RELATIVE COHOMOLOGY OF BANACH ALGEBRAS

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Abstract. Let A be a Banach algebra, not necessarily unital, and let B be a closed subalgebra of A. We establish a connection between the Banach cyclic cohomology group $\mathcal{HC}^n(A)$ of A and the Banach B-relative cyclic cohomology group $\mathcal{HC}^n_B(A)$ of A. We prove that, for a Banach algebra A with a bounded approximate identity and an amenable closed subalgebra B of A, up to topological isomorphism, $\mathcal{HC}^n(A) = \mathcal{HC}^n_B(A)$ for all $n \geq 0$. We also establish a connection between the Banach simplicial or cyclic cohomology groups of A and those of the quotient algebra A/I by an amenable closed bi-ideal I. The results are applied to the calculation of these groups for certain operator algebras, including von Neumann algebras.

1. Introduction

Much interest has attached in recent years to the computation of cyclic (co)homology groups; see [Lo] for many references. Most of the literature has been devoted to the purely algebraic context, but there have also been papers addressing the calculation of the Banach version of these groups for Banach algebras, and in particular C^*-algebras: see, for example, [ChS1], [Wo] and [He2]. There is an effective tool for computing cyclic (co)homology: the Connes-Tsygan exact sequence, which connects the cyclic (co)homology of many algebras with their simplicial (co)homology. However, it remains the case that these groups can only be calculated for a restricted range of algebras. The purpose of this paper is to describe a technique for the calculation of the Banach simplicial and cyclic cohomology groups of Banach algebras, to establish the basic properties of this technique and to apply it to some natural classes of algebras. The technique involves, for a Banach algebra A, not necessarily unital, the notion of Banach cyclic cohomology groups $\mathcal{HC}^n_B(A)$ of A relative to a closed subalgebra B of A. This concept was introduced and exploited by L. Kadison [Ka1]in the algebraic theory. We establish a connection between the Banach cyclic cohomology group $\mathcal{HC}^n(A)$ of A and the Banach B-relative cyclic cohomology group $\mathcal{HC}^n_B(A)$ of A. The key result (Theorem 4.1) is that, for a Banach algebra A with a bounded approximate identity and an amenable closed subalgebra B of A,

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$\mathcal{HC}^n(A) = \mathcal{HC}^n_B(A)$ for all $n \geq 0$. With the aid of this theorem we show, for example, that if $\mathcal{R}$ is a von Neumann algebra then, for each $n \geq 0$, $\mathcal{HC}^n(\mathcal{R})$ is the direct sum of the cyclic cohomologies $\mathcal{HC}^n(\mathcal{R}_{I_f})$ and $\mathcal{HC}^n(\mathcal{R}_{II_1})$ of the finite von Neumann algebras $\mathcal{R}_{I_f}$ and $\mathcal{R}_{II_1}$ of types I and II appearing in the standard central direct summand decomposition of $\mathcal{R}$ (Corollary 4.5). (In this paper equality of cohomology groups means topological isomorphism of seminormed spaces.)

In establishing the connection between the Banach cyclic cohomology groups and the Banach relative cyclic cohomology groups of a Banach algebra A, we need to know the connection between the Banach simplicial cohomology groups $\mathcal{H}^n(A, A^*)$ of A and its relative analogue $\mathcal{H}^n_B(A, A^*)$. Here A^* is the dual Banach space of A. In Section 1 we prove Theorem 1.6 that for a Banach algebra A, an amenable closed subalgebra B of A and a dual A-bimodule M, $\mathcal{H}^n(A, M) = \mathcal{H}^n_B(A, M)$ for all $n \geq 0$. I am grateful to B. E. Johnson for suggesting that this result should be true. He gave a proof in the special case of a Banach algebra A with an identity, an amenable closed subalgebra B of A which contains the identity of A and a unital dual A-bimodule M. He wrote that it should not be too difficult to prove the general case by known techniques (personal communication, 1993). As it transpired, the identical result (for the special case) had been proved by different methods by L. Kadison [Ka2].

E. Christensen and A. M. Sinclair in [ChS2] used the same version of relative Hochschild cohomology group for the computation of the Hochschild cohomology groups of von Neumann algebras. See also [SiSm].

In Section 3 we introduce the Banach relative cyclic cohomology of a Banach algebra and show that the relative Connes-Tsygan exact sequence exists for a Banach algebra with a bounded approximate identity. We do this using ideas of A. Ya. Helemskii [He2]. In this section we also show the existence of morphisms of certain Connes-Tsygan exact sequences.

The results of Section 1 allow us to find, in Section 2, a connection between the cohomology groups of a Banach algebra A and that of a quotient algebra A/I by an amenable closed bi-ideal I. We prove Theorem 2.1 that in this case, for a dual A/I-bimodule M, $\mathcal{H}^n(A, M) = \mathcal{H}^n(A/I, M)$ for all $n \geq 0$. Thus we can obtain, in Theorem 2.4, the following information about the Banach simplicial cohomology groups: $\mathcal{H}^n(A, A^*) = \mathcal{H}^n(A/I, (A/I)^*)$ for all $n \geq 2$ and amenable I. A connection between the Banach cyclic cohomology groups of a Banach algebra A and that of a quotient algebra A/I by an amenable closed bi-ideal I is given in Theorem 4.2 of Section 4.

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2. Definitions and notation

Let A be a Banach algebra, not necessarily unital, and let A_+ be the unitization of A. We denote by e_+ the adjoined identity and by e an identity of A when it exists.

We recall some notation and terminology used in the homological theory of Banach algebras. Let A be a Banach algebra, not necessarily unital, and let X be a Banach A-bimodule. We define *an n-cochain* to be a bounded n-linear operator of $A \times \cdots \times A$ into X and we denote the space of n-cochains by $C^n(A, X)$. For $n = 0$ the space $C^0(A, X)$ is defined to be X. Let us consider the *standard cohomological complex*

$$0 \to C^0(A,X) \xrightarrow{\delta^0} \cdots \to C^n(A,X) \xrightarrow{\delta^n} C^{n+1}(A,X) \to \dots, (\mathcal{C}(A,X))$$

where the coboundary operator δ^n is defined by

$$(\delta^n f)(a_1, ..., a_{n+1}) = a_1 \cdot f(a_2, ..., a_{n+1}) +$$

$$\sum_{i=1}^{n} (-1)^i f(a_1, ..., a_i a_{i+1}, ..., a_{n+1}) + (-1)^{n+1} f(a_1, ..., a_n) \cdot a_{n+1}.$$

The kernel of δ^n in $C^n(A, X)$ is denoted by $Z^n(A, X)$ and its elements are called *$n-$cocycles.* The image of δ^{n-1} in $C^n(A, X)$ $(n \geq 1)$ is denoted by $N^n(A, X)$ and its elements are called *$n-$coboundaries.* An easy computation yields $\delta^{n+1} \circ \delta^n = 0$, $n \geq 0$.

0.1 DEFINITION. The nth cohomology group of $\mathcal{C}(A, X)$ is called *the n-dimensional Banach cohomology group of A with coefficients in X*. It is denoted by $\mathcal{H}^n(A, X)$.

Thus $\mathcal{H}^n(A, X) = Z^n(A, X)/N^n(A, X)$; it is a complete seminormed space.

Recall that a Banach A-bimodule $M = (M_*)^*$, where M_* is a Banach A-bimodule, is called *dual.* A Banach algebra A such that $\mathcal{H}^1(A, M) = \{0\}$ for all dual A-bimodules M is called *amenable.*

Now let S be a closed subalgebra of A_+. We denote by $C^n_S(A, X)$ the closed subspace of $C^n(A, X)$ of n-cochains ρ such that

$$\rho(sa_1, a_2, ..., a_n) = s \cdot \rho(a_1, ..., a_n),$$

$$\rho(a_1, ..., a_{i-1}, a_i s, a_{i+1}, a_{i+2}, ..., a_n) = \rho(a_1, ..., a_{i-1}, a_i, s a_{i+1}, a_{i+2}, ..., a_n)$$

and

$$\rho(a_1, a_2, ..., a_n s) = \rho(a_1, ..., a_n) \cdot s$$

for all $a_1, a_2, ..., a_n \in A$, $s \in S$ and $1 \leq i \leq n$. These cochains we shall call *S-relative n-cochains.* For $n = 0$ the space $C^0_S(A, X)$ is defined to be $\mathrm{Cen}_S X \stackrel{def}{=} \{x \in X : s \cdot x = x \cdot s \text{ for all } s \in S\}$.

Note that, for each $\rho \in C^n_S(A, X)$, $\delta^n \rho$ is also an S-relative cochain. Therefore there is a subcomplex in $\mathcal{C}(A, X)$ formed by the spaces $C^n_S(A, X)$. We denote this subcomplex by $\mathcal{C}_S(A, X)$. The kernel of δ^n in $C^n_S(A, X)$ is denoted by $Z^n_S(A, X)$ and its elements are called *S-relative $n-$cocycles.* The image of $\delta^{n-1} : C^{n-1}_S(A, X) \to C^n_S(A, X)$ $(n \geq 1)$ is denoted by $N^n_S(A, X)$ and its elements are called *S-relative $n-$coboundaries.*

0.2 DEFINITION. The nth cohomology group of $\mathcal{C}_S(A, X)$ is called *the n-dimensional Banach S-relative cohomology group of A with coefficients in X*. It is denoted by $\mathcal{H}^n_S(A, X)$.

When $S = \mathbb{C}_+$ the subscript S is unnecessary and we omit it.

Throughout the paper *id* denotes the identity operator. We denote the projective tensor product of Banach spaces by $\hat{\otimes}$ and the projective tensor product of left and right Banach A-bimodules by $\hat{\otimes}_A$ [Rie].

3. Relative cohomology of Banach algebras

We need a strengthening of Theorem 4.1 of [JKR] to prove the isomorphism of the cohomology and the relative cohomology of a Banach algebra A for dual A-bimodules.

1.1 Proposition. *Let A be a Banach algebra, let B be an amenable closed subalgebra of A, let M be a dual A-bimodule and let $n \geq 1$. Suppose $\rho \in C^n(A, M)$ is such that*

$$(\delta^n \rho)(a_1, ..., a_{n+1}) = 0$$

if any one of $a_1, ..., a_{n+1}$ lies in B. Then there exists $\xi \in C^{n-1}(A, M)$ such that

$$(\rho - \delta^{n-1}\xi)(a_1, ..., a_n) = 0$$

if any one of $a_1, ..., a_n$ lies in B.

The proof is the same as that of Theorem 4.1 of [JKR]. □

The following is essentially Lemma 4.1 of [Rin], but with a weakening of the hypothesis.

