CONTINUITY PROPERTIES OF BEST ANALYTIC APPROXIMATION

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ABSTRACT. Let $\mathcal{A}$ be the operator which assigns to each $m \times n$ matrix-valued function on the unit circle with entries in $H^\infty + C$ its unique superoptimal approximant in the space of bounded analytic $m \times n$ matrix-valued functions in the open unit disc. We study the continuity of $\mathcal{A}$ with respect to various norms. Our main result is that, for a class of norms satifying certain natural axioms, $\mathcal{A}$ is continuous at any function whose superoptimal singular values are non-zero and is such that certain associated integer indices are equal to 1. We also obtain necessary conditions for continuity of $\mathcal{A}$ at point and a sufficient condition for the continuity of superoptimal singular values.

1. INTRODUCTION

The problem of finding a best uniform approximation of a given bounded function on the unit circle by an analytic function in the unit disc is a natural one from the viewpoint of pure mathematics and it also has engineering applications, for example in H^∞ control [F], broadband impedance matching [He] and robust identification [Par]. In these contexts, to effect a design or construct a model, one must compute such a best approximation, and in order that numerical computations have validity it is important that the solution to be computed depend continuously on the input data, for otherwise the imperfect precision of floating point arithmetic may lead to highly inaccurate results. It is therefore somewhat disconcerting that, with respect to the L^∞ norm, the operator of best analytic approximation is discontinuous everywhere except at points of H^∞ [M, Pa]. Nevertheless engineers regularly compute such approximations and appear to find the results reliable. A way to account for this would be to show that best analytic approximation *is* continuous on suitable Banach subspaces of $L^\infty(\mathbb{T})$ with norms which majorise the uniform norm, or at least, is continuous at most points of the space. One can expect that most functions of engineering interest will lie in one of these well-behaved subspaces, and that the errors

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introduced by computer arithmetic will result in perturbations which are small in the associated norm. We are thus led to ask for which Banach spaces $X \subset L^\infty(\mathbb{T})$ the operator $\mathcal{A}$ of best analytic approximation maps X into X and is continuous at a generic point of X (in some sense). This question has been thoroughly analysed for the case of scalar-valued functions. It was shown in [P1] that, for spaces $X \subset H^\infty + C$ satisfying some natural axioms, the restriction of $\mathcal{A}$ to X is continuous with respect to the norm of X at a function φ if and only if $\|H_\varphi\|$ is a simple singular value of the Hankel operator H_φ.

Analogous questions for matrix-valued functions are also of interest, particularly for their relevance to engineering applications. They are a good deal more complicated than in the scalar case. To begin with, there is typically no unique best analytic approximation in the matrix case, when we measure closeness by the L^∞ norm. In order to specify an approximation uniquely and so obtain a well formulated question of continuity we can use a more stringent criterion of approximation. The notion of a *superoptimal* approximation is a natural one for matrix-valued functions: by imposing the condition of the minimisation of the suprema of all singular values of the error function it gives a unique best approximant in many cases. Here is a precise definition.

Denote by $M_{m,n}$ the space of $m \times n$ complex matrices endowed with the operator norm as a space of linear operators from $\mathbb{C}^n$ to $\mathbb{C}^m$ with their standard inner products. Let $H^\infty(M_{m,n})$ denote the space of bounded analytic $M_{m,n}$-valued functions on the unit disc $\mathbb{D}$ with supremum norm:

$$||Q||_{H^\infty} \stackrel{\text{def}}{=} ||Q||_\infty \stackrel{\text{def}}{=} \sup_{z \in \mathbb{D}} ||Q(z)||_{M_{m,n}}.$$

Similarly, $L^\infty(M_{m,n})$ denotes the space of essentially bounded Lebesgue measurable $M_{m,n}$-valued functions on $\mathbb{T}$ with essential supremum norm. By Fatou's theorem [H, p.34] functions in $H^\infty(M_{m,n})$ have radial limits a.e. on $\mathbb{T}$, so that $H^\infty(M_{m,n})$ can be embedded isometrically in $L^\infty(M_{m,n})$, and we shall often tacitly regard elements of $H^\infty(M_{m,n})$ as functions on the unit circle. Where there is no risk of confusion we shall sometimes write H^∞, L^∞ for $H^\infty(M_{m,n})$, $L^\infty(M_{m,n})$. We define $H^\infty + C$ to be the space of (matrix-valued) functions on $\mathbb{T}$ which are expressible as the sum of an H^∞ function and a continuous function on $\mathbb{T}$. For any matrix A we denote the transpose of A by A^t and the singular values or s-numbers of A by

$$s_0(A) \geq s_1(A) \geq \cdots \geq 0.$$

For $F \in L^\infty(M_{m,n})$ we define, for $j = 0, 1, 2, \ldots,$

$$s_j^\infty(F) \stackrel{\text{def}}{=} \operatorname{ess} \sup_{|z|=1} s_j(F(z))$$

and

$$s^\infty(F) \stackrel{\text{def}}{=} (s_0^\infty(F), s_1^\infty(F), s_2^\infty(F), \ldots).$$

We shall say that $Q \in H^\infty(M_{m,n})$ is a superoptimal H^∞ approximant to $\Phi \in L^\infty(M_{m,n})$ if $s^\infty(\Phi - Q)$ is a minimum over $Q \in H^\infty$ with respect to the lexicographic ordering.

It was proved in [PY1] that if an $m \times n$ matrix function Φ is in $H^\infty + C$ then there is a unique superoptimal approximant to Φ in $H^\infty(M_{m,n})$. We shall denote this approximant by $\mathcal{A}\Phi$. In [PY1], in addition to proving uniqueness, we obtained detailed structural information about the "superoptimal error" $\Phi - \mathcal{A}\Phi$ and we established several heredity results (that is, theorems of the form "$\Phi \in X$ implies $\mathcal{A}\Phi \in X$" for various function spaces X). In any space which does have this heredity property it is natural to ask whether $\mathcal{A}$ acts continuously. We shall show that for a substantial class of norms there are many continuity points of $\mathcal{A}$. We cannot, however, expect $\mathcal{A}$ to be continuous everywhere: it is shown in [P1] that, for scalar functions, $\mathcal{A}$ is discontinuous with respect to virtually any norm at every φ for which $\|H_\varphi\|$ is a multiple singular value of $\|H_\varphi\|$, and it follows that (matricial) $\mathcal{A}$ is discontinuous at the matrix function $\operatorname{diag}\{\varphi, 0, \cdots\}$.

We shall study spaces $X \subset L^2(\mathbb{T})$ of functions for which the following axioms hold. Denote by $\mathbb{P}_+$, $\mathbb{P}_-$ the orthogonal projections from $L^2(\mathbb{T})$ onto the Hardy space H^2 and its orthogonal complement H^2_- in $L^2(\mathbb{T})$. For a space $X \subset L^2(\mathbb{T})$ we denote by X_+ the space $\{\mathbb{P}_+ f : f \in X\}$ and by X_- the space $\{\mathbb{P}_- f : f \in X\}$. The axioms are:
(A1) If $f \in X$ then $\bar{f} \in X$ and $\mathbb{P}_+ f \in X$;
(A2) X is a Banach algebra with respect to pointwise multiplication;
(A3) the set of trigonometric polynomials is dense in X;
(A4) every multiplicative linear functional on X is of the form $f \mapsto f(\zeta)$ for some $\zeta \in \mathbb{T}$;
(A5) if $f \in X_+$ and $h \in H^\infty$ then $\mathbb{P}_+(\bar{h} f) \in X_+$.

The following fact is well known.

Lemma 1.1.. *X_+ with the restriction of $\|\cdot\|_X$ is a commutative Banach algebra whose maximal ideal space is the closed unit disc* clos $\mathbb{D}$.

Proof. By the Closed Graph Theorem $\mathbb{P}_+$ is continuous on X, and so its range X_+ is a closed subspace of X. Functions in X_+ are continuous on $\mathbb{T}$ (the Gelfand topology of X on $\mathbb{T}$ is compact and refines the natural topology, hence coincides with it), and their negative Fourier coefficients vanish. Hence $X_+ \subset A(\mathbb{D})$, the disc algebra. It follows that $X_+ = X \cap A(\mathbb{D})$, and so X_+ is a subalgebra of X. Clearly the maximal ideal space M of X_+ contains clos $\mathbb{D}$, which is the maximal ideal space of $A(\mathbb{D})$. Since X_+ is generated as a Banach algebra by the single element z, M is naturally identified with $\sigma_{X_+}(z)$, the spectrum of z in X_+. Since X_+ is a subalgebra of X we have

$$\partial \sigma_{X_+}(z) \subset \partial \sigma_X(z) = \partial \mathbb{T} = \mathbb{T}$$

(∂ denotes boundary). That is, M contains clos $\mathbb{D}$ and $\partial M \subset \mathbb{T}$. Hence $M = \operatorname{clos} \mathbb{D}$.■

For a space X of functions and a matrix-valued function Φ we write $\Phi \in X$ to mean that each entry of Φ belongs to X. We denote by $X(M_{m,n})$ the space of $m \times n$ matrix-valued functions whose entries belong to X, endowed with the norm

$$\|\Phi\|_X \stackrel{\text{def}}{=} \sup\{\|y^*\Phi x\|_X : \|x\|_{\mathbb{C}^n} \le 1, \ \|y\|_{\mathbb{C}^m} \le 1\}.$$

$X(\mathbb{C}^n)$ is defined to be $X(M_{n,1})$. For $\Phi \in L^\infty(M_{m,n})$ we define the Hankel operator H_Φ to be the operator from $H^2(\mathbb{C}^n)$ to $H^2_-(\mathbb{C}^m)$ given by

$$H_\Phi x \stackrel{\text{def}}{=} \mathbb{P}_-(\Phi x).$$

We recall that the space QC of quasicontinuous functions is defined to be $(H^\infty + C) \cap \overline{(H^\infty + C)}$.

