SOME RESULTS ON THE ADMISSIBLE REPRESENTATIONS OF NON-CONNECTED REDUCTIVE p-ADIC GROUPS

DAVID GOLDBERG AND REBECCA A. HERB

ABSTRACT. We examine the theory of induced representations for non-connected reductive p-adic groups for which G/G^0 is abelian. We first examine the structure of those representations of the form $\mathrm{Ind}_{P^0}^{G}(\sigma)$, where P^0 is a parabolic subgroup of G^0 and σ is a discrete series representation of the Levi component of P^0. Here we develop a theory of R–groups, extending the theory in the connected case. We then prove some general results in the theory of representations of non-connected p-adic groups whose component group is abelian. We define the notion of cuspidal parabolic for G in order to give a context for this discussion. Intertwining operators for the non-connected case are examined and the notions of supercuspidal and discrete series are defined. Finally, we examine parabolic induction from a cuspidal parabolic subgroup of G. Here we also develop a theory of R–groups, and show that these groups parameterize the induced representations in a manner that is consistent with the connected case and with the first set of results as well.

1. Introduction

The theory of induced representations plays a fundamental role within representation theory in general. Within the theory of admissible representations of connected reductive algebraic groups over local fields, parabolic induction is used to complete classification theories, once certain families of representations are understood [**3**,**8**,**9**, **10**]. The theory of admissible representations on non-connected reductive groups over nonarchimedean local fields has been addressed in part in [**1**,**4**,**6**,**11**], among other places. We will study certain aspects of parabolic induction for disconnected groups whose component group is abelian.

Let F be a locally compact, non-discrete, nonarchimedean field of characteristic zero. Let G be a (not necessarily connected) reductive F-group. Thus G is the set of F-rational points of a reductive algebraic group defined over F. Let G^0 be the connected component of the identity in G. We assume that G/G^0 is finite and abelian.

Our goal is to address three major points. The first is an extension of the results of [**6**] to the case at hand. This entails a study of induction from a parabolic subgroup of G° to G. In particular, suppose that $P^\circ = M^\circ N$ is a parabolic subgroup of G°,

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and let σ_0 be an irreducible discrete series representation of M°. We are interested in the structure of $\pi_0 = \mathrm{Ind}_{P^\circ}^G(\sigma_0)$. In [**1**] Arthur suggests a construction, in terms of the conjectural local Langlands parameterization, of a finite group whose representation theory should describe the structure of π_0, when G/G° is cyclic. In [**6**] the case where G/G° is of prime order is studied, and there is a construction, on the group side, of a finite group $R_G(\sigma_0)$ which (along with an appropriate 2–cocycle) parameterizes the components of π_0. It is also shown there that $R_G(\sigma_0)$ must be isomorphic to Arthur's group R_{ψ,σ_0}, if the latter exists. One cannot confirm the existence of R_{ψ,σ_0} without proofs of both the local Langlands conjecture and Shelstad's conjecture [**12**] that R_{ψ_0,σ_0} is isomorphic to $R_{G^\circ}(\sigma_0)$. (See [**1**] for the precise definitions of R_{ψ,σ_0} and R_{ψ_0,σ_0}.) Here, by extending the definition of the standard intertwining operators (cf Section 4) we show we can construct a group $R_G(\sigma_0)$ in a manner analogous to [**6**], and show that it has the correct parameterization properties (cf Theorems 4.16 and 4.17). An argument, similar to the one given in [**6**] shows that if G/G° is cyclic, then $R_G(\sigma_0)$ must be isomorphic to R_{ψ,σ_0}, assuming the latter exists (cf Remark 4.18).

The second collection of results is an extension of some standard results in admissible representation theory to the disconnected group G. In order to develop a theory consistent with the theory for connected groups, one needs to determine an appropriate definition of parabolic subgroup. There are several definitions in the literature already, yet they do not always agree. We use a definition of parabolic subgroup which is well suited to our purposes. Among the parabolic subgroups of G we single out a collection of parabolic subgroups which we call cuspidal. They have the property that they support discrete series and supercuspidal representations and can be described as follows. Let P^0 be a parabolic subgroup of G^0 with Levi decomposition M^0N and let A be the split component of M^0. Let $M = C_G(A)$. Then $P = MN$ is a cuspidal parabolic subgroup of G lying over P^0. We also say in this case that M is a cuspidal Levi subgroup of G. Thus cuspidal parabolic subgroups of G are in one to one correspondence with parabolic subgroups of G^0.

Using our definitions we can prove the following. Let M be a Levi subgroup of G and let $M^0 = M \cap G^0$.

Lemma 1.1. *(i) If M is not cuspidal, then M has no supercuspidal representations, i.e., admissible representations with matrix coefficients which are compactly supported modulo the center of M and have zero constant term along the nil radical of any proper parabolic subgroup of M.*
(ii) If M is cuspidal and π is an irreducible admissible representation of M, then π is supercuspidal if and only if the restriction of π to M^0 is supercuspidal.
(iii) If M is not cuspidal, then M has no discrete series representations, ie. unitary representations with matrix coefficients which are square-integrable modulo the center of M.

(iv) If M is cuspidal and π is an irreducible unitary representation of M, then π is discrete series if and only if the restriction of π to M^0 is discrete series.

Using Lemma 1.1 it is easy to extend the following theorem from the connected case to our class of disconnected groups.

Theorem 1.2. *Let π be an irreducible admissible (respectively tempered) representation of G. Then there are a cuspidal parabolic subgroup $P = MN$ of G and an irreducible supercuspidal (respectively discrete series) representation σ of M such that π is a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$.*

Let $P_1 = M_1N_1$ and $P_2 = M_2N_2$ be cuspidal parabolic subgroups and let σ_i be irreducible representations of $M_i, i = 1, 2$, which are either both supercuspidal or both discrete series. By studying the orbits for the action of $P_1 \times P_2$ on G, we are able to extend the proof for the connected case to our situation and obtain the following theorem.

Theorem 1.3. *Let $P_1 = M_1N_1, P_2 = M_2N_2, \sigma_1, \sigma_2$ be as above. Then if $\pi_1 = \mathrm{Ind}_{P_1}^G(\sigma_1)$ and $\pi_2 = \mathrm{Ind}_{P_2}^G(\sigma_2)$ have a nontrivial intertwining, then there is $y \in G$ so that*

$$M_2 = yM_1y^{-1} \text{ and } \sigma_2 \simeq y\sigma_2y^{-1}.$$

The third question of study is the structure of $\pi = \mathrm{Ind}_P^G(\sigma)$ when $P = MN$ is a cuspidal parabolic subgroup of G and σ is a discrete series representation of M. We show that, as in the connected case, the components of π are naturally parameterized using a finite group R. As in the connected case we first describe a collection of standard intertwining operators $R(w, \sigma)$ which are naturally indexed by $w \in W_G(\sigma) = N_G(\sigma)/M$, where

$$N_G(\sigma) = \{x \in N_G(M) : \sigma^x \simeq \sigma\}.$$

We prove that there is a cocycle η so that

$$R(w_1w_2, \sigma) = \eta(w_1, w_2)R(w_1, \sigma)R(w_2, \sigma), w_1, w_2 \in W_G(\sigma).$$

Let σ_0 be an irreducible component of the restriction of σ to M^0, and $P^0 = M^0N = P \cap G^0$. Then $\sigma \subset \mathrm{Ind}_{M^0}^M(\sigma_0)$ so that

$$\pi = \mathrm{Ind}_P^G(\sigma) \subset \mathrm{Ind}_P^G(\mathrm{Ind}_{M^0}^M(\sigma_0)) \simeq \mathrm{Ind}_{P^0}^G(\sigma_0).$$

Using the intertwining operators and R-group theory developed earlier for the representation $\mathrm{Ind}_{P^0}^G(\sigma_0)$, we can prove the following results. First, the collection

$$\{R(w, \sigma), w \in W_G(\sigma)\}$$

spans the commuting algebra of π. Second, let Φ_1^+ be the set of positive restricted roots for which the rank one Plancherel measures of σ_0 are zero and let $W(\Phi_1)$ be the group generated by the reflections corresponding to the roots in Φ_1. Then $W(\Phi_1)$

is naturally embedded as a normal subgroup of $W_G(\sigma)$ and $W_G(\sigma)$ is the semidirect product of $W(\Phi_1)$ and the group

$$R_\sigma = \{w \in W_G(\sigma) : w\alpha > 0 \text{ for all } \alpha \in \Phi_1^+\}.$$

Finally, $R(w, \sigma)$ is scalar if $w \in W(\Phi_1)$. This proves that the operators $R(w, \sigma), w \in R_\sigma$, span the intertwining algebra. But we can compute the dimension of the space of intertwining operators for $\mathrm{Ind}_P^G(\sigma)$, again by comparison with that of $\mathrm{Ind}_{P^0}^G(\sigma_0)$, and we find that it is equal to $[R_\sigma]$. Thus we have the following theorem.

Theorem 1.4. *The $R(w, \sigma), w \in R_\sigma$, form a basis for the algebra of intertwining operators of* $\mathrm{Ind}_P^G(\sigma)$.

Just as in the connected case, we show that there are a finite central extension

$$1 \to Z_\sigma \to \tilde{R}_\sigma \to R_\sigma \to 1$$

over which η splits and a character χ_σ of Z_σ so that the irreducible constituents of $\mathrm{Ind}_P^G(\sigma)$ are naturally parameterized by the irreducible representations of $\tilde{R}_\sigma$ with Z_σ-central character χ_σ.

Finally, we give a few examples which point out some of the subtleties involved in working with disconnected groups. For instance, we show that if we do not restrict ourselves to cuspidal parabolic subgroups, then the standard disjointness theorem for induced representations fails. Examples such as these show why one must restrict to induction from cuspidal parabolic subgroups in order to develop a theory which is consistent with that for connected groups.

Many interesting problems involving disconnected groups remain. For example, the question of a Langlands classification is still unresolved, and some of the results here on intertwining operators may help in this direction. One also hopes to remove the condition that G/G° is abelian, and extend all the results herein to that case. Problems such as these we leave to further consideration.

The organization of the paper is as follows. In §2 we give the definition of parabolic subgroup and prove Lemma 1.1 and Theorem 1.2. The proof of Theorem 1.3 is in §3. The results on induction from a parabolic subgroup of G^0 to G are in §4, and the results on induction from a parabolic subgroup of G to G, including Theorem 1.4, are in §5. Finally, §6 contains examples that show what can go wrong when we induce from parabolic subgroups of G which are not cuspidal.

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2. Basic Definitions

Let F be a locally compact, non-discrete, nonarchimedean field of characteristic zero. Let G be a (not necessarily connected) reductive F-group. Thus G is the set of F-rational points of a reductive algebraic group over F. Let G^0 be the connected component of the identity in G. We assume that G/G^0 is finite and abelian.

The split component of G is defined to be the maximal F-split torus lying in the center of G. Let A be any F-split torus in G and let $M = C_G(A)$. Then M is a reductive F-group. Now A is called a **special torus** of G if A is the split component of M. (Of course A is an F-split torus lying in the center of M. The only question is whether or not A is maximal with respect to this property.)

Lemma 2.1. *Let A be a special torus of G^0. Then A is a special torus of G.*

Proof: Let $M = C_G(A)$ and $M^0 = C_{G^0}(A) = M \cap G^0$. Write $Z(M)$ and $Z(M^0)$ for the centers of M and M^0 respectively. Now A is the maximal F-split torus lying in $Z(M^0)$ and $A \subset Z(M)$. Suppose A' is the maximal F-split torus lying in $Z(M)$. Then $A \subset A'$. But A' is a torus, so it is connected. Hence $A' \subset Z(M) \cap M^0 \subset Z(M^0)$. Thus $A' \subset A$ and so $A' = A$ is the split component of M. ∎

Remark 2.2. The converse of Lemma 2.1 is not true. For example, let $G = O(2) = SO(2) \cup wSO(2)$ where $SO(2) \simeq F^\times$ is the group of 2×2 matrices

$$d(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in F^\times, \quad \text{and} \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

satisfies $wd(a)w^{-1} = d(a^{-1}), a \in F^\times$. Let $A = \{d(1)\}$. Then $M = C_G(A) = G$ and $Z(M) = \{\pm d(1)\}$. Thus A is a special torus of G. However $M^0 = C_{G^0}(A) = G^0$ and $Z(M^0) = G^0$ is an F-split torus. Hence A is not the maximal F-split torus in $Z(M^0)$ and so is not special in G^0.

If G is connected, then A is a special torus of G by the above definition if and only if A is the split component of a Levi component M of a parabolic subgroup of G. We will define parabolic subgroups in the non-connected case so that we have this property in the non-connected case also.

Let A be a special torus of G and let $M = C_G(A)$. Then M is called a Levi subgroup of G. The Lie algebra $L(G)$ can be decomposed into root spaces with respect to the roots Φ of $L(A)$:

$$L(G) = L(G)_0 \oplus \sum_{\alpha \in \Phi} L(G)_\alpha$$

where $L(G)_0$ is the Lie algebra of M. Let Φ^+ be a choice of positive roots, and let N be the connected subgroup of G corresponding to $\sum_{\alpha \in \Phi^+} L(G)_\alpha$. Since elements of M centralize A and $L(A)$, they preserve the root spaces with respect to $L(A)$. Thus M normalizes N. Now $P = MN$ is called a parabolic subgroup of G and (P, A) is called a p-pair of G. The following lemma is an immediate consequence of this definition and Lemma 2.1.

Lemma 2.3. *Let $P^0 = M^0N$ be a parabolic subgroup of G^0 and let A be the split component of M^0. Let $M = C_G(A)$. Then $P = MN$ is a parabolic subgroup of G and $P \cap G^0 = P^0$.*

Lemma 2.4. *Let P be a parabolic subgroup of G. Then $P^0 = P \cap G^0$ is a parabolic subgroup of G^0.*

Proof: Let $P = MN$ be a parabolic subgroup of G and let A be the split component of M. Let $M^0 = C_{G^0}(A) = M \cap G^0$. Let A_1 be the split component of M^0. Then $A \subset A_1$ so that $C_{G^0}(A_1) \subset C_{G^0}(A) = M^0$. But A_1 is in the center of M^0, so that $M^0 \subset C_{G^0}(A_1)$. Thus $C_{G^0}(A_1) = M^0$ so that A_1 is a special torus in G^0 and M^0 is a Levi subgroup of G^0. Let Φ and Φ_1 denote the sets of roots of $L(A)$ and $L(A_1)$ respectively. For each $\alpha_1 \in \Phi_1$, the restriction $r\alpha_1$ of α_1 to $L(A)$ is non-zero since $C_{G^0}(A_1) = C_{G^0}(A) = M^0$. Let Φ^+ be the set of positive roots used to define N. Then $\Phi_1^+ = \{\alpha_1 \in \Phi_1 : r\alpha_1 \in \Phi^+\}$ is a set of positive roots for Φ_1 and

$$\sum_{\alpha \in \Phi+} L(G)_\alpha = \sum_{\alpha_1 \in \Phi_1^+} L(G)_{\alpha_1}.$$

Thus $P^0 = M^0N$ is a parabolic subgroup of G^0. ∎

We say the parabolic subgroup P of G lies over the parabolic subgroup P^0 of G^0 if $P^0 = P \cap G^0$. We will also say the Levi subgroup M of G lies over the Levi subgroup M^0 of G^0 if $M^0 = M \cap G^0$. Lemma 2.4 and its proof show that every parabolic (resp. Levi) subgroup of G lies over a parabolic (resp. Levi) subgroup of G^0.

Remark 2.5. There can be more than one parabolic subgroup P of G lying over a parabolic subgroup P^0 of G^0. For example, define $G = O(2)$ and $G^0 = SO(2)$ as in Remark 2.2. Then $A = \{d(1)\}$ and $A_0 = SO(2)$ are special vector subgroups of G corresponding to parabolic subgroups $P = O(2)$ and $P_0 = SO(2)$ respectively. Both lie over the unique parabolic subgroup $SO(2)$ of G^0.

Lemma 2.6. *Let $P^0 = M^0N$ be a parabolic subgroup of G^0 and let A be the split component of M^0. Let $M = C_G(A)$ and let $P = MN$. Then if P_1 is any parabolic subgroup of G lying over P^0 we have $P \subset P_1$. Further, M is the unique Levi subgroup lying over M^0 such that the split component of M is equal to A.*

Proof: Write $P_1 = M_1N$ where M_1 lies over M^0. Let A_1 be the split component of M_1. Then $A_1 \subset A$ so that $M = C_G(A) \subset C_G(A_1) = M_1$. Clearly $M_1 = M$ if and only if $A_1 = A$. ∎

Remark 2.7. Lemma 2.6 shows that there is a unique smallest parabolic subgroup P of G lying over P^0. Although it is defined using a Levi decomposition $P^0 = M^0N$ of P^0, it is independent of the Levi decomposition. Recall that if M_1^0 and M_2^0 are two Levi components of P^0 with split components A_1 and A_2 respectively, then there is

$n \in N$ such that $A_2 = nA_1n^{-1}$ and $M_2^0 = nM_1^0n^{-1}$. Now if $M_i = C_G(A_i), i = 1, 2$, we have $M_2 = C_G(A_2) = nC_G(A_1)n^{-1} = nM_1n^{-1}$, and $M_2N = nM_1n^{-1}N = M_1N$ since M_1 normalizes N.

Let Z be the split component of G. We let $C_c^\infty(G, Z)$ denote the space of all smooth complex-valued functions on G which are compactly supported modulo Z. We say $f \in C_c^\infty(G, Z)$ is a cusp form if for every proper parabolic subgroup $P = MN$ of G,

$$\int_N f(xn)dn = 0 \quad \forall x \in G.$$

Let ${}^0\mathcal{A}(G)$ denote the set of cusp forms on G. We say G is cuspidal if ${}^0\mathcal{A}(G) \neq \{0\}$. We know that every connected G is cuspidal.

Lemma 2.8. *G is cuspidal if and only if the split component of G is equal to the split component of G^0. Moreover, if G is cuspidal, then a subgroup N of G is the nilradical of a proper parabolic subgroup of G if and only if N is the nilradical of a proper parabolic subgroup of G^0. If G is not cuspidal, then G has a proper parabolic subgroup G_1 with nilradical $N_1 = \{1\}$.*

Proof: First suppose that G and G^0 have the same split component Z. Let $f \neq 0 \in {}^0\mathcal{A}(G^0)$. Define $F : G \to \mathbf{C}$ by $F(x) = f(x), x \in G^0$, and $F(x) = 0, x \notin G^0$. Then $F \in C_c^\infty(G, Z)$ and is non-zero. Let $P = MN$ be any proper parabolic subgroup of G. Then $N \subset G^0$, so for all $n \in N, x \in G$, $xn \in G^0$ if and only if $x \in G^0$. Thus for $x \notin G^0$,

$$\int_N F(xn)dn = 0$$

while for $x \in G^0$,

$$\int_N F(xn)dn = \int_N f(xn)dn.$$

Now $P^0 = P \cap G^0 = M^0N$ is a parabolic subgroup of G^0. Suppose that $P^0 = G^0$. Then P lies over G^0 so that by Lemma 2.6, $G \subset P$. This contradicts the fact that P is a proper parabolic subgroup of G. Thus $P^0 = M^0N$ is a proper parabolic subgroup of G^0. Since f is a cusp form for G^0 we have

$$\int_N f(xn)dn = 0 \quad \forall x \in G^0.$$

Thus F is a non-zero cusp form for G.

The above argument also showed that if $P = MN$ is a proper parabolic subgroup of G, then $P^0 = M^0N$ is a proper parabolic subgroup of G^0. Conversely, if $P^0 = M^0N$ is a proper parabolic subgroup of G^0 and $P = MN$ is any parabolic subgroup of G lying over P^0, then clearly $P \neq G$.

Conversely, suppose that G and G^0 do not have the same split component. Let Z be the split component of G^0 and define $G_1 = C_G(Z)$. By Lemma 2.6, G_1 is a proper parabolic subgroup of G. Further since G_1 lies over G^0 its nilradical is $N_1 = \{1\}$. Now if F is any cusp form on G and $x \in G$, we have

$$F(x) = \int_{N_1} F(xn)dn = 0.$$

Thus G has no non-zero cusp forms and so is not cuspidal. ∎

Example 2.9. Let $G = O(2)$ as in Remarks 2.2 and 2.5. Then $SO(2)$ is a cuspidal parabolic subgroup of G and $O(2)$ is not cuspidal.

We can sum up the proceeding lemmas in the following proposition.

Proposition 2.10. *Let $P^0 = M^0N$ be a parabolic subgroup of G^0. Then there is a unique cuspidal parabolic subgroup $P = MN$ of G lying over P^0. It is contained in every parabolic subgroup of G lying over P^0, and is defined by $M = C_G(A)$ where A is the split component of M^0.*

Now that we have parabolic subgroups of G, we want to study parabolic induction of representations. Many of the most basic notions of representation theory are defined in [**13**, chapter 1] for any totally disconnected group. In particular, admissible representations of G are defined and the following is an easy consequence of the definition.

Lemma 2.11. *Let Π be a representation of G. Then Π is admissible if and only if $\Pi|_{G^0}$, the restriction of Π to G^0, is admissible.*

Further, the results of Gelbart and Knapp regarding induction and restriction between a totally disconnected group G and an open normal subgroup H with G/H finite abelian can be applied to G and G^0. If π is any admissible representation of G^0 on V, we will let $\mathrm{Ind}_{G^0}^G(\pi)$ denote the representation of G by left translations on $\mathcal{H} = \{f : G \to V : f(gg_0) = \pi(g_0)^{-1}f(g), \forall g \in G, g_0 \in G^0\}$.

Lemma 2.12. (Gelbart-Knapp [5]) *Let Π be an irreducible admissible representation of G. Then $\Pi|_{G^0}$ is a finite direct sum of irreducible admissible representations of G^0. Let π be an irreducible constituent of $\Pi|_{G^0}$ which occurs with multiplicity r. Then*

$$\Pi|_{G^0} \simeq r \sum_{g \in G/G_\pi} \pi^g$$

where $G_\pi = \{g \in G : \pi^g \simeq \pi\}$.

Lemma 2.13. (Gelbart Knapp [5]) *Let π be an irreducible admissible representation of G^0. Then there is an irreducible admissible representation Π of G such that π occurs in the restriction of Π to G^0 with multiplicity $r > 0$. Let X denote the group of unitary characters of G/G^0 and let $X(\Pi) = \{\chi \in X : \Pi \otimes \chi \simeq \Pi\}$. Then*

$$\mathrm{Ind}_{G^0}^G(\pi) \simeq r \sum_{\chi \in X/X(\Pi)} \Pi \otimes \chi$$

is the decomposition of $\mathrm{Ind}_{G^0}^G(\pi)$ *into irreducibles and* $r^2[X/X(\Pi)] = [G_\pi/G^0]$.

The following result was proved by Gelbart and Knapp in the case where the restriction is multiplicity one [**5**]. Tadic [**14**] refined their result in the connected case. We now prove the more general result.

