BOURGAIN ALGEBRAS, MINIMAL ENVELOPES, MINIMAL SUPPORT SETS, AND SOME APPLICATIONS

CARROLL J. GUILLORY

ABSTRACT. We explicitly compute certain Douglas algebras that are invariant under both the Bourgain map and the minimal envelope map. We also compute the Bourgain algebra and the minimal envelope of the maximal subalgebras of a certain singly generated Douglas algebra.

1. INTRODUCTION

Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disk D. By considering boundary functions, we may regard H^∞ an an essentially supremum norm-closed subalgebra of $L^\infty = L^\infty(\partial D)$. A closed subalgebra between H^∞ and L^∞ is called a *Douglas algebra*. $H^\infty + C$ is the smallest Douglas algebra, where C is the space of continuous functions on ∂D. The Chang-Marshall theorem [Ch, Ma] says that every Douglas algebra B is generated by H^∞ and complex conjugates of interpolating Blaschke products ψ with $\bar{\psi} \in B$. We denote by $M(B)$ the maximal ideal space of B. Then we may consider that $M(L^\infty) \subset M(B) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary for every Douglas algebra B. For a point x in $M(H^\infty)$, there is a representing measure μ_x on $M(L^\infty)$; $f(x) = \int_{M(L^\infty)} f \, d\mu_x$ for every $f \in H^\infty$. A Douglas algebra B with $A \subsetneq B$ is called a *minimal superalgebra* of A if there are no Douglas algebras B' such that $A \subsetneq B' \subsetneq B$. In [GI1], the authors proved that if A and B are Douglas algebras with $A \subsetneq B$, then B is a minimal superalgebra of A if and only if $\operatorname{supp} \mu_x = \operatorname{supp} \mu_y$ for every $x, y \in M(A) \setminus M(B)$.

Let Y be a Banach algebra with identity and let B be a closed subalgebra of Y. The *Bourgain algebra* B_b of B relative to Y is defined by the set of f in Y such that $\|ff_n + B\| \to 0$ for every sequence $\{f_n\}_n$ in B with $f_n \to 0$ weakly. In this note, we consider $Y = L^\infty$ and B a Douglas algebra. In [CJY], Cima, Janson, and Yale proved that $H^\infty_b = H^\infty + C$. Gorkin, Izuchi, and Martini [GIM] studied Bourgain algebras of Douglas algebras and proved that $(B_b)_b = B_b$. We denote by B_m the

1991 *Mathematics Subject Classification.* 46J15,46J20.

Key words and phrases. Maximal subalgebras, Douglas algebras, Bourgain algebras, minimal envelope, interpolating Blaschke product, minimal support points, locally thin points, support sets, Gleason parts.

smallest Douglas algebra which contains all minimal superalgebras of B. We call B_m the *minimal envelope* of B. We have $B \subseteq B_b \subseteq B_m$.

A sequence $\{z_n\}_n$ in D is called *interpolating* if for every bounded sequence $\{a_n\}_n$ there exists f in H^∞ such that $f(z_n) = a_n$ for every n. A Blaschke product

$$\psi(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in D$$

is called *interpolating* if its zeroes $\{z_n\}_n$ is interpolating. For a function f in H^∞ we put $Z(f) = \{x \in M(H^\infty + C) : f(x) = 0\}$. For $x \in M(H^\infty)$ we denote by $\operatorname{supp} \mu_x$ the support set for the representing measure μ_x. For a subset E of $M(L^\infty)$ we denote by $\operatorname{cl} E$ the closure of E in $M(L^\infty)$. Put $E = \bigcup\{\operatorname{supp} \mu_x : x \in M(H^\infty + C),\ |q(x)| < 1\}$ where q is an interpolating Blaschke product. Then $N(\bar{q}) = \operatorname{cl} E$. A closed set E in $M(L^\infty)$ is called a *peak set* for a Douglas algebra B if there is a $f \in B$ such that $f = 1$ on E and $|f| < 1$ on $M(L^\infty) \setminus E$. A closed set E in $M(L^\infty)$ is called a *weak peak* for B if E is the intersection of some family of peak sets. If E is a weak peak set for B, then the set

