C)$, set

$$Q_x = \{y \in M(L^\infty) : f(x) = f(y)\,\mathrm{for\ all\,f} \in \mathrm{QC}\}\cdot$$

Q_x is called the QC-level set for x. For $x \in M(H^\infty + C)$, we denote by u_x the representing measure for x and its support set by $\mathrm{supp\,u_x}$. By $H^\infty[\bar{q}]$ we mean the Douglas algebra generated by H^∞ and the complex conjugate of the function q. Since X is the Shilov boundary for every Douglas algebra, a closed set E contained in X is called a peak set for a Douglas algebra B if there is a function in B with $f = 1$ on E and $|f| < 1$ on $X \setminus E$. A closed set E is a weak peak set for B if E is the intersection of a family of peak sets. If the set E is a weak peak set for H^∞ and we define

$$H^\infty_E = \{f \in L^\infty : f|_E \in H^\infty|_E\},$$

then H^∞_E is a Douglas algebra. For a Douglas algebra B, B_E is similarly defined. A closed set E contained in X is called the essential set for B, denoted by ess(B), if E is the smallest set in X with the property that for f in L^∞ with $f = 0$ on E, then f is in B.

For an interpolating Blaschke product q we put $N(\bar{q})$ the closure of

$$\bigcup\{\mathrm{supp\,u_x : x \in M(H^\infty + C)\ and\ |q(x)| < 1}\}\cdot$$

$N(\bar{q})$ is a weak peak set for H^∞ and is referred to as the nonanalytic points of q. By $N_0(\bar{q})$ we denote the closure of

$$\bigcup\{\mathrm{supp\,u_x : x \in Z(q)}\}.$$

For an $x \in M(H^\infty)$ we let

$$E_x = \{y \in M(H^\infty) : \mathrm{supp\,u_y = supp\,u_x}\}$$

and call E_x the level set of x. Since the sets $\mathrm{supp\,u_x}$ and $N(\bar{q})$ are weak peak sets for H^∞, both $H^\infty_{\mathrm{supp\,u_x}}$ and $H^\infty_{N(\bar{b})}$ are Douglas algebras. For any interpolating Blaschke product q we set

$$A = \bigcap_{x \in M(H^\infty + C) \setminus M(H^\infty[\bar{q}])} H^\infty_{\mathrm{supp}\,\mathrm{u_x}}$$

and

$$A_0 = \bigcap_{y \in Z(q)} H^\infty_{\mathrm{supp}\,\mathrm{u_y}}.$$

It is easy to see that $A \subseteq A_0$ and it was shown in [11] that $A = H^\infty_{N(q)}$. For x and y in $M(H^\infty)$, the pseudo-hyperbolic distance is defined by

$$\rho(x, y) = \sup \{|h(x)| : |h| \leq 1, h \in H^\infty, h(y) = 0\} \cdot$$

For x and y in $\mathbf{D}$ we have

$$\rho(x, y) = \left| \frac{x - y}{1 - \bar{y}x} \right| \cdot$$

For any $x \in M(H^\infty)$, we define the Gleason part of x by

$$P_x = \{y \in M(H^\infty) : \rho(x, y) < 1\} \cdot$$

If $P_x \neq \{x\}$, then x is said to be a nontrivial point. We denote by G the set of nontrivial points of $M(H^\infty + C)$, and for a Douglas algebra B, we set

$$G_B = G \cap (M(H^\infty + C) \setminus M(B)) \cdot$$

A point x in G_B is called a minimal support point of G_B (or simply a minimal support point of B) if there are no $y \in G_B$ such that $\mathrm{supp}\,\mathrm{u_y} \subsetneq \mathrm{supp}\,\mathrm{u_x}$. The set $\mathrm{supp}\,\mathrm{u_x}$ is called a minimal support set for B. For Douglas algebras B and B_0 with $B_0 \subseteq B$ we let $\Omega(B, B_0)$ be all interpolating Blaschke products q such that $\bar{q} \in B$ but $\bar{q} \notin B_0$.