Lemma 2.14. *Suppose that G is a totally disconnected group, and H is a closed normal subgroup, with G/H a finite abelian group. If Π_1 and Π_2 are irreducible admissible reprsentations of G, which have a common constituent upon restriction to H, then $\Pi_2 \simeq \Pi_1 \otimes \chi$, for some character χ with $\chi|_H \equiv 1$.*

Proof: If the multiplicity of the restrictions is one, then this result holds by Gelbart-Knapp [**5**]. In particular, if $|G/H|$ is prime, the statement is true. We proceed by induction. We know the Lemma holds when $|G/H| = 2$. Suppose the statement is true whenever $|G_1/H| < n$. Suppose $|G/H| = n$. We may assume n is composite, so write $n = km$, with $1 < k < n$. Let $H \subset G_1 \subset G$ with $|G/G_1| = k$. If $\Pi_1|_{G_1}$ and $\Pi_2|_{G_2}$ have a common constituent, then, by our inductieve hypothesis, there is a χ with $\chi|_{G_1} \equiv 1$ with $\Pi_2 \simeq \Pi_1 \otimes \chi$. Since $(G/G_1\hat{)} \subset (G/H\hat{)}$, we are done, in this case.

Now suppose that τ is an irreducible subrepresentarion of both $\Pi_1|_H$ and $\Pi_2|_H$. Then, there are constituents $\Omega_i \subset \Pi_i|_{G_1}$ so that $\tau \subset \Omega_i|_H$. By the inductive hypothesis $\Omega_2 = \Omega_1 \otimes \chi$, for some χ of G_1 whose restriction to H is trivial. But, since $G_1/H \subset G/H$ is abelian we can extend χ to a character η of G/H. Note that $(\Pi_1 \otimes \eta)|G_1$ has $\Omega_1 \otimes \chi \simeq \Omega_2$ as a constituent, so, as we have seen above, $\Pi_1 \otimes \eta \otimes \omega \simeq \Pi_2$, for some character ω of G whose restriction to G_1 is trivial. Thus, the statement holds by induction. ∎

Let (π, V) be an admissible representation of G and let $\mathcal{A}(\pi)$ denote its space of matrix coefficients. We say π is supercuspidal if $\mathcal{A}(\pi) \subset {}^0\mathcal{A}(G)$. Of course if G is not cuspidal, then ${}^0\mathcal{A}(G) = \{0\}$ so that G has no supercuspidal representations. If $P = MN$ is any parabolic subgroup of G, define $V(P) = V(N)$ to be the subspace of V spanned by vectors of the form $\pi(n)v - v, v \in V, n \in N$. Then we say π is J-supercuspidal if $V(P) = V$ for every proper parabolic subgroup P of G. If G is not cuspidal, then by Lemma 2.8 there is a proper parabolic subgroup G_1 of G with nilradical $N_1 = \{1\}$. For any admissible representation (π, V) of G, $V(G_1) = V(N_1) = \{0\} \neq V$ so that π is not J-supercuspidal. Thus G has no J-supercuspidal representation.

Suppose now that G is cuspidal and let (π, V) be an irreducible admissible representation of G. Let (π_0, V) denote the restriction of π to G^0.

Lemma 2.15. *Assume that G is cuspidal. Then π is J-supercuspidal if and only if π_0 is J-supercuspidal if and only if any irreducible constituent of π_0 is J-supercuspidal.*

Proof: Since G is cuspidal, by Lemma 2.8 the set of nilradicals of proper parabolic subgroups is the same for both G and G^0. Thus π is J-supercuspidal if and only if π_0

is J-supercuspidal. Moreover, since by Lemma 2.12 the irreducible constituents of π_0 are all conjugate via elements of G, it is clear that π_0 is J-supercuspidal if and only if every irreducible constituent of π_0 is J-supercuspidal if and only if any irreducible constituent of π_0 is J-supercuspidal. ∎

Lemma 2.16. *Assume that G is cuspidal. Then π is supercuspidal if and only if π_0 is supercuspidal if and only if any irreducible constituent of π_0 is supercuspidal.*

Proof: Let Z denote the split component of G. By Lemma 2.8 it is also the split component of G^0.

Assume that π is supercuspidal. Thus $\mathcal{A}(\pi) \subset {}^0\mathcal{A}(G)$. Let f_0 be a matrix coefficient of π_0. Then there is a matrix coefficient f of π so that f_0 is the restriction of f to G^0. Now $f \in {}^0\mathcal{A}(G)$. Since f smooth and compactly supported modulo Z, so is f_0. Further, by Lemma 2.8 the nilradicals of proper parabolic subgroups are the same for both G and G^0. Thus f_0 will satisfy the integral condition necessary to be a cusp form on G^0. Hence $\mathcal{A}(\pi_0) \subset {}^0\mathcal{A}(G^0)$ so that π_0 is supercuspidal.

Conversely, suppose that π_0 is supercuspidal. Let π_1 be an irreducible constituent of π_0. Then $\pi \subset \mathrm{Ind}_{G^0}^G(\pi_1)$ so that every matrix coefficient of π is a matrix coefficient of the induced representation. But since G^0 is a normal subgroup of finite index in G, the restriction of $\mathrm{Ind}_{G^0}^G(\pi_1)$ to G^0 is equivalent to $\sum_{x\in G/G^0} \pi_1^x$. Thus matrix coefficients of the induced representation can be described as follows. Let f be a matrix coefficient of $\mathrm{Ind}_{G^0}^G(\pi_1)$ and fix $g \in G$. Then there are matrix coefficients f_x of $\pi_1^x, x \in G/G^0$, (depending on both f and g) so that for all $g_0 \in G^0$,

$$f(gg_0) = \sum_{x\in G/G^0} f_x(g_0).$$

Since π_1 is supercuspidal, so is π_1^x for any $x \in G/G^0$, and so each $f_x \in {}^0\mathcal{A}(G^0)$. Thus the restriction of f to each connected component of G is smooth and compactly supported modulo Z. Also, if N is the nilradical of any proper parabolic subgroup of G, then

$$\int_N f(gn)dn = \sum_{x\in G/G^0} \int_N f_x(n)dn = 0$$

since by Lemma 2.8, N is also the nilradical of a proper parabolic subgroup of G^0. ∎

Proposition 2.17. *Assume that G is cuspidal and let π be an irreducible admissible representation of G. Then π is supercuspidal if and only if π is J-supercuspidal.*

Proof: This is an immediate consequence of Lemmas 2.15 and 2.16 and the corresponding result in the connected case. ∎

We now drop the assumption that G is cuspidal. Let $P = MN$ be a parabolic subgroup of G and let σ be an admissible representation of M. Then we let $\mathrm{Ind}_P^G(\sigma)$ denote the representation of G by left translations on

$$\mathcal{H} = \{f \in C^\infty(G,V) : f(gmn) = \delta_P^{-\frac{1}{2}}(m)\sigma(m)^{-1}f(g), \forall g \in G, m \in M, n \in N\}.$$

Here δ_P denotes the modular function of P.

Theorem 2.18. *Let π be an irreducible admissible representation of G. Then there are a cuspidal parabolic subgroup $P = MN$ of G and an irreducible supercuspidal representation σ of M such that π is a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$.*

Remark 2.19. We will see in Corollary 3.2 that the group M and supercuspidal representation σ in Theorem 2.18 are unique up to conjugacy.

Proof: Let ρ be an irreducible constituent of the restriction of π to G^0. Then $\pi \subset \mathrm{Ind}_{G^0}^G(\rho)$. Since ρ is admissible, there are a parabolic subgroup $P^0 = M^0N$ of G^0 and an irreducible supercuspidal representation τ of M^0 such that $\rho \subset \mathrm{Ind}_{P^0}^{G^0}(\tau)$. Thus

$$\pi \subset \mathrm{Ind}_{G^0}^G(\rho) \subset \mathrm{Ind}_{G^0}^G(\mathrm{Ind}_{P^0}^{G^0}(\tau)) \simeq \mathrm{Ind}_{P^0}^G(\tau).$$

Let $P = MN$ be the unique cuspidal parabolic subgroup of G lying over P^0. Let σ be an irreducible admissible representation of M such that τ is contained in the restriction of σ to M^0. By Lemma 2.16, σ is supercuspidal. By Lemma 2.13 applied to M and M^0 we have

$$\mathrm{Ind}_{M^0}^M(\tau) \simeq s \sum_{\eta \in Y/Y(\sigma)} \sigma \otimes \eta$$

where Y is the group of unitary characters of M/M^0. Since τ is contained in the restriction of $\sigma \otimes \eta$ to M^0 for any η, all the representations $\sigma \otimes \eta$ are supercuspidal. Now

$$\pi \subset \mathrm{Ind}_{P^0}^G(\tau) \simeq s \sum_{\eta \in Y/Y(\sigma)} \mathrm{Ind}_P^G(\sigma \otimes \eta).$$

Thus $\pi \subset \mathrm{Ind}_P^G(\sigma \otimes \eta)$ for some η. ∎

Let $\mathcal{A}(G) = \cup_\pi \mathcal{A}(\pi)$ where π runs over the set of all admissible representations of G. Similarly we have $\mathcal{A}(G^0)$ and because of Lemma 2.11 it is clear that if $f \in \mathcal{A}(G)$, then $f|_{G^0} \in \mathcal{A}(G^0)$. Define the subspace $\mathcal{A}_T(G^0) \subset \mathcal{A}(G^0)$ as in [**13**,§4.5]. It is the set of functions in $\mathcal{A}(G^0)$ which satisfy the weak inequality. Define

$$\mathcal{A}_T(G) = \{f \in \mathcal{A}(G) : l(x)f|_{G^0} \in \mathcal{A}_T(G^0) \text{ for all } x \in G\}$$

where $l(x)f$ denotes the left translate of f by x. In other words, $\mathcal{A}_T(G)$ is the set of functions in $\mathcal{A}(G)$ which satisfy the weak inequality on every connected component of G. If π is an admissible representation of G, we say π is tempered if $\mathcal{A}(\pi) \subset \mathcal{A}_T(G)$. The following lemma is easy to prove using the properties of matrix coefficients of π and π_0 from the proof of Lemma 2.16.

Lemma 2.20. *Let π be an irreducible admissible representation of G. Then π is tempered if and only if π_0 is tempered if and only if any irreducible constituent of π_0 is tempered.*

Let π be an irreducible unitary representation of G and let Z be the split component of G. We say that π is discrete series if $\mathcal{A}(\pi) \subset L^2(G/Z)$. Every unitary supercuspidal representation is discrete series since its matrix coefficients are compactly supported modulo Z.

Lemma 2.21. *If G is not cuspidal, then G has no discrete series representations. If G is cuspidal, then π is discrete series if and only if π_0 is discrete series if and only if any irreducible constituent of π_0 is discrete series.*

Proof: Suppose that G is not cuspidal. Then the split component Z of G is a proper subgroup of the split component Z_0 of G^0. Let π be any irreducible unitary representation of G. Then there is an irreducible unitary representation π_1 of G^0 so that π is contained in $\mathrm{Ind}_{G^0}^G(\pi_1)$. Thus as in the proof of Lemma 2.16, for any matrix coefficient f of π and any $g \in G$, we have matrix coefficients f_x of $\pi_1^x, x \in G/G^0$, so that for all $g_0 \in G^0$,

$$f(gg_0) = \sum_{x \in G/G^0} f_x(g_0).$$

Let ω be the Z_0-character of π_1. Then for any $z \in Z_0, g_0 \in G^0$, we have

$$f(gg_0z) = \sum_{x \in G/G^0} f_x(g_0z) = \sum_{x \in G/G^0} \omega^x(z) f_x(g_0).$$

Thus $z \mapsto f(gg_0z), z \in Z_0$, is a finite linear combination of unitary characters of Z_0, and cannot be square-integrable on Z_0/Z unless it is zero. Now if f is square-integrable on G/Z, then $g_0 \mapsto f(gg_0)$ is square-integrable on G_0/Z for all coset representatives $g \in G/G^0$, and $z \mapsto f(gg_0z)$ must be square-integrable on Z_0/Z for almost all g_0, so that $f(gg_0z)$ must be zero for almost all g_0, z, and $f = 0$.

Suppose that G is cuspidal. Let π be a discrete series representation of G. Let f_0 be a matrix coefficient of π_0. Then there is a matrix coefficient f of π so that f_0 is the restriction of f to G^0. Since f is square-integrable on G/Z, certainly f_0 is square-integrable on G^0/Z.

Conversely, suppose that π_0 is discrete series. Let π_1 be an irreducible constituent of π_0. For any matrix coefficient f of π we have

$$\int_{G/Z} |f(g)|^2 d(gZ) = \sum_{g \in G/G^0} \int_{G^0/Z} |f(gg_0)|^2 d(g_0Z).$$

As above, for fixed $g \in G$, we have matrix coefficients f_x of $\pi_1^x, x \in G/G^0$, so that for all $g_0 \in G^0$,

$$f(gg_0) = \sum_{x \in G/G^0} f_x(g_0).$$

Thus

$$\left(\int_{G^0/Z} |f(gg_0)|^2 d(g_0Z)\right)^{\frac{1}{2}} \leq \sum_{x\in G/G^0} \left(\int_{G^0/Z} |f_x(g_0)|^2 d(g_0Z)\right)^{\frac{1}{2}} < \infty$$

since every f_x is square-integrable on G^0/Z. Thus f is square-integrable on G/Z. ∎

The following theorem can be proven in the same way as Theorem 2.18 using Lemmas 2.20 and 2.21.

Theorem 2.22. *Let π be an irreducible tempered representation of G. Then there are a cuspidal parabolic subgroup $P = MN$ of G and an irreducible discrete series representation σ of M such that π is a subrepresentation of* $\mathrm{Ind}_P^G(\sigma)$.

Remark 2.23. We will see in Corollary 3.2 that the group M and discrete series representation σ in Theorem 2.22 are unique up to conjugacy.

3. Intertwining Operators

Let G be a reductive F-group with G/G^0 finite and abelian as in §2. If π_1 and π_2 are admissible representations of G, we let $J(\pi_1, \pi_2)$ denote the dimension of the space of all intertwining operators from π_2 to π_1. We want to prove the following theorem.

Let $(P_i, A_i), i = 1, 2$, be cuspidal parabolic pairs of G with $P_i = M_iN_i$ the corresponding Levi decompositions. Let σ_i be an irreducible admissible representation of M_i on a vector space V_i, and let $\pi_i = \mathrm{Ind}_{P_i}^G(\sigma_i)$, where we use normalized induction as in §2. Let $W = W(A_1|A_2)$ denote the set of mappings $s : A_2 \to A_1$ such that there exists $y_s \in G$ such that $s(a_2) = y_s a_2 y_s^{-1}$ for all $a_2 \in A_2$.

Theorem 3.1. *Assume that σ_1 and σ_2 are either both supercuspidal representations, or both discrete series representations. If A_1 and A_2 are not conjugate, then $J(\pi_1, \pi_2) = 0$. Assume A_1 and A_2 are conjugate. Then*

$$J(\pi_1, \pi_2) \leq \sum_{s\in W} J(\sigma_1, \sigma_2^{y_s}).$$

Corollary 3.2. *Let σ_1 and σ_2 be as above. Then $J(\pi_1, \pi_2) = 0$ unless there is $y \in G$ such that $M_1 = M_2^y, \sigma_1 \simeq \sigma_2^y$.*

Corollary 3.3. *Let σ_1 and σ_2 be irreducible discrete series representations, and suppose that π_1 and π_2 have an irreducible constituent in common. Then $\pi_1 \simeq \pi_2$.*

Proof: In this case $J(\pi_1, \pi_2) \neq 0$ so by Corollary 3.2 there is $y \in G$ such that $M_1 = M_2^y, \sigma_1 \simeq \sigma_2^y$. Thus

$$\pi_1 = \mathrm{Ind}_{M_1N_1}^G(\sigma_1) = \mathrm{Ind}_{M_2^yN_1}^G(\sigma_2^y) \simeq \mathrm{Ind}_{M_2N_2'}^G(\sigma_2)$$

where $N_2' = N_1^{y^{-1}}$. Now $P_2 = M_2N_2$ and $P_2' = M_2N_2'$ are two cuspidal parabolic subgroups of G with the same Levi component M_2. We will see in Corollary 5.9 that there is an equivalence $R(P_2' : P_2 : \sigma_2)$ between $\pi_2 = \mathrm{Ind}_{P_2}^G(\sigma_2)$ and $\pi_2' = \mathrm{Ind}_{P_2'}^G(\sigma_2)$. Thus

$$\pi_1 \simeq \pi_2' \simeq \pi_2.$$

∎

Let $W(\sigma_1) = \{s \in W(A_1, A_1) : \sigma_1^s \simeq \sigma_1\}$. We say σ_1 is unramified if $W(\sigma_1) = \{1\}$.

Corollary 3.4. *Assume that σ_1 is discrete series. Then $J(\pi_1, \pi_1) \leq [W(\sigma_1)]$. In particular, if σ_1 is unramified, then π_1 is irreducible.*

In order to prove Theorem 3.1 for discrete series representations we will need the following results about dual exponents. Let (π, V) be an irreducible representation of G and let V' denote the algebraic dual of V. For $x \in G$, let $\pi(x)^t : V' \to V'$ denote the transpose of $\pi(x)$. Let (P, A) be a cuspidal parabolic pair in G. Then a quasi-character χ of A is called a dual exponent of π with respect to (P, A) if there is a nonzero $\phi \in V'$ such that for all $\overline{n} \in \overline{N}, a \in A$,

$$\pi(\overline{n})^t\phi = \phi \quad \text{and} \quad \pi(a)^t\phi = \delta_P^{\frac{1}{2}}(a)\chi(a)\phi. \qquad (*)$$

We will write $Y_\pi(P, A)$ for the set of all dual exponents of π with respect to (P, A).

Let $V = \sum_{i=1}^k V_i$ be the decomposition of V into G^0 invariant subspaces and let π_i be the irreducible representation of G^0 on $V_i, 1 \leq i \leq k$.

Lemma 3.5. *Let (P, A) be a cuspidal parabolic pair in G. Then*

$$Y_\pi(P, A) = \cup_{i=1}^k Y_{\pi_i}(P^0, A).$$

Proof: Let $\chi \in Y_\pi(P, A)$ and let $\phi \in V'$ be a nonzero functional satisfying (*). Then there is $1 \leq i \leq k$ such that ϕ_i, the restriction of ϕ to V_i, is nonzero. Now for any $v_i \in V_i$ and $g_0 \in G^0$ we have

$$< \pi_i(g_0)^t\phi_i, v_i >=< \pi(g_0)^t\phi, v_i > .$$

Now since $\overline{N}$ and A are both contained in G^0 it is easy to check that

$$< \pi_i(\overline{n})^t\phi_i, v_i >=< \phi_i, v_i > \quad \text{and} \quad < \pi_i(a)^t\phi_i, v_i >= \delta_{P^0}^{\frac{1}{2}}(a)\chi(a) < \phi_i, v_i > \qquad (**)$$

for all $v_i \in V_i, \overline{n} \in \overline{N}, a \in A$. Thus $\chi \in Y_{\pi_i}(P^0, A)$.

Now assume that $\chi \in Y_{\pi_i}(P^0, A)$ for some $1 \leq i \leq k$. and let $\phi_i \in V_i'$ be a nonzero functional satisfying (**). Now π is contained in the induced representation $\mathrm{Ind}_{G^0}^G(\pi_i)$ so we can realize π on a subspace V of

$$\mathcal{H} = \{f : G \to V_i : f(gg_0) = \pi_i(g_0)^{-1}f(g), g \in G, g_0 \in G^0\}$$

with the action of π given by left translation on the functions. Now define $\phi \in V'$ by

$$< \phi, f >=< \phi_i, f(1) >, f \in V.$$

Then $\phi \neq 0$ and it is easy to check that for all $g_0 \in G^0, f \in V$, we have

$$< \pi(g_0)^t\phi, f > = < \pi_i(g_0)^t\phi_i, f(1) > .$$

From this it easily follows that since ϕ_i satisfies (**), ϕ satisfies (*). Thus $\chi \in Y_\pi(P, A)$. ∎

Lemma 3.6. *Assume that G is cuspidal and let π be a discrete series representation of G. Then $Y_\pi(P, A) \cap \hat{A} = \emptyset$ for every cuspidal parabolic pair $(P, A) \neq (G, Z)$.*

Proof: This follows easily from Lemma 2.21, Lemma 3.5, and the corresponding result for connected groups. ∎

Let (P_0^0, A_0) be a minimal p-pair in G^0 and let P_0 be the cuspidal parabolic subgroup lying over P_0^0. We will call (P_0, A_0) a minimal parabolic pair in G.

Lemma 3.7. *Let (P, A) be any parabolic pair in G. Then there is $x \in G^0$ such that $P_0 \subset xPx^{-1}, xAx^{-1} \subset A_0$.*

Proof: Let $M = C_G(A)$ and let $P^0 = P \cap G^0, M^0 = M \cap G^0$. Let A' be the split component of M^0. Thus $A \subset A'$. Now (P^0, A') is a p-pair in G^0 so there is $x \in G^0$ such that $P_0^0 \subset xP^0x^{-1}$ and $xAx^{-1} \subset xA'x^{-1} \subset A_0$. Now $M_0 = C_G(A_0) \subset C_G(xAx^{-1}) = xMx^{-1}$ so that $P_0 = M_0P_0^0 \subset (xMx^{-1})(xP^0x^{-1}) = xPx^{-1}$. ∎

We will say a parabolic pair (P, A) is standard with respect to the minimal parabolic pair (P_0, A_0) if $P_0 \subset P$ and $A \subset A_0$. We say (P, A) is semi-standard with respect to (P_0, A_0) if $A \subset A_0$.

Fix a minimal parabolic pair (P_0, A_0) of G. Let $N_G(P_0, A_0)$ denote the set of all elements of G which normalize both P_0 and A_0. Write $W_G(P_0, A_0) = N_G(P_0, A_0)/M_0^0$. If (P, A) is any parabolic pair of G which is standard with respect to (P_0, A_0) and $M = C_G(A)$, write $N_M(P_0, A_0) = M \cap N_G(P_0, A_0)$ and $W_M(P_0, A_0) = N_M(P_0, A_0)/M_0^0$.

Lemma 3.8. *We can write P as a disjoint union*

$$P = \cup_{w \in W_M(P_0, A_0)} wP^0.$$

Proof: We first prove the result when $P = G$. Let $y \in G$. Then $(yP_0^0y^{-1}, yA_0y^{-1})$ is a minimal parabolic pair in G^0 and hence is conjugate to (P_0^0, A_0) via an element of G^0. Thus there is $g \in G^0$ such that gy normalizes both A_0 and P_0^0. But then gy also normalizes $M_0 = C_G(A_0)$ and $P_0 = M_0P_0^0$. Thus $gy \in N_G(P_0, A_0)$.