1.2 Lemma. *Let A be a Banach algebra, let B be a closed subalgebra of A, let X be a Banach A-bimodule and let $n \geq 1$. Suppose $\rho \in C^n(A, X)$ is such that*

$$(\delta^n \rho)(a_1, ..., a_{n+1}) = 0$$

if any one of $a_1, ..., a_{n+1}$ lies in B and

$$\rho(a_1, ..., a_n) = 0$$

if any one of $a_1, ..., a_n$ lies in B. Then $\rho \in C^n_B(A, X)$.

The proof is the same as that of Lemma 4.1 of [Rin]. □

1.3 Corollary. *Let A be a Banach algebra, let B be an amenable closed subalgebra of A, let M be a dual A-bimodule and let $n \geq 1$. Suppose $\rho \in Z^n(A, M)$. Then there exists $\xi \in C^{n-1}(A, M)$ such that*

$$(\rho - \delta^{n-1}\xi)(a_1, ..., a_n) = 0$$

if any one of $a_1, ..., a_n$ lies in B. Moreover $(\rho - \delta^{n-1}\xi) \in Z^n_B(A, M)$.

1.4 Proposition. *Let A be a Banach algebra, let B be a closed subalgebra of A with a bounded approximate identity $e_\nu, \nu \in \Lambda$, let $M = (M_*)^*$ be a dual A-bimodule and let $n \geq 1$. Suppose $\rho \in C^n_B(A, M)$ is such that*

$$(\delta^n \rho)(a_1, ..., a_{n+1}) = 0 \tag{3.1}$$

if any one of $a_1, ..., a_{n+1}$ lies in B. Then there exists $\xi_B \in C^{n-1}_B(A, M)$ such that

$$(\rho - \delta^{n-1}\xi_B)(a_1, ..., a_n) = 0$$

if any one of $a_1, ..., a_n$ *lies in* B.

Proof. For $n = 1$, by assumption, for each $b \in B$ and $\nu \in \Lambda$,

$$0 = (\delta^1 \rho)(e_\nu, b) = e_\nu \cdot \rho(b) - \rho(e_\nu b) + \rho(e_\nu) \cdot b = \rho(e_\nu) \cdot b,$$

since $\rho \in C_B^1(A, M)$ and $e_\nu \in B$. So we obtain

$$\rho(b) = \lim_\nu \rho(e_\nu b) = \lim_\nu \rho(e_\nu) \cdot b = 0.$$

Hence we can take $\xi_B = 0$.

For $n > 1$, we construct, inductively on k, $\xi_1, ..., \xi_k$ in $C_B^{n-1}(A, M)$ such that

$$(\rho - \delta^{n-1}\xi_k)(a_1, ..., a_n) = 0$$

if any one of $a_1, ..., a_k$ lies in B for $1 \leq k \leq n$. The conclusion of the proposition then follows, with $\xi_B = \xi_n$.

To construct ξ_1, we consider $g_\nu \in C^{n-1}(A, M)$ given by

$$g_\nu(a_1, ..., a_{n-1}) = \rho(e_\nu, a_1, ..., a_{n-1}),$$

for $a_1, ..., a_{n-1} \in A$ and $\nu \in \Lambda$. Since the Banach space $C^{n-1}(A, M)$ is the dual space of $A\hat{\otimes} \ldots \hat{\otimes} A \hat{\otimes} M_*$, for the bounded net $g_\nu, \nu \in \Lambda$, there exists a subnet $g_\mu, \mu \in \Lambda'$, which weak* converges to some cochain $g \in C^{n-1}(A, M)$. It is routine to check that $g \in C_B^{n-1}(A, M)$.

By assumption, for each $b \in B$ and $\nu \in \Lambda$,

$$\begin{gathered} 0 = \delta^n \rho(e_\nu, b, a_2, ..., a_n) = \\ e_\nu \cdot \rho(b, a_2, ..., a_n) - \rho(e_\nu b, a_2, ..., a_n) + \rho(e_\nu, ba_2, ..., a_n) \\ + \sum_{i=2}^{n-1} (-1)^{i+1} \rho(e_\nu, b, a_2, ..., a_i a_{i+1}, ..., a_n) + (-1)^{n+1} \rho(e_\nu, b, a_2, , ..., a_{n-1}) \cdot a_n = \\ \rho(e_\nu, ba_2, ..., a_n) \\ + \sum_{i=2}^{n-1} (-1)^{i+1} \rho(e_\nu, b, a_2, ..., a_i a_{i+1}, ..., a_n) + (-1)^{n+1} \rho(e_\nu, b, a_2, , ..., a_{n-1}) \cdot a_n. \end{gathered} \tag{3.2}$$

Thus

$$\begin{gathered} \rho(b, a_2, ..., a_n) - \delta^{n-1} g(b, a_2, ..., a_n) = \\ \rho(b, a_2, ..., a_n) - b \cdot g(a_2, ..., a_n) + g(ba_2, ..., a_n) \\ + \sum_{i=2}^{n-1} (-1)^{i+1} g(b, a_2, ..., a_i a_{i+1}, ..., a_n) + (-1)^{n+1} g(b, a_2, , ..., a_{n-1}) \cdot a_n = \\ \rho(b, a_2, ..., a_n) - \lim_\mu b \cdot \rho(e_\mu, a_2, ..., a_n) + \lim_\mu [\rho(e_\mu, ba_2, ..., a_n) \\ + \sum_{i=2}^{n-1} (-1)^{i+1} \rho(e_\mu, b, a_2, ..., a_i a_{i+1}, ..., a_n) + (-1)^{n+1} \rho(e_\mu, b, a_2, , ..., a_{n-1}) \cdot a_n]. \end{gathered}$$

The first two terms cancel, and the remaining ones add up to zero by (3.2). This proves the existence of a suitable cochain $\xi_1 = g$.

Suppose now that $1 \le k < n$, and a suitable cochain $\xi_k \in C_B^{n-1}(A, M)$ has been constructed. With $\rho - \delta^{n-1}\xi_k \in C_B^n(A, M)$ denoted by σ,

$$\sigma(a_1, ..., a_n) = 0 \tag{3.3}$$

if any one of $a_1, ..., a_k$ lies in B. In order to continue the inductive process (and so complete the proof of the theorem), it suffices to construct ζ in $C_B^{n-1}(A, M)$ such that $\sigma - \delta^{n-1}\zeta$ vanishes whenever any one of its first $k+1$ arguments lies in B. For then we have $\rho - \delta^{n-1}(\xi_k + \zeta) = \sigma - \delta^{n-1}\zeta$, and we may take $\xi_{k+1} = \xi_k + \zeta$. To this end, we consider $g_\nu \in C^{n-1}(A, M)$ given by

$$g_\nu(a_1, ..., a_{n-1}) = \sigma(a_1, \ldots, a_k, e_\nu, a_{k+1}, \ldots, a_{n-1}),$$

for $a_1, ..., a_{n-1} \in A$ and $\nu \in \Lambda$. By the same arguments as in the case $k = 1$, there exists a subnet $g_\mu, \mu \in \Lambda'$, which weak* converges to some cochain $g \in C^{n-1}(A, M)$. It can be checked that $g \in C_B^{n-1}(A, M)$ and

$$g(a_1, ..., a_{n-1}) = 0 \tag{3.4}$$

if any one of $a_1, ..., a_k$ lies in B.

In view of the assumption (3.1), for any $b \in B$ and $\nu \in \Lambda$,

$$\begin{aligned}&\delta^n\sigma(a_1, \ldots, a_k, e_\nu, b, a_{k+2}, \ldots, a_n)\\ &\qquad = (\delta^n\rho - (\delta^n \circ \delta^{n-1})(\xi_k))(a_1, \ldots, a_k, e_\nu, b, a_{k+2}, \ldots, a_n) = 0.\end{aligned}$$

Hence by the coboundary formula,

$$\begin{aligned}0 = \delta^n\sigma(a_1, \ldots, a_k, e_\nu, b, a_{k+2}, \ldots, a_n) &=\\ a_1 \cdot \sigma(a_2, \ldots, a_k, e_\nu, b, a_{k+2}, \ldots, a_n)&\\ + \sum_{i=1}^{k-1}(-1)^i\sigma(a_1, \ldots, a_i a_{i+1}, \ldots, a_k, e_\nu, b, a_{k+2}, \ldots, a_n)&\\ +(-1)^k\sigma(a_1, \ldots, a_k e_\nu, b, a_{k+2}, \ldots, a_n) + (-1)^{k+1}\sigma(a_1, \ldots, a_k, e_\nu b, a_{k+2}, \ldots, a_n)&\\ +(-1)^{k+2}\sigma(a_1, \ldots, a_k, e_\nu, b a_{k+2}, \ldots, a_n)&\\ + \sum_{i=k+2}^{n-1}(-1)^{i+1}\sigma(a_1, \ldots, a_k, e_\nu, b, a_{k+2}, \ldots, a_i a_{i+1}, \ldots, a_n)&\\ +(-1)^{n+1}\sigma(a_1, \ldots, a_k, e_\nu, b, a_{k+2}, \ldots, a_{n-1}) \cdot a_n.&\end{aligned}$$

By the inductive hypothesis (3.3) the first two terms vanish, and since $\sigma \in C_B^n(A, M)$, the third and fourth cancel. Thus

$$0 = (-1)^{k+2}\sigma(a_1, \ldots, a_k, e_\nu, b a_{k+2}, \ldots, a_n)$$

$$+\sum_{i=k+2}^{n-1}(-1)^{i+1}\sigma(a_1,\ldots,a_k,e_\nu,b,a_{k+2},\ldots,a_ia_{i+1},\ldots,a_n) \qquad (3.5)$$
$$+(-1)^{n+1}\sigma(a_1,\ldots,a_k,e_\nu,b,a_{k+2},\ldots,a_{n-1})\cdot a_n.$$

Now consider

$$(\sigma-(-1)^k\delta^{n-1}g)(a_1,\ldots,a_k,b,a_{k+2},\ldots,a_n)=$$
$$\sigma(a_1,\ldots,a_k,b,a_{k+2},\ldots,a_n)$$
$$+(-1)^{k+1}a_1\cdot g(a_2,\ldots,a_k,b,a_{k+2},\ldots,a_n)$$
$$+\sum_{i=1}^{k-1}(-1)^{i+k+1}g(a_1,\ldots,a_ia_{i+1},\ldots,a_k,b,a_{k+2},\ldots,a_n)$$
$$-g(a_1,\ldots,a_kb,a_{k+2},\ldots,a_n)+g(a_1,\ldots,a_k,ba_{k+2},\ldots,a_n)$$
$$+\sum_{i=k+2}^{n-1}(-1)^{i+k+1}g(a_1,\ldots,a_k,b,a_{k+2},\ldots,a_ia_{i+1},...,a_n)$$
$$+(-1)^{n+k+1}g(a_1,\ldots,a_k,b,a_{k+2},\ldots,a_{n-1})\cdot a_n.$$