$$B_E = \{f \in L^\infty : f|_E \in B|_E\}$$

is a Douglas algebra. The sets $\operatorname{supp} \mu_x$ and $N(\bar{q})$ are weak peak sets for H^∞, hence $H^\infty_{\operatorname{supp} \mu_x}$ and $H^\infty_{N(\bar{q})}$ are Douglas algebras. A point $x \in Z(q)$, where q is an interpolating Blaschke product, is called a *minimal element* for $H^\infty[\bar{q}]$ if there is no $y \notin M(H^\infty[\bar{q}])$ such that $\operatorname{supp} \mu_y \subsetneq \operatorname{supp} \mu_x$; that is, if $y \notin M(H^\infty[\bar{q}])$, then either $\operatorname{supp} \mu_y \cap \operatorname{supp} \mu_x = \varnothing$ or $\operatorname{supp} \mu_x \subseteq \operatorname{supp} \mu_y$. We put

$$E_x = \{\lambda \in M(H^\infty) : \operatorname{supp} \mu_\lambda = \operatorname{supp} \mu_x\}$$

and call E_x the *level set* of x. For x, y in $M(H^\infty)$, we put

$$\rho(x, y) = \sup\{|f(y)| : f \in H^\infty,\ \|f\|_\infty \leq 1,\ f(x) = 0\}.$$

The set

$$P(x) = \{\lambda \in M(H^\infty) : \rho(\lambda, x) < 1\}$$

is called the *Gleason part* containing x. We have $P(x) \subset E_x$, and if $x \in M(B)$ then $E_x \subset M(B)$. The map that assigns the algebra B_b to a Douglas algebra B is called the *Bourgain map* and the map that assigns B_m to B is called the *minimal envelope map*.

A point $x \in M(H^\infty)$ is called *locally thin* if there is an interpolating Blaschke product q such that $q(x) = 0$ and

$$(1 - |z_{n(\alpha)}|^2)|q'(z_{n(\alpha)})| \to 1$$

whenever $z_{n(\alpha)}$ is a subnet of the zero sequence $\{z_n\}_n$ of b in D converging to x. We say that q is *locally thin at* x.

We say that a minimal support point x of a Douglas algebra B corresponds to the maximal subalgebra A_x if it satisfies the equation

$$M(A_x) = M(B) \cup P_x \quad \text{or} \quad M(A_x) = M(B) \cup E_x \,.$$

We put $\mathrm{QC} = (H^\infty + C) \cap \overline{(H^\infty + C)}$. Then

$$\mathrm{QC} = \{f \in H^\infty + C : f|_{\operatorname{supp}\mu_x} \text{ is constant for every } x \in M(H^\infty + C)\} \,.$$

For $x \in M(L^\infty)$, the set

$$Q = \{y \in M(L^\infty) : f(y) = f(x) \text{ for every } f \in \mathrm{QC}\}$$

is called a QC-*level set*. For every $y \in M(H^\infty + C)$, there is a QC-level set Q_y such that $\operatorname{supp}\mu_y \subset Q_y$. For any interpolating Blaschke product q set

$$m_q = \{x \in M(H^\infty + C) : x \text{ is a minimal support point of } H^\infty[\bar{q}]\} \,.$$

In this paper we give examples (Theorem 1 and Theorem 2) of Douglas algebras that show that the Bourgain algebra of the arbitrary intersection of Douglas algebras does not equal the intersection of their corresponding Bourgain algebras. Theorems 1 and 2 also show that this is true for the minimal envelopes. We also compute the Bourgain algebra and the minimal envelope of certain maximal subalgebras. Finally we give two applications of minimal support points of certain interpolating Blaschke products.

All of the work in this paper was done while the author was at the Mathematical Science Research Institute. The author thanks the Institute for its support during this period.

Lemma 1. *Let q be an interpolating Blaschke product that is of type G, and such that $Z(q) \cap P_x$ is a finite set for all $x \in Z(q)$. Then each $x \in Z(q)$ is a locally thin point.*

Proof. Since q is of type G and the set $Z(q) \cap P_x$ is finite there is a factor q_0 of q such that $Z(z_0) \cap P_x = \{x\}$. Hence $H^\infty_{\operatorname{supp}\mu_x}[\bar{q}_0]$ is a minimal superalgebra of $H^\infty_{\operatorname{supp}\mu_x}$. Hence $H^\infty_{\operatorname{supp}\mu_x} \subset (H^\infty_{\operatorname{supp}\mu_x})_b$ and so by Theorem 5 of [MY] we have that x is a locally thin point. □

Lemma 2. *Let $B = H^\infty_{\operatorname{supp}\mu_y}$, where y is a trivial point. If $B \subset B_m$, then there is an interpolating Blaschke product q and an $x_0 \in Z(q)$ such that* $\operatorname{supp}\mu_{x_0} = \operatorname{supp}\mu_y$.