We denote by $\Omega(B)$ the set of all interpolating Blaschke products q with $\bar{q} \in B$. Let B be a Douglas algebra. The Bourgain algebra B_b of B relative to L^∞ is the set of those elements of L^∞, f, such that $\|ff_n + B\|_\infty \to 0$ for every sequence $\{f_n\}$ in B with $f_n \to 0$ weakly. The minimal envelop B_m of a Douglas algebra B is defined to be the smallest Douglas algebra which contains all minimal superalgebra of B. An algebra A is called a minimal superalgebra of B if for all $x, y \in M(B) \setminus M(A)$, $x \neq y$ implies $\mathrm{supp}\,\mathrm{u_x} = \mathrm{supp}\,\mathrm{u_y}$.

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2. A Condition for Douglas Algebras to Have Equal Essential Sets

Consider the Douglas algebras A and A_0 defined above. In [11] some conditions were given when $A \subsetneq A_0$ but yet $\mathrm{ess(A)} = \mathrm{ess(A_0)}$. This happened because $\mathrm{ess(A)} = \mathrm{N(\bar{q})}$ and $\mathrm{ess(A_0)} = \mathrm{N_0(\bar{q})}$ (this is not hard to show). Theorem 1 of [11] gives conditions when $\mathrm{ess(A)} \neq \mathrm{ess(A_0)}$. The conditions found in Theorem 5 of [11] are far more complicated then those found in Theorem 1 below. Yet $\mathrm{ess(A) = ess(A_0)}$ in that Theorem (Theorem 5 of [11]) and also satisfies the condition in Theorem 1.

Theorem 1. *Let B_0 be a subalgebra of a Douglas algebra B with*

$$\mathrm{ess(B_0) \neq X}.$$

If for every $x \in M(B_0) \setminus M(B)$ we have $\mathrm{ess(H^\infty_{supp\, u_x}) = ess(B_{supp\, u_x})}$, *then* $\mathrm{ess(B) = ess(B_0)}$.

Proof: We note that $\mathrm{ess(H^\infty_{supp\, u_x}) = supp\, u_x}$. Hence if $\mathrm{ess(H^\infty_{supp\, u_x})}$ is contained in $\mathrm{ess(B)}$ for every $x \in M(B_0) \setminus M(B)$, then $\mathrm{supp\, u_y \subset ess(B)}$ for every $y \in M(B_0)$ and so we get $\mathrm{ess(B_0) \subset ess(B)}$. Since $B_0 \subset B$ we have that $\mathrm{ess(B) \subset ess(B_0)}$, and we get $\mathrm{ess(B) = ess(B_0)}$.

Corollary 1. *Let B_0 be a maximal subalgebra of a Douglas algebra B. Then* $\mathrm{ess(B_0) = ess(B)}$.

Since $M(B_0) = M(B) \cup E_x$ for some $x \in M(B_0) \setminus M(B)$, we have that if z and y are in $M(B_0) \setminus M(B)$, then $\mathrm{supp\, u_y = supp\, u_x = supp\, u_z}$. Now $ess(B_{\mathrm{supp\, u_x}}) = ess(H^\infty_{\mathrm{supp\, u_x}})$ since the set

$$\bigcup \left\{\mathrm{supp\, u_y : y \in M(B) \cap M(H^\infty_{supp\, u_x})}\right\}$$

is dense in $\mathrm{supp\, u_x}$ (because x is a minimal support point of B). Thus $\mathrm{ess(B) = ess(B_0)}$.

Corollary 2. *Let A be any Douglas algebra and q be an interpolating Blaschke product with $\bar{q} \notin A$. Then* $\mathrm{ess(A) = ess(A[\bar{q}])}$.

Proof: Let $B = A[\bar{q}]$ and let $x \in M(A) \setminus M(B)$. We need the following two facts:

i) $M(A) = \left\{x \in M(H^\infty + C) : A_{\mathrm{supp\, u_x}} = H^\infty_{\mathrm{supp\, u_x}}\right\}$
and

ii) If $A \subset B$, then $A_{\mathrm{supp\, u_x}} \subset B_{\mathrm{supp\, u_x}}$.