By (3.4), the second and third terms vanish. So, by definition of g, we have

$$(\sigma-(-1)^k\delta^{n-1}g)(a_1,\ldots,a_k,b,a_{k+2},\ldots,a_n)=$$
$$\sigma(a_1,\ldots,a_k,b,a_{k+2},\ldots,a_n)-\lim_\mu\sigma(a_1,\ldots,a_kb,e_\mu,a_{k+2},\ldots,a_n)$$
$$+(-1)^k\lim_\mu[(-1)^{k+2}\sigma(a_1,\ldots,a_k,e_\mu,ba_{k+2},\ldots,a_n)$$
$$+\sum_{i=k+2}^{n-1}(-1)^{i+1}\sigma(a_1,\ldots,a_k,e_\mu,b,a_{k+2},\ldots,a_ia_{i+1},\ldots,a_n)$$
$$+(-1)^{n+1}\sigma(a_1,\ldots,a_k,e_\mu,b,a_{k+2},\ldots,a_{n-1})\cdot a_n].$$

The first two terms cancel, and the remaining ones add up to zero by (3.5). This shows that, if $\zeta=(-1)^kg$, then $\sigma-\delta^{n-1}\zeta$ vanishes when its $(k+1)$th argument lies in B. When $a_i=b\in B$ for some $i, 1\le i\le k$, by the inductive hypothesis (3.3) and (3.4), we obtain

$$(\sigma-(-1)^k\delta^{n-1}g)(a_1,\ldots,b,\ldots,a_n)=$$
$$(-1)^{k+i}g(a_1,\ldots,a_{i-1}b,a_{i+1},\ldots,a_n)+(-1)^{k+i+1}g(a_1,\ldots,a_{i-1},ba_{i+1},\ldots,a_n)=0,$$

since $g\in C_B^{n-1}(A,M)$. Thus $\sigma-\delta^{n-1}\zeta$ vanishes when any of its first $k+1$ arguments lies in B. As noted above, this completes the proof of the proposition. □

1.5 Proposition. *Let A be a Banach algebra, let B be an amenable closed subalgebra of A, let M be a dual A-bimodule and let $n\ge 1$. Suppose $\rho\in C_B^n(A,M)\cap N^n(A,M)$. Then there exists $\xi\in C_B^{n-1}(A,M)$ such that*

$$\delta^{n-1}\xi=\rho.$$

Proof. For $n=1$, by Proposition 1.4, there exists $\xi_B \in C^0_B(A,M) = \mathrm{Cen}_B M$ such that

$$(\rho - \delta^0 \xi_B)(a_1) = 0$$

if $a_1 \in B$. By assumption, there exists $\xi_1 \in C^0(A,M) = M$ such that $\rho = \delta^0 \xi_1$. Hence

$$(\rho - \delta^0 \xi_B)(a_1) = (\delta^0 \xi_1 - \delta^0 \xi_B)(a_1) = \delta^0(\xi_1 - \xi_B)(a_1) = a_1.(\xi_1 - \xi_B) - (\xi_1 - \xi_B).a_1 = 0$$

if $a_1 \in B$. This implies $\xi_1 - \xi_B \in \mathrm{Cen}_B M$, and so $\xi_1 \in \mathrm{Cen}_B M$.

For $n \geq 2$, by Proposition 1.4, there exists $\xi_B \in C^{n-1}_B(A,M)$ such that

$$(\rho - \delta^{n-1}\xi_B)(a_1, ..., a_n) = 0$$

if any one of $a_1, ..., a_n$ lies in B, and by Lemma 1.2, $\rho_B \stackrel{def}{=} (\rho - \delta^{n-1}\xi_B) \in C^n_B(A,M)$. By assumption, there exists $\xi_1 \in C^{n-1}(A,M)$ such that $\rho = \delta^{n-1}\xi_1$. Hence

$$\rho_B = \rho - \delta^{n-1}\xi_B = \delta^{n-1}\xi_1 - \delta^{n-1}\xi_B = \delta^{n-1}(\xi_1 - \xi_B).$$

Further, $\eta \stackrel{def}{=} \xi_1 - \xi_B$ satisfies the assumption of Proposition 1.1, and so there exists $\beta \in C^{n-2}(A,M)$ such that

$$\eta_B(a_1, ..., a_{n-1}) \stackrel{def}{=} (\eta - \delta^{n-2}\beta)(a_1, ..., a_{n-1}) = 0$$

if any one of $a_1, ..., a_{n-1}$ lies in B. Therefore

$$\rho_B = \delta^{n-1}(\xi_1 - \xi_B) = \delta^{n-1}\eta = \delta^{n-1}(\eta_B + \delta^{n-2}\beta) = \delta^{n-1}\eta_B.$$

By Lemma 1.2, $\eta_B \in C^{n-1}_B(A,M)$; this implies

$$\rho = \rho_B + \delta^{n-1}\xi_B = \delta^{n-1}\eta_B + \delta^{n-1}\xi_B = \delta^{n-1}(\eta_B + \xi_B)$$

and so, for $\xi = \eta_B + \xi_B \in C^{n-1}_B(A,M)$, we have $\delta^{n-1}\xi = \rho$. □

As was said in the introduction, the following result is based on a communication of B. E. Johnson.

1.6 Theorem. *Let A be a Banach algebra, let B be an amenable closed subalgebra of A and let M be a dual A-bimodule. Then*

$$\mathcal{H}^n(A,M) = \mathcal{H}^n_B(A,M)$$

for all $n \geq 0$.

Proof. The inclusion morphism of cochain objects $C^n_B(A,M) \to C^n(A,M)$ induces a morphism of complexes $\mathcal{C}_B(A,M) \to \mathcal{C}(A,M)$ and hence morphisms

$$\mathcal{F}_n : \mathcal{H}^n_B(A,M) \to \mathcal{H}^n(A,M)$$

given, for each $\rho \in Z^n_B(A,M)$, by $\mathcal{F}_n(\rho + N^n_B(A,M)) = \rho + N^n(A,M)$ (see, for example, [He1; Section 0.5.3]).

For $n=0$ we have $\mathcal{H}^0_B(A,M) = \mathcal{H}^0(A,M) = \mathrm{Cen}_A(M)$.

In the case where $n \geq 1$, the morphism $\mathcal{F}_n$ is injective by Proposition 1.5 and surjective by Corollary 1.3. Hence, by Lemma 0.5.9 [He1], $\mathcal{F}_n$ is a topological isomorphism. □

In the particular case when A is a unital C^*-algebra, B is the C^*-algebra generated by an amenable group of unitaries and M is a dual normal A-bimodule, the statement of Theorem 1.6 is given in [SiSm]; see Theorem 3.2.7 [SiSm].

1.7 Proposition. *Let A_i be a Banach algebra with identity $e_i, i = 1, ..., m$, let A be the Banach algebra direct sum $\bigoplus_{i=1}^m A_i$ and let M be a dual A-bimodule. Then the canonical projections from A to $A_i, i = 1, ..., m$, induce a topological isomorphism of Banach spaces $C_B^n(A, M) \to \bigoplus_{i=1}^m C^n(A_i, e_i M e_i)$, where $n \geq 0$ and B is the Banach subalgebra of A generated by $\{e_i, i = 1, ..., m\}$. Hence*

$$\mathcal{H}^n(A, M) = \bigoplus_{i=1}^m \mathcal{H}^n(A_i, e_i M e_i)$$

for all $n \geq 0$.

Here $e_i M e_i = \{e_i \cdot x \cdot e_i, x \in M\}$ is a Banach A-bimodule.

Proof. For $n = 0$ it can be checked that

$$C_B^0(A, M) = \mathrm{Cen}_B(M) = \bigoplus_{i=1}^m (e_i M e_i) = \bigoplus_{i=1}^m C^0(A_i, e_i M e_i).$$

For $n \geq 1$ and $\rho \in C_B^n(A, M)$, we have

$$\rho(a_1, ..., a_n) = \rho(\sum_{i=1}^m a_1 e_i, a_2, ..., a_n) = \sum_{i=1}^m e_i \cdot \rho(a_1 e_i, a_2 e_i, ..., a_n e_i) \cdot e_i$$

for $a_1, \ldots, a_n \in A$. We define cochain maps from $C_B^n(A, M)$ to $\bigoplus_{i=1}^m C^n(A_i, e_i M e_i)$ and back by

$$\mathcal{J}_n : \rho \mapsto (\rho_1, \ldots, \rho_m)$$

where $\rho_i(a_1^i, \ldots, a_n^i) = \rho(a_1^i, \ldots, a_n^i)$ for $a_1^i, \ldots, a_n^i \in A_i$ and for $i = 1, \ldots, m$, and

$$\mathcal{G}_n : (\rho_1, \ldots, \rho_m) \mapsto \rho$$

is given by $\rho(a_1, ..., a_n) = \sum_{i=1}^m \rho_i(a_1 e_i, a_2 e_i, ..., a_n e_i)$ for $a_1, \ldots, a_n \in A$.

It is clear that $\mathcal{J}_n \circ \mathcal{G}_n = id$, $\mathcal{G}_n \circ \mathcal{J}_n = id$ and maps $\mathcal{J}_n$ and $\mathcal{G}_n$ are bounded. Thus there is a topological isomorhism of complexes $\mathcal{C}_B(A, M) \to \bigoplus_{i=1}^m \mathcal{C}(A_i, e_i M e_i)$, and so

$$\mathcal{H}_B^n(A, M) = \bigoplus_{i=1}^m \mathcal{H}^n(A_i, e_i M e_i).$$

Note that $B = \bigoplus_{i=1}^m \mathbb{C}_\beth$ is amenable. Hence, by Theorem 1.6, $\mathcal{H}^n(A, M) = \mathcal{H}_B^n(A, M)$ for all $n \geq 0$. The result now follows directly. □

1.8 Proposition. *Let $\mathcal{R}$ be a von Neumann algebra, let*

$$\mathcal{R} = \mathcal{R}_{I_f} \oplus \mathcal{R}_{I_\infty} \oplus \mathcal{R}_{II_1} \oplus \mathcal{R}_{II_\infty} \oplus \mathcal{R}_{III}$$

be the central direct summand decomposition of $\mathcal{R}$ into von Neumann algebras of types $I_f(finite), I_\infty, II_1, II_\infty, III$ with the identity e of $\mathcal{R}$ decomposing as $e = e_{I_f} \oplus e_{I_\infty} \oplus e_{II_1} \oplus e_{II_\infty} \oplus e_{III}$ ([Sa; Section 2.2]), and let M be a dual $\mathcal{R}$-bimodule. Then

(i)

$$\mathcal{H}^n(\mathcal{R}, M) = \mathcal{H}^n(\mathcal{R}_{I_f}, e_{I_f} M e_{I_f}) \oplus \mathcal{H}^n(\mathcal{R}_{I_\infty}, e_{I_\infty} M e_{I_\infty}) \oplus$$

$$\mathcal{H}^n(\mathcal{R}_{II_1}, e_{II_1} M e_{II_1}) \oplus \mathcal{H}^n(\mathcal{R}_{II_\infty}, e_{II_\infty} M e_{II_\infty}) \oplus \mathcal{H}^n(\mathcal{R}_{III}, e_{III} M e_{III})$$

for all $n \geq 0$;

(ii) *in particular,*

$$\mathcal{H}^n(\mathcal{R}, \mathcal{R}^*) = \mathcal{H}^n(\mathcal{R}_{I_f}, \mathcal{R}^*_{I_f}) \oplus \mathcal{H}^n(\mathcal{R}_{II_1}, \mathcal{R}^*_{II_1})$$

for all $n \geq 0$.