Proof. If $B \subset B_m$, then by Theorem D of [GI2] there is an interpolating Blaschke product q such that $B[\bar{\psi}]$ is a minimal superalgebra of B. Hence $\operatorname{supp}\mu_y$ is a minimal support set of $H^\infty[\bar{\psi}]$. By Theorem 2 of [Gu] there is an $x_0 \in Z(\psi)$ such that $\operatorname{supp}\mu_{x_0} = \operatorname{supp}\mu_y$. □

It would be nice if we could prove that the converse of Lemma 2 is true. I have been unable to do so. There are special cases when the converse holds.

Theorem 1. *Let q be a sparse interpolating Blaschke product and set $B = H^\infty_{N(\bar{q})}$. Then $B = B_b = B_m$.*

Proof. First we have by Theorem C of [GI3] that there is an interpolating Blaschke product ψ (if $B \subset B_b$) such that $Z(\psi) \cap M(B) = \{x\}$, and $M(B) = M(B[\bar{\psi}]) \cup P_x$ for some x in $M(B)$. Since $N(\bar{q}) = N_0(\bar{q})$, by Proposition 1 of [GI1] we have that $B = \bigcap_{y \in Z(q)} H^\infty_{\operatorname{supp} \mu_y}$. We will show that there is a $y \in Z(q)$ with $y \in P_x$. Once this is done then we have that $P_x = E_x$ for all $x \in M(B)$ and all ψ with $Z(\psi) \cap M(B) = \{x\}$. Thus both B_b and B_m are generated by the same minimal superalgebras and we have $B_b = B_m$ (see Theorem D of [GI3]).

To see that such a y exists, note that if $x \in M(B)$ then by [Ga], p. 39, $\operatorname{supp} \mu_x \subset N(\bar{q})$. By Theorem 1 of [Iz] there is a $y \in Z(q)$ such that $\operatorname{supp} \mu_x \subset Q_y$. We show that $y \in P_x$. Since q is sparse we have that y is unique (Lemma 4 of [Iz]), $\operatorname{supp} \mu_y \subset Q_y$, and $\operatorname{supp} \mu_y$ is a maximal support set. Thus we have three possibilities: (1) $\operatorname{supp} \mu_x \subsetneq \operatorname{supp} \mu_y$, (2) $\operatorname{supp} \mu_x \cap \operatorname{supp} \mu_y = \varnothing$, and (3) $\operatorname{supp} \mu_x = \operatorname{supp} \mu_y$. We show that (3) holds.

Suppose (1) holds. Then by Theorem 2 of [GI2] there is an uncountable set $\Gamma \subset Z(\psi)$ such that $\operatorname{supp} \mu_m \subset \operatorname{supp} \mu_y$ for all $m \in \Gamma$, and $\operatorname{supp} \mu_m \cap \operatorname{supp} \mu_n = \varnothing$ for $n, m \in \Gamma$ for $n \neq m$. This implies that $Z(\psi) \cap M(B)$ is an infinite set, which is a contradiction. So (1) cannot hold.

Now suppose (2) holds. Then $\operatorname{supp} \mu_x \cap \operatorname{supp} \mu_y = \varnothing$, hence $\bar{\psi} \in H^\infty_{\operatorname{supp} \mu_y}$. But $Z(\psi) \cap M(B) = \{x\}$ implies that $\bar{\psi} \in H^\infty_{\operatorname{supp} \mu_m}$ for all $m \in Z(q)$. This implies that $\bar{\psi} \in B$, another contradiction. So (3) must hold. Since q is sparse this implies that $x \in P_y$ (or $y \in P_x$).