We use i) and ii) to show that $B_{\mathrm{supp\, u_x}} = A_{\mathrm{supp\, u_x}}[\bar{b}]$. Since $A \subset B$ by ii) $M(B_{\mathrm{supp\, u_x}}) \subset M(A_{\mathrm{supp\, u_x}})$. By the Chang–Marshall Theorem [1,16] it suffices to show that $M(B_{\mathrm{supp\, u_x}}) = M(A_{\mathrm{supp\, u_x}}[\bar{q}])$. Let $y \in M(B_{\mathrm{supp\, u_x}})$. Since $M(B_{\mathrm{supp\, u_x}}) \subset M(A_{\mathrm{supp\, u_x}})$ we have that $y \in M(A_{\mathrm{supp\, u_x}})$ and $|q(y)| = 1$ (since $M(B) = \{y \in M(A) : |q(y)| = 1\}$). Hence $y \in M(A_{\mathrm{supp\, u_x}}[\bar{q}])$. Therefore $M(B_{\mathrm{supp\, u_x}}) \subset M(A_{\mathrm{supp\, u_x}}[\bar{q}])$.

Now suppose $y \notin M(B_{\mathrm{supp\, u_x}})$. If $y \notin M(A_{\mathrm{supp\, u_x}})$, then $y \notin A_{\mathrm{supp\, u_x}}[\bar{b}]$ and we have nothing to prove. So we can assume that $y \in M(A_{\mathrm{supp\, u_x}})$. Now since $y \notin M(B_{\mathrm{supp\, u_x}})$ implies that $|q(y)| < 1$ and $y \in M(A_{\mathrm{supp\, u_x}})$. Hence $y \notin M(A_{\mathrm{supp\, u_x}}[\bar{q}])$. Thus $M(A_{\mathrm{supp\, u_x}}[\bar{q}]) \subset M(B_{\mathrm{supp\, u_x}})$. By the Chang–Marshall Theorem we have that $M(A_{\mathrm{supp\, u_x}}[\bar{q}]) = M(B_{\mathrm{supp\, u_x}})$. Hence $A_{\mathrm{supp\, u_x}}[\bar{q}] = B_{\mathrm{supp\, u_x}}$. Thus we have

$$\begin{aligned} \mathrm{ess(B_{supp\, u_x})} &= \mathrm{ess\left(A_{supp\, u_x}[\bar{q}]\right)} \\ &= \mathrm{ess\left(H^\infty_{supp\, u_x}[\bar{q}]\right)} \quad \text{by i)} \end{aligned}$$

So it suffices to show that

$$\mathrm{ess}\left(\mathrm{H}^{\infty}_{\mathrm{supp\,u_x}}[\bar{\mathrm{q}}]\right) = \mathrm{ess}\left(\mathrm{H}^{\infty}_{\mathrm{supp\,u_x}}\right).$$

To do this set $B_1 = H^{\infty}_{\mathrm{supp\,u_x}}$ and

$$E = \bigcup \left\{\mathrm{supp\,u_y} : \mathrm{supp\,u_y} \subset \mathrm{supp\,u_x},\ |\mathrm{q(y)}| = 1\right\}.$$

Assume that $\bar{E}$, the closure of E, is properly contained in $\mathrm{supp\,u_x}$. Put

$$B_2 = \bar{H}^{\infty}_{E} = \overline{\{f \in L^{\infty} : f|_E \in H^{\infty}|_E\}}.$$

By [2, page 39] $M(B_2) = \left\{m \in M(H^{\infty}) : \mathrm{supp\,u_m} \subseteq \bar{\mathrm{E}}\right\} \cup M(L^{\infty})$. Since $\bar{E} \subsetneq \mathrm{supp\,u_x}$ we have that $B_1 \subsetneq B_2$. Therefore $M(B_2) \subsetneq M(B_1)$ and so there is a nontrivial point $x_0 \in M(B_1) \setminus M(B_2)$ [5, Proposition 4.1] such that (a) $\mathrm{supp\,u_{x_0}} \subsetneq \mathrm{supp\,u_x}$, (b) $\mathrm{supp\,u_{x_0}} \not\subset \bar{\mathrm{E}}$ [otherwise $x_0 \in M(B_2)$], (c) $|q(x_0)| < 1$ and (d) $\mathrm{supp\,u_{x_0}} \cap \mathrm{E} = \varnothing$. In (a) $\mathrm{supp\,u_{x_0}} \subsetneq \mathrm{supp\,u_x}$ for if $\mathrm{supp\,u_{x_0}} = \mathrm{supp\,u_x}$ for all $x_0 \in M(B_1) \setminus M(B_2)$, then B_1 is a maximal subalgebra of B_2, so by Corollary 1 we have $\bar{E} = \mathrm{supp\,u_x} = \mathrm{ess(B_1)}$ (in fact B_2 has no maximal subalgebras). The fact that $|q(x_0)| < 1$ implies that $\bar{q} \notin H^{\infty}_{\mathrm{supp\,u_{x_0}}}$.

