COMPLETE INTERPOLATING SEQUENCES FOR PALEY-WIENER SPACES AND MUCKENHOUPT'S (A_p) CONDITION

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Abstract. We describe the complete interpolating sequences for the Paley-Wiener spaces L^p_π $(1 < p < \infty)$ in terms of Muckenhoupt's (A_p) condition. For $p = 2$, this description coincides with those given by Pavlov (1979), Nikol'skii (1980), and Minkin (1992) of the unconditional bases of complex exponentials in $L^2(-\pi, \pi)$. While the techniques of these authors are linked to the Hilbert space geometry of L^2_π, our method of proof is based on turning the problem into one about boundedness of the Hilbert transform in certain weighted L^p spaces of functions and sequences.

1. Introduction

In this paper we study interpolation in the Paley-Wiener spaces L^p_π $(1 < p < \infty)$, which consist of all entire functions of exponential type π whose restrictions to the real line are in L^p. The Paley-Wiener spaces are Banach spaces when endowed with the natural $L^p(\mathbb{R})$-norms. We want to describe those sequences $\Lambda = \{\lambda_k\}$, $\lambda_k = \xi_k + i\eta_k$, in the complex plane $\mathbb{C}$ for which the interpolation problem

$$f(\lambda_k) = a_k \tag{1}$$

has a unique solution $f \in L^p_\pi$ for every sequence $\{a_k\}$ satisfying

$$\sum_k |a_k|^p e^{-p\pi|\eta_k|}(1 + |\eta_k|) < \infty. \tag{2}$$

Such sequences Λ are termed *complete interpolating sequences for* L^p_π. A classical example of a complete interpolating sequence for L^p_π $(1 < p < \infty)$ is the sequence of integers $\mathbb{Z}$.

In the case $p = 2$ this problem is equivalent to that of describing all unconditional bases in $L^2(-\pi, \pi)$ of the form $\{\exp(i\lambda_k t)\}$. We refer to [3] for an account of this problem, including a detailed survey of its history. The unconditional basis problem was solved by Pavlov [8] under the additional restriction $\sup |\Im\lambda_k| < \infty$ and by Nikol'skii [7], assuming only $\inf \Im\lambda_k > -\infty$. Finally, Minkin [6] solved the problem without any a priori assumption on Λ.

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The methods of [3,6,7,8] are of a geometric nature and make crucial use of the Hilbert space structure of L^2_π. In this paper, we shall give a simpler proof, which works equally well for all p, $1 < p < \infty$. Incidentally, our method of proof shows that for $p = \infty$ or $0 < p \le 1$ there are no complete interpolating sequences. (See also [1], which "explains" this curious phenomenon.) The core of our approach is a careful study of properties of the Hilbert transform in weighted spaces of functions and its discrete version in weighted spaces of sequences. More precisely, we turn our problem into one about boundedness of the discrete Hilbert transform in a weighted space, defined on a subsequence of Λ located in a horisontal strip, where the weight is expressed in terms of certain infinite products involving *all* the points of Λ.

As an application of our main theorem, we prove a counterpart of the well-known Kadets 1/4 theorem.

2. Preliminary observations and statement of the main result

Suppose that Λ is a complete interpolating sequence for L^p_π. Then $f \in L^p_\pi$ implies that $\exp(i\pi z)f(z)$ belongs to the Hardy space H^p of $\mathbb{C}^+_a := \{z \in \mathbb{C} : \Im z > a\}$ for each $a \in \mathbb{R}$. It follows that the sequence $\Lambda \cap \mathbb{C}^+_a$ is H^p-interpolating in $\mathbb{C}^+_a$ (see [4], Chapter 9). Similarly, $\Lambda \cap \mathbb{C}^-_a$ is H^p-interpolating in the half-plane $\mathbb{C}^-_a := \{z \in \mathbb{C}; \Im z < a\}$. So the sequences $\Lambda \cap \mathbb{C}^+_a$ and $\Lambda \cap \mathbb{C}^-_a$ satisfy the Carleson condition in the corresponding half-planes, i.e.,

$$\sup_{\Im\lambda_j > a} \prod_{\Im\lambda_k > a, k \ne j} \left| \frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i2a} \right| > 0, \quad \sup_{\Im\lambda_j < a} \prod_{\Im\lambda_k < a, k \ne j} \left| \frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i2a} \right| > 0. \quad (3)$$