It transpires that the analysis of the continuity of $\mathcal{A}$ involves certain integer indices associated with a matrix function. These indices were introduced in [PY1], and depend on the notion of a *thematic factorization*, which is a type of diagonalization of a superoptimal error function $\Phi - \mathcal{A}\Phi$. A *thematic function* is a function $V \in L^\infty(M_{n,n})$ for some $n \in \mathbb{N}$ which is unitary-valued a.e. on $\mathbb{T}$ and of the form

$$V = \begin{pmatrix} v & \bar{\alpha} \end{pmatrix}$$

where $v \in H^\infty(\mathbb{C}^n)$ is inner and co-outer and $\alpha \in H^\infty(M_{n,n-1})$ is co-outer. Recall that an H^∞ matrix function A is *inner* if $A(z)$ is an isometry for almost all $z \in \mathbb{T}$ and is *co-outer* if $A^t H^2$ is dense in H^2. Consider $\Phi \in H^\infty + C$ of type $m \times n$. We shall assume henceforth that $m \le n$. By [PY1, Theorem 2.1] the singular values $s_j(\Phi(z) - \mathcal{A}\Phi(z))$ are constant a.e. on $\mathbb{T}$; their values $t_0 \ge t_1 \ge \cdots \ge t_{m-1}$ are the *superoptimal singular values* of Φ. Moreover, according to [PY1, Theorem 4.1], $\Phi - \mathcal{A}\Phi$ admits a factorization of the form

$$\Phi - \mathcal{A}\Phi = W_0^* W_1^* \cdots W_{m-1}^* D V_{m-1}^* V_{m-2}^* \cdots V_0^*, \tag{1.1}$$

where D of type $m \times n$ is given by

$$D \stackrel{\text{def}}{=} \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & t_1 u_1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & t_{m-1} u_{m-1} & 0 & \cdots & 0 \end{pmatrix}$$

for some unimodular functions $u_0, \dots, u_{m-1} \in QC$,

$$W_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{W}_j \end{pmatrix}, \qquad 1 \le j \le m-1,$$

$$V_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{V}_j \end{pmatrix}, \qquad 1 \le j \le m-1,$$

and W_0^t, $\tilde{W}_j^t$, V_0 and $\tilde{V}_j$ are thematic functions for $1 \le j \le m-1$. We call (1.1) a thematic factorization of $\Phi - \mathcal{A}\Phi$, and we define the *index* of t_j in this factorization to be the modulus of the winding number of u_j (or alternatively, as the Fredholm index of the Toeplitz operator T_{u_j}). Numerous properties of these indices were established in [PY3]. In Section 1 we prove continuity of $\mathcal{A}$ with respect to a wide class of norms at functions whose superoptimal singular values are nonzero and whose indices are all 1. For the Besov norm B_1^1 we obtain a continuity result even in the presence of zero superoptimal singular values. In Section 2 we consider the converse problem, and derive some necessary conditions for continuity points of $\mathcal{A}$ in the case of square matrix functions. In Section 3 we present sufficient conditions for the continuity of the superoptimal singular values themselves.

2. Sufficient conditions for continuity

Let X be a space of functions on $\mathbb{T}$ invariant under $\mathcal{A}$ (e.g. one satisfying the above axioms). As we noted above, even in the scalar case $\mathcal{A}$ is discontinuous with respect to virtually any norm at any Φ such that $\|H_\Phi\|$ is a multiple singular value of H_Φ [P1]. In the scalar case, for many spaces X the converse also holds. That is, if $\|H_\Phi\|$ is a simple singular value then Φ is a continuity point of $\mathcal{A}$ with respect to the norm of X. For matrix functions the situation is more complicated, but we do have the following sufficient condition.

Theorem 2.1.. *Let X be a space of functions on $\mathbb{T}$ satisfying Axioms* (A1) *to* (A5)*, let $\Phi \in X(M_{m,n})$, $m \le n$, and let $t_0, t_1, \cdots, t_{m-1}$ be the superoptimal singular values of Φ. Suppose that $t_{m-1} \ne 0$. If $\Phi - \mathcal{A}\Phi$ has a thematic factorization with indices*

$$k_0 = k_1 = \cdots = k_{m-1} = 1, \tag{2.2}$$

then Φ is a continuity point of the operator $\mathcal{A}$ of superoptimal approximation in $X(M_{m,n})$.

As we have observed in [PY3], (2.2) implies that all thematic factorizations of $\Phi - \mathcal{A}\Phi$ have indices equal to 1.

The proof of the theorem will be based on the recursive construction of $\mathcal{A}\Phi$ given in [PY2], which in turn was based on the proof in [PY1] that $\mathcal{A}\Phi$ is well defined. Let us briefly recall the construction of $\mathcal{A}\Phi$. The first step is to find a Schmidt pair $\{v,\ w\}$ of the compact Hankel operator H_Φ corresponding to the singular value $\|H_\Phi\|$. Then find $Q \in H^\infty(M_{m,n})$ such that

$$Qv = T_\Phi v, \qquad Q^t \bar{z}\bar{w} = T_{\Phi^t}\bar{z}\bar{w} \tag{2.3}$$

(these equations always have a solution; in fact $Q = \mathcal{A}\Phi$ satisfies them, and in the proof of the theorem below we shall even give an explicit rank two solution for Q). Next let $v_{(i)},\ w_{(i)}$ be the inner factors of $v,\ \bar{z}\bar{w}$ and let

$$V = \begin{pmatrix} v_{(i)} & \bar{\alpha} \end{pmatrix}, \qquad W^t = \begin{pmatrix} w_{(i)} & \bar{\beta} \end{pmatrix} \tag{2.4}$$

be thematic completions of $v_{(i)}$, $w_{(i)}$ respectively. Then

$$\mathcal{A}\Phi = Q + \beta\mathcal{A}\Psi\alpha^t$$

where

$$\Psi = \beta^*(\Phi - Q)\bar{\alpha}.$$

Note that Ψ is of type $(m-1)\times(n-1)$. The strategy of the proof is simply to show that α, β and Q can be chosen to depend continuously on Φ and then to use induction on m. In order to do this we have to study some properties of maximizing vectors for H_Φ.

It is easy to see from the axioms (A1)–(A5) that $H_\Phi^* H_\Phi$ is also a compact operator on $X_+(\mathbb{C}^n)$. Denote this operator on $X_+(\mathbb{C}^n)$ by R. We can identify the dual space $X_+^*(\mathbb{C}^n)$ with the space of analytic $\mathbb{C}^n$-valued functions g in $\mathbb{D}$ such that the Hermitian form

$$(f, g) = \sum_{d\ge 0}(\hat{f}(d), \hat{g}(d))_{\mathbb{C}^n}$$

defined for polynomials f in $X_+(\mathbb{C}^n)$, is continuous on $X_+(\mathbb{C}^n)$. Obviously,

$$X_+(\mathbb{C}^n) \subset H^2(\mathbb{C}^n) \subset X_+^*(\mathbb{C}^n).$$

Since R is a compact operator on $X_+(\mathbb{C}^n)$, it follows from the Riesz–Schauder theorem that R^* is compact on $X_+^*(\mathbb{C}^n)$ and if $\lambda > 0$, then λ is an eigenvalue of R if and only if λ is an eigenvalue of R^* of the same multiplicity (see [Yo], Ch. X, §5). Since $H_\Phi^* H_\Phi$ is self-adjoint, we have $R^*|H^2(\mathbb{C}^n) = H_\Phi^* H_\Phi$. Clearly, every eigenvector of R is an eigenvector of $H_\Phi^* H_\Phi$ and every eigenvector of $H_\Phi^* H_\Phi$ is an eigenvector of R^*. It follows from the Riesz–Schauder theorem that R, $H_\Phi^* H_\Phi$, and R^* have the same eigenvectors that correspond to positive eigenvalues.

Theorem 2.2.. *Let Φ be a function in $X(M_{m,n})$, $m \le n$, with superoptimal singular values $t_0, \cdots, t_{m-1}$, $t_0 \ne 0$. Suppose that $\Phi - \mathcal{A}\Phi$ has a thematic factorization whose indices k_j are equal to 1 whenever $t_j = t_0$. Let $\{v,\ w\}$ be a Schmidt pair of H_Φ corresponding to $\|H_\Phi\|$. Then v and $\bar{z}\bar{w}$ are co-outer and $v(\zeta) \ne 0$ for any $\zeta \in \mathbb{T}$.*

Clearly it is sufficient to prove that $v(1) \ne \mathbb{O}$.

We shall deduce Theorem 2.2 from the following lemma whose proof is similar to that of Lemma 3.2 of [PK].

Lemma 2.3.. *Let v be a maximizing vector for H_Φ such that $v(1) = 0$. Then $(1-z)^{-1}v \in X_+^*(\mathbb{C}^n)$ and $(1-z)^{-1}v$ is an eigenvector of R^* with eigenvalue t_0^2.*

Proof. Let us show that

$$(f, (1-z)^{-1}v) = (\mathbb{P}_+ v^* f)(1) \tag{2.5}$$

for any polynomial f in $X_+(\mathbb{C}^n)$, where $v^* f(\zeta) \stackrel{\text{def}}{=} (f(\zeta), v(\zeta))_{\mathbb{C}^n}$. Since the right-hand side of (2.5) clearly determines a continuous linear functional on $X_+(\mathbb{C}^n)$, it would follow that $(1-z)^{-1}v \in X_+^*(\mathbb{C}^n)$.

It is sufficient to establish (2.5) for $f = z^j x$, $x \in \mathbb{C}^n$. Obviously, $(1-z)^{-1}v = \sum_{j\ge 0} z^j (\sum_{d=0}^{j} \hat{v}(d))$ and so

$$(z^j x, (1-z)^{-1}v) = (x, \sum_{d=0}^{j} \hat{v}(d))_{\mathbb{C}^n} = \sum_{d=0}^{j} (x, \hat{v}(d))_{\mathbb{C}^n}.$$

On the other hand it is easy to see that

$$(\mathbb{P}_+ v^* f)(1) = \sum_{d=0}^{j} (x, \hat{v}(d))_{\mathbb{C}^n},$$

which proves (2.5).

To complete the proof of the lemma, we have to prove that $R^*(1-z)^{-1}v = t_0^2(1-z)^{-1}v$, which means that

$$(H_\Phi^* H_\Phi f, (1-z)^{-1}v) = t_0^2 (f, (1-z)^{-1}v) \tag{2.6}$$

for any $f \in X_+(\mathbb{C}^n)$. We may assume for convenience that $t_0 = 1$.