Therefore there is a $y_0 \in Z(q)$ such that $\mathrm{supp\,u_{y_0}} \subseteq \mathrm{supp\,u_{x_0}}$. By Theorem 2 of [10] there is a $z_0 \in Z(q)$ such that $\mathrm{supp\,u_{z_0}}$ is a minimal support set for $H^{\infty}[\bar{q}]$ that is contained in $\mathrm{supp\,u_{x_0}} \subseteq \mathrm{supp\,u_x}$. Since q is an interpolating Blaschke product $\mathrm{supp\,u_{z_0}}$ is not trivial. By [5, Theorem 4.2] there is $m \in M(H^{\infty}+C)$ so that $\mathrm{supp\,u_m}$ is nontrivial and $\mathrm{supp\,u_m} \subsetneq \mathrm{supp\,u_{z_0}} \subseteq \mathrm{supp\,u_{x_0}}$. Since $\mathrm{supp\,u_{z_0}}$ is a minimal support set for $H^{\infty}[\bar{q}]$ we have that $|q(m)| = 1$. Thus $\mathrm{supp\,u_{x_0}} \cap \mathrm{E} \neq \varnothing$. This contradicts (d). So $\bar{E} = \mathrm{supp\,u_x}$ and $\mathrm{ess}(\mathrm{H}^{\infty}_{\mathrm{supp\,u_x}}) = \mathrm{ess}(\mathrm{H}^{\infty}_{\mathrm{supp\,u_x}[\bar{q}]})$ and we get $\mathrm{ess}(\mathrm{B}_{\mathrm{supp\,u_x}}) = \mathrm{ess}(\mathrm{H}^{\infty}_{\mathrm{supp\,u_x}})$ for every $x \in M(A) \setminus M(B)$, so Corollary 2 follows from our Theorem.

We mention here that Corollary 2 was proved in [20, Theorem 2] by another method.

There are algebras B_0 and B that satisfy the hypothesis of Theorem 1 and are not of the form $B = B_0[\bar{q}]$ for any interpolating Blaschke product (if $B_0 \subseteq B$). To see this let Γ be the collection of sparse Blaschke products and B be the Douglas algebra $[H^{\infty} : \bar{q};\ q \in \Gamma]$. Let q_0 be any element in Γ and put $B_0 = H^{\infty}[\bar{q}]$. Then $B_0 \subset B$. By a Theorem of Hedenmaln's [13] we have that if b is a Blaschke product such that $\bar{b} \in B$, then $b = b_1 \cdots b_n$ where each b_i, $i = 1, \dots, n$, is a sparse Blaschke product. Hence if $x \in M(B_0) \setminus M(B)$, then x is the zero of some sparse Blaschke product. So x is a minimal support point of B for every $x \in M(B_0) \setminus M(B)$. This implies that $\mathrm{ess}(\mathrm{B}_{\mathrm{supp\,u_x}}) = \mathrm{ess}(\mathrm{H}^{\infty}_{\mathrm{supp\,u_x}})$. So, by Theorem 1, we have $\mathrm{ess(B)} = \mathrm{ess(B_0)}$ (Theorem 2 below shows that $H^{\infty}_{\mathrm{supp\,u_x}}$ is a maximal subalgebra of $B_{\mathrm{supp\,u_x}}$.) Now suppose there is a Blaschke product $q \in \Omega(B, B_0)$ with $B = B_0[\bar{q}]$. Again by Hedenmaln's Theorem we have $q = q_1 \cdots q_n$ with each q_i a sparse Blaschke product. Let Q be any infinite sparse Blaschke product such that $|Q| = 1$ on $\bigcup_{x \in Z(q)} P_x$. Then there is a $m \in M(H^{\infty} + C)$ such that $Q(m) = 0$ but $m \notin \bigcup_{x \in Z(q)} P_x$. Thus $|q(m)| = 1$ and so we get that

$m \in M(B_0[\bar{q}])$. Thus $\bar{Q} \notin B_0[\bar{q}]$ and yet $\bar{Q} \in B$. This implies that $B_0[\bar{q}] \subsetneq B$, which is a contradiction.