We note that, by standard manipulations (which we omit) with the Carleson condition, this is equivalent to the following condition:

$$\sup_j \sum_{k, k\ne j} \frac{(1+|\eta_j|)(1+|\eta_k|)}{|\lambda_j - \lambda_k|^2} < \infty. \quad (4)$$

In particular, for some $\varepsilon > 0$, the disks

$$K(\lambda_k) := \{z : |z - \lambda_k| < 10\varepsilon(1 + |\eta_k|)\}$$

are pairwise disjoint. (We fix this value of ε until the end of the paper.) Moreover, the measure

$$\mu^+_\Lambda := \sum_{\eta_k > 0} \eta_k \delta_{\lambda_k}$$

(δ_λ is the unit point measure at λ) is a Carleson measure, i.e.,

$$\int_{\mathbb{C}^+} |f|^s d\mu^+_\Lambda \le C\|f\|^s_{H^s}$$

for each function f in the Hardy space $H^s(\mathbb{C}^+)$, $s \ge 1$. Similarly, Λ generates a Carleson measure in the lower half-plane as well as in each of the half-planes $\mathbb{C}^\pm_a$.

If Λ is a complete interpolating sequence for L^p_π, then

$$\|f\|_{L^p(\mathbb{R})} \le C \left(\sum_k |f(\lambda_k)|^p e^{-p\pi|\eta_k|}(1+|\eta_k|) \right)^{1/p}, \quad f \in L^p_\pi. \tag{5}$$

Indeed, since Λ is interpolating for L^p_π, the operator

$$T : f \mapsto \{f(\lambda_k)e^{-\pi|\eta_k|}(1+|\eta_k|)^{1/p}\}$$

is bounded from L^p_π onto l^p. By the uniqueness of the solution of the interpolation problem, we have $\ker T = \{0\}$, and it suffices to apply the Banach theorem on inverse operators.

Given $x \in \mathbb{R}$, $r > 0$, let $Q(x,r)$ be the square with center at x, side length $2r$, and sides parallel to the coordinate axes. We say that a sequence $\Lambda \subset \mathbb{C}$ is *relatively dense* if there exists $r_0 > 0$ such that $\Lambda \cap Q(x, r_0) \neq \emptyset$ for each $x \in \mathbb{R}$. If Λ is a complete interpolating sequence for L^p_π, (5) forces Λ to be relatively dense: if this is not the case and there exist sequences $\{x_j\} \subset \mathbb{R}$ and $r_j \to \infty$ such that $Q(x_j, r_j) \cap \Lambda = \emptyset$, then, setting

$$f_j(z) = \frac{\sin \frac{\pi}{2}(z - x_j)}{z - x_j},$$

we get

$$\sum_k |f_j(\lambda_k)|^p e^{-p\pi|\eta_k|}(1+|\eta_k|) \;\to\; 0, \quad j \to \infty,$$

while $\|f_j\|_{L^p}$ is independent of j.

Suppose that Λ is a complete interpolating sequence for L^p_π. Take $r > r_0$, where r_0 is as above, define

$$Q_j = Q(4rj, r), \qquad j \in \mathbb{Z},$$

and pick a sequence $\Gamma = \{\gamma_j\} \subset \Lambda$ such that $\gamma_j \in Q_j$. Let $\Sigma = \{\sigma_j\}$ be another sequence with $|\gamma_j - \sigma_j| = \varepsilon$. Suppose $w = \{w_j\}$ is a positive weight sequence. Associate with it the weighted space l^p_w consisting of all sequences $a = \{a_k\}$ satisfying

$$\|a\|^p_{w,p} := \sum_k |a_k|^p w_k < \infty.$$

We are interested in the boundedness of the *discrete Hilbert operator* $\mathcal{H}_{\Gamma,\Sigma}$ defined by the relation

$$\mathcal{H}_{\Gamma,\Sigma} : a = \{a_j\} \mapsto \{(\mathcal{H}_{\Gamma,\Sigma}a)_j\}; \quad (\mathcal{H}_{\Gamma,\Sigma}a)_j = \sum_k \frac{a_k}{\sigma_j - \gamma_k},$$

on l^p_w. The following definitions are needed. We say that w satisfies the *discrete* (A_p) *condition* if

$$\sup_{k,n} \left(\frac{1}{n} \sum_{j=k+1}^{k+n} w_j \right) \left(\frac{1}{n} \sum_{j=k+1}^{k+n} w_j^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