Thus the coset yG_0 has a representative $n \in N_G(P_0, A_0)$ which depends only on the coset w of n in $W_G(P_0, A_0)$. Since $N_G(P_0, A_0) \cap G^0 = M_0^0$, $n_1, n_2 \in N_G(P_0, A_0)$ determine the same coset of G^0 in G just in case they are in the same coset in $W_G(P_0, A_0)$. Thus we have the disjoint union

$$G = \cup_{w \in W_G(P_0, A_0)} wG^0.$$

Now let (P, A) be arbitrary. There is a subset θ of the simple roots Δ of (P_0^0, A_0) corresponding to P^0 so that $P^0 = P_\theta^0$. Note that $W_G(P_0, A_0)$ acts on Δ and that $w^{-1}P_\theta^0 w = P_{w\theta}^0$. Let $w \in W_G(P_0, A_0)$ and suppose that $P \cap wG^0 \neq \emptyset$. Let $wg_0 \in P \cap wG^0, g_0 \in G^0$. Then wg_0 normalizes P^0 so that $w^{-1}P^0w = g_0P^0g_0^{-1}$ is conjugate to P^0 in G^0. But $w^{-1}P^0w = P_{w\theta}^0$ is a standard parabolic subgroup of G^0 and so $P_{w\theta}^0 = g_0P^0g_0^{-1} = P^0$. Hence $g_0 \in N_{G^0}(P^0) = P^0, w\theta = \theta$, and w has a representative in P. Hence w normalizes $A = \{a \in A_0 : \alpha(a) = 1 \text{ for all } \alpha \in \theta\}$. But $N_G(A) \cap P = M$. Thus $w \in W_M(P_0, A_0)$. ∎

In order to prove Theorem 3.1 we extend the proof of Silberger in [**13**, §2.5] to the disconnected case. Fortunately many of the technical results on intertwining forms needed are proven in [**13**, §1] for any totally disconnected group and so can be directly used in our case. We follow Silberger's notation. Let (P_1, A_1) and (P_2, A_2) be cuspidal parabolic pairs in G. We may as well assume that they are standard with respect to a fixed minimal parabolic pair (P_0, A_0).

We need to study the orbits for the action of $P_1 \times P_2$ on G given by $y \cdot (p_1, p_2) = p_1^{-1}yp_2$. Recall that $G^0 = \cup_v P_0^0vP_0^0$ where v runs over $W(G^0, A_0) = N_{G^0}(A_0)/M_0^0$. Thus, $G = \cup_w wG^0 = \cup_{w,v} wP_0^0vP_0^0 = \cup_{w,v} P_0^0wvP_0^0$ where $w \in W_G(P_0, A_0), v \in W(G^0, A_0)$. Now since $P_0^0 \subset P_i, i = 1, 2$, each double coset P_1yP_2 can be represented by $y = wv, w \in W_G(P_0, A_0), v \in W(G^0, A_0)$. Write $W_i = W_{M_i}(P_0, A_0), i = 1, 2$.

Lemma 3.9. *Let $\mathcal{O} = P_1w_0vP_2$ be an orbit of $P_1 \times P_2$ in G where $w_0 \in W_G(P_0, A_0)$ and $v \in W(G^0, A_0)$. Then for $w \in W_G(P_0, A_0)$, $\mathcal{O} \cap wG^0$ is empty unless $w \in W_1w_0W_2$. If $w \in W_1w_0W_2$, then $\mathcal{O} \cap wG^0 = w\mathcal{O}_w$ where $\mathcal{O}_w$ is a finite union of orbits of $(w^{-1}P_1^0w) \times P_2^0$ in G^0, all of which have fixed dimension $d_{\mathcal{O}}$ equal to the dimension of the orbit $\mathcal{O}_0 = (w_0^{-1}P_1^0w_0)vP_2^0$.*

Proof: Using Lemma 3.8 we can write

$$P_1w_0vP_2 = \cup_{w_1,w_2} w_1P_1^0w_0vw_2P_2^0$$

where $w_i \in W_i = W_{M_i}(P_0, A_0), i = 1, 2$. But

$$w_1P_1^0w_0vw_2P_2^0 = w(w^{-1}P_1^0w)w_2^{-1}vw_2P_2^0$$

where $w = w_1w_0w_2 \in W_G(P_0, A_0)$. Thus $w_1P_1^0w_0vw_2P_2^0 \subset w_1w_0w_2G^0$ and for fixed $w \in W_1w_0W_2$,

$$P_1w_0vP_2 \cap wG^0 = w \cup_{w_2} (w^{-1}P_1^0w)w_2^{-1}vw_2P_2^0$$

where w_2 runs over elements of W_2 such that $ww_2^{-1}w_0^{-1} \in W_1$. Finally,

$$(w^{-1}P_1^0w)w_2^{-1}vw_2P_2^0 = w_2^{-1}(w_0^{-1}P_1^0w_0)vP_2^0w_2$$

so that

$$\dim(w^{-1}P_1^0w)w_2^{-1}vw_2P_2^0 = \dim(w_0^{-1}P_1^0w_0)vP_2^0 = d_{\mathcal{O}}.$$

∎

Because of Lemma 3.9 each orbit $\mathcal{O}$ has a well-defined dimension $d_{\mathcal{O}}$. For each integer $\nu \geq 0$, let $\mathcal{O}(\nu)$ denote the union of all orbits of dimension less than or equal to ν. Set $\mathcal{O}(-1) = \emptyset$.

Lemma 3.10. *For each $\nu \geq 0$, $\mathcal{O}(\nu)$ is a closed set in the p-adic topology. Further, if $\mathcal{O}$ is an orbit of dimension d, then $\mathcal{O} \cup \mathcal{O}(d-1)$ is also closed in the p-adic topology.*

Proof: Using Lemma 3.9 we see that $\mathcal{O}(\nu) \cap wG^0 = w\mathcal{O}_w(\nu)$ where $\mathcal{O}_w(\nu)$ is the union of all orbits of $(w^{-1}P_1^0 w) \times P_2^0$ in G^0 having dimension less than or equal to ν. This set is closed in the p-adic topology by [**13**, pg. 93]. Thus $\mathcal{O}(\nu)$ is a finite union of closed sets, hence is closed. Similarly, if $\mathcal{O}$ has dimension d, then $[\mathcal{O} \cup \mathcal{O}(d-1)] \cap wG^0 = w(\mathcal{O}_w \cup \mathcal{O}_w(d-1))$ where $\mathcal{O}_w \cup \mathcal{O}_w(d-1)$ is a finite union of sets of the form $\mathcal{O}' \cup \mathcal{O}_w(d-1)$ where $\mathcal{O}'$ is an orbit of $(w^{-1}P_1^0 w) \times P_2^0$ in G^0 of dimension d. These are also closed by [**13**, page 94]. ∎

Let $E = V_1 \otimes V_2$ where V_i is the space on which σ_i acts, $i = 1, 2$. Let $\mathcal{T}$ be the space of all E-distributions T^0 on G such that

$$T^0(\lambda(p_1)\rho(p_2)\alpha) = T^0(\delta_1(p_1)^{\frac{1}{2}}\delta_2(p_2)^{\frac{1}{2}}\sigma_1(p_1)^{-1} \otimes \sigma_2(p_2)^{-1}\alpha)$$

for all $(p_1, p_2) \in P_1 \times P_2, \alpha \in C_c^\infty(G : E)$. Here λ and ρ denote left and right translations respectively and δ_i is the modular function of $P_i, i = 1, 2$ The first step in the proof of Theorem 3.1 is the inequality [**13**, 1.9.4]

$$I(\pi_1, \pi_2) \leq \dim \mathcal{T}. \tag{3.1}$$

Here $I(\pi_1, \pi_2)$ is the dimension of the space of "intertwining forms" defined in [**13**, §1.6]. It is related to the dimension of the space of intertwining operators by $I(\pi_1, \pi_2) = J(\tilde{\pi}_1, \pi_2)$ where $\tilde{\pi}_1$ is the contragredient of π_1. [**13**, 1.6.2]

If $\mathcal{O}$ is an orbit of dimension d, write $\mathcal{T}(\mathcal{O})$ for the vector space of $T^0 \in \mathcal{T}$ with support in $\mathcal{O} \cup \mathcal{O}(d-1)$ and $\mathcal{T}_\nu$ for the space of those with support in $\mathcal{O}(\nu)$. We have

$$\mathcal{T} = \sum_{\mathcal{O}} \mathcal{T}(\mathcal{O})$$

and

$$\dim \mathcal{T} \leq \sum_{\mathcal{O}} \dim(\mathcal{T}(\mathcal{O})/\mathcal{T}_{d(\mathcal{O})-1}). \tag{3.2}$$

Lemma 3.11. *Suppose that (P_1, A_1) and (P_2, A_2) are semi-standard cuspidal parabolic pairs in G. Then $M_1 \cap P_2 = (M_1 \cap M_2)(M_1 \cap N_2) = {}^*P_1$ is a cuspidal parabolic subgroup of M_1 with split component ${}^*A_1 = A_1 A_2$.*

Proof: We know from the connected case [**13**, p 94] that ${}^*P_1^0 = M_1^0 \cap P_2^0$ is a parabolic subgroup of M_1^0 with split component ${}^*A_1 = A_1A_2$ and Levi decomposition ${}^*P_1^0 = (M_1^0 \cap M_2^0)(M_1^0 \cap N_2)$. Thus there is a cuspidal parabolic subgroup ${}^*P_1$ of M_1 with split component ${}^*A_1$ and Levi decomposition ${}^*P_1 = {}^*M_1\, {}^*N_1$. Here ${}^*M_1 = C_{M_1}({}^*A_1) = C_{M_1}(A_1A_2) = M_1 \cap C_G(A_2) = M_1 \cap M_2$ and ${}^*N_1 = M_1^0 \cap N_2 = M_1 \cap N_2$.

Clearly ${}^*P_1 = (M_1 \cap M_2)(M_1 \cap N_2) \subset M_1 \cap P_2$. Thus we need only show that $M_1 \cap P_2 \subset {}^*P_1$.

Let $x \in M_1 \cap P_2$. Using the Levi decomposition of P_2 we can write $x = m_2 n_2$ where $m_2 \in M_2, n_2 \in N_2$. Since $m_2 n_2 \in M_1 = C_G(A_1)$ we have $m_2 n_2 a_1 n_2^{-1} m_2^{-1} = a_1$ for any $a_1 \in A_1$. This implies that $a_1^{-1} n_2 a_1 n_2^{-1} = a_1^{-1} m_2^{-1} a_1 m_2$. But since (P_1, A_1) and (P_2, A_2) are semi-standard, we have $A_1 \subset A_0 \subset C_G(A_2) = M_2$. Thus $a_1^{-1} n_2 a_1 n_2^{-1} \in N_2$ and $a_1^{-1} m_2^{-1} a_1 m_2 \in M_2$. Hence $a_1^{-1} n_2 a_1 n_2^{-1} = a_1^{-1} m_2^{-1} a_1 m_2 \in N_2 \cap M_2 = \{1\}$. Thus n_2 and m_2 both commute with a_1 so that $n_2 \in M_1 \cap N_2$ and $m_2 \in M_1 \cap M_2$. ∎

Return to the assumption that (P_1, A_1) and (P_2, A_2) are standard with respect to (P_0, A_0) and that $y \in W(A_0)$ so that (P_2^y, A_2^y) is semi-standard. Using Lemma 3.11 we know that ${}^*P_1 = M_1 \cap P_2^y$ and ${}^*P_2 = M_2 \cap P_1^{y^{-1}}$ are cuspidal parabolic subgroups of M_1 and M_2 respectively.

Let $E'(y)$ denote the space of linear functionals ϕ on $E = V_1 \otimes V_2$ such that

$$\delta_1(p)^{\frac{1}{2}} \delta_2(p^{y^{-1}})^{\frac{1}{2}} < \phi, \sigma_1(p) v_1 \otimes \sigma_2(p^{y^{-1}}) v_2 >= \delta_{P_1 \cap P_2^y}(p) < \phi, v_1 \otimes v_2 >$$

for all $p \in P_1 \cap P_2^y, v_i \in V_i, i = 1, 2$.

Lemma 3.12. *Let* $m \in {}^*M_1 = M_1 \cap M_2^y$. *Then*

$$\delta_{P_1 \cap P_2^y}(m) = \delta_{*1}(m)^{\frac{1}{2}} \delta_{*2}(m^{y^{-1}})^{\frac{1}{2}} \delta_1(m)^{\frac{1}{2}} \delta_2(m^{y^{-1}})^{\frac{1}{2}}$$

where δ_{*i} *denotes the modular function for* ${}^*P_i$ *and* δ_i *the modular function for* $P_i, i = 1, 2$.

Proof: Define the homomorphism $\delta : {}^*M_1 \to \mathbf{R}_+^*$ by

$$\delta(m) = \delta_{*1}(m)^{\frac{1}{2}} \delta_{*2}(m^{y^{-1}})^{\frac{1}{2}} \delta_1(m)^{\frac{1}{2}} \delta_2(m^{y^{-1}})^{\frac{1}{2}} \delta_{P_1 \cap P_2^y}(m)^{-1}, m \in {}^*M_1.$$

By [**13**, §2.5.2], the restriction of δ to ${}^*M_1 \cap G^0$ is trivial. Now since the quotient of ${}^*M_1$ by ${}^*M_1 \cap G^0$ is a finite group, $\delta = 1$ on all of ${}^*M_1$. ∎

The proof of the following lemma is the same as that of Lemma 2.5.1 in [**13**].

Lemma 3.13. *If* $\phi \in E'(y)$, *then* ϕ *vanishes on* $V_1({}^*P_1) \otimes V_2 + V_1 \otimes V_2({}^*P_2)$. *Let* $m \in {}^*M_1, v_1 \otimes v_2 \in E$, *and* $\phi \in E'(y)$. *Then*

$$< \phi, \sigma_1(m) v_1 \otimes \sigma_2(m^{y^{-1}}) v_2 >= \delta_{*1}(m)^{\frac{1}{2}} \delta_{*2}(m^{y^{-1}})^{\frac{1}{2}} < \phi, v_1 \otimes v_2 > .$$

Corollary 3.14. *If* $A_1 = A_2^y$, *then*

$$< \phi, \sigma_1(m) v_1 \otimes \sigma_2(m^{y^{-1}}) v_2 >=< \phi, v_1 \otimes v_2 >$$

for all $m \in {}^*M_1, v_1 \otimes v_2 \in E$, *and* $\phi \in E'(y)$.

Proof: As in [**13**, §2.5.4], $A_1 = A_2^y$ implies that ${}^*N_1 = {}^*N_2 = \{1\}$ so that ${}^*P_1$ and ${}^*P_2$ are reductive and $\delta_{*1} = \delta_{*2} = 1$. ∎

Corollary 3.15. *Assume that σ_1 and σ_2 are either both supercuspidal or both discrete series. Then $E'(y) \neq \{0\}$ implies that $A_1 = A_2^y$.*

Proof: This follows as in [**13**, §§2.5.3, 2.5.5] using Proposition 2.17 and Lemma 3.6. ∎

The following Lemma is now proven as in [**13**, §2.5.7].

Lemma 3.16. *Assume that σ_1 and σ_2 are either both supercuspidal or both discrete series. Let $\mathcal{O}$ be an orbit in G of dimension d. Then $\dim \mathcal{T}(\mathcal{O})/\mathcal{T}_{d-1} = 0$ unless there exists $y \in \mathcal{O}$ such that $A_1 = A_2^y$. If $A_1 = A_2^y$ for some $y \in \mathcal{O}$, then*

$$\dim(\mathcal{T}(\mathcal{O})/\mathcal{T}_{d-1}) \leq I(\sigma_1, \sigma_2^y).$$

Proof of Theorem 3.1. Combine equations (3.1) and (3.2) to obtain

$$I(\pi_1, \pi_2) \leq \dim \mathcal{T} \leq \sum_{\mathcal{O}} \dim(\mathcal{T}(\mathcal{O})/\mathcal{T}_{d(\mathcal{O})-1}).$$

Now using Lemma 3.16, if A_1 and A_2 are not conjugate, $\dim(\mathcal{T}(\mathcal{O})/\mathcal{T}_{d(\mathcal{O})-1}) = 0$ for every orbit $\mathcal{O}$ so that $I(\pi_1, \pi_2) = 0$. If A_1 and A_2 are conjugate we have, again using Lemma 3.16,

$$I(\pi_1, \pi_2) \leq \dim \mathcal{T} \leq \sum_{s \in W} \dim(\mathcal{T}(\mathcal{O}_s)/\mathcal{T}_{d(\mathcal{O}_s)-1}) \leq \sum_{s \in W} I(\sigma_1, \sigma_2^{y_s}).$$

To complete the proof note that $J(\pi_1, \pi_2) = I(\tilde{\pi}_1, \pi_2)$, $J(\sigma_1, \sigma_2^{y_s}) = I(\tilde{\sigma}_1, \sigma_2^{y_s})$ and $\tilde{\pi}_1 = \mathrm{Ind}_{P_1}^G(\tilde{\sigma}_1)$. Thus the statement with dimensions of spaces of intertwining operators rather than intertwining forms follows by substituting $\tilde{\pi_1}$ for π_1. ∎

4. R-groups for $\mathrm{Ind}_{P^0}^G(\sigma_0)$

In this section we will study representations of G which are induced from a parabolic subgroup $P^0 = M^0 N$ of G^0. Because in this section we will only work with parabolic subgroups of G^0, we will simplify notation by dropping the superscripts on P^0 and M^0.

Let $P = MN$ be a parabolic subgroup of G^0 and fix an irreducible discrete series representation (σ, V) of M. Define

$$\mathcal{H}_P(\sigma) = \{f \in C^\infty(G, V) : f(xmn) = \delta_P^{-\frac{1}{2}}(m)\sigma(m)^{-1} f(x) \text{ for all } x \in G, m \in M, n \in N\}$$

where δ_P is the modular function on P. Then G acts by left translation on $\mathcal{H}_P(\sigma)$ and we call this induced representation $I_P(\sigma)$. We will also need to consider $\mathcal{H}_P^0(\sigma) =$

$$\{\phi \in C^\infty(G^0, V) : \phi(xmn) = \delta_P^{-\frac{1}{2}}(m)\sigma(m)^{-1}\phi(x) \text{ for all } x \in G^0, m \in M, n \in N\}.$$

G^0 acts by left translation on $\mathcal{H}_P^0(\sigma)$ and we call this induced representation $I_P^0(\sigma)$.

It is well known that the equivalence class of $I_P^0(\sigma) = \mathrm{Ind}_P^{G^0}(\sigma)$ is independent of P. But
$$I_P(\sigma) = \mathrm{Ind}_P^G(\sigma) \simeq \mathrm{Ind}_{G^0}^G \mathrm{Ind}_P^{G^0}(\sigma),$$
so that the equivalence class of $I_P(\sigma)$ is also independent of P. We denote the equivalence classes of $I_P(\sigma)$ and $I_P^0(\sigma)$ by $i_{G,M}(\sigma)$ and $i_{G^0,M}(\sigma)$ respectively. If π is a representation of G^0 we will also write $i_{G,G^0}(\pi)$ for the equivalence class of the induced representation $\mathrm{Ind}_{G^0}^G(\pi)$.

We first want to compare the dimensions of the intertwining algebras of $i_{G,M}(\sigma)$ and $i_{G^0,M}(\sigma)$. For this we need the results of Gelbart and Knapp summarized in §2 and the following facts.

Lemma 4.1. *Suppose π_1 and π_2 are irreducible representations of G^0. Then $i_{G,G^0}(\pi_1)$ and $i_{G,G^0}(\pi_2)$ have an irreducible constituent in common if and only if $\pi_2 \simeq \pi_1^g$ for some $g \in G$. In this case they are equivalent.*

Proof: For $i = 1, 2$, write
$$i_{G,G^0}(\pi_i) = r_i \sum_{\chi \in X/X(\Pi_i)} \Pi_i \otimes \chi$$
as in Lemma 2.13. Suppose that $i_{G,G^0}(\pi_1)$ and $i_{G,G^0}(\pi_2)$ have an irreducible constituent in common. Then $\Pi_1 \otimes \chi_1 \simeq \Pi_2 \otimes \chi_2$ for some $\chi_1, \chi_2 \in X$. Now
$$\pi_2 \subset \Pi_2 \otimes \chi_2|_{G^0} \simeq \Pi_1 \otimes \chi_1|_{G^0} \subset i_{G,G^0}(\pi_1)|_{G^0} \simeq \sum_{g \in G/G^0} \pi_1^g.$$
Thus $\pi_2 \simeq \pi_1^g$ for some $g \in G$. Conversely, if $\pi_2 \simeq \pi_1^g$ for some $g \in G$, then clearly
$$i_{G,G^0}(\pi_2) \simeq i_{G,G^0}(\pi_1^g) \simeq i_{G,G^0}(\pi_1).$$
∎

Lemma 4.2. *Suppose that for some $g \in G$, both π and π^g are irreducible constituents of $i_{G^0,M}(\sigma)$. Then there is $x_0 \in G^0$ such that $gx_0 \in N_G(\sigma) = \{x \in N_G(M) : \sigma^x \simeq \sigma\}$. Conversely, if π is an irreducible constituent of $i_{G^0,M}(\sigma)$ and if $g \in N_G(\sigma)G^0$, then π^g is also an irreducible constituent of $i_{G^0,M}(\sigma)$ and π and π^g occur with the same multiplicities.*

Proof: Suppose that $\pi, \pi^g \subset i_{G^0,M}(\sigma)$. Then since $\pi^g \subset i_{G^0,M^g}(\sigma^g)$, we see that $i_{G^0,M}(\sigma)$ and $i_{G^0,M^g}(\sigma^g)$ have an irreducible constituent in common. Thus there is $x_0 \in G^0$ such that $M^{gx_0} = M$ and $\sigma^{gx_0} \simeq \sigma$, i.e. $gx_0 \in N_G(\sigma)$. Conversely, if $g \in N_G(\sigma)G^0$, then the multiplicity of π in $i_{G^0,M}(\sigma)$ is equal to the multiplicity of π^g in $i_{G^0,M}(\sigma)^g \simeq i_{G^0,M^g}(\sigma^g) \simeq i_{G^0,M}(\sigma)$. ∎

Lemma 4.3. *Let π be an irreducible constituent of $i_{G^0,M}(\sigma)$ and let $G_\pi = \{x \in G : \pi^x \simeq \pi\}$. Then*

$$[G_\pi/G^0] = [N_{G_\pi}(\sigma)/N_{G^0}(\sigma)]$$

where $N_{G^0}(\sigma) = N_G(\sigma) \cap G^0$ and $N_{G_\pi}(\sigma) = N_G(\sigma) \cap G_\pi$.

Proof: Consider the mapping from $N_{G_\pi}(\sigma)$ to G_π/G^0 given by $g \mapsto gG^0$. Its kernel is $N_{G_\pi}(\sigma) \cap G^0 = N_{G^0}(\sigma)$. Further, given $g \in G_\pi$, $\pi^g \simeq \pi$ occurs in $i_{G^0,M}(\sigma)$, so by Lemma 4.1 there is $x_0 \in G_0$ such that $gx_0 \in N_G(\sigma)$. But $\pi^{gx_0} \simeq \pi^g \simeq \pi$ so that $gx_0 \in G_\pi \cap N_G(\sigma) = N_{G_\pi}(\sigma)$ and $gx_0G^0 = gG^0$. Thus the mapping is surjective. ∎

Let $C(\sigma)$ denote the algebra of G-intertwining operators for $i_{G,M}(\sigma)$ and let $C^0(\sigma)$ denote the algebra of G^0-intertwining operators for $i_{G^0,M}(\sigma)$.

Lemma 4.4. $\dim C(\sigma) = \dim C^0(\sigma)[N_G(\sigma)/N_{G^0}(\sigma)]$.