Since such a y exists we have that $B \subset B[\bar{\psi}] \subset B[\bar{q}]$. We're going to show that $B[\bar{\psi}]$ cannot be a minimal superalgebra of B. Consider the algebras $B[\bar{q}]$ and $H^\infty[\bar{q}]$. Theorem 3.2 of [GI4] shows that any subalgebra A of $H^\infty[\bar{q}]$ is the intersection of a family of maximal subalgebras of $H^\infty[\bar{q}]$. Since the set

$$\bigcup_{y \in Z(q)} P_y$$

is the set of minimal support points of both $B[\bar{q}]$ and $H^\infty[\bar{q}]$, we can use the same proof of Theorem 3.2 to show that any subalgebra A of $B[\bar{q}]$ is also the intersection of a family of maximal subalgebras of $B[\bar{q}]$. For $m \in Z(q)$ set $B_m = B[\bar{q}] \cap H^\infty_{\operatorname{supp} \mu_m}$. Then we have that

$$B = \bigcap_{m \in Z(q)} B_m$$

and if $H^\infty_m = H^\infty[\bar{q}] \cap H^\infty_{\operatorname{supp}\mu_m}$, then

$$H^\infty + C = \bigcap_{m\in Z(q)} H_m\,.$$

Hence if $y \in Z(q)$ such that $y \in P_x$, then $M(B) = M(B[\bar{\psi}]) \cup P_y$, and we get that

$$B[\bar{\psi}] = \bigcap_{\substack{m\in Z(q)\\ m\neq y}} B_m\,.$$

Thus if $B[\bar{\psi}]$ is a minimal superalgebra of B, then the algebra

$$A^* = \bigcap_{\substack{m\in Z(q)\\ m\neq y}} H^\infty_m$$

is a minimal superalgebra of $H^\infty + C$. This is impossible since $H^\infty + C$ has no minimal subalgebra ($(H^\infty + C)_b = H^\infty + C$). Hence $B[\bar{\psi}]$ is not a minimal superalgebra for B. We get $B = B_b = B_m$. □

Theorem 2. *Let q be an interpolating Blaschke product and set $\tilde{m}_q = \{y \in m_q : \operatorname{supp}\mu_y = \operatorname{supp}\mu_t,\ t$ is a trivial point$\}$. Set $T = \bigcap_{y\in\tilde{m}_q} H^\infty_{\operatorname{supp}\mu_y}$. Then $T = T_b$. If $\tilde{m}_q = m_q$ then $T = T_m$.*

Proof. Suppose $T \neq T_b$. Then by Theorem C of [GI3] there is an interpolating Blaschke product ψ such that $Z(\psi) \cap M(T) = \{y_0\}$. By Theorem 1 of [GI1] we have $M(T) = M(T[\bar{\psi}]) \cup P_{y_0}$ and y_0 is a minimal support point of $H^\infty[\bar{\psi}]$. By Proposition 6 of [MY], $(H^\infty_{\operatorname{supp}\mu_{y_0}})_b = H^\infty_{\operatorname{supp}\mu_{y_0}}[\bar{\psi}]$. So by Theorem 5 of [MY], y_0 is a locally thin point. We show that this is not the case by showing that there is an $x_0 \in \tilde{m}_q$ with $\operatorname{supp}\mu_{x_0} = \operatorname{supp}\mu_{y_0}$. Suppose that $\operatorname{supp}\mu_{y_0} \neq \operatorname{supp}\mu_{x_0}$ for all $x_0 \in \tilde{m}_q$. Then for each $x_0 \in \tilde{m}_q$ one of the following can occur: (i) $\operatorname{supp}\mu_{x_0} \subsetneq \operatorname{supp}\mu_{y_0}$, (ii) $\operatorname{supp}\mu_{x_0} \cap \operatorname{supp}\mu_{y_0} = \varnothing$, or (iii) $\operatorname{supp}\mu_{y_0} \subsetneq \operatorname{supp}\mu_{x_0}$. We will show that none of these can actually happen. If (i) is true, then $|\psi(x_0)| = 1$ implies that $\bar{\psi} \in H^\infty_{\operatorname{supp}\mu_{x_0}}$ since y_0 is a minimal support point. If (ii) is true for all $x_0 \in \tilde{m}_q$ then again $\bar{\psi} \in H^\infty_{\operatorname{supp}\mu_{x_0}}$. So if (i) or (ii) happens for all $x_0 \in \tilde{m}_q$, then $\bar{\psi} \in T$, which implies that $T[\bar{\psi}] = T$. So $T_b = T$ here. Now if (iii) happens, then by Theorem 2 of [GI2] there is an uncountable set $\Gamma \subset Z(\psi)$ such that $\operatorname{supp}\mu_\alpha \subsetneq \operatorname{supp}\mu_{x_0}$ for all $\alpha \in \Gamma$ and $\operatorname{supp}\mu_\alpha \cap \operatorname{supp}\mu_\beta = \varnothing$ if $\alpha \neq \beta$ and $\alpha, \beta \in \Gamma$. Since $x_0 \in M(T)$, each $\alpha \in \Gamma$ is also in $M(T)$. Thus $\Gamma \subset Z(\psi) \cap M(T)$. This implies that $Z(\psi) \cap M(T) \neq \{y_0\}$, which leads to a contradiction. So (iii) cannot hold for any $x_0 \in \tilde{m}_q$. So if there is a $y_0 \in M(T) \cap Z(\psi)$ such that $T[\bar{\psi}]$ is a minimal superalgebra for T, then there is an $x_0 \in \tilde{m}_q$ with $\operatorname{supp}\mu_{y_0} = \operatorname{supp}\mu_{x_0}$. This implies that $(H^\infty_{\operatorname{supp}\mu_{y_0}})_b = (H^\infty_{\operatorname{supp}\mu_{y_0}})_b = H^\infty_{\operatorname{supp}\mu_{x_0}}[\bar{\psi}]$. By