This condition is analogous to the classical continuous (A_p) condition for a positive weight $v(x) > 0,\ x \in \mathbb{R}$:

$$\sup_I \left\{ \left(\frac{1}{|I|} \int_I v dx \right) \left(\frac{1}{|I|} \int_I v^{-1/(p-1)} dx \right)^{p-1} \right\} < \infty, \tag{6}$$

where I ranges over all intervals in $\mathbb{R}$ (see [2]). Recall that the latter condition is necessary and sufficient for boundedness of the classical Hilbert operator

$$\mathcal{H} : f \mapsto (\mathcal{H}f)(t) = \frac{1}{i\pi} \int \frac{f(\tau)}{t - \tau} d\tau$$

on the weighted space of functions $L^p(\mathbb{R}; v)$ consisting of all functions f satisfying

$$\|f\|_{v,p}^p := \int |f(t)|^p v(t) dt < \infty.$$

We shall need the following lemma.

Lemma 1. *If $\mathcal{H}_{\Gamma,\Sigma}$ is bounded from l_w^p to l_w^p, then w satisfies the discrete (A_p) condition.*

Proof. We adopt the proof for the continuous case (see [2].) Let k and n be given. For convenience, put $I_1 = \{k+1, k+2, ..., k+n\}$, $I_2 = \{k+2n+1, k+2n+2, ..., k+3n\}$. Suppose that a positive sequence a is supported on I_1. Then, for $j \in I_2$, we have

$$|(\mathcal{H}_{\Gamma,\Sigma} a)_j| \geq \sum_l a_l \frac{\Re(\sigma_j - \gamma_l)}{|\sigma_j - \gamma_l|^2} \geq \frac{C}{n} \sum_l a_l, \tag{7}$$

where C is independent of k and n. Putting $a_l = 1$, we get thus

$$\sum_{j \in I_2} w_j \leq C \sum_{l \in I_1} w_l,$$

and by symmetry

$$\sum_{j \in I_1} w_j \asymp \sum_{l \in I_2} w_l; \tag{8}$$

here and in what follows the sign $\asymp$ means that the ratio of the two sides lies between two positive constants. Now we put $a_l = w_k^\alpha$ and get by (7)

$$\left(\sum_{j \in I_2} w_j \right) \left(\frac{1}{n} \sum_{l \in I_1} w_l^\alpha \right)^p \leq C \sum_{m \in I_1} w_m^{1+\alpha p}.$$

Finally, we put $\alpha = -\frac{1}{p-1}$ and invoke (8), and the lemma is proved. $\square$

The converse of Lemma 1 is also true, but we will not need that fact. Note also that the boundedness of the operator $\mathcal{H}_{\Gamma,\Sigma}$ is independent of the choice of sequence Σ, provided the condition $|\gamma_j - \sigma_j| = \varepsilon$ holds.

Let Λ be a complete interpolating sequence for L^p_π. If the function $f_0 \in L^p_\pi$ solves the interpolation problem $f_0(\lambda_k) = \delta_{0,k},\ k \in \mathbb{Z}$, then $f_0(\mu) \neq 0$ for $\mu \in \mathbb{C} \setminus \Lambda$, since otherwise the function $(z-\lambda_0)(z-\mu)^{-1} f_0(z)$ belongs to L^p_π and vanishes on Λ. Since $f_0 \in L^p_\pi$, f_0 belongs to the Cartwright class $\mathcal{C}$ (see [5], Lecture 15) and, in particular, the limit

$$S(z) = \lim_{R\to\infty} \prod_{|\lambda_k|<R} \left(1 - \frac{z}{\lambda_k}\right) \tag{9}$$

exists and defines the *generating function* of the sequence Λ. Besides, the solution $f_k \in L^p_\pi$ of the interpolation problem $f_k(\lambda_n) = \delta_{k,n}$ has the form

$$f_k(z) = \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)}. \tag{10}$$

We may now formulate our main theorem.