To establish (2.6), we expand $(1-z)^{-1}v$ in the series $\sum_{d=0}^{\infty} z^d v$ and apply Cesàro's summation method.

Let $Q \in H^\infty$ be a best approximation to Φ, i.e. $\|\Phi - Q\|_{L^\infty} = \|H_\Phi\| = 1$. Put $\Psi = \Phi - Q$. It is well known (see [AAK], [PY1], Th. 0.2) that $H_\Phi v = \Psi v \in H^2_-(\mathbb{C}^m)$ and $\|\Psi(\zeta) v(\zeta)\|_{\mathbb{C}^n} = \|v(\zeta)\|_{\mathbb{C}^n}$. Clearly, the last equality implies that $\Psi^* \Psi v = v$.

We have

$$\begin{aligned}
(H_\Phi^* H_\Phi f, z^d v) &= (H_\Psi^* H_\Psi f, z^d v) = (H_\Psi f, \Psi z^d v) \\
&= (\mathbb{P}_- \Psi f, z^d \Psi v) = (\Psi f, \mathbb{P}_- z^d \Psi v) \\
&= (\Psi f, z^d \Psi v) - (\Psi f, \mathbb{P}_+ z^d \Psi v) \\
&= (f, z^d \Psi^* \Psi v) - (\Psi f, \mathbb{P}_+ z^d \Psi v) \\
&= (f, z^d v) - (\Psi f, \mathbb{P}_+ z^d \Psi v).
\end{aligned}$$

Let $K_N(\zeta) \stackrel{\text{def}}{=} \sum_{d=-N}^{N} (1 - \frac{|d|}{N}) \zeta^d$ be the Fejér kernel and $K_N^+ \stackrel{\text{def}}{=} \mathbb{P}_+ K_N$. Then

$$(H_\Psi^* H_\Psi f, K_N^+ v) = (f, K_N^+ v) - (\Psi f, \mathbb{P}_+ K_N \Psi v),$$

since $\Psi v \in H^2_-(\mathbb{C}^m)$. Let us prove that $\lim_{N\to\infty} (\Psi f, \mathbb{P}_+ K_N \Psi v) = 0$. Indeed

$$\begin{aligned}
(\Psi f, \mathbb{P}_+ K_N \Psi v) &= (\mathbb{P}_+ \Psi f, K_N \Psi v) = (\Psi f, K_N \Psi v) - (\mathbb{P}_- \Psi f, \Psi K_N v) \\
&= (f, K_N \Psi^* \Psi v) - (H_\Psi^* H_\Psi f, K_N v) \\
&= (f, K_N v) - (H_\Psi^* H_\Psi f, K_N v).
\end{aligned}$$

Clearly,

$$(f, K_N v) \to (f(1), v(1))_{\mathbb{C}^n} = 0;$$
$$(H_\Psi^* H_\Psi f, K_N v) \to ((H_\Psi^* H_\Psi f)(1), v(1)) = 0.$$

It remains to prove that $\lim_{N\to\infty} K_N^+ v = (1-z)^{-1}v$ in the weak topology

$$\sigma(X_+^*(\mathbb{C}^n), X_+(\mathbb{C}^n)).$$

Let $g = z^j x$, $x \in \mathbb{C}^n$. Then

$$(g, K_N^+ v) = (v^* g, K_N^+) = (\mathbb{P}_+ v^* g, K_N) \to (\mathbb{P}_+ v^* g)(1)$$

as $N \to \infty$. The result follows now from (2.5). ■

Corollary 2.4.. *Let v be a maximizing vector for H_Φ such that $v(1) = \mathbb{O}$. Then $(1-z)^{-1}v \in X_+(\mathbb{C}^n)$ and $(1-z)^{-1}v$ is also a maximizing vector for H_Φ.*

Proof of Theorem 2.2. Suppose that $v(1) = \mathbb{O}$. By Corollary 1.4, $v = (1-z)q$, where $q \in H^2$. Let

$$w = \frac{1}{t_0} H_\Phi v.$$

Then as we have already mentioned in the proof of Lemma 2.3, $\|v(\zeta)\|_{\mathbb{C}^n} = \|w(\zeta)\|_{\mathbb{C}^n}$, $\zeta \in \mathbb{T}$. Let h be a scalar outer function such that $|h(\zeta)| = \|v(\zeta)\|$, $\zeta \in \mathbb{T}$, and let $h_1 = (1-z)^{-1}h$. Clearly, h_1 is also a scalar outer function and $|h_1(\zeta)| = \|q(\zeta)\|_{\mathbb{C}^n}$, $\zeta \in \mathbb{T}$. Now there exist scalar inner functions ϑ_1, ϑ_2 such that v and $\bar{z}\bar{w}$ admit factorizations $v = \vartheta_1 h v^{(i)}$, $\bar{z}\bar{w} = \vartheta_2 h w^{(i)}$, where $v^{(i)}$ and $w^{(i)}$ are inner and co-outer in $H^2(\mathbb{C}^n)$. Then $hv^{(i)}$ is also a maximizing vector for H_Φ and

$$\frac{1}{t_0} H_\Phi h v^{(i)} = \bar{\vartheta}_1 w$$

(see the proof of Theorem 4.1 of [PY1]).

Let $V = \begin{pmatrix} v^{(i)} & \bar{\alpha} \end{pmatrix}$, $W^T = \begin{pmatrix} w^{(i)} & \bar{\beta} \end{pmatrix}$ be thematic matrices. It follows from Lemma 2.3 of [PY1] that

$$W(\Phi - \mathcal{A}\Phi)V = \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & F \end{pmatrix}$$

and $\Phi - \mathcal{A}\Phi$ has a thematic factorization with index equal to $\dim \operatorname{Ker} T_{u_0}$, where

$$u_0 = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}/h. \tag{2.7}$$

Since $h = (1-z)q$, we have

$$u_0 = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2 \frac{1-\bar{z}}{1-z}\frac{\bar{q}}{q} = -\bar{z}^2\bar{\vartheta}_1\bar{\vartheta}_2\frac{\bar{q}}{q},$$

and so $k_0 = \dim \operatorname{Ker} T_{u_0} \geq 2$, since obviously q and zq belong to $\operatorname{Ker} T_{u_0}$. This contradicts the hypotheses of Theorem 1.2, and so $v(1) \neq 0$. In similar fashion, the relation (2.7) shows that Ker T_{u_0} contains h, $\bar{\vartheta}_1 h$ and $\bar{\vartheta}_2 h$. Thus, if $v^{(i)}$ or $w^{(i)}$ is not co-outer, we have again contradicted dim Ker $T_{u_0} = 1$. Hence v, w are co-outer. ■

Lemma 2.5.. *Let $n > 1$ and let φ be an inner function in $X_+(\mathbb{C}^n)$. Then 0 is an isolated spectral point of the operators $T_{\bar{\varphi}}^X T_{\varphi^t}^X$ on $X_+(\mathbb{C}^n)$ and $T_{\bar{\varphi}} T_{\varphi^t}$ on $H^2(\mathbb{C}^n)$.*

Proof. Let us prove the lemma for the operator $T^X_{\bar\varphi}T^X_{\varphi^t}$. The proof for $T_{\bar\varphi}T_{\varphi^t}$ is exactly the same.

Let us observe that we may assume that φ is co-outer. Indeed if $\varphi = \vartheta\tau$, where ϑ is a scalar inner function and τ is an inner co-outer function, then it follows from the axiom (A5) that $\tau \in X_+(\mathbb{C}^n)$ and clearly $T^X_{\bar\varphi}T^X_{\varphi^t} = T^X_{\bar\tau}T^X_{\tau^t}$.

Consider the operator $T^X_{\varphi^t}T^X_{\bar\varphi}$ on X_+. It is well known that a nonzero point $\lambda \in \mathbb{C}$ belongs to the spectrum of $T^X_{\bar\varphi}T^X_{\varphi^t}$ if and only if it belongs to the spectrum of $T^X_{\varphi^t}T^X_{\bar\varphi}$. Therefore to prove the lemma it is sufficient to show that $T^X_{\varphi^t}T^X_{\bar\varphi}$ is invertible.

We have

$$T^X_{\varphi^t}T^X_{\bar\varphi} = I - H^{*X}_{\bar\varphi}H^X_{\bar\varphi}.$$

It follows easily from the axioms (A1)–(A5) that the operator $H^{*X}_{\bar\varphi}H^X_{\bar\varphi}$ is compact. Hence it is sufficient to show that $\operatorname{Ker} T^X_{\varphi^t}T^X_{\bar\varphi} = \{\mathbb{O}\}$. Let $f \in \operatorname{Ker} T^X_{\varphi^t}T^X_{\bar\varphi}$. Then $H^{*X}_{\bar\varphi}H^X_{\bar\varphi}f = f$, which clearly means that $H^*_{\bar\varphi}H_{\bar\varphi}f = f$. Since $\|H_{\bar\varphi}\| = 1$ and $\|\bar\varphi\|_{L^\infty(\mathbb{C}^n)} = 1$, it follows that $\bar\varphi f \in H^2_-(\mathbb{C}^n)$. Thus $\bar f\varphi^t H^2(\mathbb{C}^n) \subset zH^1$, and since $\varphi^t H^2(\mathbb{C}^n)$ is dense in H^2, it follows that $\bar f H^2 \subset zH^1$, and hence that $\bar f \in zH^2$. Thus $f = \mathbb{O}$. ■

For an inner function $\varphi \in H^\infty(\mathbb{C}^n)$ we denote by L_φ the kernel of T_{φ^t} and by P_φ the orthogonal projection from $H^2(\mathbb{C}^n)$ onto L_φ. Similarly, we denote by L^X_φ the kernel of $T^X_{\varphi^t}$. Clearly, $L_\varphi = \operatorname{Ker} T_{\bar\varphi}T_{\varphi^t}$ and $L^X_\varphi = \operatorname{Ker} T^X_{\bar\varphi}T^X_{\varphi^t}$.