Proof: Let

$$i_{G^0,M}(\sigma) = \sum_{\pi \in S(\sigma)} m_\pi \pi$$

be the decomposition of $i_{G^0,M}(\sigma)$ into irreducible constituents. For $\pi_1, \pi_2 \in S(\sigma)$, we will say that $\pi_1 \sim \pi_2$ if there is $g \in G$ such that $\pi_2 \simeq \pi_1^g$. Then using Lemma 4.2, we have $\pi_1 \sim \pi_2$ if and only if $\pi_2 \simeq \pi_1^g$ for some $g \in N_G(\sigma)$. We can write

$$i_{G^0,M}(\sigma) = \sum_{\pi \in S(\sigma)/\sim} m_\pi \sum_{g \in N_G(\sigma)/N_{G_\pi}(\sigma)} \pi^g.$$

Now

$$\begin{aligned} i_{G,M}(\sigma) \simeq i_{G,G^0}(i_{G^0,M}(\sigma)) &\simeq \sum_{\pi \in S(\sigma)/\sim} m_\pi \sum_{g \in N_G(\sigma)/N_{G_\pi}(\sigma)} i_{G,G^0}(\pi^g) \\ &= \sum_{\pi \in S(\sigma)/\sim} m_\pi [N_G(\sigma)/N_{G_\pi}(\sigma)] r_\pi \sum_{\chi \in X/X(\Pi_\pi)} \Pi_\pi \otimes \chi \end{aligned}$$

where Π_π is an irreducible representation of G such that $\pi \subset \Pi_\pi|_{G^0}$. Because of Lemma 4.1, the representations $\Pi_\pi \otimes \chi$ are pairwise inequivalent as π ranges over $S(\sigma)/\sim$ and χ ranges over $X/X(\Pi_\pi)$. Thus

$$\dim C(\sigma) = \sum_{\pi \in S(\sigma)/\sim} m_\pi^2 [N_G(\sigma)/N_{G_\pi}(\sigma)]^2 r_\pi^2 [X/X(\Pi_\pi)].$$

But by Lemmas 2.13 and 4.3,

$$r_\pi^2 [X/X(\Pi_\pi)] = [G_\pi/G^0] = [N_{G_\pi}(\sigma)/N_{G^0}(\sigma)].$$

Thus

$$\begin{aligned} \dim C(\sigma) &= \sum_{\pi \in S(\sigma)/\sim} m_\pi^2 [N_G(\sigma)/N_{G_\pi}(\sigma)]^2 [N_{G_\pi}(\sigma)/N_{G^0}(\sigma)] \\ &= [N_G(\sigma)/N_{G^0}(\sigma)] \sum_{\pi \in S(\sigma)/\sim} m_\pi^2 [N_G(\sigma)/N_{G_\pi}(\sigma)] \end{aligned}$$

$$= [N_G(\sigma)/N_{G^0}(\sigma)] \dim C^0(\sigma).$$

∎

We now want to find a basis for $C(\sigma)$. We proceed as in the connected case. Let A be the split component of the center of M and write $\underline{a}^*_{\mathbf{C}}$ for the dual of its complex Lie algebra. Each $\nu \in \underline{a}^*_{\mathbf{C}}$ determines a one-dimensional character χ_ν of M which is defined by

$$\chi_\nu(m) = q^{<H_P(m),\nu>}, m \in M.$$

We write $(I_P(\sigma : \nu), \mathcal{H}_P(\sigma : \nu))$ and $(I^0_P(\sigma : \nu), \mathcal{H}^0_P(\sigma : \nu))$ for the induced representations of G and G^0 as above corresponding to $\sigma_\nu = \sigma \otimes \chi_\nu$. Let K be a good maximal compact subgroup of G^0 with respect to a minimal parabolic pair (P_0, A_0) of G^0 such that $P_0 \subset P, A \subseteq A_0$. Then $G^0 = KP$ and we also have the usual compact realization of $I^0_P(\sigma)$ on $\mathcal{H}^K_P(\sigma) =$

$$\{f_K \in C^\infty(K, V) : f_K(kmn) = \sigma^{-1}(m) f_K(k) \text{ for all } k \in K, m \in M \cap K, n \in N \cap K\}.$$

The intertwining operators between $\mathcal{H}^0_P(\sigma, \nu)$ and $\mathcal{H}^K_P(\sigma)$ are given by

$$F^K_P(\nu) : \mathcal{H}^0_P(\sigma : \nu) \to \mathcal{H}^K_P(\sigma),$$

$$F^K_P(\nu)\phi(k) = \phi(k), \phi \in \mathcal{H}^0_P(\sigma : \nu), k \in K.$$

For all $f_K \in \mathcal{H}^K_P(\sigma), x \in G^0$,

$$F^K_P(\nu)^{-1} f_K(x) = \delta_P^{-\frac{1}{2}}(\mu(x))\sigma_\nu^{-1}(\mu(x)) f_K(\kappa(x)).$$

Here for any $x \in G^0, \kappa(x) \in K, \mu(x) \in M$ are chosen so that $x \in \kappa(x)\mu(x)N$.

Since we don't know whether there is a "good" maximal compact subgroup for G, we don't have a single compact realization for $I_P(\sigma)$. However we can proceed one coset at a time as follows. Write G as a disjoint union of cosets, $G = \cup_{i=1}^k x_i G^0$. For any $f \in \mathcal{H}_P(\sigma, \nu), 1 \le i \le k$, we can define

$$f_i(x) = \begin{cases} f(x), & \text{if } x \in x_i G^0; \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_i \in \mathcal{H}_P(\sigma : \nu)$ for each i and $f = \sum_{i=1}^k f_i$. For each $1 \le i \le k$, define

$$F^i_P(\nu) : \mathcal{H}_P(\sigma : \nu) \to \mathcal{H}^K_P(\sigma)$$

by

$$F^i_P(\nu) f(k) = f(x_i k), f \in \mathcal{H}_P(\sigma : \nu), k \in K.$$

Define

$$F^i_P(\nu)^{-1} : \mathcal{H}^K_P(\sigma) \to \mathcal{H}_P(\sigma : \nu)$$

by

$$F^i_P(\nu)^{-1} f_K(x) = \begin{cases} \delta_P^{-\frac{1}{2}}(\mu(x_0))\sigma_\nu^{-1}(\mu(x_0)) f_K(\kappa(x_0)), & \text{if } x = x_i x_0, x_0 \in G^0; \\ 0, & \text{otherwise.} \end{cases}$$

Then $F_P^i(\nu)F_P^i(\nu)^{-1}f_K = f_K$ for all $f_K \in \mathcal{H}_P^K(\sigma)$ and $F_P^i(\nu)^{-1}F_P^i(\nu)f = f_i$ for all $f \in \mathcal{H}_P(\sigma : \nu)$.

If $P = MN$ and $P' = MN'$ are two parabolic subgroups of G^0 with Levi component M, then we have the formal intertwining operators

$$J^0(P' : P : \sigma : \nu) : \mathcal{H}_P^0(\sigma : \nu) \to \mathcal{H}_{P'}^0(\sigma : \nu)$$

given by the standard integral formula

$$J^0(P' : P : \sigma : \nu)\phi(x) = \int_{\overline{N}\cap N'} \phi(x\overline{n})d\overline{n}, \quad x \in G^0.$$

Here $M\overline{N}$ is the opposite parabolic to P and $d\overline{n}$ is normalized Haar measure on $\overline{N}\cap N'$. We can define

$$J(P' : P : \sigma : \nu) : \mathcal{H}_P(\sigma : \nu) \to \mathcal{H}_{P'}(\sigma : \nu)$$

by the same formula

$$J(P' : P : \sigma : \nu)f(x) = \int_{\overline{N}\cap N'} f(x\overline{n})d\overline{n}, \quad x \in G.$$

In order to talk rigorously about holomorphicity and analytic continuation of these operators we transfer them to the compact realizations. Thus we have

$$J_K^0(P' : P : \sigma : \nu) : \mathcal{H}_P^K(\sigma) \to \mathcal{H}_P^K(\sigma)$$

defined for $f_K \in \mathcal{H}_P^K(\sigma), k \in K$, by

$$J_K^0(P' : P : \sigma : \nu)f_K(k) = F_{P'}^K(\nu)J^0(P' : P : \sigma : \nu)F_P^K(\nu)^{-1}f_K(k) =$$

$$\int_{\overline{N}\cap N'} \delta_P^{-\frac{1}{2}}(\mu(k\overline{n}))\sigma_\nu^{-1}(\mu(k\overline{n}))f_K(\kappa(k\overline{n}))d\overline{n}.$$

Theorem 4.5. *(Harish-Chandra [**7**,**13**]) Suppose σ is an irreducible discrete series representation of M. There is a chamber $\underline{a}_{\mathbf{C}}^*(P' : P)$ in $\underline{a}_{\mathbf{C}}^*$ such that for $\nu \in \underline{a}_{\mathbf{C}}^*(P' : P)$ the formal intertwining operator $J^0(P' : P : \sigma : \nu)$ converges and defines a bounded operator. For a fixed $f_K \in \mathcal{H}_P^K(\sigma)$, the mapping*

$$\nu \mapsto J_K^0(P' : P : \sigma : \nu)f_K$$

from $\underline{a}_{\mathbf{C}}^*(P' : P)$ to $\mathcal{H}_P^K(\sigma)$ is holomorphic. Further, it extends to a meromorphic function on all of $\underline{a}_{\mathbf{C}}^*$.

Because of Harish-Chandra's theorem we can define $J^0(P' : P : \sigma : \nu)$ for all $\nu \in \underline{a}_{\mathbf{C}}^*$ by

$$J^0(P' : P : \sigma : \nu) = F_{P'}^K(\nu)^{-1}J_K^0(P' : P : \sigma : \nu)F_P^K(\nu).$$

Corollary 4.6. *Suppose σ is an irreducible discrete series representation of M. Then the formal intertwining operator $J(P' : P : \sigma : \nu)$ converges and defines a bounded operator for $\nu \in \underline{a}^*_{\mathbf{C}}(P' : P)$. Further, for all $\nu \in \underline{a}^*_{\mathbf{C}}(P' : P)$, we have*

$$J(P' : P : \sigma : \nu) = \sum_{i=1}^{k} F^i_{P'}(\nu)^{-1} J^0_K(P' : P : \sigma : \nu) F^i_P(\nu).$$

Proof: It is clear from the definitions that for $\nu \in \underline{a}^*_{\mathbf{C}}(P' : P), f \in \mathcal{H}_P(\sigma : \nu), x \in G, x_0 \in G^0$,

$$J(P' : P : \sigma : \nu) f(xx_0) = J^0(P' : P : \sigma : \nu)\phi(x_0)$$

where $\phi = l(x^{-1})f|_{G^0}$ is the restriction to G^0 of the left translate of f by x^{-1}. Thus the integral converges. Fix $1 \leq i \leq k$. An elementary calculation shows that

$$F^i_{P'}(\nu) J(P' : P : \sigma : \nu) F^i_P(\nu)^{-1} = J^0_K(P' : P : \sigma : \nu).$$

Thus for any $f \in \mathcal{H}_P(\sigma)$, since clearly $J(P' : P : \sigma : \nu)(f_i) = (J(P' : P : \sigma : \nu)f)_i$ we have

$$F^i_{P'}(\nu)^{-1} J^0_K(P' : P : \sigma : \nu) F^i_P(\nu) f$$

$$= F^i_{P'}(\nu)^{-1} F^i_{P'}(\nu) J(P' : P : \sigma : \nu) F^i_P(\nu)^{-1} F^i_P(\nu)\phi = (J(P' : P : \sigma : \nu)f)_i.$$

■

Because of Corollary 4.6 we can define $J(P' : P : \sigma : \nu)$ for all $\nu \in \underline{a}^*_{\mathbf{C}}$ by the formula

$$J(P' : P : \sigma : \nu) = \sum_{i=1}^{k} F^i_{P'}(\nu)^{-1} J^0_K(P' : P : \sigma : \nu) F^i_P(\nu).$$

Corollary 4.7. *Let $\nu \in \underline{a}^*_{\mathbf{C}}, f \in \mathcal{H}_P(\sigma : \nu), x \in G, x_0 \in G^0$. Then*

$$J(P' : P : \sigma : \nu) f(xx_0) = J^0(P' : P : \sigma : \nu)\phi(x_0)$$

where $\phi = l(x^{-1})f|_{G^0}$.

Proof: Suppose that $1 \leq j \leq k$. Because $J^0(P' : P : \sigma : \nu)$ commutes with the left action of G^0, it is enough to prove the result for $x = x_j$. For any $\nu \in \underline{a}^*_{\mathbf{C}}, f \in \mathcal{H}_P(\sigma : \nu), x_0 \in G^0, J(P' : P : \sigma : \nu) f(x_j x_0) =$

$$\sum_{i=1}^{k} F^i_{P'}(\nu)^{-1} J^0_K(P' : P : \sigma : \nu) F^i_P(\nu) f(x_j x_0) = F^j_{P'}(\nu)^{-1} J^0_K(P' : P : \sigma : \nu) F^j_P(\nu) f(x_j x_0)$$

$$= F^K_{P'}(\nu)^{-1} J^0_K(P' : P : \sigma : \nu) F^K_P(\nu)\phi_j(x_0)$$

where $\phi_j = L(x_j^{-1})f|_{G^0}$. ■

Fix scalar normalizing factors $r(P' : P : \sigma : \nu)$ as in [**2**] used to define the normalized intertwining operators

$$R^0(P' : P : \sigma) = r(P' : P : \sigma : 0)^{-1} J^0(P' : P : \sigma : 0).$$

Using Corollary 4.7, the fact that $r(P' : P : \sigma : \nu)^{-1} J^0(P' : P : \sigma : \nu)$ is holomorphic and non-zero at $\nu = 0$ will imply that $r(P' : P : \sigma : \nu)^{-1} J(P' : P : \sigma : \nu)$ is also holomorphic and non-zero at $\nu = 0$. Thus we can use the same normalizing factors to define

$$R(P' : P : \sigma) = r(P' : P : \sigma : 0)^{-1} J(P' : P : \sigma : 0).$$

For the normalized intertwining operators we will also have the formula

$$R(P' : P : \sigma) f(xx_0) = R^0(P' : P : \sigma)\phi(x_0)$$

where notation is as in Corollary 4.7.

Lemma 4.8. *Suppose P_1, P_2, and P_3 are parabolic subgroups of G^0 with Levi component M. Then $R(P_1 : P_3 : \sigma) = R(P_1 : P_2 : \sigma) R(P_2 : P_3 : \sigma)$.*

Proof: This follows easily using Corollary 4.7 from the corresponding formula for the connected case. ∎

Lemma 4.9. *Suppose P_1 and P_2 are parabolic subgroups of G^0 with Levi component M. Then $R(P_2 : P_1 : \sigma)$ is an equivalence from $\mathcal{H}_{P_1}(\sigma)$ onto $\mathcal{H}_{P_2}(\sigma)$.*

Proof: It follows from Lemma 4.8 that

$$R(P_1 : P_2 : \sigma) R(P_2 : P_1 : \sigma) = R(P_1 : P_1 : \sigma).$$

But it follows from the integral formula that $J(P_1 : P_1 : \sigma : \nu)$ is the identity operator for ν in the region of convergence, and hence for all ν. Thus $R(P_1 : P_1 : \sigma)$ is a non-zero constant times the identity operator and so $R(P_2 : P_1 : \sigma)$ is invertible. ∎

Let $x \in N_G(M)$. Then if $P_1 = MN_1$ is a parabolic subgroup of G^0 with Levi component M, so is $P_2 = xPx^{-1} = xMx^{-1}xN_1x^{-1} = MN_2$. Let dn_1 and dn_2 denote normalized Haar measure on N_1 and N_2 respectively. (Thus dn_i assigns measure one to $K \cap N_i, i = 1, 2$.) Define $\alpha_{P_1}(x) \in \mathbf{R}^+$ by

$$\int_{N_2} \phi(n_2) dn_2 = \alpha_{P_1}(x) \int_{N_1} \phi(xn_1x^{-1}) dn_1, \phi \in C_c^\infty(N_2).$$

For all $m \in M$,

$$\alpha_{P_1}(m) = \delta_{P_1}(m).$$

Further, if $x \in N_G(K) \cap N_G(M)$, then $\alpha_{P_1}(x) = 1$. The following lemma is an easy consequence of the definition.

Lemma 4.10. *Let $x, y \in N_G(M)$. Then*
$$\alpha_P(yx) = \alpha_{xPx^{-1}}(y)\alpha_P(x).$$
Moreover if $m \in M, y \in N_G(M)$, then
$$\alpha_P(ym) = \alpha_P(y)\delta_P(m).$$

Now let $N_G(\sigma) = \{g \in N_G(M) : \sigma^g \simeq \sigma\}$ and let $W_G(\sigma) = N_G(\sigma)/M$. Let $W_{G^0}(\sigma) = (N_G(\sigma) \cap G^0)/M$. If $w \in W_G(\sigma)$, σ can be extended to a representation of the group M_w generated by M and any representative n_w for w in $N_G(\sigma)$. Fix such an extension and denote it by σ_w. Now we can define an intertwining operator
$$A_P(w) : \mathcal{H}_{w^{-1}Pw}(\sigma) \to \mathcal{H}_P(\sigma)$$
by
$$(A_P(w)f)(x) = \sigma_w(n_w)\alpha_{w^{-1}Pw}(n_w)^{\frac{1}{2}} f(xn_w).$$
The $\alpha_{w^{-1}Pw}$ term is not used in the connected case because coset representatives n_w can be chosen in K where $\alpha_{w^{-1}Pw} = 1$. In the general case we don't know if there is a natural choice of coset representatives with $\alpha_{w^{-1}Pw} = 1$. Thus we add the $\alpha_{w^{-1}Pw}$ term so that $A_P(w)$ is independent of the coset representative n_w for w. $A_P(w)$ does however depend on the choice of the extension σ_w.

Lemma 4.11. *The intertwining operator $A_P(w)$ is independent of the choice of coset representative n_w for $w \in W_G(\sigma)$. For $w_1, w_2 \in W_G(\sigma)$ there is a non-zero complex constant $c_P(w_1, w_2)$ so that*
$$A_P(w_1w_2) = c_P(w_1, w_2)A_P(w_1)A_{w_1^{-1}Pw_1}(w_2).$$

Proof: Using Lemma 4.10 we have for any $m \in M, f \in \mathcal{H}_{w^{-1}Pw}(\sigma)$,
$$\begin{aligned} &\sigma_w(n_wm)\alpha_{w^{-1}Pw}(n_wm)^{\frac{1}{2}} f(xn_wm) \\ &= \sigma_w(n_w)\sigma(m)\alpha_{w^{-1}Pw}(n_w)^{\frac{1}{2}}\delta_{w^{-1}Pw}(m)^{\frac{1}{2}}\sigma(m)^{-1}\delta_{w^{-1}Pw}(m)^{-\frac{1}{2}} f(xn_w) \\ &= \sigma_w(n_w)\alpha_{w^{-1}Pw}(n_w)^{\frac{1}{2}} f(xn_w). \end{aligned}$$
Thus the intertwining operator is independent of the choice of coset representative.

Let $w_1, w_2 \in W_G(\sigma)$ and fix coset representatives n_1, n_2 for w_1, w_2 respectively. Then we can use n_1n_2 as a coset representative for w_1w_2. We will also write $P_1 = w_1^{-1}Pw_1$ and $P_{12} = w_2^{-1}w_1^{-1}Pw_1w_2$. For any $w \in W_G(\sigma)$ and representative n_w for w we have
$$\sigma_w(n_w)\sigma(m)\sigma_w(n_w)^{-1} = \sigma(n_wmn_w^{-1})$$
for all $m \in M$. Thus
$$\sigma_{w_1w_2}(n_1n_2)\sigma_{w_2}(n_2)^{-1}\sigma_{w_1}(n_1)^{-1}$$
is a non-zero self-intertwining operator for σ and hence there is a non-zero constant c so that
$$\sigma_{w_1w_2}(n_1n_2) = c\sigma_{w_1}(n_1)\sigma_{w_2}(n_2).$$

Further, from Lemma 4.10 we have
$$\alpha_{P_{12}}(n_1n_2) = \alpha_{P_1}(n_1)\alpha_{P_{12}}(n_2).$$
By definition, for any $x \in G, f \in \mathcal{H}_{P_{12}}(\sigma)$,
$$\begin{aligned} A_P(w_1w_2)f(x) &= \sigma_{w_1w_2}(n_1n_2)\alpha_{P_{12}}(n_1n_2)^{\frac{1}{2}}f(xn_1n_2) \\ &= c\sigma_{w_1}(n_1)\alpha_{P_1}(n_1)^{\frac{1}{2}}\sigma_{w_2}(n_2)\alpha_{P_{12}}(n_2)^{\frac{1}{2}}f(xn_1n_2) \\ &= cA_P(w_1)A_{w_1^{-1}Pw_1}(w_2)f(x). \end{aligned}$$

■

If $w \in W_{G^0}(\sigma)$ and n_w is chosen to be in K, then we also have
$$A^0(w) : \mathcal{H}^0_{w^{-1}Pw}(\sigma) \to \mathcal{H}^0_P(\sigma)$$
given by
$$(A^0(w)\phi)(x) = \sigma_w(n_w)\phi(xn_w).$$
The compositions
$$R^0(w_0,\sigma) = A^0(w_0)R^0(w_0^{-1}Pw_0 : P : \sigma), w_0 \in W_{G^0}(\sigma),$$
$$R(w,\sigma) = A_P(w)R(w^{-1}Pw : P : \sigma), w \in W_G(\sigma),$$
give self-intertwining operators for $I^0_P(\sigma)$ and $I_P(\sigma)$ respectively.

Lemma 4.12. *There is a cocycle η so that*
$$R(w_1w_2,\sigma) = \eta(w_1,w_2)R(w_1,\sigma)R(w_2,\sigma)$$
for all $w_1, w_2 \in W_G(\sigma)$.

Proof: Using Lemmas 4.8 and 4.11, we have
$$R(w_1w_2,\sigma) = A_P(w_1w_2)R(w_2^{-1}w_1^{-1}Pw_1w_2 : P : \sigma)$$
$$= c_P(w_1,w_2)A_P(w_1)A_{w_1^{-1}Pw_1}(w_2)R(w_2^{-1}w_1^{-1}Pw_1w_2 : w_2^{-1}Pw_2 : \sigma)R(w_2^{-1}Pw_2 : P : \sigma).$$
We will show that there is a non-zero constant $c'_P(w_1,w_2)$ so that
$$A_{w_1^{-1}Pw_1}(w_2)R(w_2^{-1}w_1^{-1}Pw_1w_2 : w_2^{-1}Pw_2 : \sigma)A_P(w_2)^{-1}$$
$$= c'_P(w_1,w_2)R(w_1^{-1}Pw_1 : P : \sigma).$$
When this is established we will have $R(w_1w_2,\sigma) =$
$$c_P(w_1,w_2)c'_P(w_1,w_2)A_P(w_1)R(w_1^{-1}Pw_1 : P : \sigma)A_P(w_2)R(w_2^{-1}Pw_2 : P : \sigma)$$
$$= c_P(w_1,w_2)c'_P(w_1,w_2)R(w_1,\sigma)R(w_2,\sigma).$$
Thus η can be defined by $\eta(w_1,w_2) = c_P(w_1,w_2)c'_P(w_1,w_2)$. It is immediate from the formulas for the composition of the operators $R(w_i,\sigma)$ that η is a 2–cocycle.