Proof. Part (i) follows from Proposition 1.7. In part (ii) we apply (i) to $M = \mathcal{R}^*$. By Proposition 2.2.4 [Sa], for a properly infinite von Neumann algebra $\mathcal{U}$ there exists a sequence (p_m) of mutually orthogonal, equivalent projections in $\mathcal{U}$ with $p_m \sim e$. Thus, by Theorem 2.1 [Fa], each hermitian element of $\mathcal{U}$ is the sum of five commutators. Hence there are no non-zero bounded traces on $\mathcal{U}$. Thus, by virtue of Corollary 3.3 [ChS1], for a von Neumann algebra $\mathcal{U}$ of one of the types I_∞, II_∞ or III, the simplicial cohomology groups $\mathcal{H}^n(\mathcal{U}, \mathcal{U}^*) = \{0\}$ for all $n \geq 0$. □

1.9 Remark. As for finite von Neumann algebras of type I, by [Sa; Theorem 2.3.2], they are the l_∞-direct sum of type I_m von Neumann algebras $\mathcal{R}_{I_m}$, where $m < \infty$. By Theorem 2.3.3 [Sa] and results of Section 1.22 [Sa], $\mathcal{R}_{I_m}$ is $*$-isomorphic to the C^*-tensor product $Z \otimes_{min} \mathcal{B}(H)$, where Z is the centre of $\mathcal{R}_{I_m}$ and $\dim(H) = m$. Hence, by Theorem 7.9 [Jo1], finite von Neumann algebras $\mathcal{R}_{I_m}$ of type I_m are amenable, and so their simplicial cohomology groups vanish $\mathcal{H}^n(\mathcal{R}_{I_m}, \mathcal{R}^*_{I_m}) = \{0\}$ for all $n \geq 1$. It is still not clear to the author whether $\mathcal{H}^n(\mathcal{R}_{I_f}, \mathcal{R}^*_{I_f}) = \{0\}$ for all $n \geq 2$.

Note that the statement of Proposition 1.8 (i) is proved in [SiSm], Corollary 3.3.8, for the case of a dual normal $\mathcal{R}$-bimodule M. However, this result does not apply to $M = \mathcal{R}^*$, since $\mathcal{R}^*$ is not in general a normal dual module.

Now let us consider two unital Banach algebras A_1 and A_2, a unital Banach A_1–A_2-bimodule Y, and the natural triangular matrix algebra

$$\mathcal{U} = \begin{bmatrix} A_1 & Y \\ 0 & A_2 \end{bmatrix}$$

with matrix multiplication and norm

$$\| \begin{bmatrix} r_1 & y \\ 0 & r_2 \end{bmatrix} \| = \sup_{\|d_1\| \leq 1, \|d_2\| \leq 1} \max\{\|r_1 d_1 + y \cdot d_2\|, \|r_2 d_2\|\},$$

where $d_1 \in A_1, d_2 \in A_2$. Let $e_{ii}, i = 1, 2$, denote the idempotents $e_{11} = \begin{bmatrix} e_{A_1} & 0 \\ 0 & 0 \end{bmatrix}$ and $e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & e_{A_2} \end{bmatrix}$ respectively. Let us also consider the Banach subalgebra B of $\mathcal{U}$ generated by $\{e_{ii}, i = 1, 2\}$.

1.10 Proposition. *Let A_1 and A_2 be unital Banach algebras, let Y be a unital Banach $A_1 - A_2$-bimodule, and let*

$$\mathcal{U} = \begin{bmatrix} A_1 & Y \\ 0 & A_2 \end{bmatrix}$$

be the natural triangular matrix algebra. Then the two canonical projections from $\mathcal{U}$ to A_1 and A_2 induce a topological isomorphism of Banach spaces $C^n_B(\mathcal{U}, \mathcal{U}^*) \to \bigoplus_{i=1}^2 C^n(A_i, A_i^*), n \geq 0$, and hence

$$\mathcal{H}^n(\mathcal{U}, \mathcal{U}^*) = \mathcal{H}^n(A_1, A_1^*) \oplus \mathcal{H}^n(A_2, A_2^*)$$

for all $n \geq 0$.

The same assertion in a purely algebraic context was proved by L. Kadison in [Ka1]. A key step in Kadison's proof is the equality $\mathcal{H}^n(\mathcal{U}, \mathcal{U}^*) = \mathcal{H}^n_B(\mathcal{U}, \mathcal{U}^*)$ for all $n \geq 0$, where B is the Banach algebra generated by e_{11} and e_{22}. This equality remains valid in the present context by Theorem 1.6. The rest of the proof is as in [Ka1], with insignificant changes.

4. The connection between the cohomologies of A and A/I

Recall Proposition 5.1 [Jo1] that a quotient algebra of an amenable algebra is amenable, and that an extension of an amenable algebra by an amenable bi-ideal is an amenable algebra (this can also be found in Corollary 35 and Proposition 39 [He3]). The following theorem gives some additional information about the cohomology of Banach algebras A and A/I without the assumption that A be amenable.

2.1 Theorem. *Let A be a Banach algebra and let I be a closed two-sided ideal of A. Suppose that I is an amenable Banach algebra and M is a dual A/I-bimodule. Then*

$$\mathcal{H}^n(A, M) = \mathcal{H}^n(A/I, M)$$

for all $n \geq 0$.

Proof. For $n = 0$ we have $\mathcal{H}^0(A, M) = \mathcal{H}^0(A/I, M) = \mathrm{Cen}_{A/I}(M)$. In the case where $n \geq 1$ the inclusion morphism of cochain objects $C^n(A/I, M) \to C^n(A, M)$ induces a morphism of complexes $\mathcal{C}(A/I, M) \to \mathcal{C}(A, M)$ and hence morphisms

$$\mathcal{L}_n : \mathcal{H}^n(A/I, M) \to \mathcal{H}^n(A, M)$$

given, for each $\hat{\rho} \in Z^n(A/I, M)$, by

$$\mathcal{L}_n(\hat{\rho} + N^n(A/I, M)) = \rho + N^n(A, M),$$

where $\rho(a_1,...,a_n)=\hat\rho(\theta(a_1),...,\theta(a_n))$ and $\theta: A\to A/I$ is the natural epimorphism. It is straight-forward to check that if $\hat\rho=\delta^{n-1}\hat\xi$ for some $\hat\xi\in C^{n-1}(A/I,M)$ then $\rho=\delta^{n-1}\xi$ where $\xi(a_1,...,a_{n-1})=\hat\xi(\theta(a_1),...,\theta(a_{n-1}))$.

By Corollary 1.3, for $\eta\in Z^n(A,M)$, there exists $\xi\in C^{n-1}(A,M)$ such that

$$(\eta-\delta^{n-1}\xi)(a_1,...,a_n)=0$$

if any one of $a_1,...,a_n$ lies in I. We can therefore define

$$\eta_I\overset{def}{=}\eta-\delta^{n-1}\xi \text{ and } \hat\eta_I(a_1+I,...,a_n+I)\overset{def}{=}\eta_I(a_1,...,a_n).$$

Hence for each $\eta\in Z^n(A,M)$ there exists $\hat\eta_I\in Z^n(A/I,M)$ such that

$$\mathcal{L}_n(\hat\eta_I+N^n(A/I,M))=\eta+N^n(A,M),$$

and so $\mathcal{L}_n$ is surjective.

Let $\mathcal{L}_n(\hat\rho+N^n(A/I,M))=0$, that is, $\rho(a_1,...,a_n)=\hat\rho(\theta(a_1),...,\theta(a_n))\in N^n(A,M)$. This implies that there is $\beta\in C^{n-1}(A,M)$ such that $\rho=\delta^{n-1}\beta$. Further, β satisfies the assumption of Proposition 1.1, and so there exists $\alpha\in C^{n-2}(A,M)$ such that

$$(\beta-\delta^{n-2}\alpha)(a_1,...,a_{n-1})=0$$

if any one of $a_1,...,a_{n-1}$ lies in I. We can define

$$\beta_I\overset{def}{=}\beta-\delta^{n-2}\alpha$$

and see that $\rho=\delta^{n-1}\beta=\delta^{n-1}\beta_I$. Therefore $\hat\rho=\delta^{n-1}\hat\beta_I$, where

$$\hat\beta_I(a_1+I,...,a_n+I)=\beta_I(a_1,...,a_n).$$

This proves the injectivity of $\mathcal{L}_n$. Hence, by Lemma 0.5.9 [He1], $\mathcal{L}_n$ is a topological isomorphism. □

2.2 Proposition. *Let A be a Banach algebra and let I be a closed two-sided ideal of A. Suppose that I has a bounded approximate identity. Then $\mathcal{H}^0_I(A,I^*)=\mathrm{Cen}_I I^*$ and*

$$\mathcal{H}^n_I(A,I^*)=\{0\}$$

for all $n\ge 1$.

Proof. Let us consider the Banach space $C^n_I(A,I^*)$ which is isometrically isomorphic to the Banach space ${}_Ih_I(A\hat\otimes_I\dots\hat\otimes_I A,I^*)$ of all Banach I-bimodule morphisms from $A\hat\otimes_I\dots\hat\otimes_I A$ into I^*. The latter Banach space is isometricaly isomorphic to $\mathrm{Cen}_I(A\hat\otimes_I\dots\hat\otimes_I A\hat\otimes_I I)^*$ by Proposition VII.2.17 [He1]. By virtue of the assumption, I has a bounded approximate identity, and so Proposition II.3.13 [He1] gives us an isomophism of Banach I-bimodules $A\hat\otimes_I\dots\hat\otimes_I A\hat\otimes_I I=I$. Therefore, there exists an isometric isomorphism of Banach spaces

$$\mathcal{F}_n: C^n_I(A,I^*)\to\mathrm{Cen}_I I^*$$

for all $n\ge 0$.