Consider a simple closed positively oriented Jordan curve Ω which lies in the resolvent sets of $T_{\bar\varphi}T_{\varphi^t}$ and $T^X_{\bar\varphi}T^X_{\varphi^t}$, encircles zero but does not wind round any other point of the spectra of $T_{\bar\varphi}T_{\varphi^t}$ and $T^X_{\bar\varphi}T^X_{\varphi^t}$. Clearly

$$P_\varphi = \frac{1}{2\pi \mathrm{i}}\oint_\Omega (\zeta I - T_{\bar\varphi}T_{\varphi^t})^{-1}d\zeta.$$

Consider the projection P^X_φ from $X_+(\mathbb{C}^n)$ onto L^X_φ defined by

$$P^X_\varphi = \frac{1}{2\pi \mathrm{i}}\oint_\Omega (\zeta I - T^X_{\bar\varphi}T^X_{\varphi^t})^{-1}d\zeta. \tag{2.8}$$

Obviously, $P^X_\varphi f = P_\varphi f$ for $f \in X_+(\mathbb{C}^n)$.

Suppose now that $\{\varphi^{(k)}\}_{k\ge1}$ is a sequence of inner functions in $X_+(\mathbb{C}^n)$, which converges to φ in the norm. Then $T^X_{(\varphi^{(k)})^t}T^X_{\bar\varphi^{(k)}} \to T^X_{\varphi^t}T^X_{\bar\varphi}$ in the norm of $\mathcal{L}(X_+(\mathbb{C}^n))$. As in the proof of Lemma 1.5, $T^X_{\varphi^t}T^X_{\bar\varphi}$ is invertible, and hence there is a neighbourhood U of zero which lies in the resolvent set of $T^X_{\varphi^t}T^X_{\bar\varphi}$ and of $T^X_{(\varphi^{(k)})^t}T^X_{\bar\varphi^{(k)}}$ for all sufficiently large k. Without loss of generality we may assume that this holds for all values of k. Choose a simple closed contour Ω lying in U and winding round 0. Then 0 is the only point inside or on Ω of the spectra of $T_{\bar\varphi^{(k)}}T_{(\varphi^{(k)})^t}$ and $T^X_{\bar\varphi^{(k)}}T^X_{(\varphi^{(k)})^t}$. We can therefore define projections P_φ, P^X_φ, $P_{\varphi^{(k)}}$, $P^X_{\varphi^{(k)}}$ by integrals as above, all using the same contour Ω. It is then easy to see from (2.8) that $P^X_{\varphi^{(k)}} \to P^X_\varphi$ in the operator norm.

Lemma 2.6.. *Let* $V = \begin{pmatrix} \varphi & \overline{\varphi_c} \end{pmatrix}$ *be unitary-valued on* $\mathbb{T}$, *where* φ_c *is inner and co-outer. There exist inner co-outer functions* $\varphi_c^{(k)}$ *such that* $V^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} \varphi^{(k)} & \overline{\varphi_c^{(k)}} \end{pmatrix}$ *is unitary-valued on* $\mathbb{T}$ *and* $\|V - V^{(k)}\|_{X(M_{n,n})} \to 0$.

Proof. It was shown in [PY1] (see the proof of Theorem 1.1) that, for a given inner column φ, one can construct an inner co-outer α such that $\begin{pmatrix} \varphi & \overline{\alpha} \end{pmatrix}$ is unitary-valued on $\mathbb{T}$ and the columns of α have the form $P_\varphi C_1, P_\varphi C_2, \cdots, P_\varphi C_{n-1}$, where $C_1, C_2, \cdots, C_{n-1}$ are constant column functions. By [PY1, Corollary 1.6], $\varphi_c = \alpha U$ for some constant unitary U. Hence the columns of φ_c also have the form $P_\varphi C_j$ for some constants C_j. Consider the subspace of $H^2(\mathbb{C}^n)$

$$P_\varphi \mathbb{C}^n \stackrel{\text{def}}{=} \{P_\varphi C : C \in \mathbb{C}^n\},$$

where we identify $C \in \mathbb{C}^n$ with a constant function in $H^2(\mathbb{C}^n)$. This space has the remarkable property that the pointwise and H^2 inner products coincide on it. That is, if $f_j = P_\varphi C_j,\ j = 1, 2$, where $C_1,\ C_2 \in \mathbb{C}^n$, then

$$(f_1, f_2)_{H^2(\mathbb{C}^n)} = (f_1(z), f_2(z))_{\mathbb{C}^n} \tag{2.9}$$

for almost all $z \in \mathbb{T}$. To see this note that L_φ is a closed z-invariant subspace of $H^2(\mathbb{C}^n)$, and so is of the form $\Theta H^2(\mathbb{C}^p)$ for some natural number p and some $n \times p$ inner function Θ. Then for any $C \in \mathbb{C}^n$,

$$P_\varphi C = \Theta \mathbb{P}_+ \Theta^* C = \Theta\Theta(0)^* C,$$

and so

$$\begin{aligned}
(f_1, f_2)_{H^2(\mathbb{C}^n)} &= (P_\varphi C_1, P_\varphi C_2)_{H^2} = (\Theta\Theta(0)^* C_1, \Theta\Theta(0)^* C_2)_{H^2} \\
&= (\Theta(0)^* C_1, \Theta(0)^* C_2)_{H^2} = (\Theta(0)^* C_1, \Theta(0)^* C_2)_{\mathbb{C}^p} \\
&= (\Theta(z)\Theta(0)^* C_1, \Theta(z)\Theta(0)^* C_2)_{\mathbb{C}^n} \\
&= (f_1(z), f_2(z))_{\mathbb{C}^n}
\end{aligned}$$

for almost all $z \in \mathbb{T}$. It follows that any unit vector in $P_\varphi \mathbb{C}^n$ is an inner column function, and any orthonormal sequence (with respect to the inner product of $H^2(\mathbb{C}^n)$) of vectors in $P_\varphi \mathbb{C}^n$ constitutes the columns of an inner function.

Now let $P_\varphi C_j,\ 1 \le j \le n-1$, be the columns of φ_c as above, and consider the functions $P_{\varphi^{(k)}} C_1, P_{\varphi^{(k)}} C_2, \cdots, P_{\varphi^{(k)}} C_{n-1}$. Clearly

$$\|P_\varphi C_j - P_{\varphi^{(k)}} C_j\|_{X(\mathbb{C}^n)} = \|P^X_\varphi C_j - P^X_{\varphi^{(k)}} C_j\|_{X(\mathbb{C}^n)} \to 0 \quad \text{as} \quad k \to \infty.$$

It follows that for large values of k the inner products $(P_{\varphi^{(k)}} C_{j_1}, P_{\varphi^{(k)}} C_{j_2})_{H^2(\mathbb{C}^n)}$ are small for $j_1 \ne j_2$ and are close to 1 if $j_1 = j_2$. We shall show that the desired $\varphi_c^{(k)}$ can be obtained by orthonormalising the $P_{\varphi^{(k)}} C_j$.

Pick $M > 1$ such that $\|P_{\varphi^{(k)}} C_j\|_{X(\mathbb{C}^n)} \leq M$ for all $k \in \mathbb{N}$ and $1 \leq j < n$. By the equivalence of norms on finite-dimensional spaces there exists $K > 0$ such that, for any $(n-1)$-square matrix $T = (t_{ij})$,

$$\max \mid t_{ij} \mid \leq \ \|T\| \ \leq \ K \max \mid t_{ij} \mid \tag{2.10}$$

(here $\|.\|$ is the operator norm on $\mathcal{L}(\mathbb{C}^{n-1})$).

Let $0 < \varepsilon < 1$. Choose k_0 such that $k \geq k_0$ implies

$$\|P_{\varphi^{(k)}} C_j - P_\varphi C_j\|_{X(\mathbb{C}^n)} < \frac{\varepsilon}{2}, \qquad j = 1, \ldots, n-1, \tag{2.11}$$

and

$$\mid (P_{\varphi^{(k)}} C_i,\ P_{\varphi^{(k)}} C_j) - \delta_{ij} \mid < \frac{\varepsilon}{2KnM}, \qquad i, j = 1, \ldots, n-1. \tag{2.12}$$

Fix $k \geq k_0$ and let $T : \mathbb{C}^{n-1} \to P_{\varphi^{(k)}} \mathbb{C}^n$ be the operator which maps the jth standard basis vector e_j of $\mathbb{C}^{n-1}$ to $P_{\varphi^{(k)}} C_j$. The matrix of $T^*T \in \mathcal{L}(\mathbb{C}^{n-1})$ is the Gram matrix $(P_{\varphi^{(k)}} C_j,\ P_{\varphi^{(k)}} C_i)$, and so by (2.10) and (2.12) we have

$$\|T^*T - I\| < \frac{\varepsilon}{2nM} < \frac{1}{2}.$$

By diagonalisation,

$$\|(T^*T)^{-\frac{1}{2}} - I\| < \frac{\varepsilon}{2nM}.$$

Let $(T^*T)^{-\frac{1}{2}} = (t_{ij})$: then

$$\mid t_{ij} - \delta_{ij} \mid < \frac{\varepsilon}{2nM}.$$

Let the polar decomposition of T be $T = U(T^*T)^{\frac{1}{2}}$, so that $U = T(T^*T)^{-\frac{1}{2}}$. Then U is unitary, so that $Ue_1, \ldots, Ue_{n-1}$ are orthonormal in $P_{\varphi^{(k)}} \mathbb{C}^n$. Let $\varphi_c^{(k)}$ be the $n \times (n-1)$ matrix with columns $Ue_1, \ldots, Ue_{n-1}$. By the remark above, $\varphi_c^{(k)}$ is inner. By the fact that $P_{\varphi^{(k)}} \mathbb{C}^n \subset L_{\varphi^{(k)}}$, the columns of $\bar{\varphi}_c^{(k)}$ are pointwise orthogonal to $\varphi^{(k)}$. Hence

$$V^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} \varphi^{(k)} & \bar{\varphi}_c^{(k)} \end{pmatrix}$$

is unitary-valued. Furthermore, the jth column Ue_j of $\varphi_c^{(k)}$ satisfies

$$\begin{aligned} \|P_{\varphi^{(k)}} C_j - Ue_j\|_{X(\mathbb{C}^n)} &= \|Te_j - T(T^*T)^{-\frac{1}{2}} e_j\|_X \qquad (2.13) \\ &= \|Te_j - (Te_1 \ldots Te_{n-1}) \begin{pmatrix} t_{1j} \\ \vdots \\ t_{n-1,j} \end{pmatrix} \|_X \\ &\leq \mid t_{1j} \mid \|Te_1\|_X + \cdots + \mid t_{jj} - 1 \mid \|Te_j\|_X + \\ &\quad \cdots + \mid t_{n-1,j} \mid \|Te_{n-1}\|_X \\ &\leq (n-1) \frac{\varepsilon}{2nM} M < \frac{\varepsilon}{2}. \end{aligned}$$