In order to prove the above identity we need to go back to the original definition of the standard intertwining operators. $W_G(\sigma)$ acts on $\underline{a}^*_{\mathbf{C}}$ and $\chi_\nu(n_w^{-1}mn_w) = \chi_{w\nu}(m)$

if n_w is any representative of $w \in W_G(\sigma)$. Now for any $\nu \in \underline{a}^*_{\mathbf{C}}, f \in \mathcal{H}_{w^{-1}Pw}(\sigma,\nu), x \in G, m \in M, n \in N$, if $A_P(w)$ is defined exactly as above, we have

$$A_P(w)f(xmn) = \sigma_w(n_w)\alpha_{w^{-1}Pw}(n_w)^{\frac{1}{2}} f(xmnn_w)$$
$$= \sigma_w(n_w)\alpha_{w^{-1}Pw}(n_w)^{\frac{1}{2}}\delta_{w^{-1}Pw}(n_w^{-1}m^{-1}n_w)^{-\frac{1}{2}}\sigma(n_w^{-1}m^{-1}n_w)\chi_\nu(n_w^{-1}m^{-1}n_w)f(xn_w)$$
$$= \delta_P(m)^{-\frac{1}{2}}\sigma(m)^{-1}\chi_{w\nu}(m)^{-1}\sigma_w(n_w)\alpha_{w^{-1}Pw}(n_w)^{\frac{1}{2}}f(xn_w)$$
$$= \delta_P(m)^{-\frac{1}{2}}\sigma(m)^{-1}\chi_{w\nu}(m)^{-1}A_P(w)f(x).$$

Thus $A_P(w)$ maps $\mathcal{H}_{w^{-1}Pw}(\sigma,\nu)$ to $\mathcal{H}_P(\sigma,w\nu)$. Now write $P_1 = w_1^{-1}Pw_1, P_2 = w_2^{-1}Pw_2$,
$P_{12} = w_2^{-1}w_1^{-1}Pw_1w_2$ and consider the composition

$$A_{P_1}(w_2)J(P_{12} : P_2 : \sigma : w_2^{-1}\nu)A_P(w_2)^{-1}$$

It maps

$$\mathcal{H}_P(\sigma,\nu) \to \mathcal{H}_{P_2}(\sigma,w_2^{-1}\nu) \to \mathcal{H}_{P_{12}}(\sigma,w_2^{-1}\nu) \to \mathcal{H}_{P_1}(\sigma,\nu).$$

If ν is in the region of convergence for the integral formula for $J(P_{12} : P_2 : \sigma : w_2^{-1}\nu)$ and $d\overline{n}_2$ denotes normalized Haar measure on $\overline{N}_2 \cap N_{12}$, then we have

$$A_{P_1}(w_2)J(P_{12} : P_2 : \sigma : w_2^{-1}\nu)A_P(w_2)^{-1}f(x)$$
$$= \sigma_{w_2}(n_{w_2})\alpha_{P_{12}}(n_{w_2})^{\frac{1}{2}}J(P_{12} : P_2 : \sigma : w_2^{-1}\nu)A_P(w_2)^{-1}f(xn_{w_2})$$
$$= \sigma_{w_2}(n_{w_2})\alpha_{P_{12}}(n_{w_2})^{\frac{1}{2}}\int_{\overline{N}_2\cap N_{12}} A_P(w_2)^{-1}f(xn_{w_2}\overline{n}_2)d\overline{n}_2$$
$$= \sigma_{w_2}(n_{w_2})\alpha_{P_{12}}(n_{w_2})^{\frac{1}{2}}\int_{\overline{N}_2\cap N_{12}} \alpha_{P_2}(n_{w_2})^{-\frac{1}{2}}\sigma_{w_2}(n_{w_2})^{-1}f(xn_{w_2}\overline{n}_2n_{w_2}^{-1})d\overline{n}_2$$
$$= \alpha_{P_{12}}(n_{w_2})^{\frac{1}{2}}\alpha_{P_2}(n_{w_2})^{-\frac{1}{2}}\int_{\overline{N}_2\cap N_{12}} f(xn_{w_2}\overline{n}_2n_{w_2}^{-1})d\overline{n}_2.$$

But $\overline{N}_2 \cap N_{12} = w_2^{-1}(\overline{N}\cap N_1)w_2$ so if $d\overline{n}$ denotes normalized Haar measure on $\overline{N}\cap N_1$, there is a positive real number r so that

$$\int_{\overline{N}_2\cap N_{12}} \phi(n_{w_2}\overline{n}_2n_{w_2}^{-1})d\overline{n}_2 = r\int_{\overline{N}\cap N_1}\phi(\overline{n})d\overline{n}$$

for all $\phi \in C_c^\infty(\overline{N}\cap N_1)$. Thus

$$A_{P_1}(w_2)J(P_{12} : P_2 : \sigma : w_2^{-1}\nu)A_P(w_2)^{-1}f(x)$$
$$= \alpha_{P_{12}}(n_{w_2})^{\frac{1}{2}}\alpha_{P_2}(n_{w_2})^{-\frac{1}{2}}r\int_{\overline{N}\cap N_1} f(x\overline{n})d\overline{n}$$
$$= \alpha_{P_{12}}(n_{w_2})^{\frac{1}{2}}\alpha_{P_2}(n_{w_2})^{-\frac{1}{2}}rJ(P_1 : P : \sigma : \nu)f(x).$$

Setting

$$c''_P(w_1,w_2) = \alpha_{P_{12}}(n_{w_2})^{\frac{1}{2}}\alpha_{P_2}(n_{w_2})^{-\frac{1}{2}}r$$

we have

$$A_{P_1}(w_2)J(P_{12} : P_2 : \sigma : w_2^{-1}\nu)A_P(w_2)^{-1} = c''_P(w_1,w_2)J(P_1 : P : \sigma : \nu)$$

for all ν in the region of convergence for the integral formula for $J(P_{12} : P_2 : \sigma : w_2^{-1}\nu)$. By analytic continuation, the identity is valid for all ν. Now divide both sides of the equation by $r(P_{12} : P_2 : \sigma : w_2^{-1}\nu)$ and evaluate at $\nu = 0$. We obtain

$$A_{P_1}(w_2)R(P_{12} : P_2 : \sigma)A_P(w_2)^{-1} = c'_P(w_1, w_2)R(P_1 : P : \sigma)$$

where

$$c'_P(w_1, w_2) = c''_P(w_1, w_2)r(P_1 : P : \sigma : 0)r(P_{12} : P_2 : \sigma : 0)^{-1}.$$

∎

For $\phi \in \mathcal{H}^0_P(\sigma)$ define $f = \Phi(\phi) \in \mathcal{H}_P(\sigma)$ such that $f(x) = 0$ if $x \notin G^0$ and $f(x_0) = \phi(x_0)$ if $x_0 \in G^0$.

Lemma 4.13. *Let $f = \Phi(\phi)$ as above. Then for all $w, w_1 \in W_G(\sigma), x_0 \in G^0$,*

$$R(w, \sigma)f(x_0 n_{w_1}^{-1}) = 0$$

unless $w = w_1 w_0, w_0 \in W_{G^0}(\sigma)$. If $w = w_1 w_0, w_0 \in W_{G^0}(\sigma)$, then $R(w, \sigma)f(x_0 n_{w_1}^{-1}) =$

$$\eta(w_1, w_0)\sigma_{w_1}(n_{w_1})\alpha_{w_1^{-1}Pw_1}(n_{w_1})^{\frac{1}{2}}R^0(w_1^{-1}Pw_1 : P : \sigma)R^0(w_0 : \sigma)\phi(x_0).$$

Proof: For any $x \in G, w \in W_G(\sigma)$, using Corollary 4.7, $R(w^{-1}Pw : P : \sigma)f(x) = R^0(w^{-1}Pw : P : \sigma)\phi'(1)$ where ϕ' is the restriction to G^0 of $l(x^{-1})f$. Now since $f = \Phi(\phi)$ is supported on G^0, $\phi' = 0$ unless $x \in G^0$. If $x_0 \in G^0$, then $R(w^{-1}Pw : P : \sigma)f(x_0) = R^0(w^{-1}Pw : P : \sigma)\phi(x_0)$.

Now by definition, $R(w, \sigma)f(x_0 n_{w_1}^{-1}) =$

$$\sigma_w(n_w)\alpha_{w^{-1}Pw}(n_w)^{\frac{1}{2}}R(w^{-1}Pw : P : \sigma)f(x_0 n_{w_1}^{-1}n_w).$$

By the above this is zero unless $n_{w_1}^{-1}n_w \in G^0$, that is unless $w = w_1 w_0, w_0 \in W_{G^0}(\sigma)$. In this case, using Lemma 4.12,

$$R(w_1 w_0, \sigma)f(x_0 n_{w_1}^{-1}) = \eta(w_1, w_0)R(w_1 : \sigma)R(w_0 : \sigma)f(x_0 n_{w_1}^{-1})$$

$$= \eta(w_1, w_0)\sigma_{w_1}(n_{w_1})\alpha_{w_1^{-1}Pw_1}(n_{w_1})^{\frac{1}{2}}R(w_1 P w_1^{-1} : P : \sigma)R(w_0 : \sigma)f(x_0)$$

$$= \eta(w_1, w_0)\sigma_{w_1}(n_{w_1})\alpha_{w_1^{-1}Pw_1}(n_{w_1})^{\frac{1}{2}}R^0(w_1 P w_1^{-1} : P : \sigma)R^0(w_0 : \sigma)\phi(x_0).$$

∎

Recall that if we write $W^0_{G^0}(\sigma)$ for the subgroup of elements $w \in W_{G^0}(\sigma)$ such that $R^0(w, \sigma)$ is scalar, then $W^0_{G^0}(\sigma) = W(\Phi_1)$ is generated by reflections in a set Φ_1 of reduced roots of (G, A). Let Φ^+ be the positive system of reduced roots of (G, A) determined by P and let $\Phi_1^+ = \Phi_1 \cap \Phi^+$. If we define

$$R^0_\sigma = \{w \in W_{G^0}(\sigma) : w\beta \in \Phi^+ \text{ for all } \beta \in \Phi_1^+\},$$

then $W_{G^0}(\sigma)$ is the semidirect product of R^0_σ and $W(\Phi_1)$. R^0_σ is called the R-group for $I^0_P(\sigma)$ and the operators

$$\{R^0(r, \sigma), r \in R^0_\sigma\}$$

form a basis for the algebra of intertwining operators of $I_P^0(\sigma)$. We will define

$$R_\sigma = \{w \in W_G(\sigma) : w\beta \in \Phi^+ \text{ for all } \beta \in \Phi_1^+\}.$$

Clearly $R_\sigma \cap G^0 = R_\sigma^0$.

Lemma 4.14. *$R(w,\sigma)$ is scalar if $w \in W(\Phi_1)$, and $W_G(\sigma)$ is the semidirect product of $W(\Phi_1)$ and R_σ.*

Proof: Fix $w \in W(\Phi_1), n_w \in K$ a representative for w, and $f \in \mathcal{H}_P(\sigma)$. Then for all $x \in G$,

$R(w,\sigma)f(x) = \sigma_w(n_w)R(w^{-1}Pw : P : \sigma)f(xn_w) = \sigma_w(n_w)R^0(w^{-1}Pw : P : \sigma)\phi(n_w) =$

where $\phi(x_0) = f(xx_0)$ for all $x_0 \in G^0$. Thus

$$R(w,\sigma)f(x) = R^0(w,\sigma)\phi(1).$$

But since $w \in W(\Phi_1)$ there is a constant c_w such that $R^0(w,\sigma)\phi = c_w\phi$ for all $\phi \in \mathcal{H}_P^0(\sigma)$. Thus

$$R(w,\sigma)f(x) = c_w\phi(1) = c_w f(x)$$

so $R(w,\sigma)$ is scalar.

We must show that for any $w \in W_G(\sigma), w\Phi_1 = \Phi_1$ so that

$$R_\sigma = \{w \in W_G(\sigma) : w\Phi_1^+ = \Phi_1^+\}.$$

Then as in the connected case it will be clear that $W_G(\sigma)$ is the semidirect product of R_σ and $W(\Phi_1)$. Let $w \in W_G(\sigma), \alpha \in \Phi_1$, and let $s_\alpha \in W(\Phi_1)$ denote the reflection corresponding to α. Then $ws_\alpha w^{-1} = s_{w\alpha} \in W_{G^0}(\sigma)$. But $R(s_\alpha : \sigma)$ is scalar so that using Lemma 4.12, so is $R(s_{w\alpha},\sigma) = R(ws_\alpha w^{-1} : \sigma)$. This implies as above that $R^0(s_{w\alpha},\sigma)$ is scalar. Hence $s_{w\alpha} \in W(\Phi_1)$ and $w\alpha \in \Phi_1$. ∎

Lemma 4.15. *The dimension of $C(\sigma)$ is equal to $[R_\sigma]$.*

Proof: By Lemma 4.4,

$$\dim C(\sigma) = [N_G(\sigma)/N_{G^0}(\sigma)] \dim C^0(\sigma) = [W_G(\sigma)/W_{G^0}(\sigma)][R_\sigma^0]$$

$$= [W_G(\sigma)/W_{G^0}(\sigma)][W_{G^0}(\sigma)/W(\Phi_1)] = [W_G(\sigma)/W(\Phi_1)] = [R_\sigma].$$

∎

Theorem 4.16. *The operators $\{R(r,\sigma), r \in R_\sigma\}$ form a basis for the algebra of intertwining operators of $I_P(\sigma)$.*

Proof: By Lemma 4.15 it suffices to show that the operators are linearly independent. Suppose that $c_w, w \in R_\sigma$, are constants so that

$$\sum_{w \in R_\sigma} c_w R(w, \sigma) f(x) = 0$$

for all $f \in \mathcal{H}_P(\sigma), x \in G$. Fix $w_1 \in R_\sigma$. Then for all $f = \Phi(\phi)$ with $\phi \in \mathcal{H}_P^0(\sigma)$ and all $x_0 \in G^0$, we have

$$\sum_{w \in R_\sigma} c_w R(w, \sigma) f(x_0 n_{w_1}^{-1}) = 0.$$

Now by Lemma 4.13, $R(w, \sigma) f(x_0 n_{w_1}^{-1}) = 0$ unless $w = w_1 w_0$ where $w_0 \in W_{G^0}(\sigma) \cap R_\sigma = R_\sigma^0$. Now again using Lemma 4.13,

$$0 = \sum_{w_0 \in R_\sigma^0} c_{w_1 w_0} R(w_1 w_0, \sigma) f(x_0 n_{w_1}^{-1})$$

$$= \sum_{w_0 \in R_\sigma^0} c_{w_1 w_0} \eta(w_1, w_0) \sigma_{w_1}(n_{w_1}) \alpha_{w_1^{-1} P w_1}(n_{w_1})^{\frac{1}{2}} R^0(w_1^{-1} P w_1 : P : \sigma) R^0(w_0 : \sigma) \phi(x_0)$$

$$= \sigma_{w_1}(n_{w_1}) \alpha_{w_1^{-1} P w_1}(n_{w_1})^{\frac{1}{2}} R^0(w_1^{-1} P w_1 : P : \sigma) \sum_{w_0 \in R_\sigma^0} c_{w_1 w_0} \eta(w_1, w_0) R^0(w_0 : \sigma) \phi(x_0).$$

Thus

$$\sum_{w_0 \in R_\sigma^0} c_{w_1 w_0} \eta(w_1, w_0) R^0(w_0 : \sigma) \phi(x_0) = 0$$

for all $\phi \in \mathcal{H}_P^0(\sigma)$ and all $x_0 \in G^0$. Now since we know that the operators $R^0(w_0 : \sigma)$ are linearly independent on $\mathcal{H}_P^0(\sigma)$, we can conclude that the $c_{w_1 w_0} \eta(w_1, w_0)$ and hence the $c_{w_1 w_0}$ are all zero. ∎

As in Arthur [**2**] we now have to deal with the cocycle η of Lemma 4.12. Fix a finite central extension

$$1 \to Z_\sigma \to \tilde{R}_\sigma \to R_\sigma \to 1$$

over which η splits. Also define the functions $\xi_\sigma : \tilde{R}_\sigma \to \mathbf{C}^*$ and the character χ_σ of Z_σ as in Arthur [**2**]. Then we obtain a homomorphism

$$\tilde{R}(r, \sigma) = \xi_\sigma^{-1}(r) R(r, \sigma), r \in \tilde{R}_\sigma,$$

of $\tilde{R}_\sigma$ into the group of unitary intertwining operators for $I_P(\sigma)$ which transforms by

$$\tilde{R}(zr, \sigma) = \chi_\sigma(z)^{-1} \tilde{R}(r, \sigma), z \in Z_\sigma, r \in \tilde{R}_\sigma.$$

Now we can define a representation $\mathcal{R}$ of $\tilde{R}_\sigma \times G$ on $\mathcal{H}_P(\sigma)$ given by

$$\mathcal{R}(r, x) = \tilde{R}(r, \sigma) I_P(\sigma, x), r \in \tilde{R}_\sigma, x \in G.$$

Let $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ denote the set of irreducible representations of $\tilde{R}_\sigma$ with Z_σ central character χ_σ and let $\Pi_\sigma(G)$ denote the set of irreducible constituents of $I_P(\sigma)$.

Theorem 4.17. *There is a bijection $\rho \mapsto \pi_\rho$ of $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ onto $\Pi_\sigma(G)$ such that*

$$\mathcal{R} = \oplus_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \quad (\rho^\vee \otimes \pi_\rho).$$

Proof: Write the decomposition of $\mathcal{R}$ into irreducibles as

$$\mathcal{R} = \sum_{\rho,\pi} m_{\rho,\pi} \ (\rho^\vee \otimes \pi)$$

where ρ runs over $\Pi(\tilde{R}_\sigma, \chi_\sigma)$, π runs over $\Pi_\sigma(G)$, and each $m_{\rho,\pi} \geq 0$. This corresponds to a decomposition

$$\mathcal{H}_P(\sigma) \simeq \sum_{\rho,\pi} (V_{\rho^\vee} \otimes W_\pi)^{m_{\rho,\pi}}$$

where $V_{\rho^\vee}$ and W_π denote the representation spaces for the irreducible representations $\rho^\vee$ and π. For each ρ, we also write

$$\mathcal{H}_\rho \simeq V_{\rho^\vee} \otimes \sum_\pi W_\pi^{m_{\rho,\pi}}$$

for the ρ-isotypic component of $\mathcal{H}_P(\sigma)$. Each subspace $\mathcal{H}_\rho$ is invariant under the action of $\mathcal{R}$, in particular by all of the intertwining operators $\tilde{R}(r, \sigma), r \in \tilde{R}_\sigma$. Since these intertwining operators span the space $C(\sigma)$ of G-intertwining operators for $\mathcal{H}_P(\sigma)$, each $T \in C(\sigma)$ must satisfy $T(\mathcal{H}_\rho) \subset \mathcal{H}_\rho$ for every ρ.

We will first show that given π, there is at most one ρ such that $m_{\rho,\pi} > 0$. So fix π and suppose that there are ρ_1 and ρ_2 such that $m_{\rho_i,\pi} > 0, i = 1, 2$. Then W_π occurs as a G-summand of both $\mathcal{H}\rho_1$ and $\mathcal{H}\rho_2$. Thus there is $T_{12} \neq 0$ in $Hom_G(\mathcal{H}\rho_1, \mathcal{H}\rho_2)$. We can extend T_{12} to an element T of $C(\sigma)$ by setting $T = T_{12}$ on $\mathcal{H}_{\rho_1}$ and $T = 0$ on $\mathcal{H}_\rho, \rho \neq \rho_1$. Thus there is T in $C(\sigma)$ such that $T(\mathcal{H}_{\rho_1}) \subset \mathcal{H}_{\rho_2}$. But by the remark in the previous paragraph, $T(\mathcal{H}_{\rho_1}) \subset \mathcal{H}_{\rho_1}$. Now since $T(\mathcal{H}_{\rho_1}) \neq 0$, we must have $\rho_1 \simeq \rho_2$.

Now fix ρ and look at $\mathcal{H}_\rho \simeq V_{\rho^\vee} \otimes W$ where $W = \sum_\pi W_\pi^{m_{\rho,\pi}}$. We will show that W is irreducible as a G-module. Thus suppose that $W = W_1 \oplus W_2$ where W_1, W_2 are G-submodules of W. Then we can define $T \in C(\sigma)$ by $T(v \otimes (w_1 + w_2)) = v \otimes w_1$ if $v \in V_{\rho^\vee}, w_1 \in W_1, w_2 \in W_2$, and $T = 0$ on $\mathcal{H}_{\rho'}$ if $\rho' \not\simeq \rho$. Now since $T \in C(\sigma)$, we can write $T = \sum_r c_r \tilde{\mathcal{R}}(r, \sigma)$ where r runs over $\tilde{R}_\sigma$. Thus for all $v \in V_{\rho^\vee}, w_1 \in W_1, w_2 \in W_2$, we have

$$v \otimes w_1 = T(v \otimes (w_1 + w_2)) = (\sum_r c_r \rho^\vee(r)v) \otimes w_1 + (\sum_r c_r \rho^\vee(r)v) \otimes w_2.$$

Suppose $W_2 \neq 0$. This implies that $\sum_r c_r \rho^\vee(r)v = 0$ for all v so that $v \otimes w_1 = 0$ for all w_1. Thus $W_1 = \{0\}$ and hence W is irreducible. But this implies that $m_{\rho,\pi} \leq 1$ for all π and that there is at most one π such that $m_{\rho,\pi} = 1$.

Define

$$\Pi_1 = \{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma) : m_{\rho,\pi} = 1 \text{ for some } \pi\}.$$

For each $\rho \in \Pi_1$ we have shown that the representation π such that $m_{\rho,\pi} = 1$ is unique. Thus we will call it π_ρ. Further, we have shown that $\pi_{\rho_1} \simeq \pi_{\rho_2}$ just in case $\rho_1 \simeq \rho_2$.

Further, by definition of $\Pi_\sigma(G)$, each $\pi \in \Pi_\sigma(G)$ occurs in $\mathcal{H}_P(\sigma)$ and so must be of the form π_ρ for some $\rho \in \Pi_1$. Thus to complete the proof of the theorem we need only show that $\Pi_1 = \Pi(\tilde{R}_\sigma, \chi_\sigma)$.

Since

$$\mathcal{H}_P(\sigma) \simeq \sum_{\rho\in\Pi_1} (V_{\rho^\vee} \otimes W_{\pi_\rho}),$$

we must have $\dim C(\sigma) = \sum_{\rho\in\Pi_1}(deg\ \rho)^2$. But since the $R(r,\sigma), r \in R_\sigma$, form a basis for $C(\sigma)$, we know that

$$\dim C(\sigma) = [R_\sigma] = \sum_{\rho\in\Pi(\tilde{R}_\sigma,\chi_\sigma)} (deg\ \rho)^2.$$

Thus $\Pi_1 = \Pi(\tilde{R}_\sigma, \chi_\sigma)$. ∎

Remark 4.18 Suppose now that G/G^0 is cyclic. Then in [**1**], Arthur predicts a dual group construction of a group $R_{\psi,\sigma}$, in terms of the conjectural parameter ψ for the L–packet of σ, which should also describe the components of $\mathrm{Ind}_P^G(\sigma)$. In particular, $R_{\psi,\sigma} = W_{\psi,\sigma}/W^0_{\psi,\sigma}$, where these groups are defined in terms of centralizers of the image of ψ. Furthermore, Arthur has conjecturally identified $W^0_{\psi,\sigma}$ with $W(\Phi_1)$ and $W_{\psi,\sigma}$ with $W_G(\sigma)$. Thus, if the conjectural parameterization exists in the connected case, and Shelstad's Theorem [**12**] extends to the p-adic case, then it must be the case that $R_\sigma \simeq R_{\psi,\sigma}$. That is, we have shown that there is a group side construction of Arthur's R–group, if such an object exists. (For more details see [**1**] and [**6**], particularly Sections 1 and 4.)