3. Maximal Subalgebras That Have No Maximal Subalgebra

We begin by extending Proposition 1 of [9]. There the authors showed that if $x \in Z(q)$ with q a sparse Blaschke product, then the algebra $H^\infty_{\mathrm{supp\,u_x}}$ is a maximal subalgebra of $H^\infty_{\mathrm{supp\,u_x}}[\bar{q}]$. Below we show that this is true for a larger class.

Theorem 2. *Let A be any Douglas algebra with maximal subalgebra and x be a minimal support point of G_A. Then $H^\infty_{\mathrm{supp\,u_x}}$ is a maximal subalgebra of $A_{\mathrm{supp\,u_x}}$ and $A_{\mathrm{supp\,u_x}} = H^\infty_{\mathrm{supp\,u_x}}[\bar{q}]$ for some $q \in \Omega(A)$.*

Proof: Let $B_0 = H^\infty_{\mathrm{supp\,u_x}}$ and $B = A_{\mathrm{supp\,u_x}}$. Suppose x is a minimal support set for G_A. Then we have for any interpolating Blaschke product $\psi \in \Omega(A)$ with $|\psi(x)| < 1$ and any $y \in M(H^\infty + C)$ with $\mathrm{supp\,u_y} \subsetneq \mathrm{supp\,u_x}$, $|\psi(\mathrm{y})| = 1$. Thus if $\psi_0 \in \Omega(B)$ and $|\psi_0(x)| < 1$ there is a $\psi \in \Omega(A)$ such that $\psi|_{\mathrm{supp\,u_x}} = \psi_0|_{\mathrm{supp\,u_x}}$. This implies that $|\psi_0(y)| = 1$ for every such y. Hence x is a minimal support point for G_B. Note that this implies that $M(B) = M(B_0) \setminus E_x$ where E_x is the level set for x. Hence $M(B_0) = M(B) \cup E_x$, so by Theorem 1 of [10], B_0 is a maximal subalgebra of B. Let q be any element in $\Omega(A)$ with $q(x) = 0$. Then $q \in \Omega(B, B_0)$ and we have that $B = B_0[\bar{q}]$.

Theorem 3. *A Douglas algebra A has no maximal subalgebra if and only if $H^\infty_{\mathrm{supp\,u_x}}$ is not a maximal subalgebra of $A_{\mathrm{supp\,u_x}}$ for every $x \in G_A$.*

Proof: Suppose A has no maximal subalgebra and let $x \in G_A$. Since x is not minimal support set of G_A there is a $y \in G_A$ with $\mathrm{supp\,u_y} \subsetneq \mathrm{supp\,u_x}$, and a $\psi \in \Omega(A)$ such that $|\psi(y)| < 1$. Since $\bar{\psi} \notin H^\infty_{\mathrm{supp\,u_y}}$, we can assume that $\psi(y) = 0$. Hence $y \notin M(A_{\mathrm{supp\,u_x}})$. By Lemma 4 of [8] there is a $\psi_0 \in \Omega(A_{\mathrm{supp\,u_x}})$ such that $|\psi_0(y)| = 1$ and $\psi_0(x) = 0$. Then we have $H^\infty_{\mathrm{supp\,u_x}} \subsetneq H^\infty_{\mathrm{supp\,u_x}}[\bar{\psi}_0] \subsetneq A_{\mathrm{supp\,u_x}}$. So $H^\infty_{\mathrm{supp\,u_x}}$ is not a maximal subalgebra of $A_{\mathrm{supp\,u_x}}$.

Suppose that for all $x \in G_A$, $H^\infty_{\mathrm{supp\,u_x}}$ is not a maximal subalgebra of $A_{\mathrm{supp\,u_x}}$. Then there is an algebra B with $H^\infty_{\mathrm{supp\,u_x}} \subsetneq B \subseteq A_{\mathrm{supp\,u_x}}$. Thus we can find a $y \in M(H^\infty + C)$ such that $\mathrm{supp\,u_y} \subsetneq \mathrm{supp\,u_x}$ and $y \in M(B) \setminus M(A_{\mathrm{supp\,u_x}})$. This implies that there is an interpolating Blaschke product q with $\bar{q} \in B \subset A_{\mathrm{supp\,u_x}}$ such that $|q(y)| = 1$ and $|q(x)| < 1$. Hence there is a $q_0 \in \Omega(A)$ with $q_0|_{\mathrm{supp\,u_x}} = q|_{\mathrm{supp\,u_x}}$. So $|q_0(y)| = 1$ and $|q_0(x)| < 1$. This implies that x is not a minimal support point of G_A for every $x \in G_A$. So by Theorem 1 of [10] A has no maximal subalgebra.