On combining this inequality with (2.11) we obtain

$$\begin{aligned}\|P_\varphi C_j - Ue_j\|_X &\le \|P_\varphi C_j - P_{\varphi^{(k)}} C_j\|_X + \|P_{\varphi^{(k)}} C_j - Ue_j\|_X \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

That is, the jth column of $\varphi_c^{(k)}$ tends to the jth column of φ_c with respect to the norm of $X(\mathbb{C}^n)$. Hence $V^{(k)} \to V$ in $X(M_{n,n})$. Finally, it follows from [P3, Lemma 1.2] that $\varphi_c^{(k)}$ is co-outer. ■

Corollary 2.7.. *Suppose φ is co-outer and $V^{(k)}$ is constructed as in Lemma 1.6. For sufficiently large k, $\varphi^{(k)}$ is co-outer and so $V^{(k)}$ is thematic.*

Proof. By [PY1, Theorem 1.2], $\det V$ is constant, hence has zero winding number about 0. Since $\det V^{(k)} \to \det V$ uniformly on $\mathbb{T}$, $\det V^{(k)}$ also has zero winding number about 0 for sufficiently large k. Again by [PY1, Theorem 1.2], $\varphi^{(k)}$ is co-outer. ■

Lemma 2.8.. *Let E, F be Banach spaces, let $T : E \to F$ be a surjective continuous linear mapping and let $x \in E$, $y \in F$ be such that $Tx = y$. Let $\varepsilon > 0$ and let $T' \in \mathcal{L}(E, F)$. There exists $\delta > 0$ such that, whenever $\|T' - T\| < \delta$, the equation $T'x' = y$ has a solution x' satisfying $\|x' - x\| < \varepsilon$.*

Proof. We can suppose $\|x\| = 1$. By the Open Mapping Theorem there exists $c > 0$ such that the ball of radius c in F is contained in the image under T of the unit ball in E. Then

$$\|T^* f\| \ge c\|f\| \qquad \text{for all } f \in F^*.$$

Let $\delta = \frac{c}{2}\min\{1, \varepsilon\}$. Suppose $\|T' - T\| < \delta$. For any $f \in F^*$

$$\begin{aligned}\|T'^* f\| &= \|T^* f + (T' - T)^* f\| \ge \|T^* f\| - \|T' - T\| \cdot \|f\| \\ &\ge c\|f\| - \frac{c}{2}\|f\| = \frac{c}{2}\|f\|. \qquad (2.14)\end{aligned}$$

Thus T' maps the closed unit ball of E to a superset of the closed ball of radius $\frac{c}{2}$ in F. Since

$$\|(T - T')x\| \le \|T - T'\| < \delta,$$

it follows that there exists $\xi \in E$ such that

$$\|\xi\| < \frac{2\delta}{c} \le \varepsilon$$

and $T'\xi = (T - T')x$. Then $x' \stackrel{\text{def}}{=} x + \xi$ has the stated properties:

$$T'x' = T'x + (T - T')x = Tx = y,$$

$$\|x - x'\| = \|\xi\| < \varepsilon.$$

■

Lemma 2.9.. *Let $f,\ \varphi \in X_+(\mathbb{C}^n)$ be such that $\varphi^t f = 1$ and let $\varepsilon > 0$. There exists $\delta > 0$ such that, for any $\psi \in X_+(\mathbb{C}^n)$ satisfying $\|\varphi - \psi\|_X < \delta$, there is a $g \in X_+(\mathbb{C}^n)$ such that $\|f - g\|_X < \varepsilon$ and $\psi^t g = 1$.*

Proof. Let $T = T^X_{\varphi^t} : X_+(\mathbb{C}^n) \to X_+$, so that $Tx = \varphi^t x$ for $x \in X_+(\mathbb{C}^n)$. Then T is a surjective continuous linear mapping and $Tf = 1$. By Lemma 1.7 there exists δ such that $\|T' - T\| < \delta$ implies that the equation $T'g = 1$ has a solution $g \in X_+(\mathbb{C}^n)$ satisfying $\|f - g\|_X < \varepsilon$. If $\psi \in X_+(\mathbb{C}^n)$ is such that $\|\varphi - \psi\|_X < \delta$ then $\|T^X_{\psi^t} - T^X_{\varphi^t}\| < \delta$, and so the lemma applies to $T' = T^X_{\psi^t}$; that is, there exists $g \in X_+(\mathbb{C}^n)$ such that $\psi^t g = 1$ and $\|f - g\|_X < \varepsilon$. ∎

Proof of Theorem 2.1. We proceed by induction on m.

Let $\{\Phi^{(k)}\}_{k\ge1}$ be a sequence of functions in X such that $\|\Phi - \Phi^{(k)}\|_{X(M_{m,n})} \to 0$. We shall show that some subsequence of $\mathcal{A}\Phi^{(k)}$ converges to $\mathcal{A}\Phi$ in the norm of X: this will suffice to establish the continuity of $\mathcal{A}$ at Φ. Let $v^{(k)}$ be a co-outer maximizing vector for the operator $H_{\Phi^{(k)}}$ on $H^2(\mathbb{C}^n)$. We can take it that the norm of $v^{(k)}$ in $X_+(\mathbb{C}^n)$ is equal to 1:

$$\|v^{(k)}\|_{X(\mathbb{C}^n)} = 1,\ \|H_{\Phi^{(k)}} v^{(k)}\|_{H^2_-(\mathbb{C}^m)} = \|H_{\Phi^{(k)}}\| \cdot \|v^{(k)}\|_{H^2(\mathbb{C}^n)}.$$

Let Ω be a positively oriented Jordan contour which winds once round the largest eigenvalue t_0^2 of $H^*_\Phi H_\Phi$, contains no eigenvalues and encircles no other eigenvalues.

It is easy to see from the axioms $(A1)$–$(A5)$ that the operators $H^{*X}_{\Phi^{(k)}} H^X_{\Phi^{(k)}}$ converge to $H^{*X}_\Phi H^X_\Phi$ in the operator norm of $X_+(\mathbb{C}^n)$. It follows that for large values of k there are no points of the spectrum of $H^{*X}_{\Phi^{(k)}} H^X_{\Phi^{(k)}}$ on Ω. Let

$$\mathcal{P} = \frac{1}{2\pi \mathrm{i}} \oint_\Omega (\zeta I - H^{*X}_\Phi H^X_\Phi)^{-1} d\zeta$$

and

$$\mathcal{P}^{(k)} = \frac{1}{2\pi \mathrm{i}} \oint_\Omega (\zeta I - H^{*X}_{\Phi^{(k)}} H^X_{\Phi^{(k)}})^{-1} d\zeta.$$

Clearly, $\mathcal{P}v^{(k)}$ is a maximizing vector of $H^{*X}_\Phi H^X_\Phi$ and

$$\|v^{(k)} - \mathcal{P}v^{(k)}\|_{X(\mathbb{C}^n)} = \|\mathcal{P}^{(k)} v^{(k)} - \mathcal{P}v^{(k)}\|_{X(\mathbb{C}^n)} \to 0, \quad k \to \infty.$$

The vectors $\mathcal{P}v^{(k)}$ belong to the finite-dimensional subspace of maximizing vectors of $H^*_\Phi H_\Phi$. Therefore there exists a convergent subsequence of the sequence $\{\mathcal{P}v^{(k)}\}_{k\ge0}$. Without loss of generality we may assume that the sequence $\{\mathcal{P}v^{(k)}\}_{k\ge0}$ converges in $X(\mathbb{C}^n)$ to a vector v, which is a maximizing vector of $H^*_\Phi H_\Phi$. Obviously, $\|v^{(k)} - v\|_{X(\mathbb{C}^n)} \to 0$ as $k \to \infty$.

We also need the other Schmidt vectors corresponding to v and $v^{(k)}$. We may assume that $\|H_{\Phi^{(k)}}\| \ne 0$ for all k. Let

$$w = t_0^{-1} H_\Phi v, \qquad w^{(k)} = H_{\Phi^{(k)}} v^{(k)} / \|H_{\Phi^{(k)}}\|$$

in $X_-(\mathbb{C}^m)$. Since $H_{\Phi^{(k)}} \to H_\Phi$ in the norm of $\mathcal{L}(X_+(\mathbb{C}^n), X_-(\mathbb{C}^m))$ and $v^{(k)} \to v$ in X_+ it follows that $w^{(k)} \to w$ in X_-. The $v^{(k)}$ are co-outer by choice; the same is true of $w^{(k)}$ for sufficiently large k by Corollary 1.7.

Now let us show that Theorem 2.1 holds when $m = 1$. In this case w and $w^{(k)}$ are scalar functions in X. By [AAK], $|w(z)| = \|v(z)\|$ a.e. on $\mathbb{T}$. By continuity, equality holds at all points of $\mathbb{T}$. By Theorem 2.2, v (and hence also w) is non-zero at every point of the maximal ideal space $\mathbb{T}$ of X. Thus $1/w \in X$. By virtue of the continuity of inversion in Banach algebras we deduce that $1/w^{(k)} \in X$ for sufficiently large k, and $1/w^{(k)} \to 1/w$ in X. Again by [AAK],

$$w^*(\Phi - \mathcal{A}\Phi) = \|H_\Phi\| v^*$$

and hence

$$\Phi - \mathcal{A}\Phi = \|H_\Phi\| \frac{v^*}{w^*}, \qquad \Phi^{(k)} - \mathcal{A}\Phi^{(k)} = \|H_{\Phi^{(k)}}\| \frac{v^{(k)*}}{w^{(k)*}},$$

the latter for large k. From these equations it is clear that $\mathcal{A}\Phi^{(k)} \to \mathcal{A}\Phi$ in X. Thus the case $m = 1$ is established.