5. R-groups for $\mathrm{Ind}_P^G(\sigma)$

In this section we will study representations of G which are induced from discrete series representations of a parabolic subgroup P of G. Thus we revert to the notation that parabolic subgroups of G^0 are denoted by P^0.

Let $P = MN$ be a cuspidal parabolic subgroup of G. Let σ be an irreducible discrete series representation of M and let σ_0 be an irreducible constituent of the restriction of σ to M^0. We want to find a basis for the intertwining algebra $C(\sigma)$ of the induced representation $\mathrm{Ind}_P^G(\sigma)$. Since σ is contained in $\mathrm{Ind}_{M^0}^M(\sigma_0)$ we know $\mathrm{Ind}_P^G(\sigma)$ is contained in $\mathrm{Ind}_P^G(\mathrm{Ind}_{M^0}^M(\sigma_0)) \simeq \mathrm{Ind}_{P^0}^G(\sigma_0)$. In §4 we found a basis for the intertwining algebra $C(\sigma_0)$ of $\mathrm{Ind}_{P^0}^G(\sigma_0)$. We will see how to obtain a basis for $C(\sigma)$ by restricting the intertwining operators defined in §4.

We first need to embed σ in a family $\sigma_\nu, \nu \in \underline{a}^*_{\mathbf{C}}$, where $\underline{a}$ is the real Lie algebra of the split component A of M. Write $X(M), X(A)$ for the groups of rational characters of M, A respectively. Let

$$r : X(M) \otimes_{\mathbf{Z}} \mathbf{R} \to X(A) \otimes_{\mathbf{Z}} \mathbf{R}$$

be the map given by restriction. That is $r(\chi \otimes t) = \chi|_A \otimes t$ for $\chi \in X(M), t \in \mathbf{R}$.

Lemma 5.1. *The homomorphism $r : X(M) \otimes_{\mathbf{Z}} \mathbf{R} \to X(A) \otimes_{\mathbf{Z}} \mathbf{R}$ is surjective.*

Proof: Since G is a linear group we have an embedding of G in $L = GL(V)$, where V is a finite dimensional F–vector space. Since A is a split torus, the action of A on V can be diagonalized. For any $\chi \in X(A)$ let $V(\chi) = \{v \in V : av = \chi(a)v \text{ for all } a \in A\}$. Let $\chi_i, 1 \leq i \leq k$, denote the distinct elements of $X(A)$ such that $V_i = V(\chi_i) \neq \{0\}$. We can identify $a \in A$ with the block diagonal matrix with diagonal entries $\chi_i(a)I_{d_i}$ where $d_i = \dim V_i$ and I_{d_i} denotes the identity matrix of size $d_i, 1 \leq i \leq k$.

Since $M = C_G(A)$, we have $M \subset C_L(A) \simeq GL(V_1) \times GL(V_2) \times ... \times GL(V_k)$. For each $1 \leq i \leq k$ we can define $\det_i \in X(M)$ by $\det_i(m_1, ..., m_k) = \det m_i$. Now $\det_i \otimes d_i^{-1} \in X(M) \otimes_{\mathbf{Z}} \mathbf{R}$ and for $a \in A$, $\det_i \otimes d_i^{-1}(a) = \chi_i(a)^{d_i} \otimes d_i^{-1} = \chi_i(a) \otimes 1$. Thus $r(\det_i \otimes d_i^{-1}) = \chi_i \otimes 1$. The $\chi_i, 1 \leq i \leq k$, are generators of $X(A)$, although they need not be independent. Thus r is surjective. ∎

Let $X_0(M) = \{\chi \in X(M) : \chi|_{M^0} = 1\}$.

Lemma 5.2. *The kernel of r is $X_0(M) \otimes_{\mathbf{Z}} \mathbf{R}$.*

Proof: Suppose $\chi \otimes t$ is in the kernel of r where $\chi \in X(M)$ and $t \in \mathbf{R}$. If $t = 0$ then $\chi \otimes t$ is the identity element. Assume $t \neq 0$. Then $|\chi(a)|_F^t = 1$ for all $a \in A$ implies that $|\chi(a)|_F = 1$ for all $a \in A$. Since $\chi|_A$ is a rational character of a split torus this implies that $\chi|_A = 1$. But restriction from $X(M^0)$ to $X(A)$ is injective [**13**, Lemma 0.4.1], so that $\chi|_{M^0} = 1$. Thus $\chi \in X_0(M)$. ∎

Recall the homomorphism $H_{M^0} : M^0 \to Hom(X(M^0), \mathbf{Z})$ defined by

$$< H_{M^0}(m), \chi >= log_q|\chi(m)|_F, m \in M^0, \chi \in X(M^0).$$

Define an analogous homomorphism $H_M : M \to Hom(X(M), \mathbf{Z})$ by

$$< H_M(m), \chi >= log_q|\chi(m)|_F, m \in M, \chi \in X(M).$$

Lemma 5.3. *Suppose that $\chi \in X_0(M)$. Then $< H_M(m), \chi >= 0$ for all $m \in M$.*

.

Proof: Let $\chi \in X_0(M)$. Thus $\chi(m_0) = 1$ for all $m_0 \in M^0$. Let d be the index of M^0 in M. Thus $m^d \in M^0$ for all $m \in M$ so that $\chi(m^d) = 1$ for all $m \in M$. Thus $\chi(m)$ is a d^{th} root of unity and $|\chi(m)|_F = 1$ for all $m \in M$. Thus $< H_M(m), \chi >= log_q|\chi(m)|_F = 0$ for all $m \in M$. ∎

Recall that $Hom(X(M^0), \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{R} \simeq Hom(X(A), \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{R} = \underline{a}$ is the real Lie algebra of A, $\underline{a}^* = X(A) \otimes_{\mathbf{Z}} \mathbf{R}$ is its real dual, and $\underline{a}^*_{\mathbf{C}} = \underline{a}^* \otimes_{\mathbf{R}} \mathbf{C}$ is its complex dual. For each $\nu \in \underline{a}^*_{\mathbf{C}}$ we have a character χ^0_ν of M^0 defined by $\chi^0_\nu(m) = q^{<H_{M^0}(m), \nu>}, m \in M^0$. By Lemmas 5.1 and 5.2 the mapping r above induces an isomomorphism

$$r_* : \frac{X(M) \otimes_{\mathbf{Z}} \mathbf{C}}{X_0(M) \otimes_{\mathbf{Z}} \mathbf{C}} \simeq X(A) \otimes_{\mathbf{Z}} \mathbf{C} \simeq \underline{a}^*_{\mathbf{C}}.$$

By Lemma 5.3, for each $m \in M$, $H_M(m)$ is an element of the complex dual of $\frac{X(M)\otimes_{\mathbf{Z}}\mathbf{C}}{X_0(M)\otimes_{\mathbf{Z}}\mathbf{C}}$. Thus for each $\nu \in \underline{a}^*_{\mathbf{C}}$, we can define a character χ_ν of M by

$$\chi_\nu(m) = q^{<H_M(m), r_*^{-1}(\nu)>}, m \in M.$$

Lemma 5.4. *For all $\nu \in \underline{a}^*_{\mathbf{C}}$, the restriction of χ_ν to M^0 is χ^0_ν.*

Proof: For $m_0 \in M^0, \chi \in X(M)$ we have

$$< H_M(m_0), \chi >= log_q|\chi(m_0)|_F = log_q|\chi_0(m_0)|_F =< H_{M_0}(m_0), \chi_0 >$$

where χ_0 denotes the restriction of χ to M^0. Since the isomorphism r_* comes from the restriction map it is easy to see that $< H_M(m_0), r_*^{-1}(\nu) >=< H_{M^0}(m_0), \nu >$ for all $m_0 \in M^0, \nu \in \underline{a}^*_{\mathbf{C}}$. ∎

As above, let σ be an irreducible discrete series representation of M and let σ_0 be an irreducible constituent of the restriction of σ to M^0. Let V_0 be the representation space for σ_0 and let

$$W = \{f : M \to V_0 : f(mm_0) = \sigma_0(m_0)^{-1}f(m) \text{ for all } m \in M, m_0 \in M^0\}.$$

Then M acts on W by left translation and we will call this induced representation (I_M, W). Let V denote the representation space for σ and fix a non-zero intertwining operator $S : V \to W$ so that $S\sigma(m) = I_M(m)S$ for all $m \in M$. Since I_M is unitary, we can also define a projection operator $P : W \to V$ so that $PI_M(m) = \sigma(m)P$ for all $m \in M$ and $PSv = v$ for all $v \in V$. We also define representation spaces

$$\mathcal{H}_P(\sigma) = \{\phi \in C^\infty(G, V) : \phi(xmn) = \delta_P^{-\frac{1}{2}}(m)\sigma(m)^{-1}\phi(x) \text{ for all } x \in G, m \in M, n \in N\};$$

$$\mathcal{H}_{P^0}(\sigma_0) =$$

$$\{\psi \in C^\infty(G, V_0) : \psi(xm_0n) = \delta_{P^0}^{-\frac{1}{2}}(m_0)\sigma_0(m_0)^{-1}\psi(x) \text{ for all } x \in G, m_0 \in M^0, n \in N\};$$

$$\mathcal{H}_P(I_M) =$$

$$\{\psi \in C^\infty(G, W) : \psi(xmn) = \delta_P^{-\frac{1}{2}}(m)I_M(m)^{-1}\psi(x) \text{ for all } x \in G, m \in M, n \in N\}.$$

In each case G acts on the representation space by left translations and we call the induced representations $I_P(\sigma), I_{P^0}(\sigma_0)$, and $I_P(I_M)$ respectively. They are the representations $\mathrm{Ind}_P^G(\sigma), \mathrm{Ind}_{P^0}^G(\sigma_0)$, and $\mathrm{Ind}_P^G(\mathrm{Ind}_{M^0}^M(\sigma_0))$ respectively.

The intertwining operators $S : V \to W$ and $P : W \to V$ induce intertwining operators S^* from $(I_P(\sigma), \mathcal{H}_P(\sigma))$ to $(I_P(I_M), \mathcal{H}_P(I_M))$ and P^* from $(I_P(I_M), \mathcal{H}_P(I_M))$ to $(I_P(\sigma), \mathcal{H}_P(\sigma))$ given by

$$(S^*\phi)(x) = S\phi(x) \text{ for all } \phi \in \mathcal{H}_P(\sigma), x \in G;$$

$$(P^*\psi)(x) = P\psi(x) \text{ for all } \psi \in \mathcal{H}_P(I_M), x \in G.$$

There is also an equivalence T between $(I_P(I_M), \mathcal{H}_P(I_M))$ and $(I_{P^0}(\sigma_0), \mathcal{H}_{P^0}(\sigma_0))$ given by

$$(T\psi)(x) = \psi(x)(1) \text{ for all } \psi \in \mathcal{H}_P(I_M), x \in G.$$

Its inverse is given by

$$(T^{-1}\psi')(x)(m) = \delta_P^{\frac{1}{2}}(m)\psi'(xm) \text{ for all } \psi' \in \mathcal{H}_{P^0}(\sigma_0), x \in G, m \in M.$$

Recall for each $\nu \in \underline{a}_{\mathbf{C}}^*$ we have defined characters χ_ν^0 of M^0 and χ_ν of M such that χ_ν^0 is the restriction of χ_ν to M^0. We use these characters to define representations $\sigma(\nu) = \sigma \otimes \chi_\nu$ and $I_M(\nu) = I_M \otimes \chi_\nu$ of M and $\sigma_0(\nu) = \sigma_0 \otimes \chi_0(\nu)$ of M^0. As above we use these to form induced representation spaces $\mathcal{H}_P(\sigma, \nu) = \mathcal{H}_P(\sigma(\nu)), \mathcal{H}_P(I_M, \nu) = \mathcal{H}_P(I_M(\nu))$, and $\mathcal{H}_{P^0}(\sigma, \nu) = \mathcal{H}_{P^0}(\sigma_0(\nu))$. The intertwining operators $S : V \to W$ and $P : W \to V$ also intertwine $\sigma(\nu)$ and $I_M(\nu)$ and so as above define induced intertwining operators $S_\nu^* : \mathcal{H}_P(\sigma, \nu) \to \mathcal{H}_P(I_M, \nu)$ and $P_\nu^* : \mathcal{H}_P(I_M, \nu) \to \mathcal{H}_P(\sigma, \nu)$. There are also equivalences $T_\nu : \mathcal{H}_P(I_M, \nu) \to \mathcal{H}_{P^0}(\sigma_0, \nu)$ given by

$$(T_\nu\psi)(x) = \psi(x)(1) \text{ for all } \psi \in \mathcal{H}_P(I_M, \nu), x \in G.$$

The inverses are given by

$$(T_\nu^{-1}\psi')(x)(m) = \delta_P^{\frac{1}{2}}(m)\chi_\nu(m)\psi'(xm) \text{ for all } \psi' \in \mathcal{H}_{P^0}(\sigma_0, \nu), x \in G, m \in M.$$

Suppose $P_1 = MN_1$ and $P_2 = MN_2$ are two cuspidal parabolic subgroups of G with Levi component M. In §4 we defined a meromorphic family of intertwining operators

$$J(P_2^0 : P_1^0 : \sigma_0 : \nu) : \mathcal{H}_{P_1^0}(\sigma_0, \nu) \to \mathcal{H}_{P_2^0}(\sigma_0, \nu).$$

We can transfer these intertwining operators to the equivalent spaces $\mathcal{H}_{P_i}(I_M, \nu), i = 1, 2$, by means of the equivalences T_{ν, P_i}. Thus we define

$$J(P_2 : P_1 : I_M : \nu) : \mathcal{H}_{P_1}(I_M, \nu) \to \mathcal{H}_{P_2}(I_M, \nu)$$

by

$$J(P_2 : P_1 : I_M : \nu) = T_{\nu, P_2}^{-1} J(P_2^0 : P_1^0 : \sigma_0 : \nu) T_{\nu, P_1}.$$

We can also define

$$J(P_2 : P_1 : \sigma : \nu) : \mathcal{H}_{P_1}(\sigma, \nu) \to \mathcal{H}_{P_2}(\sigma, \nu)$$

by

$$J(P_2 : P_1 : \sigma : \nu) = P_{\nu, P_2}^* J(P_2 : P_1 : I_M : \nu) S_{\nu, P_1}^*.$$

Lemma 5.5. *Suppose that $\nu \in \underline{a}_{\mathbf{C}}^*(P_2^0 : P_1^0)$ so that $J(P_2^0 : P_1^0 : \sigma_0 : \nu)$ is given by the convergent integral*

$$J(P_2^0 : P_1^0 : \sigma_0 : \nu)\psi'(x) = \int_{\overline{N}_1 \cap N_2} \psi'(x\overline{n}) d\overline{n}, x \in G, \psi' \in \mathcal{H}_{P_1^0}(\sigma_0, \nu).$$

Then $J(P_2 : P_1 : I_M : \nu)$ is given by the convergent integral

$$J(P_2 : P_1 : I_M : \nu)\psi(x) = \int_{\overline{N}_1 \cap N_2} \psi(x\overline{n}) d\overline{n}, x \in G, \psi \in \mathcal{H}_{P_1}(I_M, \nu)$$

and $J(P_2 : P_1 : \sigma : \nu)$ is also given by the convergent integral

$$J(P_2 : P_1 : \sigma : \nu)\phi(x) = \int_{\overline{N}_1 \cap N_2} \phi(x\overline{n})d\overline{n}, x \in G, \phi \in \mathcal{H}_{P_1}(\sigma, \nu).$$

Proof: Using the definitions of the operators and the transformation property of the representation space $\mathcal{H}_{P_1}(I_M, \nu)$, we have for all $x \in G, m \in M, \psi \in \mathcal{H}_{P_1}(I_M, \nu)$,

$$J(P_2 : P_1 : I_M : \nu)\psi(x)(m) = T^{-1}_{\nu,P_2} J(P^0_2 : P^0_1 : \sigma_0 : \nu)T_{\nu,P_1}\psi(x)(m)$$

$$= \chi_\nu(m)\delta^{\frac{1}{2}}_{P_2}(m)(J(P^0_2 : P^0_1 : \sigma_0 : \nu)T_{\nu,P_1}\psi)(xm)$$

$$= \chi_\nu(m)\delta^{\frac{1}{2}}_{P_2}(m) \int_{\overline{N}_1 \cap N_2} (T_{\nu,P_1}\psi)(xm\overline{n})d\overline{n}$$

$$= \chi_\nu(m)\delta^{\frac{1}{2}}_{P_2}(m) \int_{\overline{N}_1 \cap N_2} \psi(xm\overline{n})(1)d\overline{n}$$

$$= \delta^{\frac{1}{2}}_{P_2}(m) \int_{\overline{N}_1 \cap N_2} \delta^{-\frac{1}{2}}_{P_1}(m)\psi(xm\overline{n}m^{-1})(m)d\overline{n}.$$

Now there is a homomorphism $\beta : M \to \mathbf{R}^+$ so that for all $m \in M, f \in C^\infty_c(\overline{N}_1 \cap N_2)$,

$$\int_{\overline{N}_1 \cap N_2} f(m\overline{n}m^{-1})d\overline{n} = \beta(m) \int_{\overline{N}_1 \cap N_2} f(\overline{n})d\overline{n}.$$

For $m \in M^0$ we know that

$$\beta(m) = \delta^{\frac{1}{2}}_{P_1}(m)\delta^{-\frac{1}{2}}_{P_2}(m).$$

Since $\beta\delta^{-\frac{1}{2}}_{P_1}\delta^{\frac{1}{2}}_{P_2}$ is a homomorphism from the finite group M/M^0 into $\mathbf{R}^+$, it must be identically one. Hence for all $m \in M$ we have

$$\int_{\overline{N}_1 \cap N_2} \psi(xm\overline{n}m^{-1})(m)d\overline{n} = \delta^{\frac{1}{2}}_{P_1}(m)\delta^{-\frac{1}{2}}_{P_2}(m) \int_{\overline{N}_1 \cap N_2} \psi(x\overline{n})(m)d\overline{n}.$$

Thus

$$J(P_2 : P_1 : I_M : \nu)\psi(x)(m) = \int_{\overline{N}_1 \cap N_2} \psi(x\overline{n})(m)d\overline{n}.$$

Now for $x \in G, m \in M, \phi \in \mathcal{H}_{P_1}(\sigma, \nu)$, we have

$$J(P_2 : P_1 : \sigma : \nu)\phi(x) = [P^*_{\nu,P_2} J(P_2 : P_1 : I_M : \nu)S^*_{\nu,P_1}\phi](x)$$

$$= P \cdot [J(P_2 : P_1 : I_M : \nu)S^*_{\nu,P_1}\phi](x)$$

$$= P \cdot \int_{\overline{N}_1 \cap N_2} [S^*_{\nu,P_1}\phi](x\overline{n})d\overline{n}$$

$$= PS \cdot \int_{\overline{N}_1 \cap N_2} \phi(x\overline{n})d\overline{n}$$

$$= \int_{\overline{N}_1 \cap N_2} \phi(x\overline{n})d\overline{n}.$$

∎

Let $r(P_2^0 : P_1^0 : \sigma_0 : \nu)$ be the scalar normalizing factors used in §4 to define the normalized intertwining operators

$$R(P_2^0 : P_1^0 : \sigma_0) = r(P_2^0 : P_1^0 : \sigma_0 : 0)^{-1} J(P_2^0 : P_1^0 : \sigma_0 : 0).$$

The fact that

$$r(P_2^0 : P_1^0 : \sigma_0 : \nu)^{-1} J(P_2^0 : P_1^0 : \sigma_0 : \nu)$$

is holomorphic and non-zero at $\nu = 0$ and that $T_{\nu,P_i}, i = 1, 2$, are equivalences, will imply that

$$r(P_2^0 : P_1^0 : \sigma_0 : \nu)^{-1} J(P_2 : P_1 : I_M : \nu)$$

is also holomorphic and non-zero at $\nu = 0$. Thus we can define

$$R(P_2 : P_1 : I_M) = r(P_2^0 : P_1^0 : \sigma_0 : 0)^{-1} J(P_2 : P_1 : I_M : 0).$$

We also define

$$R(P_2 : P_1 : \sigma) = r(P_2^0 : P_1^0 : \sigma_0 : 0)^{-1} J(P_2 : P_1 : \sigma : 0) = P^* R(P_2 : P_1 : I_M) S^*.$$

Lemma 5.6. *Let $\phi \in \mathcal{H}_{P_1}(\sigma)$. Then $R(P_2 : P_1 : I_M) S^* \phi(x) \in S(V)$ for all $x \in G$.*

Proof: For every ν we have an intertwining operator $J(P_2 : P_1 : I_M : \nu) S^*_\nu : \mathcal{H}_{P_1}(\sigma, \nu) \to \mathcal{H}_{P_2}(I_M : \nu)$. In order to carry out arguments using the integral formula and meromorphic extension of the intertwining operator we want a compact realization of the representation. Since we do not know if there is a maximal compact subgroup of G which meets every connected component, we proceed one coset at a time. Let $G_M = G^0 M$ and write $G = \cup_{i=1}^k x_i G_M$. Then $P \subset G_M$ for any parabolic subgroup P of G with Levi component M. Let K^0 be a good maximal compact subgroup of G^0 so that $G^0 = K^0 P^0$. Thus $G_M = K^0 P$. Let $\mathcal{H}_{K^0}(\sigma) =$

$$\{f_K \in C^\infty(K^0, V) : f_K(kmn) = \sigma^{-1}(m) f_K(k) \text{ for all } m \in K^0 \cap M, n \in K^0 \cap N, k \in K^0\}.$$

For any $\phi \in \mathcal{H}_P(\sigma : \nu)$ and $1 \le i \le k$ we can define

$$\phi_i(x) = \begin{cases} \phi(x), & \text{if } x \in x_i G_M; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\phi_i \in \mathcal{H}_P(\sigma : \nu)$ for each i and $\phi = \sum_{i=1}^k \phi_i$. Thus every element of $\mathcal{H}_P(\sigma : \nu)$ is a sum of elements supported on a single coset of G_M in G and so it is enough to prove the lemma for $\phi \in \mathcal{H}_{P_1}(\sigma)$ supported on a single coset of G_M in G.

Fix $1 \le i \le k$ and define

$$F_i(\nu) : \mathcal{H}_{P_1}(\sigma : \nu) \to \mathcal{H}_{K^0}(\sigma)$$

by $F_i(\nu)\phi(k) = \phi(x_i k)$ for all $k \in K^0$. Define

$$F_i^{-1}(\nu) : \mathcal{H}_{K^0}(\sigma) \to \mathcal{H}_{P_1}(\sigma : \nu)$$

by

$$F_i^{-1}(\nu)f_K(x) = \begin{cases} \delta_{P_1}^{-\frac{1}{2}}(m)\sigma^{-1}(m)\chi_\nu^{-1}(m)f_K(k), & \text{if } x = x_ikmn, k \in K^0, m \in M, n \in N; \\ 0, & \text{otherwise.} \end{cases}$$

Then $F_i(\nu)F_i^{-1}(\nu)f_K = f_K$ for all $f_K \in \mathcal{H}_{K^0}(\sigma)$ and $F_i^{-1}(\nu)F_i(\nu)\phi = \phi_i$ for all $\phi \in \mathcal{H}_{P_1}(\sigma : \nu)$.