Proposition 1. *Let $x \in M(H^\infty) \backslash M(L^\infty)$. Then $H^\infty_{\mathrm{supp\,u_x}}$ has no maximal subalgebra.*

Proof: Now $\mathrm{ess}(\mathrm{H}^\infty_{\mathrm{supp\,u_x}}) = \mathrm{supp\,u_x}$. Hence if $y \in G_{H^\infty_{\mathrm{supp\,u_x}}}$, then $\mathrm{supp\,u_y} \cap \mathrm{supp\,u_x} = \varnothing$. Hence if A is a subalgebra of $H^\infty_{\mathrm{supp\,u_x}}$, then there is a $y \in M(A) \setminus M(H^\infty_{\mathrm{supp\,u_x}})$ with

$\mathrm{supp}\,\mathrm{u_y} \cap \mathrm{supp}\,\mathrm{u_x} = \varnothing$. Hence $\mathrm{ess(A)} \supseteq \mathrm{supp}\,\mathrm{u_y} \cup \mathrm{supp}\,\mathrm{u_x} \supsetneq \mathrm{supp}\,\mathrm{u_x} = \mathrm{ess}(\mathrm{H}^\infty_{\mathrm{supp}\,\mathrm{u_x}})$. By Corollary 1, A is not a maximal subalgebra of $H^\infty_{\mathrm{supp}\,\mathrm{u_x}}$.

Proposition 2. *Let A be a Douglas algebra that has only one maximal subalgebra A_0. Then A_0 has no maximal subalgebra.*

Proof: Suppose there is a subalgebra $B_0 \subsetneq A_0$ such that B_0 is a maximal subalgebra of A_0. Then by Theorem 1 of [10] there is an $x_0 \in G_{A_0}$ such that

$$M(B_0) = M(A_0) \cup E_{x_0} \tag{1}$$

Since A_0 is a maximal subalgebra of A there is an $x \in G_A \cap M(B_0)$ such that

$$M(A_0) = M(A) \cup E_x \tag{2}$$

By (1) and (2) we have that $M(B_0) = M(A) \cup E_x \cup E_{x_0}$. Since $x \in M(A_0)$ we have that $\mathrm{supp}\,\mathrm{u_x} \neq \mathrm{supp}\,\mathrm{u_{x_0}}$. Also since $x_0 \notin E_x$ and $x_0 \in G_{A_0}$ implies that $\mathrm{supp}\,\mathrm{u_{x_0}} \not\subset \mathrm{supp}\,\mathrm{u_x}$ (otherwise $x_0 \in M(A)$ by (2). We show that x_0 is a minimal support point of G_A, and hence get a contradiction. Let $y \in M(H^\infty + C)$ such that $\mathrm{supp}\,\mathrm{u_y} \subsetneq \mathrm{supp}\,\mathrm{u_{x_0}}$. Since x_0 is a minimal support point of G_{A_0} we have that $y \in M(A_0) = M(A) \cup E_x$. If $y \in M(A)$ then we are done. So we can assume $y \in E_x$. If $y \in E_x$, then $\mathrm{supp}\,\mathrm{u_y} = \mathrm{supp}\,\mathrm{u_x}$, so we have that $\mathrm{supp}\,\mathrm{u_x} \subsetneq \mathrm{supp}\,\mathrm{u_{x_0}}$. Since $x \notin M(A)$ there is an interpolating Blaschke product q with $\bar{q} \in A$ and such that $q(x) = 0$. By Theorem 2 of [11] there is an uncountable set U of $Z(q)$ such that (a) $\mathrm{supp}\,\mathrm{u_m} \subsetneq \mathrm{supp}\,\mathrm{u_{x_0}}$ for all $m \in U$ and (b) $\mathrm{supp}\,\mathrm{u_m} \cap \mathrm{supp}\,\mathrm{u_k}$ for all $m, k \in U, m \neq k$. By (1) each such $m \in U$ is in $M(A_0)$. Since for all $m \in U$ (except if $m = x$) we have $\mathrm{supp}\,\mathrm{u_x} \cap \mathrm{supp}\,\mathrm{u_m} = \varnothing$, hence by (2), $m \in M(A)$. But $\bar{q} \in A$ and $U \subset Z(q) \cap M(A)$. This is a contradiction, and we get $y \notin E_x$. So $y \in M(A)$ and since $\mathrm{supp}\,\mathrm{u_{x_0}} \neq \mathrm{supp}\,\mathrm{u_x}$ we have that x_0 is a minimal support point of G_A. This is a contradiction. So A_0 has no maximal subalgebra.