Now consider $m > 1$ and suppose the theorem true for $m - 1$. We prove the induction step by block-diagonalisation of $\Phi - \mathcal{A}\Phi$. Let v, w be as above and let h be the outer factor of v. Once again by [AAK], h is also the outer factor of $\bar{z}\bar{w}$. It is given explicitly by the formula [H]

$$h = e^{u + \mathrm{i}\tilde{u}}$$

where

$$u = \log \|v(\cdot)\|$$

and $\tilde{u}$ is the harmonic conjugate of u,

$$\tilde{u} = -\mathrm{i}(2\mathbb{P}_+ - I)u.$$

Since $v \in X_+(\mathbb{C}^n)$ it is clear from axioms A1 and A2 that $\|v(\cdot)\|^2 \in X$. By Theorem 1.2, $\|v(\cdot)\|^2$ does not vanish on $\mathbb{T}$, and so its spectrum in the Banach algebra X is a compact interval of the positive real numbers. By the analytic functional calculus, $u = \frac{1}{2}\log \|v(\cdot)\|^2 \in X$. By A1 we have also $\tilde{u} \in X$. Thus $h = e^{u+\mathrm{i}\tilde{u}} \in X$. The above construction also makes it clear that if $v^{(k)}$, $h^{(k)}$ are the corresponding entities for $\Phi^{(k)}$, so that $v^{(k)} \to v$ in X, then $h^{(k)} \to h$ in X. Indeed, since $\mathbb{P}_+$ maps X into itself, it follows from the Closed Graph Theorem that $\mathbb{P}_+$ is continuous on X, and hence the Hilbert transform $u \to \tilde{u}$ is continuous on X.

Note also that since $|h| = \|v(\cdot)\|$ is bounded away from zero, h is invertible in X and $1/h^{(k)} \to 1/h$ in X. Let $v_{(i)}$, $w_{(i)}$, $v_{(i)}^{(k)}$, $w_{(i)}^{(k)}$ be the inner factors of v, $\bar{z}\bar{w}$, $v^{(k)}$, $\bar{z}\bar{w}^{(k)}$ respectively, so that

$$v_{(i)} = v/h, \qquad v_{(i)}^{(k)} = v^{(k)}/h^{(k)}$$

etc. Then $v_{(i)}^{(k)} \to v_{(i)}$ and $w_{(i)}^{(k)} \to w_{(i)}$ in X_+ as $k \to \infty$. By Theorem 2.2, $v_{(i)}$ and $w_{(i)}$ are co-outer.

By Lemma 1.6 we can find thematic functions

$$V = \begin{pmatrix} v_{(i)} & \bar{\alpha} \end{pmatrix}, \qquad W^t = \begin{pmatrix} w_{(i)} & \bar{\beta} \end{pmatrix},$$

$$V^{(k)} = \begin{pmatrix} v_{(i)}^{(k)} & \bar{\alpha}^{(k)} \end{pmatrix}, \qquad W^{(k)t} = \begin{pmatrix} w_{(i)}^{(k)} & \bar{\beta}^{(k)} \end{pmatrix}$$

such that $V^{(k)} \to V$ and $W^{(k)} \to W$ in X. *A fortiori*,

$$\alpha^{(k)} \to \alpha, \qquad \beta^{(k)} \to \beta \tag{2.15}$$

in $X(M_{n,n-1}),\ X(M_{m,m-1})$ respectively.

Now we construct $Q,\ Q^{(k)} \in X_+(M_{m,n})$ such that (cf (2.3))

$$Qv = T_\Phi v, \qquad Q^t \bar{z}\bar{w} = T_{\Phi^t}(\bar{z}\bar{w}), \tag{2.16}$$

$$Q^{(k)} v^{(k)} = T_{\Phi^{(k)}} v^{(k)}, \qquad Q^{(k)t} \bar{z}\bar{w}^{(k)} = T_{\Phi^{(k)t}}(\bar{z}\bar{w}^{(k)}) \tag{2.17}$$

and $Q^{(k)} \to Q$ in X. We can do this using a formula for Q which we gave in [PY2, Sec. 2, Remark 3]. Let

$$y_1 = T_\Phi v/h, \qquad y_2 = T_{\Phi^t}(\bar{z}\bar{w})/h.$$

Then $y_1,\ y_2 \in X$ and from the fact that the equations (2.16) are consistent (they hold with $Q = \mathcal{A}\Phi$) we have $y_2^t v_{(i)} = w_{(i)}^t y_1$ $(= w_{(i)}^t Q v_{(i)})$. The components of $v_{(i)}$ are elements of the Banach algebra X_+. By Theorem 1.2 they do not vanish simultaneously at any point of $\mathbb{T}$, nor (since $v_{(i)}$ is co-outer) do they at any point of $\mathbb{D}$. Hence they do not all belong to any maximal ideal of X_+ (see Lemma 0.1), and so the ideal they generate in X_+ is the whole algebra. Thus there exists $f_1 \in X_+(\mathbb{C}^n)$ such that $f_1^t v_{(i)} = 1$. Likewise there exists $f_2 \in X_+(\mathbb{C}^m)$ such that $f_2^t w_{(i)} = 1$. It is simple to verify that a solution of (2.16) is

$$Q = y_1 f_1^T + f_2 y_2^t - f_2 y_2^t v_{(i)} f_1^t.$$

Now perform a similar construction to obtain $Q^{(k)}$. Let

$$y_1^{(k)} = T_{\Phi^{(k)}} v^{(k)}/h^{(k)}, \qquad y_2^{(k)} = T_{\Phi^{(k)t}}(\bar{z}\bar{w}^{(k)})/h^{(k)}.$$

Then $y_1^{(k)} \to y_1$ and $y_2^{(k)} \to y_2$ in X.

Apply Lemma 1.8 to $f = f_1,\ \varphi = v_{(i)}$. For any $N \in \mathbb{N}$ there exists $\delta_N > 0$ such that $\|v_{(i)} - \psi\|_X < \delta_N$ implies that there exists $g \in X_+(\mathbb{C}^n)$ with $g^t\psi = 1$ and $\|f_1 - g\|_X < \frac{1}{N}$. Define a sequence of integers (k_N) and $f_1^{(k_N)} \in X_+(\mathbb{C}^n)$ inductively as follows. Let $k_1 = 1,\quad f_1^{(1)} = 0$. Choose $k_N > k_{N-1}$ so that $\|v_{(i)} - v_{(i)}^{(k_N)}\| < \delta_N$. Then there exists $f_1^{(k_N)} \in X_+(\mathbb{C}^n)$ such that $f_1^{(k_N)t} v_{(i)}^{(k_N)} = 1$ and $\|f_1^{(k_N)} - f_1\| < \frac{1}{N}$. Passing to the subsequence $(\Phi^{(k_N)})$ of $(\Phi^{(k)})$, we may assume that $f_1^{(k)t} v_{(i)}^{(k)} = 1$ and

$f_1^{(k)} \to f_1$ in X. In a similar way we construct $f_2^{(k)} \in X_+(\mathbb{C}^m)$ such that $f_2^{(k)t} w_{(i)}^{(k)} = 1$ and $f_2^{(k)} \to f_2$ in X. Now let

$$Q^{(k)} = y_1^{(k)} f_1^{(k)t} + f_2^{(k)} y_2^{(k)t} - f_2^{(k)} y_2^{(k)t} v_{(i)}^{(k)} f_1^{(k)t}.$$

Then $Q^{(k)}$ satisfies (2.17) and $Q^{(k)} \to Q$ in X. Let

$$\Psi \stackrel{\text{def}}{=} \beta^*(\Phi - Q)\bar{\alpha}, \qquad \Psi^{(k)} \stackrel{\text{def}}{=} \beta^{(k)*}(\Phi^{(k)} - Q^{(k)})\bar{\alpha}^{(k)}.$$

Then $\Psi^{(k)} \to \Psi$ in $X(M_{m-1,n-1})$. It is shown in [PY1,PY2] that

$$\Phi - \mathcal{A}\Phi = W^* \begin{pmatrix} t_0 u_0 & 0 \\ 0 & \Psi - \mathcal{A}\Psi \end{pmatrix} V^* \tag{2.18}$$

where u_0 is a badly approximable unimodular function. It follows that the superoptimal singular values of Ψ are $t_1, \dots, t_{m-1}$ and are non-zero. Furthermore, every thematic factorization of $\Psi - \mathcal{A}\Psi$ gives rise to one of $\Phi - \mathcal{A}\Phi$, and hence the indices in any thematic factorization of $\Psi - \mathcal{A}\Psi$ are all equal to 1. By the inductive hypothesis $\mathcal{A}$ is continuous at Ψ, and hence $\mathcal{A}\Psi^{(k)} \to \mathcal{A}\Psi$ in $X(M_{m-1,n-1})$. By [PY2],

$$\mathcal{A}\Phi = Q + \beta \mathcal{A}\Psi \alpha^t, \qquad \mathcal{A}\Phi^{(k)} = Q^{(k)} + \beta^{(k)} \mathcal{A}\Psi^{(k)} \alpha^{(k)t} \tag{2.19}$$

and hence $\mathcal{A}\Phi^{(k)} \to \mathcal{A}\Phi$ in $X(M_{m,n})$ as $k \to \infty$. Thus $\mathcal{A}$ is continuous at Φ. ∎

What if one of the superoptimal singular values t_j of Φ is 0? One can see by considering diagonal examples such as $\operatorname{diag}\{\bar{z}, 0\}$ that it is important whether $\mathcal{A}$ is continuous at 0 in (scalar) X, or equivalently whether $\mathcal{A}$ is bounded. This is not always so for spaces satisfying A1 to A5 (see [P2]), and so the conclusion of Theorem 1.1 does not follow if the condition $t_{m-1} \neq 0$ is relaxed. There is one case when it does.