the remark following Theorem 5 of [MY] we have that x_0 is not a locally thin point. So no such y_0 exist and we have $T = T_b$.

To show that $T = T_m$ if $\tilde{m}_q = m_q$ we proceed as follows. By the argument above, if $T \neq T_m$ there is a $x_0 \in \tilde{m}_q$ such that $\{\lambda \in M(T) : |\psi(\lambda)| < 1\} = E_{x_0}$, $y_0 \in E_{x_0}$, and $\{y_0\} = Z(\psi) \cap M(T)$. We show that this is a contradiction by showing the set $Z(\psi) \cap \tilde{m}_q$ contains an uncountable set. This will suffice since $\tilde{m}_q \subset M(T)$. Without loss of generality we can assume that if $x, y \in \tilde{m}_q$, $x \neq y$, then $\operatorname{supp} \mu_x \cap \operatorname{supp} \mu_y = \varnothing$. Let $\{z_n\}_{n=1}^{\infty}$ be the zero sequence of q in D. Then there is a subnet $\{z_{n_\alpha}\}_{\alpha \in A}$ such that $z_{n_\alpha} \to x_0$. Let $\epsilon > 0$ and set $B = A \setminus \{n_\alpha : |\psi(z_{n_\alpha})| > 1 - \epsilon\}$. Since $|\psi(x_0)| < 1$ and the subnet $\{|\psi(z_{n_\alpha})|\}$ converges to $|\psi(x_0)|$, there is an $\alpha_0 \in A$ such that if $\alpha \geq \alpha_0$, we have $n_\alpha \in B$. Hence if $\alpha_1 > \alpha_0$ we have that $|\psi(z_{n_\alpha})| < 1 - \epsilon$ for all $\alpha \geq \alpha_1$. Take the subnet $\{z_{n_\alpha}\}_{\alpha \in B}$. Then on the set $\overline{\{z_{n_\alpha}\}_{\alpha \in B}} \setminus \{z_{n_\alpha}\}_{\alpha \in B}$ we have that $|\psi| < 1 - \epsilon$. Take a subsequence $\{z_{n_k}\}$ of the subnet $\{z_{n_\alpha}\}_{\alpha \in B}$. Let q_0 be the factor of q with zero sequence $\{z_{n_k}\}$. Then $Z(q_0) = \overline{\{z_{n_k}\}} \setminus \{z_{n_k}\}$ and $|\psi(u)| < 1$ for all $u \in Z(q_0)$. By Theorem 2 of [GI1] there is an uncountable set $\Gamma \subset Z(q_0)$ such that $\Gamma \subset \tilde{m}_q$. By Theorem 2 of [GI1] we can assume that if $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$, then $\operatorname{supp} \mu_\alpha \cap \operatorname{supp} \mu_\beta = \varnothing$. For each $\alpha \in \Gamma$ there is an $x_\alpha \in Z(\psi)$ such that $\operatorname{supp} \mu_{x_\alpha} \subseteq \operatorname{supp} \mu_\alpha$. This implies that $x_\alpha \in M(T)$ since $M(T) = \overline{\bigcup_{y \in \tilde{m}_q} M(H^\infty_{\operatorname{supp} \mu_y})}$. Thus we have that $x_\alpha \in Z(\psi) \cap M(T)$. So the set $Z(\psi) \cap M(T)$ is uncountable. This shows that $T = T_m$. □

Both B_b and B_m are generated by a special type of minimal superalgebras (determined by the character of the minimal support point).