Theorem 1. *$\Lambda = \{\lambda_k\}$, where $\lambda_k = \xi_k + i\eta_k$, is a complete interpolating sequence for L^p_π if and only if the following three conditions hold.*

(i) *The sequences $\Lambda \cap \mathbb{C}^+$ and $\Lambda \cap \mathbb{C}^-$ satisfy the Carleson condition in $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively, i.e.* (3) *holds with $a = 0$, and also $\inf_{k\neq j} |\lambda_k - \lambda_j| > 0$.*

(ii) *The limit $S(z)$ in* (9) *exists and represents an entire function of exponential type π.*

(iii) *There exists a relatively dense subsequence $\Gamma = \{\gamma_j\} \subset \Lambda$ such that the sequence $\{|S'(\gamma_j)|^p\}$ satisfies the discrete (A_p) condition.*

Defining $F(x) = |S(x)|/\mathrm{dist}(x,\Lambda)$, we may replace statement (iii) *by the following:*

(iii') *F^p satisfies the (continuous) (A_p) condition.*

Note that that condition (i) is equivalent to the statement that, for each $a \in \mathbb{R}$, the sequences $\Lambda \cap \mathbb{C}^\pm_a$ satisfy the Carleson condition (3). Another, more compact way of expressing (i), is given by (4).

3. Proof of Theorem 1: Necessity

We have already proved the necessity of (i) and (ii), and also the existence of a relatively dense sequence $\Gamma = \{\gamma_j\} \subset \Lambda$. We prove now that (iii) is necessary as well. Let ε be as above. Then, for every j, we can find a point σ_j with $|\sigma_j - \gamma_j| = \varepsilon$ and

$$|S(\sigma_j)| = \varepsilon |S'(\gamma_j)|.$$

This follows from the fact that $S(z)(z-\gamma_j)^{-1} \neq 0$ for $|z-\gamma_j| \le \varepsilon$, hence

$$\min_{|z-\gamma_j|=\varepsilon} |S(z)(z-\gamma_j)^{-1}| \le |S'(\gamma_j)| \le \max_{|z-\gamma_j|=\varepsilon} |S(z)(z-\gamma_j)^{-1}|.$$

Set $\Sigma = \{\sigma_j\}$. The Plancherel-Pólya inequality (see [5], Lecture 20) yields

$$\sum_j |f(\sigma_j)|^p \le C\|f\|^p_{L^p}, \quad f \in L^p_\pi. \tag{11}$$

Now let $a = \{a_j\}$ be a finite sequence. By (10), the unique solution of the interpolation problem $f(\gamma_j) = a_j$, $f(\lambda_k) = 0$, $\lambda_j \notin \Gamma$ has the form

$$f(z) = \sum_j \frac{a_j}{S'(\sigma_j)} \frac{S(z)}{(z - \gamma_j)}.$$

By (5) and (11), we have

$$\sum_j |f(\sigma_j)|^p \le C \sum_j |a_j|^p.$$

Now, by our particular choice of the sequence Σ, we obtain (iii) by observing that Lemma 1 applies with $w_j = |S'(\gamma_j)|^p$.

To prove that (iii) implies (iii'), we need the following lemma.

Lemma 2. *Suppose $x \in \mathbb{R}$ and $\Re\gamma_j \le x \le \Re\gamma_{j+1}$. Then there exists an $\alpha = \alpha(x) \in [0,1]$ such that*

$$|S'(\gamma_j)|^\alpha |S'(\gamma_{j+1})|^{1-\alpha} \asymp |S(x)|/\mathrm{dist}(x, \Lambda),$$

uniformly with respect to $x \in \mathbb{R}$.

In fact, assuming this lemma to hold, we see that (9) with $v = F^p$ follows from (iii) and the inequality $t^\alpha s^{1-\alpha} \le t + s, \quad t, s > 0, \quad \alpha \in [0,1]$.

Proof of Lemma 2. We assume that $x \in [\Re\gamma_j, \Re\gamma_{j+1}]$ and, for simplicity, $x \notin \Lambda$. Set $\Lambda(x) = \{\lambda \in \Lambda : |\lambda - x| < 30r\}$. (Here r is the number used for constructing Γ.) For $\alpha \in [0,1]$ we have