Theorem 2.10.. *Let X be the Besov space B_1^1 and let $\Phi \in X(M_{m,n})$. If $\Phi - \mathcal{A}\Phi$ has a thematic factorization in which the indices corresponding to non-zero superoptimal singular values are all equal to 1 then Φ is a continuity point of the operator $\mathcal{A}$ of superoptimal approximation in $X(M_{m,n})$.*

Proof. The fact that this statement is true in the case $\Phi = \mathbb{O}$ is Theorem 5.6 of [PY1]. Note that X satisfies axioms A1 to A5. Let Φ have superoptimal singular values $t_0, \dots, t_{m-1}$. Let r be the number of nonzero superoptimal singular values of Φ: $r = \inf\{j : t_j = 0\}$. We prove the result by induction on r. As in the proof of Theorem 2.1, let $\{\Phi^{(k)}\}_{k \geq 1}$ be a sequence of functions in X such that $\|\Phi - \Phi^{(k)}\|_{X(M_{m,n})} \to 0$.

If $r = 0$ then $\Phi = \mathcal{A}\Phi \in H^\infty$. Since $\Phi^{(k)} \to \Phi$ in X, by the cited theorem, $\mathcal{A}(\Phi^{(k)} - \Phi) \to \mathbb{O}$, and since $\Phi \in H^\infty$, $\mathcal{A}(\Phi^{(k)} - \Phi) = \mathcal{A}\Phi^{(k)} - \Phi$. Thus $\mathcal{A}\Phi^{(k)} \to \Phi$. Hence $\mathcal{A}$ is continuous at Φ.

Now consider $r \geq 1$ and suppose the assertion holds for $r - 1$. Since $t_0 \neq 0$ the compact operator H_Φ is not zero and so $H_\Phi^* H_\Phi$ has finite-dimensional eigenspace corresponding to t_0^2. We now proceed as in the proof of Theorem 2.1: pick Schmidt vectors v, $v^{(k)}$, w, $w^{(k)}$, thematic functions V, $V^{(k)}$, W^t, $W^{(k)t}$ and L^∞ functions Q, $Q^{(k)}$, Ψ, $\Psi^{(k)}$ exactly as described above. Once again (2.18) holds and the indices corresponding to any nonzero superoptimal singular value in any thematic factorization of $\Psi - \mathcal{A}\Psi$ are all 1. Moreover, the superoptimal singular values of Ψ are $t_1, \dots, t_{m-1}$, so that Ψ has $r-1$ nonzero superoptimal singular values. By the inductive hypothesis $\mathcal{A}\Psi^{(k)} \to \mathcal{A}\Psi$ in $X(M_{m-1,n-1})$. The relations (2.19) now show that $\mathcal{A}\Phi^{(k)} \to \Phi$ in $X(M_{m,n})$ as $k \to \infty$. Thus $\mathcal{A}$ is continuous at Φ. ■

3. Necessary conditions for continuity

It is conceivable that the sufficient condition for continuity of $\mathcal{A}$ which we established in Theorem 2.1 is also necessary for functions belonging to a space X satisfying our axioms A1 to A5. We can prove it for square matrix functions whose superoptimal singular values are all nonzero.

Lemma 2.1.. *Let $\Phi \in X$ be of type $n \times n$, and let $\varepsilon > 0$. Suppose that all n superoptimal singular values of Φ are nonzero and that $\mathcal{A}$ is continuous at Φ with respect to the norm of X. Then there exists $\Psi \in X$ such that $\|\Phi - \Psi\|_X < \varepsilon$, all n superoptimal singular values of Ψ are nonzero and all n indices of Ψ are equal to 1.*

Proof. Since $\mathcal{A}$ is continuous at Φ the same is true for the mapping $G \mapsto \det(G - \mathcal{A}G)$, which maps $X(M_{n,n})$ to the space of constant functions in X; it maps G to the product of the superoptimal singular values of G. The latter mapping is nonzero at Φ, by hypothesis, and hence there exists $\varepsilon_1 > 0$ such that the product of the superoptimal singular values of G is nonzero whenever $\|\Phi - G\|_X < \varepsilon_1$. It will therefore suffice to prove by induction on n the following

Assertion: *Let $\Phi \in X$ be of type $n \times n$, and let $\varepsilon, \varepsilon_1 > 0$. Suppose that all n superoptimal singular values of G are nonzero whenever $\|\Phi - G\|_X < \varepsilon_1$. Then there exists $\Psi \in X$ such that $\|\Phi - \Psi\|_X < \varepsilon$, all n superoptimal singular values of Ψ are nonzero and all n indices of Ψ are equal to 1.*

To prove this we show first that there exists $\Upsilon \in X$ such that $\|H_\Upsilon\| > \|H_\Phi\|$, $\|\Upsilon - \Phi\|_X$ is arbitrarily small and $H_{\Upsilon-\Phi}$ has rank one. Indeed, if H_Φ has maximising vector v, $H_\Phi v = \bar{z}\bar{g}$ for some $g \in H^2$ and $\zeta \in \mathbb{D}$ is a point at which v is non-zero, then it suffices to take

$$\Upsilon(z) = \Phi(z) + (z-\zeta)^{-1}\eta \otimes v(\zeta)$$

where $\eta \in \mathbb{C}^n$ is a non-zero vector of suitably small norm satisfying $\eta^t g(\zeta) > 0$. We have

$$\begin{aligned}
(H_\Upsilon v, \bar{z}\bar{g}) &= ((\Phi + (z-\zeta)^{-1}\eta \otimes v(\zeta))v, \bar{z}\bar{g}) \\
&= (\Phi v, \bar{z}\bar{g}) + (\mathbb{P}_-(z-\zeta)^{-1}\eta \otimes v(\zeta)v, \bar{z}\bar{g}) \\
&= (H_\Phi v, \bar{z}\bar{g}) + \|v(\zeta)\|^2((z-\zeta)^{-1}\eta, \bar{z}\bar{g}) = \|H_\Phi v\|^2 + \|v(\zeta)\|^2\eta^t g(\zeta) \\
&> \|H_\Phi v\|^2 = \|H_\Phi\| \, \|v\| \, \|\bar{z}\bar{g}\|.
\end{aligned}$$

Thus $\|H_\Upsilon\| > \|H_\Phi\|$. By choosing η small we can ensure that Υ and Φ are close in any norm, in particular the X norm. Υ thus has the properties claimed.

Since H_Υ is a rank one perturbation of H_Φ

$$s_1(H_\Upsilon) \le s_0(H_\Phi) < s_0(H_\Upsilon),$$

so that the maximising subspace of H_Υ is one-dimensional. Let the superoptimal singular values of Υ be $t_j^\sharp$, $j \ge 0$. The index of $t_0^\sharp = s_0(H_\Upsilon)$ in any thematic factorisation of Υ is 1; for suppose otherwise. Then we have

$$\Upsilon - \mathcal{A}\Upsilon = W^* \begin{pmatrix} t_0^\sharp u & 0 \\ 0 & F \end{pmatrix} V^* \tag{2.1}$$

where V, W^t are thematic functions, u is a badly approximable unimodular function and the Toeplitz operator T_u has index less than –1. Thus dim Ker $T_u > 1$. It is easy to see that $\{Vf : f \in \operatorname{Ker} T_u\}$ is a space of maximising vectors of H_Υ, and this contradicts the simplicity of the singular value $s_0(H_\Upsilon)$. Thus the index of $t_0^\sharp$ is 1. Moreover, by [PY3, Theorem 1.1], $t_1^\sharp \le s_1(H_\Upsilon)$, so that $t_1^\sharp < s_0(H_\Upsilon)$. That is, in (2.1) $\|F\|_\infty < t_0^\sharp$.

The case $n = 1$ of Assertion is established by choice of Ψ equal to Υ. Now consider $n > 1$ and suppose it true for $n - 1$. Pick Υ as above with

$$\|\Upsilon - \Phi\|_X < \frac{1}{2}\min\{\varepsilon, \varepsilon_1\},$$

and pick a thematic factorization (2.1) of $\Upsilon - \mathcal{A}\Upsilon$, so that $\|F\|_\infty < t_0^\sharp$. Since multiplication is continuous in the normed algebra $X(M_{n,n})$ there exists $K > 1$ such that

$$\|W^* G V^*\|_X \le K\|G\|_X$$

for all $G \in X(M_{n,n})$. Let

$$\delta \stackrel{\text{def}}{=} \min\{\frac{\varepsilon}{2K}, \frac{\varepsilon_1}{2K}, t_0^\sharp - \|F\|_\infty\}.$$

In the notation of (2.4) we have

$$F = \beta^*(\Upsilon - \mathcal{A}\Upsilon)\bar{\alpha} \in X(M_{n-1,n-1}).$$

We claim that, for any $E \in X(M_{n-1,n-1})$ such that $\|F - E\|_X < \delta$, the superoptimal singular values of E are all nonzero. We have

$$\begin{aligned} \|E\|_\infty &= \|F\|_\infty + \|E - F\|_\infty \le \|F\|_\infty + \|E - F\|_X \\ &< \|F\|_\infty + t_0^\sharp - \|F\|_\infty = t_0^\sharp, \end{aligned}$$

and hence

$$\|W^* \begin{pmatrix} t_0^\sharp u & 0 \\ 0 & E \end{pmatrix} V^*\|_\infty = t_0^\sharp.$$

Now let

$$\Phi_E = \mathcal{A}\Upsilon + W^* \begin{pmatrix} t_0^\sharp u & 0 \\ 0 & E \end{pmatrix} V^*. \tag{2.2}$$

Then $\|H_{\Phi_E}\| \le \|\Phi_E - \mathcal{A}\Upsilon\|_\infty = t_0^\sharp$. Now $V,\ u$ have the form

$$V = (v_{(i)} \quad \bar{\alpha}), \quad u = \bar{z}\bar{h}/h$$

where $v = v_{(i)}h$ is the inner-outer factorization of a maximising vector v of H_Φ (see [PY1, Section 2, or PY2]). This v satisfies

$$\|H_{\Phi_E} v\| = t_0^\sharp \|v\|,$$

and hence we have $\|H_{\Phi_E}\| = t_0^\sharp$. Thus $\mathcal{A}\Upsilon$ is a best (though typically *not* a superoptimal) analytic approximation to Φ_E, and (2.2) is a first stage thematic factorization of $\Phi_E - \mathcal{A}\Upsilon$. It follows from [PY1, Lemma 2.4] that the superoptimal singular values of E are those of Φ_E, all but the first. However,

$$\begin{aligned} \|\Phi - \Phi_E\| &< \|\Phi - \Upsilon\|_X + \|\Upsilon - \Phi_E\|_X < \frac{1}{2}\min\{\varepsilon, \varepsilon_1\} + \|W^* \begin{pmatrix} 0 & 0 \\ 0 & E - F \end{pmatrix} V^*\|_X \\ &\le \frac{1}{2}\min\{\varepsilon, \varepsilon_1\} + K\delta < \min\{\varepsilon, \varepsilon_1\}. \end{aligned} \tag{2.3}$$

By hypothesis the superoptimal singular values of Φ_E are nonzero, and hence those of E are also. This establishes the claim.