Under certain conditions we can determine the Bourgain algebras and the minimal envelope algebras of maximal subalgebras of a Douglas algebra B (here B will have a maximal subalgebra).

Theorem 3. *Let B be a Douglas algebra such that $B_b[\bar{q}] = B[\bar{q}]$ or $B_m[\bar{q}] = B[\bar{q}]$ for some interpolating Blaschke product q. Let A_x be the maximal subalgebra of $B[\bar{q}]$ for which x is the corresponding minimal support point of $B[\bar{q}]$. Then either:*

(i). *$A_x \subset (A_x)_b$ and $(A_x)_b = (A_x)_m = B[\bar{q}]$, or*
(ii). *$A_x = (A_x)_b$ and $(A_x)_m = B[\bar{q}]$.*

Proof. By using Theorem 3 of [MY], Theorem 3 of [GIM], and Theorems 4 and 5 of [GI3], our hypothesis implies that $B[\bar{q}] = (B[\bar{q}])_b = (B[\bar{q}])_m$.

Now let A_x be any maximal subalgebra of $B[\bar{q}]$ associated with the minimal support point x. First we assume that x is a locally thin point (see Theorem 5 of [MY]). Then the maximal ideal space of $B[\bar{q}]$ and A_x are related by the equation

$$M(A_x) = M(B[\bar{q}]) \cup P_x \,.$$

Then $B[\bar{q}]$ is a minimal superalgebra of A_x, hence $B[\bar{q}] \subseteq (A_x)_b$. Now, using Theorem 3 of [MY] again, we get

$$\begin{aligned} B[\bar{q}] &= (B[\bar{q}])_b \\ &= (A_x[\bar{q}])_b \\ &= (A_x)_b[\bar{q}] \\ &= (A_x)_b \quad \text{since } \bar{q} \in (A_x)_b\,. \end{aligned}$$

Similarly $B[\bar{q}] = (A_x)_m = (A_x)_b$ if x is a locally thin point. This proves (i).

Now suppose that x is not a locally thin point (for example, P_x is not a homeomorphic disk, see [Ga]). Then $M(A_x)$ and $M(B[\bar{q}])$ are related by

$$M(A_x) = M(B[\bar{q}]) \cup E_x$$

with $P_x \subsetneq E_x$ (or $Z(q) \cap P_x$ has infinitely many points). Using Theorem 4 of [GI3] we have

$$\begin{aligned} B[\bar{q}] &= (B[\bar{q}])_m \\ &= (A_x[\bar{q}])_m \\ &= (A_x)_m[\bar{q}] \quad \text{by Theorem 4 of [Gu]} \\ &= (A_x)_m \quad \text{since } \bar{q} \in (A_x)_m\,. \end{aligned}$$

Now, for any Douglas algebra A we have that $A \subseteq A_b \subseteq A_m$. Thus

$$A_x \subseteq (A_x)_b \subsetneq B[\bar{q}] = (A_x)_m\,.$$

Since A_x is a maximal subalgebra of $B[\bar{q}]$, we have that $A_x = (A_x)_b$ if x is not locally thin. This proves (ii). □

Corollary 1. *Let q be any interpolating Blaschke product and set $B = H^\infty[\bar{q}]$. Let A_x be any maximal subalgebra of B that corresponds to the minimal support point of B. Then either*

(i). *$A_x \subset (A_x)_b$ and $(A_x)_b = (A_x)_m = B$, or*
(ii). *$A_x = (A_x)_b$ and $(A_x)_m = B$.*

Corollary 2. *Let A be any Douglas algebra such that $A = A_b$ or $A = A_m$, and let q be any interpolating Blaschke product such that $\bar{q} \notin A$. Set $B = A[\bar{q}]$ and let B_x be any maximal subalgebra of B corresponding to the minimal support point of $A[\bar{q}]$. Then either*

(i). *$B_x \subset (B_x)_b$ and $(B_x)_b = (B_x)_m = B$, or*
(ii). *$B_x = (B_x)_b$ and $(B_x)_m = B$.*

Theorem 4. *Let B be a Douglas algebra that has a maximal subalgebra A_x, where x is the minimal support point of B corresponding to A_x. Then $(A_x)_m = B_m$.*