Now it is routine to check that the diagram

$$\begin{array}{ccccccccc}
0 \to C_I^0(A,I^*) & \xrightarrow{\delta^0} & C_I^1(A,I^*) & \xrightarrow{\delta^1} \dots \to & C_I^n(A,I^*) & \xrightarrow{\delta^n} & C_I^{n+1}(A,I^*) & \to \dots \\
\downarrow \mathcal{F}_0 & & \downarrow \mathcal{F}_1 & & \downarrow \mathcal{F}_n & & \downarrow \mathcal{F}_{n+1} & \\
0 \to \ \mathrm{Cen}_I I^* & \xrightarrow{\eta^0} & \mathrm{Cen}_I I^* & \xrightarrow{\eta^1} \dots \to & \mathrm{Cen}_I I^* & \xrightarrow{\eta^n} & \mathrm{Cen}_I I^* & \to \dots,
\end{array}$$

is commutative, where $\eta^n(f) = 0$ for all even n and $\eta^n(f) = f$ for all odd n. The cohomology of the upper complex is, by definition, $\mathcal{H}_I^n(A, I^*)$. Thus the result now follows directly. □

2.3 Corollary. *Let A be a Banach algebra and let I be a closed two-sided ideal of A. Suppose that I is an amenable Banach algebra. Then $\mathcal{H}^0(A,I^*) = \mathrm{Cen}_I I^*$ and*

$$\mathcal{H}^n(A,I^*) = \{0\}$$

for all $n \geq 1$.

Proof. By Theorem 1.6,

$$\mathcal{H}^n(A,I^*) = \mathcal{H}_I^n(A,I^*)$$

for all $n \geq 0$. Hence the result follows from Proposition 2.2. □

2.4 Theorem. *Let A be a Banach algebra and let I be a closed two-sided ideal of A. Suppose that I is an amenable Banach algebra. Then*

(i)

$$\mathcal{H}^n(A,A^*) = \mathcal{H}^n(A/I,(A/I)^*)$$

for all $n \geq 2$, and the natural map from $\mathcal{H}^1(A/I,(A/I)^*)$ into $\mathcal{H}^1(A,A^*)$ is surjective;

(ii) *if $\mathrm{Cen}_I I^* = \{0\}$ then*

$$\mathcal{H}^n(A,A^*) = \mathcal{H}^n(A/I,(A/I)^*)$$

for all $n \geq 0$.

Proof. We consider the short exact sequence of Banach A-bimodules

$$0 \leftarrow A/I \xleftarrow{j} A \xleftarrow{i} I \leftarrow 0, \qquad (\mathcal{I})$$

where i and j are the natural embedding and quotient mapping respectively, and its dual complex

$$0 \to (A/I)^* \xrightarrow{j^*} A^* \xrightarrow{i^*} I^* \to 0. \qquad (\mathcal{I}^*)$$

By virtue of its amenability, I has a bounded approximate identity and so the complex $(\mathcal{I}^*)$ is admissible. Hence, by Corollary III.4.11 [He1], there exists a long exact sequence

$$0 \to \mathcal{H}^0(A,(A/I)^*) \to \mathcal{H}^0(A,A^*) \to \mathcal{H}^0(A,I^*) \to \mathcal{H}^1(A,(A/I)^*) \to \mathcal{H}^1(A,A^*) \to$$

$$\mathcal{H}^1(A,I^*) \to \mathcal{H}^2(A,(A/I)^*) \to \mathcal{H}^2(A,A^*) \to \mathcal{H}^2(A,I^*) \to \dots$$

$$\dots \to \mathcal{H}^{n-1}(A,I^*) \to \mathcal{H}^n(A,(A/I)^*) \to \mathcal{H}^n(A,A^*) \to \mathcal{H}^n(A,I^*) \to \dots.$$

Recall that, by Corollary 2.3, $\mathcal{H}^n(A, I^*) = \{0\}$ for all $n \geq 1$. Thus $\mathcal{H}^n(A, (A/I)^*) = \mathcal{H}^n(A, A^*)$ (see Lemma 0.5.9 [He1]) for all $n \geq 2$. Therefore, by Theorem 2.1,

$$\mathcal{H}^n(A/I, (A/I)^*) = \mathcal{H}^n(A, (A/I)^*) = \mathcal{H}^n(A, A^*)$$

for all $n \geq 2$. □

Note that $\mathrm{Cen}_A A^*$ coincides with the space

$$A^{tr} = \{f \in A^* : f(ab) = f(ba) \text{ for all } a, b \in A\}$$

of continuous traces on A.

Theorem 2.4 applies whenever I is a nuclear C^*-algebra. Other examples are given by the Banach algebra $A = \mathcal{B}(E)$ of all bounded operators on a Banach space E with the property $(\mathbb{A})$, which was defined in [GJW], and the closed ideal $I = \mathcal{K}(E)$ of compact operators on E. In this case $\mathcal{K}(E)$ is amenable [GJW]. The property $(\mathbb{A})$ implies that $\mathcal{K}(E)$ contains a bounded sequence of projections of unbounded finite rank, and from this it is easy to show (via embedding of matrix algebras) that there is no non-zero bounded trace on $\mathcal{K}(E)$. Thus we can see from Theorem 2.4 (ii) that, for a Banach space E with the property $(\mathbb{A})$, $\mathcal{H}^n(\mathcal{B}(E), \mathcal{B}(E)^*) = \mathcal{HC}^n(\mathcal{B}(E)/\mathcal{K}(E), (\mathcal{B}(E)/\mathcal{K}(E))^*)$ for all $n \geq 0$. Several classes of Banach spaces have the property $(\mathbb{A})$: $l_p; 1 < p < \infty; C(K)$, where K is a compact Hausdorff space; $L_p(\Omega, \mu); 1 < p < \infty$, where (Ω, μ) is a measure space (for details and more examples see [Jo1] and [GJW]).

In the case of C^*-algebras, we know that the Banach simplicial cohomology groups vanish for C^*-algebras without non-zero bounded traces [ChS1; Corollary 3.3]. Therefore, for an infinite-dimensional Hilbert space H, we obtain

$$\mathcal{H}^n(\mathcal{B}(H)/\mathcal{K}(H), (\mathcal{B}(H)/\mathcal{K}(H))^*) = \mathcal{H}^n(\mathcal{B}(H), \mathcal{B}(H)^*) = \{0\}$$

for all $n \geq 0$, since $\mathcal{K}(H)^{tr} = \{0\}$ by [An; Theorem 2] and $\mathcal{B}(H)^{tr} = \{0\}$ by [Hal]. One can also see directly that the Calkin algebra has no non-zero bounded trace, and hence has trivial Banach simplicial cohomology.

Notice that, for every Banach algebra A, the vanishing of $\mathcal{H}^n(A, A^*)$ for all $n \geq 0$ is equivalent to the vanishing of the Banach simplicial homology groups $\mathcal{H}_n(A, A)$ for all $n \geq 0$ [He1, Proposition 2.5.28]. The latter relation was established by M. Wodzicki [Wo] in the case $A = \mathcal{B}(H)$. He also proved the vanishing of the (Banach) simplicial homology groups for stable C^*-algebras, i.e., for algebras isomorphic to their C^*-tensor product with $\mathcal{K}(H)$ for a separable H, and the vanishing of the algebraic simplicial homology groups for $A = \mathcal{B}(H)/\mathcal{K}(H)$ and a separable H.

Recall from [Di; Sections 4.2, 4.3] that a C^*-algebra A is called CCR (or *liminary*) if $\pi(A) = \mathcal{K}(H)$ for each irreducible representation (π, H) of A. A C^*-algebra A is called GCR (or *postliminary*) if each non-zero quotient of A has a non-zero closed two-sided CCR-ideal. Finally we say that A is NGCR (or *antiliminary*) if it contains no non-zero closed two-sided CCR-ideal. By [Di; Proposition 4.3.3 and 4.3.6], each C^*-algebra A has a largest closed two-sided GCR-ideal I_α, and A/I_α is NGCR. The following result allow us to reduce the computation of the simplicial cohomology groups of C^*-algebras to the case of NGCR-algebras.

2.5 PROPOSITION. *Let A be a C^*-algebra. Then*

$$\mathcal{H}^n(A, A^*) = \mathcal{H}^n(A/I_\alpha, (A/I_\alpha)^*)$$

for all $n \geq 1$.

Proof. By [Ha; Corollary 4.2], $\mathcal{H}^1(A, A^*) = \{0\}$ for every C^*-algebra A. By Theorem 7.9 [Jo1], I_α is amenable. Thus the result directly follows from Theorem 2.4 (i). □

5. THE EXISTENCE OF THE CONNES-TSYGAN EXACT SEQUENCE

Let A be a Banach algebra, not necessarily unital, and let S be a closed subalgebra of A_+. In this section we introduce the Banach version of the concept of S-relative cyclic cohomology $\mathcal{HC}_S^n(A)$ (compare with [Ka1]). We also show that the S-relative Connes-Tsygan exact sequence exists for every Banach algebra A with a bounded approximate identity. This is accomplished with the aid of ideas from [He2].

When $S = \mathbb{C}_+$ the subscript S is unnecessary and we omit it.