By the inductive hypothesis there exists $G \in X(M_{n-1,n-1})$ such that

$$\|F - G\|_X < \delta,$$

all superoptimal singular values of G are nonzero and all $n-1$ indices of G are 1. Let

$$\Psi \stackrel{\text{def}}{=} \mathcal{A}\Upsilon + W^* \begin{pmatrix} t_0^\sharp u & 0 \\ 0 & G \end{pmatrix} V^* \in X(M_{n,n}). \tag{2.4}$$

In other words, $\Psi = \Phi_G$, and so by the above, the superoptimal singular values of Ψ consist of $t_0^\sharp$ and those of G, hence are all nonzero. By (2.3), $\|\Phi - \Psi\|_X < \varepsilon$. Any thematic factorisation of $G - \mathcal{A}G$ induces one of $\Psi - \mathcal{A}\Psi$ through the relation

$$\Psi - \mathcal{A}\Upsilon - \beta\mathcal{A}G\alpha^t = W^* \begin{pmatrix} t_0^\sharp u & 0 \\ 0 & G - \mathcal{A}G \end{pmatrix} V^*,$$

where we use the notation (2.4) for $V,\ W$. Since the indices of $t_0^\sharp u$ and $G - \mathcal{A}G$ are all 1, so are those of $\Psi - \mathcal{A}\Psi$. The Assertion follows by induction. ■

Theorem 2.2.. *Let X be a space of functions on $\mathbb{T}$ satisfying Axioms* (A1) *to* (A5)*, let $\Phi \in X$ be of type $n \times n$ and suppose that the superoptimal singular values of Φ are all nonzero. If $\mathcal{A}$ is continuous at Φ then all indices in any thematic factorisation of $\Phi - \mathcal{A}\Phi$ are equal to 1.*

Proof. Thematic functions have constant determinant [PY1, Theorem 1.2]. Hence $\det(\Phi - \mathcal{A}\Phi)$ is a function of nonzero constant modulus on $\mathbb{T}$ whose winding number about 0 is the sum of the indices in any thematic factorisation of $\Phi - \mathcal{A}\Phi$. Thus the winding number is n if and only if all the indices in any thematic factorisation are equal to 1. By Lemma 2.1, $\Phi - \mathcal{A}\Phi$ is a limit in the norm of X of a sequence of functions Ψ such that $\Psi - \mathcal{A}\Psi$ has all indices defined and equal to 1, hence such that $\det(\Psi - \mathcal{A}\Psi)$ has winding number n. It follows that $\det(\Phi - \mathcal{A}\Phi)$ has winding number n. ■

Remark. The proof shows a slightly stronger statement: if $\mathcal{A}$ is continuous at Φ as a mapping from X to BMO (which is a weaker hypothesis than continuity from X to X) then the same conclusion holds.

As we mentioned in our discussion of sufficiency, continuity of $\mathcal{A}$ at functions which have some superoptimal singular value equal to zero is related to the boundedness properties of scalar $\mathcal{A}$ on X.

Theorem 2.3.. *Let X be one of the Besov spaces B_p^s, $s > 1/p$ or the Holder-Zygmund spaces λ_α, Λ_α, $\alpha > 0$. Then $\mathcal{A}$ is discontinuous at any matrix-valued function in X which has a zero superoptimal singular value.*

Proof. It is shown in [P2] that $\mathcal{A}$ is unbounded on these spaces. Let $\Phi \in X(M_{m,n})$. We can suppose that $m \le n$. Let $t_r = 0$, some $r \le m$, but $t_j \ne 0$ for $j < r$. We suppose $r \ge 1$: the modifications for the case $r = 0$ (i.e. $\Phi \in H^\infty$) are easy. Consider a thematic factorisation

$$\Phi - \mathcal{A}\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & t_{r-1}u_{r-1} & 0 \\ 0 & \cdot & 0 & 0 \end{pmatrix} V_{r-1}^* \cdots V_0^*.$$

By [P1], for $0 < \delta < t_0$ we may pick a scalar function $\psi_\delta \in X$ such that $\|\psi_\delta\|_X < \delta$ and $\|\mathcal{A}\psi_\delta\|_X \ge 1$. Let

$$\Phi_\delta = \mathcal{A}\Phi + W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & t_{r-1}u_{r-1} & 0 & 0 \\ 0 & \cdot & 0 & \psi_\delta & 0 \\ 0 & \cdot & 0 & 0 & 0 \end{pmatrix} V_{r-1}^* \cdots V_0^*.$$

Clearly $\|\Phi - \Phi_\delta\|_X \to 0$ as $\delta \to 0$. If we solve the superoptimal analytic approximation problem for Φ_δ by successive diagonalisation then for the first r stages it proceeds exactly as for Φ (a detailed proof of this statement would be along the same lines as the proof of Lemma 2.1). It follows that

$$\Phi_\delta - \mathcal{A}\Phi_\delta = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & t_{r-1}u_{r-1} & 0 & 0 \\ 0 & \cdot & 0 & \psi_\delta - \mathcal{A}\psi_\delta & 0 \\ 0 & \cdot & 0 & 0 & 0 \end{pmatrix} V_{r-1}^* \cdots V_0^*.$$

Thus

$$\mathcal{A}\Phi - \mathcal{A}\Phi_\delta = W_0^* \cdots W_{r-1}^* \begin{pmatrix} 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & 0 & 0 \\ 0 & \cdot & 0 & \mathcal{A}\psi_\delta & 0 \\ 0 & \cdot & 0 & 0 & 0 \end{pmatrix} V_{r-1}^* \cdots V_0^*.$$

Since $\|\mathcal{A}\psi_\delta\|_X \geq 1$, it cannot be true that $\mathcal{A}\Phi_\delta \to \mathcal{A}\Phi$ in X. Thus $\mathcal{A}$ is discontinuous on X at Φ. ■

3. Continuity of superoptimal singular values

The first superoptimal singular value t_0 of $\Phi \in H^\infty + C$ is equal to $\|H_\Phi\|$, hence is continuous with respect to the L^∞ norm. Is the same true for the other superoptimal singular values? Or at least with respect to one of the norms $\|\cdot\|_X$ discussed above? We will not venture a guess as to the answer to this question, but we can at least prove continuity with respect to $\|\cdot\|_X$ under the same hypothesis as in Theorem 1.1. For $\Phi \in H^\infty + C$ we shall denote by $t_j(\Phi)$ the jth superoptimal singular value of Φ.

Lemma 3.1.. *Let $X \subset H^\infty + C$ be a normed algebra of functions on $\mathbb{T}$ whose norm majorises the L^∞ norm and which is invariant under $\mathcal{A}$. If $\Phi \in X$ is a point of continuity of $\mathcal{A}$ in X then Φ is also a point of continuity of each of the superoptimal singular values $t_j(\cdot)$ with respect to $\|\cdot\|$.*

Proof. We recall that, for any matrix A of type $m \times n$ and any integer p, $2 \leq p \leq m$, the pth exterior power $\wedge^p A$ is defined to be the matrix of type $\binom{m}{p} \times \binom{n}{p}$ whose entries are the $p \times p$ minors of A. Consider an $m \times n$ matrix function $G \in X$, $m \leq n$, and any integer p, $2 \leq p \leq m$. Define $(\wedge^p G)(z)$ to be $\wedge^p(G(z))$. Since the entries of $\wedge^p G$ are polynomials in those of G we have $\wedge^p G \in X$ and the mapping $G \mapsto \wedge^p G$ is continuous with respect to the X norms. Thus, if $\mathcal{A}$ is continuous at Φ, so is the mapping $G \mapsto \| \wedge^p (G - \mathcal{A}G)\|_\infty$. It is immediate from consideration of thematic factorisations that $\| \wedge^p (G - \mathcal{A}G)\|_\infty$ equals the product of the first p superoptimal singular values of G. Hence $t_0(\cdot)$, $t_0(\cdot)t_1(\cdot)$, $t_0(\cdot)t_1(\cdot)t_2(\cdot), \cdots$ are all continuous at Φ. The result now follows from the following simple observation which is valid for any topological space. If $f_0 \geq f_1 \geq f_2 \geq \cdots \geq 0$ are real-valued functions such that f_0, $f_0 f_1$, $f_0 f_1 f_2, \cdots$ are all continuous at a point x then each f_j is continuous at x (consider separately the two cases $f_{j-1}(x) \neq 0$ and $f_{j-1}(x) = 0$). ■

Theorem 3.2.. *Let X be a space of functions on $\mathbb{T}$ satisfying Axioms* (A1) *to* (A5) *and let $\Phi \in X(M_{m,n})$, $m \le n$. Suppose that either $t_{m-1} \ne 0$ or X is the Besov space B_1^1. If $\Phi - \mathcal{A}\Phi$ has a thematic factorisation with indices corresponding to nonzero superoptimal singular values all equal to 1 then $t_j(\cdot)$ is continuous at Φ with respect to $\|\cdot\|_X$ for $0 \le j < m$.*

The proof is immediate from Theorems 1.1 and 1.10 and the foregoing Lemma.

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