Proof. By Theorem 4 of [GI3] we have that $(A_x)_m \subseteq B_m$ since $A_x \subseteq B$. Since A_x is a maximal subalgebra of B there is an $x_0 \in M(A) \setminus M(B)$ and an interpolating Blaschke product ψ_0 such that

$$\begin{aligned} M(A_x) &= M(A[\bar{\psi}_0]) \cup E_{x_0} \\ &= M(B) \cup E_{x_0}\,. \end{aligned}$$

So by Theorem D of [GI3] we have $B \subseteq (A_x)_m$. If B_0 is another minimal superalgebra containing A_x, then there is some $y_0 \in M(A_x)$ such that $M(A_x) = M(B_0) \cup E_{y_0}$. Hence we have that $B_0 \subseteq (A_x)_m$ and $y_0 \in M(B)$, otherwise $E_{y_0} = E_{x_0}$. To show that $B_m \subseteq (A_x)_m$, let ψ be any interpolating Blaschke product such that $\bar{\psi} \in B_m$. Then by Theorem D of [GI3] we can assume that

$$\{\lambda \in M(B) : |\psi(\lambda)| < 1\} = E_x$$

for some $x \in M(B)$. But

$$\{m \in M(A_x) : |\psi(m)| < 1\} = E_x \cup \{m \in M(A_x) : |\psi(m)| < 1\} \cap E_{x_0}.$$

The set on the right hand side is either E_x or $E_x \cup E_{x_0}$. Hence by Theorem 4 of [GI3] we have that $\bar{\psi} \in (A_x)_m$. Hence $B_m \subseteq (A_x)_m$. □

For $(A_x)_b$ we have the following special result if we assume an additional assumption.

Theorem 5. *Let B be a Douglas algebra that has a maximal subalgebra A_x, where x is the minimal support point of B corresponding to A_x. Assume that P_x is a nonhomeomorphic disk. Then $(A_x)_b \subsetneq B_b$.*

Proof. Since $B \subseteq B_b$ and $(A_x)_b \subseteq B_b$, it suffices to show that $B \nsubseteq (A_x)_b$. Since A_x is a maximal subalgebra of B corresponding to x, by Theorem 1 of [GI1] we have

$$M(A_x) = M(B) \cup E_x\,.$$

Note that $P_x \subset E_x$. Hence if ψ is any interpolating Blaschke product, we have by Corollary 1.5 of [GLM] the set $M(A_x) \cap Z(\psi) \supseteq P_x \cap Z(\psi)$ is an infinite set. By Theorem 2 of [GIM], $\bar{\psi} \notin (A_x)_b$. Hence $B \nsubseteq (A_x)_b$. □

The following two propositions on minimal support points seem to indicate that the sets given in them are smaller in some sense than the set in the following two well-known facts.

Fact 1. *Let B be any Douglas algebra. Then an interpolating Blaschke product q is invertible in B if and only if $Z(q) \cap M(B) = \varnothing$.*

Fact 2. *For any Douglas algebra B we have*

$$B = \bigcap_{x \in M(B)} H^\infty_{\operatorname{supp} \mu_x}.$$

Proposition 1. *An interpolating Blaschke product q in invertible in a Douglas algebra B if and only if $M(B) \cap m_q = \varnothing$.*

Proof. By Theorem 2 of [GI1] we have that $m_q \subseteq Z(q)$, hence if $m_q \cap M(B) \neq \varnothing$ then $\bar{q} \notin B$.

To prove the converse, suppose $\bar{q} \notin B$. Then by Fact 1, $Z(q) \cap M(B) \neq \varnothing$. By the proof of Theorem 2 of [GI1], there is a $y_0 \in Z(q)$ such that $\operatorname{supp} \mu_{y_0} \subseteq \operatorname{supp} \mu_x$ for any $x \in Z(q) \cap M(B)$ and $y_0 \in m_q$. Since $x \in M(B)$ we have that $M(H^\infty_{\operatorname{supp} \mu_x}) \subset M(B)$. Since $M(H^\infty_{\operatorname{supp} \mu_x}) = M(L^\infty) \cup \{\lambda \in M(H^\infty + C) : \operatorname{supp} \mu_\lambda \subseteq \operatorname{supp} \mu_x\}$ we have that $y_0 \in M(B)$. Hence $m_q \cap M(B) \neq \varnothing$. □

Let B be any Douglas algebra and set $m_q(B) = m_q \cap M(B)$. Let

$$M_B = \bigcup \{m_q(B) : q \text{ is an interpolating Blaschke product, } \bar{q} \notin B\}.$$

Let $\Gamma(B) = \{x_\alpha\}_{\alpha \in \Lambda}$ be the family of all minimal support points from M_B such that $\operatorname{supp} \mu_{x_\alpha} \cap \operatorname{supp} \mu_{x_\beta} = \varnothing$ if $\alpha \neq \beta$. Then

Proposition 2. $B = \bigcap_{x_\alpha \in \Gamma(B)} H^\infty_{\operatorname{supp} \mu_{x_\alpha}}$.