$$\begin{aligned}\rho := &\frac{|S'(\gamma_j)|^\alpha |S'(\gamma_{j+1})|^{1-\alpha}}{|S(x)|\mathrm{dist}(x,\Lambda)^{-1}} \\ = &\left\{ \frac{\left|\frac{1}{\gamma_j}\prod_{\lambda_k \in \Lambda(x)\setminus\{\gamma_j\}}\left(1 - \frac{\gamma_j}{\lambda_k}\right)\right|^\alpha \left|\frac{1}{\gamma_{j+1}}\prod_{\lambda_k \in \Lambda(x)\setminus\{\gamma_{j+1}\}}\left(1 - \frac{\gamma_{j+1}}{\lambda_k}\right)\right|^{1-\alpha}}{\left|\prod_{\lambda \in \Lambda(x)}\left(1 - \frac{x}{\lambda}\right)\right|} \mathrm{dist}(x,\Lambda) \right\} \\ &\times \left\{ \prod_{\lambda \in \Lambda \setminus \Lambda(x)} \frac{|\gamma_j - \lambda_k|^\alpha |\gamma_{j+1} - \lambda_k|^{1-\alpha}}{|x - \lambda_k|} \right\} = \Pi_1(x) \times \Pi_2(x).\end{aligned}$$

A simple estimation shows that $\Pi_1(x) \asymp 1$ uniformly with respect to $\alpha \in [0,1]$ so we need only estimate $\Pi_2(x)$.

Let us put

$$\gamma_j = x - x_j + iy_j, \;\; \gamma_{j+1} = x + x_{j+1} + iy_{j+1}.$$

The values x_j and x_{j+1} depend upon x and also satisfy the inequalities $0 \le x_j, x_{j+1} \le 8r$. We may then write

$$\begin{aligned}\rho^2 \asymp &\prod_{\lambda_k \notin \Lambda(x)} \frac{((x - x_j - \xi_k)^2 + (y_j - \eta_k)^2)^\alpha ((x + x_{j+1} - \xi_k)^2 + (y_{j+1} - \eta_k)^2)^{1-\alpha}}{(x - \xi_k)^2 + \eta_k^2} \\ &= \prod_{\lambda_k \notin \Lambda(x)} \left(1 - \frac{2x_j(x - \xi_k) + 2y_j\eta_k + O(1)}{(x - \xi_k)^2 + \eta_k^2}\right)^\alpha \\ &\qquad \times \left(1 + \frac{2x_{j+1}(x - \xi_k) - 2y_{j+1}\eta_k + O(1)}{(x - \xi_k)^2 + \eta_k^2}\right)^{1-\alpha}.\end{aligned}$$

Choosing $\alpha = \alpha(x)$ so that $\alpha x_j - (1-\alpha)x_{j+1} = 0$, i.e., $\alpha = x_{j+1}/(x_j + x_{j+1})$, we find that

$$\rho^2 \asymp \exp\left(c \sum_{\lambda_k \notin \Lambda(x)} \frac{|\eta_k|}{(x-\xi_k)^2 + \eta_k^2}\right).$$

By Carleson's condition (4), the sum is uniformly bounded, and we are done. $\square$

4. Proof of Theorem 1: Sufficiency

We will now prove that (i), (ii), (iii') imply that Λ is a complete interpolating sequence.

To begin with, note that

$$\int [F(x)]^p \frac{dx}{1+|x|^p} < \infty \tag{12}$$

and

$$\int [F(x)]^p dx = \infty. \tag{13}$$

The first relation follows from the fact that $\int [F(x)]^p |\mathcal{H} f(x)|^p dx < \infty$ for each bounded finite function f; it suffices to take $f = \chi_{[0,1]}$. To obtain (12), we may apply the operator $\mathcal{H}$ to a δ-sequence $\{\delta_n(x)\}$.

First, we check that Λ is a uniqueness set. To this end, we need to estimate $|S(z)|$ from below.

Lemma 3. *Let ε be the number from relation* (3)*. Then*

$$|S(z)| \geq C(1-|z|)^{-1/p} e^{\pi|\Im z|} \quad \textit{for} \ \operatorname{dist}(z,\Lambda) > \varepsilon(1+|\Im z|). \tag{14}$$

Proof of Lemma 3. Put $\Lambda' = \Lambda \cap \{z : |\Im z| < \varepsilon\}$ and consider the auxiliary function

$$S_1(z) = S(z) \prod_{\lambda \in \Lambda'} \frac{z - \lambda + 2i\varepsilon}{z - \lambda}.$$

It is plain that

$$|S_1(z)| \asymp |S(z)|, \quad |\Im z| > 3\varepsilon, \tag{15}$$

and, besides, $|S_1(x)|^p$ satisfies the (A_p) condition. Consider the inner-outer factorization of S_1 in the upper half-plane,

$$S_1(z) = e^{-i\pi z} G(z) B_1(z), \quad \Im z > 0. \tag{16}$$

Here the Blaschke product B_1 corresponds to the Carleson sequence $(\Lambda \cap \mathbb{C}^+) \setminus \Lambda'$ and, in particular,

$$|B_1(z)| > c > 0 \ \text{ for } \operatorname{dist}(z,\Lambda) > \varepsilon|\Im z|. \tag{17}$$