We denote by $C_S^n(A), n = 0, 1, ...,$ the Banach space of continuous $(n+1)$-linear functionals on A such that

$$f(sa_0, a_1, ..., a_n) = f(a_0, a_1, ..., a_n s)$$

and, for $j = 0, 1, ..., n-1$,

$$f(a_0, ..., a_j s, a_{j+1}, ..., a_n) = f(a_0, ..., a_j, sa_{j+1}, ..., a_n)$$

for all $s \in S$ and $a_0, ..., a_n \in A$; these functionals we shall call *n-dimensional S-relative cochains*. We let

$$t_n : C_S^n(A) \to C_S^n(A), \qquad n = 0, 1, ...,$$

denote the operator given by

$$t_n f(a_0, a_1, ..., a_n) = (-1)^n f(a_1, ..., a_n, a_0),$$

and we set $t_0 = id$. The important point is that $t_n f$ is an S-relative cochain since

$$t_n f(sa_0, a_1, ..., a_n) = (-1)^n f(a_1, ..., a_n, sa_0) = (-1)^n f(a_1, ..., a_n s, a_0) = t_n f(a_0, ..., a_n s)$$

and the other identities follow just as readily. A cochain $f \in C_S^n(A)$ satisfying $t_n f = f$ is called *cyclic*. We let $CC_S^n(A)$ denote the closed subspace of $C_S^n(A)$ formed by the cyclic cochains. In particular,

$$CC_S^0(A) = C_S^0(A) = \mathrm{Cen}_S A^* = \{f \in A^* : f(sa) = f(as) \text{ for all } a \in A, s \in S\}.$$

From the S-relative cochains we form the standard cohomology complex $\tilde{\mathcal{C}}_S(A)$:

$$0 \to C_S^0(A) \xrightarrow{\delta^0} \cdots \to C_S^n(A) \xrightarrow{\delta^n} C_S^{n+1}(A) \to \ldots,$$

where the continuous operator δ^n is given by the formula

$$(\delta^n f)(a_0, a_1, ..., a_{n+1}) =$$

$$\sum_{i=0}^{n}(-1)^i f(a_0, ..., a_i a_{i+1}, ..., a_{n+1}) + (-1)^{n+1} f(a_{n+1}a_0, ..., a_n).$$

One can easily check that $\delta^{n+1} \circ \delta^n$ is indeed 0 for all n and that each $\delta^n f$ is again an S-relative cochain. It is not difficult to verify that every δ^n sends a cyclic cochain again to a cyclic one. Therefore there is a subcomplex in $\tilde{\mathcal{C}}_S(A)$ formed by the spaces $CC^n_S(A)$. We denote this subcomplex by $\tilde{\mathcal{CC}}_S(A)$, and its differentials are denoted by

$$\delta c^n : CC^n_S(A) \to CC^{n+1}_S(A).$$

Note that the complex $\tilde{\mathcal{C}}_S(A)$ is a subcomplex of $\tilde{\mathcal{C}}(A)$ and $\tilde{\mathcal{CC}}_S(A)$ is a subcomplex of $\tilde{\mathcal{CC}}(A)$ respectively.

3.1 DEFINITION. The nth cohomology of $\tilde{\mathcal{C}}_S(A)$, denoted by $\mathcal{H}^n_S(A)$, is called the *n-dimensional Banach S-relative simplicial*, or *Hochschild, cohomology group* of the Banach algebra A. The nth cohomology of $\tilde{\mathcal{CC}}_S(A)$, denoted by $\mathcal{HC}^n_S(A)$, is called the *n-dimensional Banach S-relative cyclic cohomology group* of A.

Note that, by definition, $\delta c^0 = \delta^0$, so that $\mathcal{HC}^0_S(A) = \mathcal{H}^0_S(A)$ coincides with the space $A^{tr} = \{f \in A^* : f(ab) = f(ba) \text{ for all } a, b \in A\}$.

We define $\mathcal{HC}_S^{-1}(A)$ to be $\{0\}$.

3.2 REMARK. The canonical identification of $(n+1)-$linear functionals on A and $n-$linear operators from A to A^* shows that $\mathcal{H}^n_S(A)$ is just another way of writing $\mathcal{H}^n_S(A, A^*)$.

Further, we need the following complex $\tilde{\mathcal{CR}}_S(A)$:

$$0 \to C^0_S(A) \stackrel{\delta r^0}{\to} \cdots \to C^n_S(A) \stackrel{\delta r^n}{\to} C^{n+1}_S(A) \to \dots,$$

where the continuous operator δr^n is given by the formula

$$(\delta r^n f)(a_0, a_1, ..., a_{n+1}) = \sum_{i=0}^{n}(-1)^i f(a_0, ..., a_i a_{i+1}, ..., a_{n+1}).$$

The nth cohomology of $\tilde{\mathcal{CR}}_S(A)$ is denoted by $\mathcal{HR}^n_S(A)$.

Following [He2] we consider the sequence

$$0 \to \tilde{\mathcal{CC}}_S(A) \stackrel{\bar{i}}{\to} \tilde{\mathcal{C}}_S(A) \stackrel{\bar{M}}{\to} \tilde{\mathcal{CR}}_S(A) \stackrel{\bar{N}}{\to} \tilde{\mathcal{CC}}_S(A) \to 0$$

of complexes in the category of Banach spaces and continuous operators, where $\bar{i}$ denotes the natural inclusion

$$i_n : CC^n_S(A) \to C^n_S(A),$$
$$M_n = id - t_n : C^n_S(A) \to C^n_S(A)$$

and

$$N_n = id + t_n + \cdots + t^n_n : C^n_S(A) \to CC^n_S(A).$$

3.3 PROPOSITION. *Let A be a Banach algebra and let S be a closed subalgebra of A_+. Then the sequence*

$$0 \to \tilde{\mathcal{CC}}_S(A) \xrightarrow{\tilde{i}} \tilde{\mathcal{C}}_S(A) \xrightarrow{\tilde{M}} \tilde{\mathcal{CR}}_S(A) \xrightarrow{\tilde{N}} \tilde{\mathcal{CC}}_S(A) \to 0$$

is exact.

The proof is the same as that of Proposition 4 of [He2]. □

3.4 PROPOSITION. *Let A be a Banach algebra with a bounded approximate identity $e_\nu, \nu \in \Lambda$, and let S be a closed subalgebra of A_+. Then $\mathcal{HR}^n_S(A) = \{0\}$ for all $n \geq 0$.*

Proof. For $f \in C^n_S(A)$ we define $g_\nu \in C^{n-1}(A)$ by

$$g_\nu(a_0, ..., a_{n-1}) = f(e_\nu, a_0, ..., a_{n-1}),$$

for $a_0, ..., a_{n-1} \in A$ and $\nu \in \Lambda$. Since the Banach space $C^{n-1}(A)$ is the dual space of $A \hat{\otimes} \dots \hat{\otimes} A$, for the bounded net $g_\nu, \nu \in \Lambda$, there exists a subnet $g_\mu, \mu \in \Lambda'$, which weak* converges to some cochain $g \in C^{n-1}(A)$. It can be checked that $g \in C^{n-1}_S(A)$.

For each $f \in C^n_S(A)$ such that $\delta r^n(f) = 0$ and for each $\nu \in \Lambda$,

$$\begin{aligned} 0 &= \delta r^n f(e_\nu, a_0, ..., a_n) \\ &= f(e_\nu a_0, a_1, ..., a_n) + \sum_{i=0}^{n-1} (-1)^{i+1} f(e_\nu, a_0, ..., a_i a_{i+1}, ..., a_n). \end{aligned} \tag{5.1}$$

Thus we obtain

$$\begin{aligned} \delta r^{n-1} g(a_0, ..., a_n) &= \sum_{i=0}^{n-1} (-1)^i g(a_0, ..., a_i a_{i+1}, ..., a_n) \\ &= \lim_\mu \sum_{i=0}^{n-1} (-1)^i f(e_\mu, a_0, ..., a_i a_{i+1}, ..., a_n) = f(a_0, a_1, ..., a_n), \end{aligned}$$

by (5.1). □

Note that in the case where $S = \mathbb{C}_+$ it is easy to see that the statement of Proposition 3.4 is true for every Banach algebra with left or right bounded approximate identity. This and other conditions are given in detail in [He2, Section 2].

3.5 PROPOSITION. *Let A be a Banach algebra with a bounded approximate identity and let S be a closed subalgebra of A_+. Then the S-relative Connes-Tsygan exact sequence for A*

$$\dots \to \mathcal{H}^n_S(A) \xrightarrow{B^n} \mathcal{HC}^{n-1}_S(A) \xrightarrow{S^n} \mathcal{HC}^{n+1}_S(A) \xrightarrow{I^{n+1}} \mathcal{H}^{n+1}_S(A) \xrightarrow{B^{n+1}} \mathcal{HC}^n_S(A) \to \dots$$

exists.

Proof. By Proposition 3.3, there are two short exact sequences of complexes

$$0 \to \tilde{\mathcal{CC}}_S(A) \xrightarrow{\tilde{i}} \tilde{\mathcal{C}}_S(A) \xrightarrow{\tilde{M}} \tilde{\mathcal{CS}}_S(A) \to 0$$

and

$$0 \to \tilde{\mathcal{CS}}_S(A) \xrightarrow{\tilde{j}} \tilde{\mathcal{CR}}_S(A) \xrightarrow{\tilde{N}} \tilde{\mathcal{CC}}_S(A) \to 0$$

where $\tilde{\mathcal{CS}}_S(A)$ is the subcomplex $\mathrm{Im}(\bar{M}) = \mathrm{Ker}(\bar{N})$ of $\tilde{\mathcal{CR}}_S(A)$. Hence, by [He1, Chapter 0, Section 5.5], there exist two long exact sequences (*)

$$\cdots \to \mathcal{HC}^n_S(A) \stackrel{H^n(\bar{i})}{\to} \mathcal{H}^n_S(A) \stackrel{H^n(\bar{M})}{\to} \mathcal{HS}^n_S(A) \stackrel{\zeta^n}{\to} \mathcal{HC}^{n+1}_S(A) \stackrel{H^{n+1}(\bar{i})}{\to} \mathcal{H}^{n+1}_S(A) \to \dots$$

and (**)

$$\cdots \to \mathcal{HS}^n_S(A) \stackrel{H^n(\bar{j})}{\to} \mathcal{HR}^n_S(A) \stackrel{H^{n+1}(\bar{N})}{\to} \mathcal{HC}^n_S(A) \stackrel{\eta^n}{\to} \mathcal{HS}^{n+1}_S(A) \stackrel{H^{n+1}(\bar{j})}{\to} \mathcal{HR}^{n+1}_S(A) \to \dots.$$

By Proposition 3.4, $\mathcal{HR}^n_S(A) = \{0\}$ for all $n \geq 0$. Thus we can see from (**) and Proposition 8 [He2] that η^n is a topological isomorphism, and so that $\mathcal{HC}^n_S(A) = \mathcal{HS}^{n+1}_S(A)$ for all $n \geq 0$. By setting $\mathcal{HC}^{n-1}_S(A)$ instead of $\mathcal{HS}^n_S(A)$ in (*), we get the required exact sequence, where $I^n = H^n(\bar{i}), S^n = \zeta^{n-1} \circ \eta^n$ and $B^n = (\eta^n)^{-1} \circ H^n(\bar{M})$. □