Note that Proposition 1 follows from Proposition 2 if x is a minimal support point for some interpolating Blaschke product.

Let q be an interpolating Blaschke product. We consider the algebra $H^\infty_{N(\bar{q})}$. Certainly $H^\infty_{N(\bar{q})}$ is not known to be a maximal subalgebra of any Douglas algebra, but does have some of the same properties of $H^\infty_{\mathrm{supp}\,\mathrm{u_x}}$. For example

Proposition 3. *The algebra $H^\infty_{N(\bar{b})}$ has no maximal subalgebra.*

Proof: Set $B = H^\infty_{N(\bar{q})}$. Let $x \in G_B$ and suppose x is a minimal support point for G_B. Then if $y \in M(H^\infty + C)$ such that $\mathrm{supp}\,\mathrm{u_y} \subsetneq \mathrm{supp}\,\mathrm{u_x}$ we have that $y \in M(B)$. By [2, page 39] we must have that $\mathrm{supp}\,\mathrm{u_y} \subseteq \mathrm{N}(\bar{\mathrm{q}}) = \mathrm{ess(B)}$. Thus we have that $\mathrm{ess(B)} \cap \mathrm{supp}\,\mathrm{u_x} = \mathrm{N}(\bar{\mathrm{q}}) \cap \mathrm{supp}\,\mathrm{u_x} \neq \varnothing$. By [14, Theorem 1] $N(\bar{q}) = \bigcup_{x \in Z(q)} Q_x$. So there is an $x_0 \in Z(q)$ such that $\mathrm{supp}\,\mathrm{u_x} \cap \mathrm{Q_{x_0}} \neq \varnothing$. By the definition of Q_{x_0}, we have that $\mathrm{supp}\,\mathrm{u_x} \subset \mathrm{Q_{x_0}}$. By [2, page 39] this implies that $x \in M(B)$, which is a contradiction. So if $x \in G_B$, then $\mathrm{supp}\,\mathrm{u_x} \cap \mathrm{N}(\bar{\mathrm{q}}) = \varnothing$, which implies that x is not a minimal support point for G_B. B has no maximal subalgebra.

4. Minimal Superalgebras of $H^\infty_{\operatorname{supp} \mathrm{u_x}}$

We will compute the Bourgain algebras and the Minimal Envelopes of the Douglas algebra $H^\infty_{\operatorname{supp} \mathrm{u_x}}$ for any $x \in M(H^\infty + C)$. We have the following

Theorem 4. *Let $x \in M(H^\infty+C) \setminus M(L^\infty$ such that $|q(x)| < 1$ for some interpolating Blaschke product q and set $B = H^\infty_{S_{\operatorname{supp} \mathrm{u_x}}}$. Then*

i) *Either $B_b = B$ or $B_b = B[\bar\psi]$ for some interpolating Blaschke product ψ.*
ii) *Either $B_m = B_b = B$ or $B_m = B[\bar\psi]$ for some interpolating Blaschke product ψ.*

Proof: We will use Theorem 2 of [4] which says that for any interpolating Blaschke product ψ with $\bar\psi \in B_b$, we have that the set $Z(\psi) \cap M(B)$ is a finite set and the fact that

$$M(B) = M\left(L^\infty\right) \cup \{m \in M(H^\infty) : \operatorname{supp} \mathrm{u_m} \subseteq \operatorname{supp} \mathrm{u_x}\} \tag{3}$$