Fix $1 \leq i \leq k$ and $\phi \in \mathcal{H}_{P_1}(\sigma) = \mathcal{H}_{P_1}(\sigma : 0)$ such that $\phi = \phi_i$ is supported on x_iG_M. Let $f_K = F_i(0)\phi \in \mathcal{H}_{K^0}(\sigma)$ and for each ν define $\phi_i(\nu) \in \mathcal{H}_{P_1}(\sigma : \nu)$ by $\phi_i(\nu : x) = F_i^{-1}(\nu)f_K(x), x \in G$. We have $\phi_i(0) = F_i^{-1}(0)F_i(0)\phi = \phi_i = \phi$.

Fix $w^* \in S(V)^\perp$ and define $\Phi(\nu : x) =< J(P_2 : P_1 : I_M : \nu)S_\nu^*\phi_i(\nu : x), w^* >$. Then $\nu \mapsto \Phi(\nu : x)$ is a meromorphic function of $\nu \in \underline{a}_{\mathbf{C}}^*$ for each $x \in G$. If $\nu \in \underline{a}_{\mathbf{C}}^*(P_2^0 : P_1^0)$, by Lemma 5.5 we have

$$J(P_2 : P_1 : I_M : \nu)S_\nu^*\phi_i(\nu : x) = \int_{\overline{N}_1 \cap N_2} S\phi_i(\nu : x\overline{n})d\overline{n} \in S(V).$$

Thus $\Phi(x : \nu) = 0$ for all $\nu \in \underline{a}_{\mathbf{C}}^*(P_2^0 : P_1^0)$ and hence for all ν. Thus

$$J(P_2 : P_1 : I_M : \nu)S_\nu^*\phi_i(\nu : x) \in S(V)$$

for all ν and so

$$r(P_2^0 : P_1^0 : \sigma_0 : \nu)^{-1}J(P_2 : P_1 : I_M : \nu)S_\nu^*\phi_i(\nu : x) \in S(V)$$

for all $\nu \in \underline{a}_{\mathbf{C}}^*, x \in G$. In particular for $\nu = 0$ we have $R(P_2 : P_1 : I_M)S^*\phi(x) \in S(V)$ for all $x \in G$. ∎

Corollary 5.7. *Let $\phi \in \mathcal{H}_{P_1}(\sigma)$. Then $R(P_2 : P_1 : I_M)S^*\phi = S^*R(P_2 : P_1 : \sigma)\phi$.*

Proof: This follows from Lemma 5.6 since SP is the identity on $S(V)$. ∎

Lemma 5.8. *Suppose P_1, P_2, and P_3 are cuspidal parabolic subgroups of G with Levi component M. Then*

$$R(P_1 : P_3 : I_M) = R(P_1 : P_2 : I_M)R(P_2 : P_3 : I_M)$$

and

$$R(P_1 : P_3 : \sigma) = R(P_1 : P_2 : \sigma)R(P_2 : P_3 : \sigma).$$

Proof: The statement for I_M follows easily from Lemma 4.8 since the intertwining operators T_ν are equivalences. Now using Corollary 5.7 we have

$$R(P_1 : P_3 : \sigma) = P^*R(P_1 : P_3 : I_M)S^* = P^*R(P_1 : P_2 : I_M)R(P_2 : P_3 : I_M)S^*$$
$$P^*R(P_1 : P_2 : I_M)S^*R(P_2 : P_3 : \sigma) = R(P_1 : P_2 : \sigma)R(P_2 : P_3 : \sigma).$$

∎

Lemma 5.9. *Suppose P_1 and P_2 are parabolic subgroups of G with Levi component M. Then $R(P_2 : P_1 : \sigma)$ is an equivalence from $\mathcal{H}_{P_1}(\sigma)$ onto $\mathcal{H}_{P_2}(\sigma)$.*

Proof: The proof is the same as that of Lemma 4.9. ∎

Now as above we define $N_G(\sigma_0) = \{g \in N_G(A) : \sigma_0^g \simeq \sigma_0\}$. If $w \in W_G(\sigma_0) = N_G(\sigma_0)/M^0$, σ_0 can be extended to a representation $\sigma_{0,w}$ of the group M_w^0 generated by M^0 and any representative n_w for w. Define $T(n_w) : W \to W$ by

$$T(n_w)f(m) = \sigma_{0,w}(n_w)f(n_w^{-1}mn_w), m \in M.$$

It is easy to check that

$$T(n_w)I_M(m) = I_M(n_w m n_w^{-1})T(n_w), m \in M.$$

Next we define intertwining operators $B_P(w) : \mathcal{H}_{w^{-1}Pw}(I_M) \to \mathcal{H}_P(I_M)$ by

$$B_P(w)\psi(g) = \alpha_{w^{-1}Pw}(n_w)^{\frac{1}{2}}T(n_w)\psi(gn_w), g \in G.$$

Finally we define self-intertwining operators $R(w, I_M) : \mathcal{H}_P(I_M) \to \mathcal{H}_P(I_M)$ by

$$R(w, I_M) = B_P(w)R(w^{-1}Pw : P : I_M).$$

Recall that in §4 we defined intertwining operators

$$A_P(w) : \mathcal{H}_{w^{-1}P^0w}(\sigma_0) \to \mathcal{H}_{P^0}(\sigma_0) \text{ and } R(w, \sigma_0) : \mathcal{H}_{P^0}(\sigma_0) \to \mathcal{H}_{P^0}(\sigma_0).$$

It is easy to check that

$$B_P(w) = T_P^{-1}A_P(w)T_{w^{-1}Pw} \text{ and } R(w, I_M) = T_P^{-1}R(w, \sigma_0)T_P$$

where $T_P : \mathcal{H}_P(I_M) \to \mathcal{H}_{P^0}(\sigma_0)$ is the equivalence defined above. In particular this implies that $B_P(w)$ and $R(w, I_M)$ are independent of the coset representatives chosen.

Define $N_G(\sigma) = \{g \in N_G(M) : \sigma^g \simeq \sigma\}$. If $w \in W_G(\sigma) = N_G(\sigma)/M$, σ can be extended to a representation of the group M_w generated by M and any representative n_w for w. Denote such an extension by σ_w and define $A'_P(w) : \mathcal{H}_{w^{-1}Pw}(\sigma) \to \mathcal{H}_P(\sigma)$ by

$$(A'_P(w)\phi)(x) = \sigma_w(n_w)\alpha_{w^{-1}Pw}(n_w)^{\frac{1}{2}}\phi(xn_w).$$

Lemma 5.10. *The intertwining operator $A'_P(w)$ is independent of the choice of coset representative n_w for $w \in W_G(\sigma)$. For $w_1, w_2 \in W_G(\sigma)$ there is a non-zero constant $c_P(w_1, w_2)$ so that*

$$A'_P(w_1, w_2) = c_P(w_1, w_2)A'_P(w_1)A'_{w_1^{-1}Pw_1}(w_2).$$

Proof: The proof is exactly the same as that of Lemma 4.11. ∎

Finally, for $w \in W_G(\sigma)$, we define $R(w, \sigma) : \mathcal{H}_P(\sigma) \to \mathcal{H}_P(\sigma)$ by

$$R(w, \sigma) = A'_P(w)R(w^{-1}Pw : P : \sigma).$$

Note that for $u \in W_G(\sigma_0)$ we could also have defined an intertwining operator

$$R'(u, \sigma) : \mathcal{H}_P(\sigma) \to \mathcal{H}_P(\sigma)$$

by

$$R'(u, \sigma) = P^* R(u, I_M)S^*.$$

We want to relate these two definitions. Let

$$W_G(\sigma_0,\sigma) = [N_G(\sigma_0)\cap N_G(\sigma)]/M^0 \subset W_G(\sigma_0).$$

Suppose that $x \in N_G(\sigma_0)\cap N_G(\sigma)$. Then x represents an element $xM^0 \in W_G(\sigma_0,\sigma)$ and an element $xM \in W_G(\sigma)$. Let

$$p : W_G(\sigma_0,\sigma) \to W_G(\sigma)$$

be given by $p(xM^0) = xM, x \in N_G(\sigma_0)\cap N_G(\sigma)$.

Lemma 5.11. *The mapping p is surjective. Its kernel is $W_M(\sigma_0) = N_M(\sigma_0)/M^0$.*

Proof: Let $w \in W_G(\sigma)$ and let $x \in N_G(\sigma)$ be a representative for w. Then σ_0^x is contained in the restriction of $\sigma^x \simeq \sigma$ to M^0 so that there is $m \in M$ such that $\sigma_0^x \simeq \sigma_0^m$. Hence w has a representative $xm^{-1} \in N_G(\sigma)\cap N_G(\sigma_0)$. Thus p is surjective. Clearly $p(xM^0) = M$ just in case $x \in N_G(\sigma_0)\cap N_G(\sigma)\cap M = N_M(\sigma_0)$. ∎

Lemma 5.12. *Suppose that $u \in W_G(\sigma_0)$ is in the complement of $W_G(\sigma_0,\sigma)$. Then*

$$R'(u,\sigma) = 0.$$

If $u \in W_G(\sigma_0,\sigma)$, then there is a complex constant c so that

$$R'(u,\sigma) = cR(p(u),\sigma).$$

Proof: Using Corollary 5.7, for any $u \in W_G(\sigma_0)$,

$$\begin{aligned} R'(u,\sigma) &= P^*R(u,I_M)S^* = P^*B_P(u)R(u^{-1}Pu : P : I_M)S^* \\ &= P^*B_P(u)S^*R(u^{-1}Pu : P : \sigma). \end{aligned}$$

But for any $g \in G, \phi \in \mathcal{H}_{u^{-1}Pu}(\sigma)$, if $x \in N_G(\sigma_0)$ is a representative for u,

$$P^*B_P(u)S^*\phi(g) = \alpha_{u^{-1}Pu}(x)^{\frac{1}{2}}(PT(x)S)\phi(gx).$$

Since $T(x)$ intertwines I_M and I_M^x, we see that $PT(x)S$ intertwines σ and $\sigma^x \simeq \sigma^u$. Thus $PT(x)S = 0$ and hence $R'(u,\sigma) = 0$ unless $\sigma^u \simeq \sigma$.

Suppose $u \in W_G(\sigma_0,\sigma)$. Write $u = u_x, w = w_x = p(u), x \in N_G(\sigma_0)\cap N_G(\sigma)$. Then $PT(x)S$ and $\sigma_w(x)$ both intertwine σ and σ^x, and $\sigma_w(x) \neq 0$. Thus there is a complex constant c' so that $PT(x)S = c'\sigma_w(x)$. Thus for any $g \in G$,

$$\begin{aligned} R'(u,\sigma)\phi(g) &= \alpha_{u^{-1}Pu}(x)^{\frac{1}{2}}(PT(x)S)[R(u^{-1}Pu : P : \sigma)\phi](gx) \\ &= c'\alpha_{u^{-1}Pu}(x)^{\frac{1}{2}}\sigma_w(x)[R(u^{-1}Pu : P : \sigma)\phi](gx) \\ &= c'A'_P(w)R(w^{-1}Pw : P : \sigma)\phi(g) = c'R(w,\sigma)\phi(g). \end{aligned}$$

∎

Lemma 5.13. *The $R(w,\sigma), w \in W_G(\sigma)$, span the algebra $C(\sigma)$ of self-intertwining operators on $\mathcal{H}_P(\sigma)$.*

Proof: Let R be a self-intertwining operator for $\mathcal{H}_P(\sigma)$. Then S^*RP^* is a self-intertwining operator for $\mathcal{H}_P(I_M)$, hence in the span of the $R(u, I_M), u \in W_G(\sigma_0)$. But then $R = P^*S^*RP^*S^*$ is in the span of the $P^*R(u, I_M)S^* = R'(u, \sigma), u \in W_G(\sigma_0)$. But by Lemma 5.12, each $R'(u, \sigma)$ is either zero or a multiple of one of the operators $R(w, \sigma), w \in W_G(\sigma)$. ∎

Lemma 5.14. *Let $u \in W_G(\sigma_0)$ and suppose that $R(u, I_M)$ is scalar. Then $u \in W_G(\sigma_0, \sigma)$ and $R(p(u), \sigma)$ is scalar.*

Proof: Suppose that there is a constant $s \in \mathbf{C}$ such $R(u, I_M)\psi = s\psi$ for all $\psi \in \mathcal{H}_P(I_M)$. Since $R(u, I_M) \neq 0, s \neq 0$. Now for all $\phi \in \mathcal{H}_P(\sigma)$,

$$R'(u, \sigma)\phi = P^*R(u, I_M)S^*\phi = sP^*S^*\phi = s\phi.$$

Thus $R'(u, \sigma)$ is scalar and non-zero. Thus by Lemma 5.12 we have $u \in W_G(\sigma_0, \sigma)$. Further, by Lemma 5.12, there is a constant c so that $R'(u, \sigma) = cR(p(u), \sigma)$. Since $R'(u, \sigma) \neq 0, c \neq 0$. Thus $R(p(u), \sigma) = c^{-1}R'(u, \sigma)$ is scalar. ∎

Lemma 5.15. *There is a cocycle η so that*

$$R(w_1w_2, \sigma) = \eta(w_1, w_2)R(w_1, \sigma)R(w_2, \sigma)$$

for all $w_1, w_2 \in W_G(\sigma)$.

Proof: The proof is similar to Lemma 4.12. ∎

As in §4, if $W^0_{G^0}(\sigma_0)$ is the subgroup of elements $u \in W_{G^0}(\sigma_0)$ such that $R^0(u, \sigma_0)$ is scalar, then $W^0_{G^0}(\sigma_0) = W(\Phi_1)$ is generated by reflections in a set Φ_1 of reduced roots of (G, A). Let Φ^+, Φ_1^+ be defined as in §4. Since M centralizes A, $W_G(\sigma) \subset N_G(A)/M$ acts on roots of A and we can define

$$R_\sigma = \{w \in W_G(\sigma) : w\beta \in \Phi^+ \text{ for all } \beta \in \Phi_1^+\}.$$

We want to prove the following.

Theorem 5.16. *The $R(w, \sigma), w \in R_\sigma$, form a basis for the algebra of intertwining operators of $I_P(\sigma)$.*

In order to prove Theorem 5.16, we will first compute the dimension of $C(\sigma)$ using our knowledge of the dimension of $C(\sigma_0)$. We denote the equivalence classes of $\mathrm{Ind}_P^G(\sigma)$ and $\mathrm{Ind}_{P^0}^G(\sigma_0)$ by $i_{G,M}(\sigma)$ and $i_{G,M^0}(\sigma_0)$ respectively. Let X and Y denote the groups of unitary characters of G/G^0 and M/M^0 respectively. For $\chi \in X$, let $\chi_M \in Y$ denote the restriction of χ to M. Define

$$X(\sigma) = \{\chi \in X : \chi_M \otimes \sigma \simeq \sigma\};$$
$$X_1(\sigma) = \{\chi \in X : \chi \otimes i_{G,M}(\sigma) \simeq i_{G,M}(\sigma)\};$$
$$Y(\sigma) = \{\eta \in Y : \sigma \otimes \eta \simeq \sigma\}.$$

Let s denote the multiplicity of σ_0 in the restriction of σ to M^0.

Lemma 5.17.

$$\dim C(\sigma_0) = \dim C(\sigma) s^2 [X/X(\sigma)][X_1(\sigma)/X(\sigma)].$$

Proof: Using Lemma 2.13,

$$i_{M,M^0}(\sigma_0) = s \sum_{\eta \in Y/Y(\sigma)} \sigma \otimes \eta.$$

This implies that

$$i_{G,M}(i_{M,M^0}(\sigma_0)) = s \sum_{\eta \in Y/Y(\sigma)} i_{G,M}(\sigma \otimes \eta).$$

Since both G/G^0 and M/M^0 are finite abelian, it is clear that the map $\chi \mapsto \chi_M$ induces an isomorphism between $X/X(\sigma)$ and $Y/Y(\sigma)$. Thus we can rewrite

$$i_{G,M}(i_{M,M^0}(\sigma_0)) = s \sum_{\chi \in X/X(\sigma)} i_{G,M}(\sigma \otimes \chi_M).$$

But by Corollary 3.3 the induced representations $i_{G,M}(\sigma \otimes \chi_M)$ are either disjoint or equal. Further,

$$i_{G,M}(\sigma \otimes \chi_M) = i_{G,M}(\sigma) \otimes \chi = i_{G,M}(\sigma)$$

just in case $\chi \in X_1(\sigma)$. Thus we have

$$i_{G,M}(i_{M,M^0}(\sigma_0)) = s[X_1(\sigma)/X(\sigma)] \sum_{\chi \in X/X_1(\sigma)} i_{G,M}(\sigma) \otimes \chi$$

where the representations $i_{G,M}(\sigma) \otimes \chi$ are disjoint for $\chi \in X/X_1(\sigma)$. Thus

$$\begin{aligned} \dim C(\sigma_0) &= \dim C(\sigma) s^2 [X_1(\sigma)/X(\sigma)]^2 [X/X_1(\sigma)] \\ &= \dim C(\sigma) s^2 [X/X(\sigma)][X_1(\sigma)/X(\sigma)]. \end{aligned}$$

■

Lemma 5.18.

$$s^2[X/X(\sigma)][X_1(\sigma)/X(\sigma)] = [W_G(\sigma_0)]/[W_G(\sigma)].$$

Proof: First, using Lemma 2.13 we have

$$s^2[X/X(\sigma)] = s^2[Y/Y(\sigma)] = [N_M(\sigma_0)/M^0] = [W_M(\sigma_0)].$$

We claim that

$$[X_1(\sigma)/X(\sigma)] = [W_G(\sigma_0)/W_G(\sigma_0, \sigma)].$$

This would establish the lemma since by Lemma 5.11 we have

$$[W_G(\sigma)] = [W_G(\sigma_0, \sigma)/W_M(\sigma_0)].$$

We will define a bijection between $X_1(\sigma)/X(\sigma)$ and

$$(N_G(\sigma_0) \cap N_G(\sigma)) \backslash N_G(\sigma_0) \simeq W_G(\sigma_0, \sigma) \backslash W_G(\sigma_0).$$

Let $\chi \in X_1(\sigma)$. The equivalence class of $\sigma \otimes \chi_M$ depends only on the coset $\overline{\chi}$ of χ in $X_1(\sigma)/X(\sigma)$. Further, by definition of $X_1(\sigma)$, we have

$$i_{G,M}(\sigma \otimes \chi_M) = i_{G,M}(\sigma).$$

By Corollary 3.2 there is $x \in N_G(A)$ such that $\sigma \otimes \chi_M \simeq \sigma^x$. Thus $\sigma|_{M^0} \simeq \sigma^x|_{M^0}$. Thus σ_0^x occurs in $\sigma|_{M^0}$ and so there is $m \in M$ such that $\sigma_0^x \simeq \sigma_0^m$. Then $y = xm^{-1} \in N_G(\sigma_0)$. Although $y \in N_G(\sigma_0)$ is not uniquely determined by χ,

$$\sigma \otimes \chi_M \simeq \sigma^{y_1} \simeq \sigma^{y_2}$$

if and only if $y_1 y_2^{-1} \in N_G(\sigma_0) \cap N_G(\sigma)$. Thus for each $\overline{\chi} \in X_1(\sigma)/X(\sigma)$ there is a unique coset $\overline{x}(\overline{\chi}) = (N_G(\sigma_0) \cap N_G(\sigma))x$ in $(N_G(\sigma_0) \cap N_G(\sigma)) \backslash N_G(\sigma_0)$ such that $\chi_M \otimes \sigma \simeq \sigma^x$. Finally, given $x \in N_G(\sigma_0), \sigma^x$ is a constituent of $i_{M,M^0}(\sigma_0^x) \simeq i_{M,M^0}(\sigma_0)$ so that there is $\eta \in Y$ such that $\sigma^x \simeq \sigma \otimes \eta$. Now let $\chi \in X$ such that $\chi_M = \eta$. Then $\sigma \otimes \chi_M \simeq \sigma^x$ so that $\overline{x} = \overline{x}(\overline{\chi})$. ∎

Recall from §4 that $W_G(\sigma_0)$ is the semidirect product of subgroups R_{σ_0} and $W(\Phi_1)$ where $R(w, \sigma_0)$ is scalar for $w \in W(\Phi_1)$ and the $R(r, \sigma_0), r \in R_{\sigma_0}$, give a basis for $C(\sigma_0)$.

Lemma 5.19.

$$\dim C(\sigma) = [W_G(\sigma)]/[W(\Phi_1)].$$

Proof: Combining Lemmas 5.17 and 5.18 we have

$$\dim C(\sigma_0) = \dim C(\sigma) \cdot \frac{[W_G(\sigma_0)]}{[W_G(\sigma)]}.$$

But from Lemma 4.15, $\dim C(\sigma_0) = [R_{\sigma_0}] = [W_G(\sigma_0)]/[W(\Phi_1)]$. ∎

Since $W(\Phi_1) \subset N_{G^0}(A)/M^0$, it can be naturally embedded in $W_G(A) = N_G(A)/M$.

Lemma 5.20. *$W_G(\sigma)$ is the semidirect product of $W(\Phi_1)$ and R_σ. For $w \in W_G(\sigma)$, $R(w, \sigma)$ is scalar if and only if $w \in W(\Phi_1)$.*

Proof: If $x \in N_{G^0}(\sigma_0)$ represents an element of $W(\Phi_1)$, then by Lemma 4.14 $R(u_x, I_M)$ is scalar. Thus by Lemma 5.14, $w_x \in W_G(\sigma)$ and $R(w_x, \sigma)$ is scalar. Let $W_G^0(\sigma)$ denote the set of all $w \in W_G(\sigma)$ such that $R(w, \sigma)$ is scalar. By the above $W(\Phi_1) \subset W_G^0(\sigma)$. Using Lemmas 5.13 and 5.19 we see that

$$[W_G(\sigma)]/[W(\Phi_1)] = \dim C(\sigma) \leq [W_G(\sigma)]/[W_G^0(\sigma)].$$

Thus $W(\Phi_1) = W_G^0(\sigma)$.

Now as in the proof Lemma 4.14, $W(\Phi_1)$ is a normal subgroup of $W_G(\sigma)$ and so $w\Phi_1 = \Phi_1$ for all $w \in W_G(\sigma)$. This implies that

$$R_\sigma = \{w \in W_G(\sigma) : w\Phi_1^+ = \Phi_1^+\}$$

which yields the semidirect product decomposition. ∎

Proof of Theorem 5.16. It follows from Lemmas 5.13,5.15, and 5.20 that the $R(w,\sigma)$,
$w \in R_\sigma$, span the algebra $C(\sigma)$. Further, by Lemmas 5.19 and 5.20,

$$\dim C(\sigma) = [W_G(\sigma)]/W(\Phi_1)] = [R_\sigma].$$

■

Let η be the cocycle of Lemma 5.15. Exactly as in §4 we can fix a finite central extension

$$1 \to Z_\sigma \to \tilde{R}_\sigma \to R_\sigma \to 1$$

over which η splits, a character χ_σ of Z_σ, and a representation $\mathcal{R}$ of $\tilde{R}_\sigma \times G$ on $\mathcal{H}_P(\sigma)$. Let $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ denote the set of irreducible representations of $\tilde{R}_\sigma$ with Z_σ central character χ_σ, and let $\Pi_\sigma(G)$ denote the set of irreducible constituents of $I_P(\sigma)$.