3.6 Proposition. *Let A be a Banach algebra with a bounded approximate identity and let S be a closed subalgebra of A_+. Then the inclusion morphism of cochain objects $C^n_S(A, A^*) \to C^n(A, A^*)$ induces a morphism of Connes-Tsygan exact sequences for A, that is, a commutative diagram*

$$\begin{array}{ccccccccccc}
\dots \to & \mathcal{H}^n_S(A) & \stackrel{B^n}{\to} & \mathcal{HC}^{n-1}_S(A) & \stackrel{S^n}{\to} & \mathcal{HC}^{n+1}_S(A) & \stackrel{I^{n+1}}{\to} & \mathcal{H}^{n+1}_S(A) & \stackrel{B^{n+1}}{\to} & \mathcal{HC}^n_S(A) & \to \dots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \to & \mathcal{H}^n(A) & \stackrel{B^n}{\to} & \mathcal{HC}^{n-1}(A) & \stackrel{S^n}{\to} & \mathcal{HC}^{n+1}(A) & \stackrel{I^{n+1}}{\to} & \mathcal{H}^{n+1}(A) & \stackrel{B^{n+1}}{\to} & \mathcal{HC}^n(A) & \to \dots.
\end{array}$$

Proof. Note that the inclusion morphism of cochain objects $C^n_S(A, A^*) \to C^n(A, A^*)$ gives morphisms of two pairs short exact sequences of complexes

$$\begin{array}{ccccccccc}
0 & \to & \tilde{\mathcal{CC}}_S(A) & \stackrel{\bar{i}}{\to} & \tilde{\mathcal{C}}_S(A) & \stackrel{\bar{M}}{\to} & \tilde{\mathcal{CS}}_S(A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tilde{\mathcal{CC}}(A) & \stackrel{\bar{i}}{\to} & \tilde{\mathcal{C}}(A) & \stackrel{\bar{M}}{\to} & \tilde{\mathcal{CS}}(A) & \to & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \to & \tilde{\mathcal{CS}}_S(A) & \stackrel{\bar{j}}{\to} & \tilde{\mathcal{CR}}_S(A) & \stackrel{\bar{N}}{\to} & \tilde{\mathcal{CC}}_S(A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tilde{\mathcal{CS}}(A) & \stackrel{\bar{j}}{\to} & \tilde{\mathcal{CR}}(A) & \stackrel{\bar{N}}{\to} & \tilde{\mathcal{CC}}(A) & \to & 0.
\end{array}$$

By the cohomology analogue of Proposition II.4.2 [Ma], a morphism of two short exact sequences of complexes induces a morphism of long exact cohomology sequences. Hence we have two commutative diagrams (*)

$$\begin{array}{ccccccccccc}
\dots \to & \mathcal{HC}^n_S(A) & \to & \mathcal{H}^n_S(A) & \to & \mathcal{HS}^n_S(A) & \to & \mathcal{HC}^{n+1}_S(A) & \to & \mathcal{H}^{n+1}_S(A) & \to \dots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \to & \mathcal{HC}^n(A) & \to & \mathcal{H}^n(A) & \to & \mathcal{HS}^n(A) & \to & \mathcal{HC}^{n+1}(A) & \to & \mathcal{H}^{n+1}(A) & \to \dots
\end{array}$$

and (**)

$$\begin{array}{ccccccccccc} \ldots \to \mathcal{HS}^n_S(A) & \to & \mathcal{HR}^n_S(A) & \to & \mathcal{HC}^n_S(A) & \to & \mathcal{HS}^{n+1}_S(A) & \to & \mathcal{HR}^{n+1}_S(A) & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to \mathcal{HS}^n(A) & \to & \mathcal{HR}^n(A) & \to & \mathcal{HC}^n(A) & \to & \mathcal{HS}^{n+1}(A) & \to & \mathcal{HR}^{n+1}(A) & \ldots . \end{array}$$

By Proposition 3.4, $\mathcal{HR}^n_S(A) = \mathcal{HR}^n(A) = \{0\}$ for all $n \geq 0$. Thus we can see from (**) and Propositon 8 [He2] that there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{HC}^n_S(A) & = & \mathcal{HS}^{n+1}_S(A) \\ \downarrow & & \downarrow \\ \mathcal{HC}^n(A) & = & \mathcal{HS}^{n+1}(A) \end{array}$$

for all $n \geq 0$. By setting $\mathcal{HC}^{n-1}_S(A)$ instead of $\mathcal{HS}^n_S(A)$ and $\mathcal{HC}^{n-1}(A)$ instead of $\mathcal{HS}^n(A)$ in (*), we get the required commutative diagram. □

3.7 Proposition. *Let A and D be Banach algebras with right or left bounded approximate identities. Suppose there exists a continuous homomorphism $\kappa : A \to D$. Then the associated morphism of cochain objects $C^n(D, D^*) \to C^n(A, A^*)$ induces a morphism of Connes-Tzygan exact sequences for A, that is, a commutative diagram*

$$\begin{array}{ccccccccccc} \ldots \to \mathcal{H}^n(D) & \stackrel{B^n}{\to} & \mathcal{HC}^{n-1}(D) & \stackrel{S^n}{\to} & \mathcal{HC}^{n+1}(D) & \stackrel{I^{n+1}}{\to} & \mathcal{H}^{n+1}(D) & \stackrel{B^{n+1}}{\to} & \mathcal{HC}^n(D) & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to \mathcal{H}^n(A) & \stackrel{B^n}{\to} & \mathcal{HC}^{n-1}(A) & \stackrel{S^n}{\to} & \mathcal{HC}^{n+1}(A) & \stackrel{I^{n+1}}{\to} & \mathcal{H}^{n+1}(A) & \stackrel{B^{n+1}}{\to} & \mathcal{HC}^n(A) & \ldots . \end{array}$$

The proof requires only minor modifications of that of Proposition 3.6. □

3.8 Proposition. *Let A be a Banach algebra for which one of the following conditions is satisfied:*

(i) *A has a left or right bounded approximate identity;*

(ii) *A coincides with the topological square $\overline{A^2}$ of A, and that either A is a flat right Banach A-module or $\mathbb{C}$ is a flat left Banach A-module.*

Then the vanishing of $\mathcal{HC}^n(A)$ for all $n \geq 0$ is equivalent to the vanishing of the Banach cyclic homology $\mathcal{HC}_n(A)$ for all $n \geq 0$.

The definition of flat module can be found in [He1], and the definition of Banach cyclic homology of a Banach algebra can be found, for example, in [He2; Section 5].

Proof. The assumption gives the existence the Connes-Tsygan exact sequence for cohomology of A [He2, Theorems 15, 16]. We can see from this exact sequence that the vanishing of $\mathcal{HC}^n(A)$ for all $n \geq 0$ is equivalent to the vanishing of the simplicial cohomology $\mathcal{H}^n(A, A^*)$ for all $n \geq 0$. The latter relation is equvalent to the vanishing of the simplicial homology $\mathcal{H}_n(A, A)$ for all $n \geq 0$ [He1; Proposition 2.5.28]. As was noted in [He2; Section 5], for the given assumption, the canonical Connes-Tsygan exact sequence for homology of A also exists. Thus it is easy to see from the Connes-Tsygan exact sequence for homology of A that all these relations are equivalent to the vanishing of the cyclic homology $\mathcal{HC}_n(A)$ for all $n \geq 0$. □

6. Relative cyclic cohomology of Banach algebras

4.1 Theorem. *Let A be a Banach algebra with a bounded approximate identity, and let B be an amenable closed subalgebra of A. Then*

$$\mathcal{HC}^n(A) = \mathcal{HC}^n_B(A)$$

for all $n \geq 0$.

Proof. Consider the commutative diagram of Proposition 3.6

$$\begin{array}{ccccccccccccc}
0 \to & \mathcal{HC}^0_B(A) & \stackrel{I^0}{\to} & \mathcal{H}^0_B(A) & \stackrel{B^0}{\to} & \mathcal{HC}^{-1}_B(A) & \stackrel{S^0}{\to} & \mathcal{HC}^1_B(A) & \stackrel{I^1}{\to} & \mathcal{H}^1_B(A) & \stackrel{B^1}{\to} & \mathcal{HC}^0_B(A) \dots \\
& \downarrow \mathcal{G}_0 & & \downarrow \mathcal{F}_0 & & \downarrow \mathcal{G}_{-1} & & \downarrow \mathcal{G}_1 & & \downarrow \mathcal{F}_1 & & \downarrow \mathcal{G}_0 \\
0 \to & \mathcal{HC}^0(A) & \stackrel{I^0}{\to} & \mathcal{H}^0(A) & \stackrel{B^0}{\to} & \mathcal{HC}^{-1}(A) & \stackrel{S^0}{\to} & \mathcal{HC}^1(A) & \stackrel{I^1}{\to} & \mathcal{H}^1(A) & \stackrel{B^1}{\to} & \mathcal{HC}^0(A) \dots
\end{array}$$

$$\begin{array}{ccccccccccc}
\dots \to & \mathcal{H}^n_B(A) & \stackrel{B^n}{\to} & \mathcal{HC}^{n-1}_B(A) & \stackrel{S^n}{\to} & \mathcal{HC}^{n+1}_B(A) & \stackrel{I^{n+1}}{\to} & \mathcal{H}^{n+1}_B(A) & \stackrel{B^{n+1}}{\to} & \mathcal{HC}^n_B(A) & \to \dots \\
& \downarrow \mathcal{F}_n & & \downarrow \mathcal{G}_{n-1} & & \downarrow \mathcal{G}_{n+1} & & \downarrow \mathcal{F}_{n+1} & & \downarrow \mathcal{G}_n \\
\dots \to & \mathcal{H}^n(A) & \stackrel{B^n}{\to} & \mathcal{HC}^{n-1}(A) & \stackrel{S^n}{\to} & \mathcal{HC}^{n+1}(A) & \stackrel{I^{n+1}}{\to} & \mathcal{H}^{n+1}(A) & \stackrel{B^{n+1}}{\to} & \mathcal{HC}^n(A) & \to \dots .
\end{array}$$

Note that $\mathcal{HC}^0(A) = \mathcal{HC}^0_B(A) = A^{tr}$. By Theorem 1.6,

$$\mathcal{H}^n(A) = \mathcal{H}^n(A, A^*) = \mathcal{H}^n_B(A, A^*) = \mathcal{H}^n_B(A)$$

for all $n \geq 0$.

As an induction hypothesis suppose that the vertical map

$$\mathcal{G}_i : \mathcal{HC}^i_B(A) \to \mathcal{HC}^i(A)$$

is an isomorphism for each $i \leq n$. Then it follows from the five lemma [Ma; Lemma 1.3.3] that the middle vertical map $\mathcal{G}_{n+1}$ above is an isomorphism. Hence by Lemma 0.5.9 [He1], $\mathcal{G}_{n+1}$ is topological isomorphism. □

We note that the assumptions of Theorem 4.1 are obviously satisfied by all C^*-algebras A and all nuclear C^*-subalgebras B of A_+. Recall that all nuclear C^*-algebras and only these C^*-algebras are amenable. For example, all GCR-algebras, in particular, all commutative C^*-algebras are amenable. Other examples are given by the Banach algebra $A = \mathcal{B}(E)$ and $B = \mathcal{K}(E)$, where E is a Banach space with the property $(\mathbb{A})$ (see [GJW]).