Moreover, G is an outer function and $|G(x)|^p$ satisfies the (A_p) condition. Therefore, $|G(x)|^{-q}$ is an (A_q) weight (here $1/p + 1/q = 1$), $G(x)^{-1}(1+|x|)^{-1} \in L^q(\mathbb{R})$, and thus

$$\frac{1}{(z+i)G(z)} = \frac{1}{2\pi i}\int \frac{1}{(t+i)G(t)}\frac{dt}{t-z}, \quad \Im z > 0.$$

It follows that

$$\frac{1}{|G(z)|} \le C(1+|z|)^{1/p}. \tag{18}$$

Combining relations (15)–(18), we obtain (14) for $\Im z > 3\varepsilon$. The estimate for $\Im z < -3\varepsilon$ is similar, and to fill the gap $-3\varepsilon < \Im z < 3\varepsilon$, we may repeat the construction, taking another horisontal line instead of $\mathbb{R}$. $\square$

Note that, since $|S_1(x)|^p$ is an (A_p) weight, we have

$$\int |S_1(x)|^p dx = \infty.$$

The Phragmén-Lindelöf theorem (see [5], Lecture 20) yields

$$\int |S_1(x+ia)|^p dx = \infty, \quad a \in \mathbb{R},$$

and, by (15),

$$\int |S(x+i)|^p dx = \infty.$$

Again applying the Phragmén-Lindelöf theorem, we get

$$\int |S(x)|^p dx = \infty. \tag{19}$$

We are now in position to prove the uniqueness. Indeed, if $f \in L^P_\pi$ and $f(\lambda) = 0$, $\lambda \in \Lambda$, then $\phi(z) = f(z)/S(z)$ is an entire function of exponential type 0, and (14) yields that $|\phi(z)|$ is uniformly bounded for z satisfying $\operatorname{dist}(z,\Lambda) > \varepsilon(|\Im z|+1)$. Therefore $\phi(z) \equiv C$, which is incompatible with (19), unless $C = 0$.

It remains only to check that we can actually solve the interpolation problem (1) for each sequence $a = \{a_k\}$ satisfying (2). It suffices to consider a finite sequence a and bound the norm of the solution by a constant times the left-hand side of (2). After doing so, we can apply a limit procedure. If a is a finite sequence, then, by (12), the unique solution of the interpolation problem (1) has the form

$$f(z) = \sum_j \frac{a_k}{S'(\lambda_k)}\frac{S(z)}{(z-\lambda_k)}. \tag{20}$$

We split the sum (20) into two parts, corresponding to points lying in $\mathbb{C}^+ \cup \mathbb{R}$ and in $\mathbb{C}^-$, respectively. We may estimate the norm of each sum separately, so let us assume that all the λ_k corresponding to $a_k \neq 0$ are in $\mathbb{C}^+ \cup \mathbb{R}$. Clearly, we may estimate the L^p integral along $\Im(z) = -\frac{1}{2}$. Let us, however, for conventional reasons, estimate it along $\mathbb{R}$ and assume all the points λ_k satisfy $\eta_k \geq \frac{1}{2}$. Now let

$$B(z) = \prod_k \frac{z + \frac{i}{2} - (\lambda_k + \frac{i}{2})}{z + \frac{i}{2} - (\overline{\lambda_k} - \frac{i}{2})}.$$

Writing $S(z) = B(z)e^{-i\pi z}G(z)$, where G is an outer function in $\mathbb{C}^+$, we observe that (iii') is equivalent to $|G(x)|^p$ satisfying the (A_p) condition. We note that we have

$$|S'(\lambda_k)| \asymp |G(\lambda_k)|\frac{e^{\pi\eta_k}}{\eta_k}.$$

Thus it is enough to consider the L^p boundedness of

$$\tilde{f}(x) = \sum_k \frac{a_k \eta_k e^{-\pi\eta_k}}{G(\lambda_k)} \frac{G(x)}{x - \lambda_k}.$$