Proof. If $x \in M_B$, then there is an $x_\alpha \in \Gamma(B)$ such that $\operatorname{supp} \mu_{x_\alpha} = \operatorname{supp} \mu_x$. So it suffices to show that

$$B = \bigcap_{x \in M_B} H^\infty_{\operatorname{supp} \mu_x}.$$

Set $B_0 = \bigcap_{x \in M_B} H^\infty_{\operatorname{supp} \mu_x}$. Since $M_B \subseteq M(B)$, by Fact 2 we have that $B \subseteq B_0$. Suppose $B \subsetneq B_0$. Then by the Chang-Marshall Theorem [Ch, Ma] there is an interpolating Blaschke product q such that $\bar{q} \in B_0$ but $\bar{q} \notin B$. Hence there is a $y \in M(B)$ such that $q(y) = 0$. Hence $\bar{q} \notin H^\infty_{\operatorname{supp} \mu_y}$. By Theorem 2 of [GI1] (or Proposition 1) there is a $y_0 \in Z(q)$ such that $\operatorname{supp} \mu_{y_0} \subseteq \operatorname{supp} \mu_y$ and $y_0 \in M_B$. This implies that $\bar{q} \notin H^\infty_{\operatorname{supp} \mu_{y_0}}$, so $\bar{q}$ cannot be invertible in B_0. Thus $B_0 = B$. We are done. □

References

[Ch] Chang, S. Y., "A characterization of Douglas subalgebras", Acta. Math. 137 (1976), 81–89.

[CJY] Cima, J., Janson, S., and Yale, K., "Completely continuous Hankel operators on H^∞ and Bourgain algebras", Proc. Amer. Math. Soc., 105 (1989), 121–125.

[Ga] Gamelin, T. W., "Uniform Algebras", New York: Chelsea Publishing Company, 1984.

[GLM] Gorkin, P., Lingenberg, H. M., and Mortini, R., "Homeomorphic disks in the spectrum of H^∞", Indiana Univ. Math. J., 39 (1990), 961–983.

[GM] Gorkin, P., and Mortini, R., "Interpolating Blaschke products and factorization in Douglas algebras", Michigan Math. J., 38 (1991), 147–160.

[GIM] Gorkin, P., Izuchi, K., and Mortini, R., "Bourgain algebras of Douglas algebras", Canad. J. Math, 44 (1992), 797–804.

[Gu] Guillory, C. J., "Douglas algebras that have no maximal subalgebra and no minimal superalgebra", MSRI Preprint series 1995-083 (available at http://www.msri.org/MSRI-preprints/online/1995-083).

[GI1] Guillory, C. J. and Izuchi, K., "Maximal Douglas subalgebras and minimal support points", Proc. Amer. Math. Soc., 116 (1992), 477–481.

[GI2] Guillory, C. J. and Izuchi, K., "Interpolating Blaschke products and nonanalytic sets", J. Complex Variable, 23 (1993), 163–175.

[GI3] Guillory, C. J. and Izuchi, K., "Minimal envelopes of Douglas algebras and Bourgain algebras", Houston J. Math., 19 (1993), 201–222.

[GI4] Guillory, C. J. and Izuchi, K., "Interpolating Blaschke products of type G", to appear in J. Complex Variables.

[Iz] Izuchi, K., "QC-level sets and quotients of Douglas algebras", J. Funct. Anal., 65 (1986), 293–308.

[MY] Mortini, R. and Younis, R., "Douglas algebras which are invariant under the Bourgain map", Arch. Math., vol. 59 (1992), 371–378.

[Ma] Marshall, D., "Subalgebras of L^∞ containing H^∞", Acta. Math., 137 (1976), 91–98.

Department of Mathematics, University of Southwestern Louisiana, Lafayette, LA 70504

Current address: Mathematical Sciences Research Institute, 100 Centennial Drive, Berkeley, CA 94720

E-mail address: cjg@msri.org