Theorem 5.21. *There is a bijection $\rho \mapsto \pi_\rho$ of $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ onto $\Pi_\sigma(G)$ such that*

$$\mathcal{R} = \oplus_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \quad (\rho^\vee \otimes \pi_\rho).$$

Proof: The proof is exactly the same as that of Theorem 4.17. ■

6. Examples

For applications involving comparisons of representations between groups and twisted trace formulas it is customary to use the following definition of parabolic subgroup. Let P^0 be a parabolic subgroup of G^0. Then $P = N_G(P^0)$ is a parabolic subgroup of G. Thus, using this definition, parabolic subgroups of G are in one to one correspondence with parabolic subgroups of G^0. We will show that the parabolic subgroups obtained using this definition are also parabolic subgroups using the definition of §2. However they are not cuspidal in general. Indeed, recall from Proposition 2.10 that if P^0 is a parabolic subgroup of G, then the corresponding cuspidal parabolic subgroup is the smallest parabolic subgroup of G lying over P^0. On the other hand, if P is any subgroup of G with $P \cap G^0 = P^0$, then $P \subset N_G(P^0)$. Thus $N_G(P^0)$ will be the largest parabolic subgroup of G lying over P^0. We will give examples to show that this class of parabolic subgroups, which we call N-parabolic subgroups (N for normalizer), do not yield a nice theory of parabolically induced representations of G.

Lemma 6.1. *Let P^0 be a parabolic subgroup of G^0. Then $P = N_G(P^0)$ is a parabolic subgroup of G. It is the largest parabolic subgroup lying over P^0. If M^0 is a Levi component for P^0, then $M = N_G(M^0) \cap P$ is a Levi component for P.*

Proof: Let $P^0 = M^0 N$ be a Levi decomposition of P^0 and define $M = N_G(M^0) \cap P$. Then $M \cap G^0 = M^0$ and $MN \subset P$. Let $x \in P = N_G(P^0)$. Then xM^0x^{-1} is a Levi subgroup of P^0 and so there is $n \in N$ such that $xM^0x^{-1} = nM^0n^{-1}$. Now $n^{-1}x \in N_G(M^0) \cap N_G(P^0) = M$ and so $x \in NM = MN$. Thus $P = MN$.

Let A be the split component of M^0. Then M normalizes A and we define a Weyl group $W = M/C_M(A)$ where $C_M(A)$ denotes the centralizer of A in M. Since $M^0 \subset C_M(A)$, we know that W is a finite abelian group. The split component of M is $A' = \{a \in A : xax^{-1} = a \text{ for all } x \in M\} = \{a \in A : wa = a \text{ for all } a \in W\}$. Let $M' = C_G(A')$. If we can show that A' is the split component of M', then A' is a special vector subgroup.

Let $\Phi^+ = \Phi(P^0, A)$ denote the set of roots of A in P^0, $\underline{a}$ the real Lie algebra of A, and $\underline{a}^+$ the positive chamber of $\underline{a}$ with respect to Φ^+. Fix $w \in W$ and define $\underline{a}_w = \{H \in \underline{a} : wH = H\}$. Since M normalizes P^0, we have $w\Phi^+ = \Phi^+$ and $w\underline{a}^+ = \underline{a}^+$. Let k be the order of w. Then since $\underline{a}^+$ is convex, for any $H \in \underline{a}^+$ we have $H_w = H + wH + w^2H + ... + w^{k-1}H \in \underline{a}^+$ with $wH_w = H_w$. Thus $\underline{a}_w^+ = \underline{a}_w \cap \underline{a}^+ \neq \emptyset$ and so for any $\alpha \in \Phi^+$, the restriction of α to $\underline{a}_w$ is non-zero. Since W is a finite abelian group, an easy induction argument shows in fact that the restriction of α to $\underline{a}'$, the real Lie algebra of A', is non-zero for every $\alpha \in \Phi^+$. Thus $M' \cap G^0 = C_{G^0}(A') = C_{G^0}(A) = M^0$.

Let A'' be the split component of M'. Thus $A' \subset A''$. But since $M' \cap G^0 = M^0$ and $M \subset M'$, we have $A'' = \{a \in A : xax^{-1} = a \text{ for all } x \in M'\} \subset A'$. Thus $A' = A''$ is the split component of M'. This implies that A' is a special vector subgroup and that $M' = C_G(A')$ is a Levi subgroup of G. But since the restriction of α to $\underline{a}'$, is non-zero for every $\alpha \in \Phi^+$, we can choose a set of positive roots $(\Phi')^+$ of $L(G)$ with respect to $L(A')$ by restricting the roots in Φ^+. With this choice of positive roots, we obtain a parabolic subgroup $P' = M'N'$ of G with $N' = N$. Thus M' normalizes N. It also normalizes M^0 since $M' \cap G^0 = M^0$. Thus $M' \subset N_G(M^0) \cap N_G(P^0) = M$. Now $M' = M$ so $P' = MN = P$. ∎

Let (P_0^0, A_0) be a minimal p-pair in G^0 and let Δ denote the set of simple roots of A_0 in P_0^0. Then as usual the standard parabolic subgroups of G^0 are indexed by subsets Θ of Δ. Write (P_Θ^0, A_Θ) for the standard parabolic pair of G^0 corresponding to $\Theta \subset \Delta$ and write $P_\Theta = N_G(P_\Theta^0)$, the N-parabolic subgroup of G lying over P_Θ^0. Let $N_G(P_0^0, A_0)$ be the set of elements in G that normalize both A_0 and P_0^0. Clearly $N_G(P_0^0, A_0) \cap G^0 = P_0^0 \cap N_{G^0}(A_0) = C_{G^0}(A_0) = M_0^0$. Write $W_G(P_0^0, A_0) = N_G(P_0^0, A_0)/M_0^0$. Then $W_G(P_0^0, A_0)$ acts on Δ and for each $\Theta \subset \Delta$ we write

$$W(\Theta) = \{w \in W_G(P_0^0, A_0) : w\Theta = \Theta\}.$$

Lemma 6.2. *For all* $\Theta \subset \Delta$,

$$P_\Theta = \cup_{w \in W(\Theta)} w P_\Theta^0.$$

Proof: This follows from Lemma 3.8. ∎

Now that we have a simple method of computing the groups P_Θ, we will give examples to show the following unpleasant facts.

Fact 6.3. Let P_1^0 and P_2^0 be parabolic subgroups of G^0 and let $P_i = N_G(P_i^0)$, i = 1,2. Then $P_1^0 \subset P_2^0$ does not imply that $P_1 \subset P_2$.

Example 6.4. Let $G = O(2n), n \geq 2$. Then $G^0 = SO(2n)$ and the minimal parabolic subgroup B^0 of G^0 is the group of upper triangular matrices in $SO(2n)$ with A_0 the subgroup of diagonal matrices. The simple roots are

$$\Delta = \{e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n, e_{n-1} + e_n\}.$$

The Weyl group $W_G(B^0, A_0)$ has order two and is generated by the sign change c_n : $e_n \mapsto -e_n$ that interchanges $e_{n-1} - e_n$ and $e_{n-1} + e_n$. Thus a subset Θ of Δ is stable under c_n just in case neither or both of $e_{n-1} \pm e_n$ belong to Θ. So for example if $\Theta = \{e_{n-1} - e_n\}$, then $B^0 \subset P_\Theta^0$, but $B = B^0 \cup c_n B^0$ is not contained in $P_\Theta = P_\Theta^0$.

We call P_0 a minimal N-parabolic subgroup of G if given any N-parabolic subgroup P of G there is $x \in G$ such that $P_0 \subset xPx^{-1}$.

Fact 6.5. G need not have a minimal N-parabolic subgroup.

Suppose P_0 is a minimal N-parabolic subgroup of G. Then it is easy to see that $P_0^0 = P_0 \cap G^0$ is a minimal parabolic subgroup of G^0. Now by Lemma 6.2, P_0 meets every connected component of G. But as Example 6.4 shows, there are N-parabolic subgroups of $O(2n)$ which are contained in the identity component $SO(2n)$. Thus no conjugate could contain the minimal N-parabolic subgroup.

If $P^0 = M^0 N$ is a Levi decomposition for P^0, then we obtained a Levi decomposition $P = MN$ for $P = N_G(P^0)$ by defining $M = N_G(M^0) \cap P$. Thus M depends on both M^0 and P.

Fact 6.6. Suppose that $P_1 = M_1 N_1$ and $P_2 = M_2 N_2$ are N-parabolic subgroups of G such that $M_1^0 = M_2^0$, ie. P_1^0 and P_2^0 have the same Levi subgroup. Then it need not be true that $M_1 = M_2$, or even that M_1 and M_2 have the same number of connected components.

Example 6.7. Let $G = O(8)$ as in Example 6.4. Let $P_1 = P_{\Theta_1}$ where $\Theta_1 = \{e_3 - e_4\}$ and let $P_2' = P_{\Theta_2}$ where $\Theta_2 = \{e_1 - e_2\}$. Then $P_1 = P_1^0$ is connected and $P_2' = (P_2')^0 \cup c_4 (P_2')^0$ meets both components of G. Let $w = (13)(24) \in N_G(A_0)/A_0$ be the Weyl group element that permutes the pairs (e_1, e_3) and (e_2, e_4) and define $P_2 = wP_2'w^{-1}$. Then $wA_{\Theta_2}w^{-1} = A_{\Theta_1}$ and so $P_2^0 = w(P_2')^0 w^{-1}$ and P_1^0 both have Levi component $M_1^0 = C_{G^0}(A_{\Theta_1})$. However $M_1 = M_1^0$ is connected and $M_2 = M_2^0 \cup c_2 M_2^0$ meets both components of G.

In addition to structural problems, the class of N-parabolic subgroups does not yield a nice theory of parabolic induction. One of the basic cornerstones of representation theory in the connected case is that every irreducible admissible representation is contained in a representation which is parabolically induced from a supercuspidal representation and every tempered representation is a subrepresentation of a representation which is parabolically induced from a discrete series representation. But if supercuspidal and discrete series representations are defined as in the connected case and in §2, then the Levi component M of a parabolic subgroup P has no supercuspidal or discrete series representations unless P is cuspidal. Thus we will not in general be able to obtain all irreducible admissible or tempered representations of G via induction from supercuspidal or discrete series representations of N-parabolic subgroups.

Example 6.8. Define $G = O(2) = SO(2) \cup wSO(2)$ as in Remark 2.2. The only parabolic subgroup of $G^0 = SO(2) \simeq F^\times$ is itself. Thus the only N-parabolic subgroup of G is $N_G(G^0) = G$ itself which has split component $Z = \{1\}$. But G has no representations with compactly supported matrix coefficients, hence no supercuspidal representations. It also has no representations with square-integrable matrix coefficients, hence no discrete series representations.

As can be seen in Example 6.8, the problem with using the standard definitions for supercuspidal and discrete series representations with N-parabolic subgroups $P = MN$ is that the split component of M may be smaller than the split component of M^0. In order to guarantee the existence of enough supercuspidal and discrete series representations we could define a representation of M to be supercuspidal (respectively discrete series) just in case its restriction to M^0 is supercuspidal (respectively discrete series). Then it would be easy to prove as in Theorem 2.18 that every irreducible admissible (respectively tempered) representation of G is contained in a representation which is induced from a supercuspidal (respectively discrete series) representation of an N-parabolic subgroup.

Another basic property of parabolic induction in the connected case is the following. Suppose that $P_1 = M_1N_1$ and $P_2 = M_2N_2$ are parabolic subgroups and $\sigma_i, i = 1, 2$, are irreducible representations of M_i which are both either supercuspidal or discrete series. Then if the induced representations $\mathrm{Ind}_{P_i}^G(\sigma_i)$ are not disjoint, then the pairs (M_1, σ_1) and (M_2, σ_2) are conjugate. Further, in the discrete series case, the induced representations are equivalent. These properties fail in the disconnected case when the P_i are N-parabolic subgroups of G and supercuspidal and discrete series representations are defined as above.

Example 6.9. Let $G = O(8)$ and define P_1 and P_2 as in Example 6.7. Recall in this case that $M_1 = M_1^0 \simeq GL(2)\times GL(1)^2$ while $M_2 = M_2^0 \cup c_2M_2^0 \simeq GL(2)\times GL(1)\times O(2)$ with $M_2^0 = GL(2) \times GL(1) \times SO(2) = M_1^0$. Let $\sigma_0 = \rho \otimes \chi_1 \otimes \chi_2$ be an irreducible unitary supercuspidal representation of $M_1^0 = M_2^0$ where ρ is an irreducible unitary supercuspidal representation of $GL(2)$, χ_1 is a unitary character of $GL(1)$, and χ_2 is a non-trivial unitary character of $GL(1)$ with $\chi_2^2 = 1$. Then $\sigma_0^{c_2} = \rho \otimes \chi_1 \otimes \chi_2^{-1} = \sigma_0$ so there is an irreducible representation σ_2 of M_2 which extends σ_0. Further,

$$\mathrm{Ind}_{M_2^0}^{M_2}(\sigma_0) = \sigma_2 \oplus (\sigma_2 \otimes \eta)$$

where η is the non-trivial character of M_2/M_2^0. The representations σ_2 and $\sigma_2 \otimes \eta$ of M_2 would both be supercuspidal (and discrete series) since they both restrict to σ_0

on M_2^0. Now

$$\mathrm{Ind}_{P_2^0}^G(\sigma_0) \simeq \mathrm{Ind}_{P_2}^G \mathrm{Ind}_{M_2^0}^{M_2}(\sigma_0) = \mathrm{Ind}_{P_2}^G(\sigma_2) \oplus \mathrm{Ind}_{P_2}^G(\sigma_2 \otimes \eta).$$

Since P_1^0 and P_2^0 are parabolic subgroups of G^0 with the same Levi component $M_1^0 = M_2^0$ we have

$$\mathrm{Ind}_{P_1^0}^{G^0}(\sigma_0) \simeq \mathrm{Ind}_{P_2^0}^{G^0}(\sigma_0).$$

But $\sigma_1 = \sigma_0$ is an irreducible supercuspidal (and discrete series) representation of $M_1 = M_1^0$ and

$$\mathrm{Ind}_{P_1}^G(\sigma_1) = \mathrm{Ind}_{P_1^0}^G(\sigma_0) \simeq \mathrm{Ind}_{G^0}^G \mathrm{Ind}_{P_1^0}^{G^0}(\sigma_0)$$

$$\simeq \mathrm{Ind}_{G^0}^G \mathrm{Ind}_{P_2^0}^{G^0}(\sigma_0) \simeq \mathrm{Ind}_{P_2^0}^G(\sigma_0) \simeq \mathrm{Ind}_{P_2}^G(\sigma_2) \oplus \mathrm{Ind}_{P_2}^G(\sigma_2 \otimes \eta).$$

Thus we have irreducible supercuspidal (and discrete series) representations σ_1 of M_1 and σ_2 of M_2 so that

$$\mathrm{Ind}_{P_1}^G(\sigma_1) \simeq \mathrm{Ind}_{P_2}^G(\sigma_2) \oplus \mathrm{Ind}_{P_2}^G(\sigma_2 \otimes \eta).$$

Clearly M_1 and M_2 cannot be conjugate in G since M_1 is connected while M_2 is not. Further, the representations $\mathrm{Ind}_{P_i}^G(\sigma_i)$ have a nontrivial intertwining, but are not equivalent.

A final nice property of parabolic induction in the connected case is the theory of R-groups. If $P = MN$ is a parabolic subgroup and σ is an irreducible discrete series representation of M, then R is a subgroup of $W_G(\sigma)$, the group of Weyl group elements fixing σ, which determines the reducibility of $\mathrm{Ind}_P^G(\sigma)$. Its most basic property is that the dimension of the algebra of self-intertwining operators of $\mathrm{Ind}_P^G(\sigma)$ is equal to the number of elements in R. The following example shows that there could not be such a simple R-group theory in the disconnected case for N-parabolic subgroups.

Example 6.10. Let $G^0 = SL_2(F) \times SL_2(F) = \{(x,y) : x, y \in SL_2(F)\}$. Let $G = G^0 \cup \gamma G_0$ where $\gamma(x,y)\gamma^{-1} = (y,x)$. Let $B_1 = A_1N_1$ denote the usual Borel subgroup of $G_1 = SL_2(F)$ where A_1 is the subgroup of diagonal elements and N_1 is the subgroup of upper triangular matrices with ones on the diagonal. Then $B^0 = B_1 \times B_1$ is a Borel subgroup of G^0 and $B = N_G(B^0) = B^0 \cup \gamma B^0$. B^0 and B have Levi decompositions $B^0 = M^0N$ and $B = MN$ where $M^0 = A^0 = A_1 \times A_1, M = M^0 \cup \gamma M^0$, and $N = N_1 \times N_1$. We have Weyl groups $W(G^0, A^0) = W(G_1, A_1) \times W(G_1, A_1) \simeq Z_2 \times Z_2$ and $W(G, A^0) = W(G^0, A^0) \cup \gamma W(G^0, A^0) \simeq D_8$, the dihedral group of order 8.

Let χ_1 be a non-trivial character of $A_1 \simeq F^\times$ of order two so that $\mathrm{Ind}_{B_1}^{G_1}(\chi_1) = \pi_1 \oplus \pi_s$ is a reducible principal series representation. Note that the R-group here is $R_1 = W(G_1, A_1) \simeq Z_2$ and we denote the irreducible constituents of the induced representation by π_1, π_s to indicate that they are parameterized by the trivial and sign characters ρ_1 and ρ_s of R_1 respectively. Now let $\chi = \chi_1 \otimes \chi_1$. Then

$$\mathrm{Ind}_{B^0}^{G^0}(\chi) \simeq (\pi_1 \oplus \pi_s) \otimes (\pi_1 \oplus \pi_s) = \pi_{11} \oplus \pi_{1s} \oplus \pi_{s1} \oplus \pi_{ss}$$

where for $i, j \in \{1, s\}$, we write $\pi_{ij} = \pi_i \otimes \pi_j$. Note that $\pi_{ij}^\gamma \simeq \pi_{ji}$, so that $\pi_{ij}^\gamma \simeq \pi_{ij}$ if and only if $i = j$. Let Π_{ij} be an irreducible representation of G such that π_{ij} is contained in the restriction of Π_{ij} to G^0. Then if $i = j$ we have

$$\mathrm{Ind}_{G^0}^G(\pi_{ij}) = \Pi_{ij} \oplus (\Pi_{ij} \otimes \eta)$$

where η is the non-trivial character of G/G^0 and $\Pi_{ij} \otimes \eta \not\simeq \Pi_{ij}$. If $i \neq j$ we have

$$\mathrm{Ind}_{G^0}^G(\pi_{ij}) = \Pi_{ij} \simeq \Pi_{ji} = \mathrm{Ind}_{G^0}^G(\pi_{ji}).$$

In this case we also have $\Pi_{ij} \otimes \eta \simeq \Pi_{ij}$. Thus we have

$$\begin{aligned}\mathrm{Ind}_{B^0}^G(\chi) &= \mathrm{Ind}_{G^0}^G(\pi_{11} \oplus \pi_{1s} \oplus \pi_{s1} \oplus \pi_{ss})\\ &= \Pi_{11} \oplus (\Pi_{11} \otimes \eta) \oplus \Pi_{ss} \oplus (\Pi_{ss} \otimes \eta) \oplus 2\Pi_{1s}.\end{aligned}$$

Note that the above decompositions are reflected in the R-groups as follows. First, the R-group for $\mathrm{Ind}_{B^0}^{G^0}(\chi)$ is given by $R^0 = R_1 \times R_1 = W(G^0, A^0) \simeq Z_2 \times Z_2$. It has 4 characters $\rho_{ij}^0 = \rho_i \otimes \rho_j, i, j \in \{1, s\}$ corresponding to the irreducible representations π_{ij}. The R-group for $\mathrm{Ind}_{B^0}^G(\chi)$ is $R = W(G, A_0) = R^0 \cup \gamma R^0$. By abuse of notation we will denote the non-trivial character of R/R^0 by the same letter η as used above for the non-trivial character of G/G^0 and below for the non-trivial character of M/M^0. The irreducible representations of R are the characters ρ_{ii} and $\rho_{ii} \otimes \eta$ where

$$\mathrm{Ind}_{R^0}^R(\rho_{ii}^0) = \rho_{ii} \oplus (\rho_{ii} \otimes \eta)$$

and the two-dimensional irreducible representation

$$\mathrm{Ind}_{R^0}^R(\rho_{1s}^0) = \rho_{1s} \simeq \mathrm{Ind}_{R^0}^R(\rho_{s1}^0).$$

Thus we have the irreducible constituents of $\mathrm{Ind}_{B^0}^G(\chi)$ parameterized by the irreducible representations of R and occurring with multiplicities given by the degrees of the corresponding representations.

Now we consider $\mathrm{Ind}_{M^0}^M(\chi)$. Since $M = M^0 \cup \gamma M^0$ and $\chi^\gamma = \chi$, we have

$$\mathrm{Ind}_{M^0}^M(\chi) = \sigma \oplus (\sigma \otimes \eta)$$

where $\sigma, \sigma \otimes \eta$ are distinct one-dimensional unitary representations of M which restrict to χ on M^0. Now, using transitivity of induction and properties of tensor products, we can write

$$\begin{aligned}\mathrm{Ind}_{B^0}^G(\chi) &\simeq \mathrm{Ind}_B^G \mathrm{Ind}_{M^0}^M(\chi)\\ &\simeq \mathrm{Ind}_B^G(\sigma) \oplus \mathrm{Ind}_B^G(\sigma \otimes \eta).\end{aligned}$$

Now since $\mathrm{Ind}_B^G(\sigma \otimes \eta) \simeq \mathrm{Ind}_B^G(\sigma) \otimes \eta$, we see that

$$\mathrm{Ind}_B^G(\sigma) \oplus (\mathrm{Ind}_B^G(\sigma) \otimes \eta) \simeq$$

$$\Pi_{11} \oplus (\Pi_{11} \otimes \eta) \oplus \Pi_{ss} \oplus (\Pi_{ss} \otimes \eta) \oplus 2\Pi_{1s}.$$

Thus we can assume that Π_{11}, Π_{ss} were chosen so that

$$\mathrm{Ind}_B^G(\sigma) \simeq \Pi_{11} \oplus \Pi_{ss} \oplus \Pi_{1s}$$

and

$$\mathrm{Ind}_B^G(\sigma\otimes\eta)\simeq(\Pi_{11}\otimes\eta)\oplus(\Pi_{ss}\otimes\eta)\oplus\Pi_{1s}.$$

This example exhibits a number of unpleasant features. First, we have irreducible discrete series representations $\sigma_1=\sigma$ and $\sigma_2=\sigma\otimes\eta$ of B such that $\mathrm{Ind}_B^G(\sigma_1)$ and $\mathrm{Ind}_B^G(\sigma_2)$ have a non-trivial intertwining, but are not equivalent. Second, $\mathrm{Ind}_B^G(\sigma)$ has 3 inequivalent irreducible subrepresentations, each occuring with multiplicity one, so that the dimension of its space of intertwining operators is 3. There are no subgroups of any possible Weyl groups here with order 3.

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David Goldberg, Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: goldberg@math.purdue.edu

Rebecca A. Herb, Department of Mathematics, University of Maryland, College Park, MD 20742
E-mail address: rah@math.umd.edu