We claim that if ψ is an interpolating Blaschke product such that $\bar\psi \in B_b$, then $Z(\psi) \cap M(B) \subset E_x$, the level set of x. Suppose not. Then there is an $x_0 \in Z(\psi) \cap M(B)$ such that $\operatorname{supp} \mathrm{u_{x_0}} \subsetneq \operatorname{supp} \mathrm{u_x}$. By Theorem 2 of [11] there is an uncountable set Γ of $Z(\psi)$ such that (a) $\operatorname{supp} \mathrm{u}_\gamma \subsetneq \operatorname{supp} \mathrm{u_x}$ for all $\gamma \in \Gamma$ and (b) $\operatorname{supp} \mathrm{u_{x_m}} \cap \operatorname{supp} \mathrm{u}_\gamma = \varnothing$ for all $m, \gamma \in \Gamma$, $m \neq \gamma$. By (a) and (3) each $\gamma \in M(B)$ and so $\Gamma \subset Z(\psi) \cap M(B)$. This implies that the set $Z(\psi) \cap M(B)$ is infinite. This is a contradiction. Hence if $x_0 \in Z(\psi) \cap M(B)$ then $\operatorname{supp} \mathrm{u_{x_0}} = \operatorname{supp} \mathrm{u_x}$, so we get $Z(\psi) \cap M(B) \subset E_x$. There are two possibilities (1) the set $Z(\psi) \cap M(B) = \varnothing$ for which $\bar\psi \in B$, so $B_b \subseteq B$. This gives us the case when $B_b = B$. (2) If $Z(\psi) \cap M(B) \neq \varnothing$ but finite. Then the algebra $B[\bar\psi] \subseteq B_b$ To show that $B_b = B[\bar\psi]$, let ψ_0 be other interpolating Blaschke product with $\bar\psi_0 \in B_b$. Since both sets $Z(\psi_0) \cap M(B)$ and $Z(\psi) \cap M(B)$ are contained in E_x, we have that $M(B) \setminus M(B[\bar\psi_0]) = M(B) \setminus M(B[\bar\psi]) = E_x$. Thus $M(B[\bar\psi_0]) = M(B[\bar\psi])$ and by the Chang–Marshall Theorem [1, 16] we have $B[\bar\psi_0] = B[\bar\psi]$, since this is true for all ψ, ψ_0 we have by Theorem C of [12], $B_b = B[\bar\psi]$ for any such ψ $B[\bar\psi]$ is a minimal superalgebra of B.) This proves (i).

To prove (ii) let $\bar\psi \in B_m$. Then, by Theorem 3 of [12] there is a finite set $\{x_1, \ldots, x_n\} \subset Z(\psi) \cap M(B)$ such that $\{u \in M(B) : |\psi(u)| < 1\} = E_{x_1} \cup \cdots \cup E_{x_n}$. Again we claim that $E_{x_1} = E_{x_2} = \cdots = E_{x_n} = E_x$. Suppose that $E_{x_1} \neq E_{x_2}$. Then $\operatorname{supp} \mathrm{u_{x_1}} \neq \operatorname{supp} \mathrm{u_{x_2}}$. By (3) either $\operatorname{supp} \mathrm{u_{x_1}} \subsetneq \operatorname{supp} \mathrm{u_x}$ or $\operatorname{supp} \mathrm{u_{x_1}} \subsetneq \operatorname{supp} \mathrm{u_x}$ or both. Let us suppose that $\operatorname{supp} \mathrm{u_{x_1}} \subsetneq \operatorname{supp} \mathrm{u_x}$. Then by Theorem 2 of [11] there is an uncountable set Γ such that $E_\alpha \neq E_\beta$ for all $\alpha, \beta \in \Gamma$ and $\bigcup_{\alpha \in \Gamma} E_\alpha \subset \{u \in M(B) : |\psi(u)| < 1\}$. This contradicts Theorem 3 of [12]. Thus $E_{x_1} = E_{x_2} = \cdots = E_{x_n} = E_x$. As before we have that for $\bar\psi \in B_m$, $Z(\psi) \cap M(B) \subset E_x$ and $B_m = B[\bar\psi]$ if $Z(\psi) \cap M(B) \neq \varnothing$. This proves ii).

Corollary 3. i) *Let $x \in M(H^\infty + C) \setminus M(L^\infty)$ and $B = H^\infty_{S_{u_x}}$. Then $B \subset B_m$ if and only if x is a minimal support point of $H^\infty[\bar\psi]$ for some interpolating Blaschke product ψ.*

ii) $B = B_b = B_m$ *if and only if* x *is not a minimal support point of* $H^\infty[\bar{\psi}]$ *for any interpolating Blaschke product.*

Theorem 4 i) has also appeared in [17].

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