By duality,

$$\begin{aligned}
\|\tilde{f}\|_p &\asymp \sup_{\|h\|_q = 1, h \in H^q} \left| \sum_k \frac{a_k \eta_k e^{-\pi\eta_k}}{G(\lambda_k)} \int_{\mathbb{R}} \frac{G(x)h(x)}{x - \lambda_k} dx \right| \\
&\leq \sup_{\|h\|_q = 1, h \in H^q} \left| \sum_k \frac{a_k \eta_k e^{-\pi\eta_k}}{G(\lambda_k)} \mathcal{H}Gh(\lambda_k) \right| \\
&\leq \sup_{\|h\|_q = 1, h \in H^q} \left(\sum_k |a_k|^p \eta_k e^{-p\pi\eta_k} \right)^{1/p} \left(\sum_k \left| \frac{\mathcal{H}Gh(\lambda_k)}{G(\lambda_k)} \right|^q \eta_k \right)^{1/q}.
\end{aligned}$$

Since $|G(x)|^{-q}$ satisfies the (A_q) condition, G is an outer function in $\mathbb{C}^+$, and $h \in H^q$, $\|h\|_q \leq 1$, we have $\mathcal{H}Gh(z)/G(z) \in H^q$, and $\|\mathcal{H}Gh(z)/G(z)\|_q \leq C$. Since $\sum_k \eta_k \delta_{\lambda_k}$ is a Carleson measure, we get the desired conclusion.

The sum corresponding to points in $\mathbb{C}^-$ is treated similarly.

5. A stability result

We will now show how Theorem 1 can be used to obtain a result similar to the Kadets 1/4 theorem. The same technique implies more sophisticated stability results similar to the theorems of Avdonin and Katsnelson; for these results we refer to [3].

For $1 < p < \infty$ we denote by q the conjugate exponent, $1/p + 1/q = 1$, and put

$$p' = \max(p, q).$$

We may now prove:

Theorem 2. *Suppose that $\lambda_k = k + \delta_k$, $k \in \mathbb{Z}$. If $|\delta_k| \le d < 1/(2p')$ for every k, then $\Lambda = \{\lambda_k\}$ is a complete interpolating sequence for L^p_π. If merely $|\delta_k| < 1/(2p')$ for every k, then $\Lambda = \{\lambda_k\}$ is not necessarily a complete interpolating sequence for L^p_π.*

Note that for $p = 2$ this is precisely the Kadets theorem (see [3]).

Proof of Theorem 2. We prove first that the inequality $|\delta_k| < 1/(2p')$ is not sufficient. If $\delta_0 = 1$ and otherwise $\delta_k = \operatorname{sgn}(k)\delta$, $-1 < \delta < 1$, standard estimates of infinite products yield

$$F(x) \asymp (1 + |x|)^{-2\delta}.$$

For $1 < p < 2$ we choose $\delta = 1/(2q)$. Then

$$\frac{1}{|x|}\int_0^x F^p dt \left(\frac{1}{|x|}\int_0^x F^{-q} dt\right)^{p-1} \ge C(\log(1+|x|))^{p-1},$$

and the (A_p) condition fails. We obtain the same conclusion if $|\delta_k| < 1/(2q)$ and δ_k tends sufficiently fast to $\operatorname{sgn}(k)/(2q)$ as k tends to $\pm\infty$. If $2 < p < \infty$, we put $\delta = -1/(2p)$, and argue similarly.

With Λ as above, define $\lambda_\alpha = (k + \alpha\delta_k)$ and $\Lambda_\alpha = \{\lambda_\alpha\}$, where α is a real number. Suppose that $\delta < 1/2$ and $|\alpha|\delta < 1/2$, so that the distance between any two distinct points of Λ, and likewise the distance between any two distinct numbers of Λ_α, exceeds a certain positive number. Then estimates of infinite products show that

$$F_\alpha(x) \asymp [F(x)]^\alpha, \tag{21}$$

where $F_\alpha(x) = |S_\alpha(x)|/\operatorname{dist}(x, \Lambda_\alpha)$ and S_α is the generating function of Λ_α.

Suppose first that $1 < p < 2$. If $d < 1/(2q)$, then $F^2_{q/2}$ satisfies the (A_2) condition, according to the classical 1/4 theorem. By (21), it means that F^q satisfies the (A_2) condition, which implies, by Hölder's inequality, that F^p satisfies the (A_p) condition.

If $2 < p < \infty$, put $\alpha = p/2$ and argue similarly. $\square$

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