We noted in remark after Theorem 2.4 that $\mathcal{K}(E)^{tr} = \{0\}$ for a Banach space with the property $(\mathbb{A})$. Thus one can see from the following theorem that $\mathcal{HC}^n(\mathcal{B}(E)) = \mathcal{HC}^n(\mathcal{B}(E)/\mathcal{K}(E))$ for all $n \geq 0$. In particular, for an infinite-dimensional Hilbert space H, we have $\mathcal{HC}^n(\mathcal{B}(H)/\mathcal{K}(H)) = \mathcal{HC}^n(\mathcal{B}(H)) = \{0\}$ for all $n \geq 0$ by Theorem 4.1 [ChS1], since $(\mathcal{B}(H))^{tr} = \{0\}$ by [Hal]. Hence, by Proposition 3.8, the Banach cyclic homology of these algebras $\mathcal{HC}_n(\mathcal{B}(H)/\mathcal{K}(H)) = \mathcal{HC}_n(\mathcal{B}(H)) = \{0\}$ for all $n \geq 0$.

4.2 THEOREM. *Let A be a Banach algebra with a right or left bounded approximate identity and let I be a closed two-sided ideal of A. Suppose that I is an amenable Banach algebra. Then*

(i) *for all even $n \geq 0$ the natural map from $\mathcal{HC}^n(A/I)$ into $\mathcal{HC}^n(A)$ is injective, and for all odd $n \geq 1$ the natural map from $\mathcal{HC}^n(A/I)$ into $\mathcal{HC}^n(A)$ is surjective;*

(ii) *if $I^{tr} = \{0\}$ (that is, $\mathrm{Cen}_I I^* = \{0\}$) then*

$$\mathcal{HC}^n(A) = \mathcal{HC}^n(A/I)$$

for all $n \geq 0$.

Proof. Consider the commutative diagram of Proposition 3.7 for the Banach algebras A and A/I

$$\begin{array}{ccccccccccccc}
0 \to \mathcal{HC}^0(A/I) & \overset{I^0}{\to} & \mathcal{H}^0(A/I) & \overset{B^0}{\to} & \mathcal{HC}^{-1}(A/I) & \overset{S^0}{\to} & \mathcal{HC}^1(A/I) & \overset{I^1}{\to} & \mathcal{H}^1(A/I) & \overset{B^1}{\to} & \mathcal{HC}^0(A/I) \\
\downarrow \mathcal{G}_0 & & \downarrow \mathcal{L}_0 & & \downarrow \mathcal{G}_{-1} & & \downarrow \mathcal{G}_1 & & \downarrow \mathcal{L}_1 & & \downarrow \mathcal{G}_0 \\
0 \to \mathcal{HC}^0(A) & \overset{I^0}{\to} & \mathcal{H}^0(A) & \overset{B^0}{\to} & \mathcal{HC}^{-1}(A) & \overset{S^0}{\to} & \mathcal{HC}^1(A) & \overset{I^1}{\to} & \mathcal{H}^1(A) & \overset{B^1}{\to} & \mathcal{HC}^0(A)
\end{array}$$

$$\begin{array}{ccccccccccc}
\ldots \to \mathcal{H}^n(A/I) & \overset{B^n}{\to} & \mathcal{HC}^{n-1}(A/I) & \overset{S^n}{\to} & \mathcal{HC}^{n+1}(A/I) & \overset{I^{n+1}}{\to} & \mathcal{H}^{n+1}(A/I) & \overset{B^{n+1}}{\to} & \mathcal{HC}^n(A/I) \ \ldots \\
\downarrow \mathcal{L}_n & & \downarrow \mathcal{G}_{n-1} & & \downarrow \mathcal{G}_{n+1} & & \downarrow \mathcal{L}_{n+1} & & \downarrow \mathcal{G}_n \\
\ldots \to \mathcal{H}^n(A) & \overset{B^n}{\to} & \mathcal{HC}^{n-1}(A) & \overset{S^n}{\to} & \mathcal{HC}^{n+1}(A) & \overset{I^{n+1}}{\to} & \mathcal{H}^{n+1}(A) & \overset{B^{n+1}}{\to} & \mathcal{HC}^n(A) \ \ldots .
\end{array}$$

Note that the maps $\mathcal{G}_0 : \mathcal{HC}^0(A/I) \to \mathcal{HC}^0(A) : f \mapsto f \circ g$ and $\mathcal{L}_0 : \mathcal{H}^0(A/I) \to \mathcal{H}^0(A) : f \mapsto f \circ g$ are injective. By Theorem 2.4,

$$\mathcal{H}^n(A) = \mathcal{H}^n(A, A^*) = \mathcal{H}^n(A/I, (A/I)^*) = \mathcal{H}^n(A/I),$$

so that $\mathcal{L}_n$ is an isomorphism, for all $n \geq 2$ and the natural $\mathcal{L}_1$ map from

$$\mathcal{H}^1(A/I, (A/I)^*)$$

into $\mathcal{H}^1(A, A^*)$ is surjective. Then it follows from the five lemma that the middle vertical map $\mathcal{G}_1$ above is surjective.

Suppose, inductively, that the vertical map, for each $i \leq n$,

$$\mathcal{G}_i : \mathcal{HC}^i(A/I) \to \mathcal{HC}^i(A)$$

is injective if i is even and surjective if i is odd. The result then follows from the five lemma. □

Note that in particular Theorem 4.2 applies whenever A is a C^*-algebra and I is an amenable closed ideal.

4.3 PROPOSITION. *Let A and D be Banach algebras with right or left bounded approximate identities, and let $\kappa : A \to D$ be a continuous homomorphism. If κ induces a topological isomorphism*

$$\mathcal{H}^n(D, D^*) \to \mathcal{H}^n(A, A^*)$$

for all $n \geq 0$, then it induces a topological isomorphism

$$\mathcal{HC}^n(D) \to \mathcal{HC}^n(A)$$

for all $n \geq 0$, and conversely.

Proof. The forward implication is a repetition of that of Theorem 4.1 with the commutative diagram of Proposition 3.7. The converse statement follows easily from the five lemma. $\square$

4.4 Proposition. *Let A_i be a Banach algebra with identity $e_i, i = 1, ..., m$, and let A be the Banach algebra direct sum $\bigoplus_{i=1}^m A_i$. Then*

$$\mathcal{HC}^n(A) = \bigoplus_{i=1}^{m} \mathcal{HC}^n(A_i)$$

for all $n \geq 0$.

Proof. By Theorem 4.1, $\mathcal{HC}^n(A) = \mathcal{HC}_B^n(A)$, where $n \geq 0$ and B is the Banach subalgebra of A generated by $\{e_i, i = 1, ..., m\}$. By Proposition 1.7, the canonical projections from A to $A_i, i = 1, ..., m$, induce a topological isomorphism of complexes $\mathcal{C}_B(A, A^*) \to \bigoplus_{i=1}^m \mathcal{C}(A_i, A_i^*)$. Thus $\mathcal{HC}_B^n(A) = \bigoplus_{i=1}^m \mathcal{HC}^n(A_i)$. The result now follows directly. $\square$

4.5 Corollary. *Let $\mathcal{R}$ be a von Neumann algebra, let*

$$\mathcal{R} = \mathcal{R}_{I_f} \oplus \mathcal{R}_{I_\infty} \oplus \mathcal{R}_{II_1} \oplus \mathcal{R}_{II_\infty} \oplus \mathcal{R}_{III}$$

be the central direct summand decomposition of $\mathcal{R}$ into von Neumann algebras of types $I_f, I_\infty, II_1, II_\infty, III$. Then

$$\mathcal{HC}^n(\mathcal{R}) = \mathcal{HC}^n(\mathcal{R}_{I_f}) \oplus \mathcal{HC}^n(\mathcal{R}_{II_1})$$

for all $n \geq 0$.

Proof. As we noted in the proof of Proposition 1.8, there are no non-zero bounded traces on $\mathcal{R}_{I_\infty}, \mathcal{R}_{II_\infty}$ and $\mathcal{R}_{III}$. Thus, by Theorem 4.1 [ChS1], their Banach cyclic cohomology groups vanish for all $n \geq 0$. $\square$

Note that, by Theorem 25 [He2], for a von Neumann algebra $\mathcal{R}_{I_m}$ of type I_m, where $m < \infty$, we have $\mathcal{HC}^n(\mathcal{R}_{I_m}) = \mathcal{R}_{I_m}^{tr}$ for all even n, and $\mathcal{HC}^n(\mathcal{R}_{I_m}) = \{0\}$ for all odd n.

4.6 Corollary. *Let A be a C^*-algebra without non-zero bounded traces. Then the Banach cyclic homology $\mathcal{HC}_n(A) = \{0\}$ for all $n \geq 0$.*

Proof. It follows from Proposition 3.8 and Theorem 4.1 [ChS1]. $\square$

Note that in particular Corollary 4.6 applies whenever A is a properly infinite von Neumann algebra (see Proposition 1.8), or a stable C^*-algebra [Fa; Theorem 1.1]. The calculation of the (Banach) cyclic homology groups for stable C^*-algebras, for $\mathcal{B}(H)$ and for the Calkin algebra on a separable Hilbert space was given in [Wo].

Now we give the Banach version of the algebraic statement of Theorem 9 [Ka1] of L. Kadison.

4.7 Proposition. *Let A_1 and A_2 be unital Banach algebras, let Y be a unital Banach $A_1 - A_2$-bimodule, and let*

$$\mathcal{U} = \begin{bmatrix} A_1 & Y \\ 0 & A_2 \end{bmatrix}$$

the natural triangular matrix algebra. Then the two canonical projections from $\mathcal{U}$ to A_1 and A_2 induce a topological isomorphism

$$\mathcal{HC}^n(\mathcal{U}) = \mathcal{HC}^n(A_1) \oplus \mathcal{HC}^n(A_2)$$

for all $n \geq 0$.

Proof. By Theorem 4.1 and Proposition 1.10, we obtain

$$\mathcal{HC}^n(\mathcal{U}) = \mathcal{HC}^n_B(\mathcal{U}) = \mathcal{HC}^n(A_1) \oplus \mathcal{HC}^n(A_2),$$

where the amenable subalgebra B of $\mathcal{U}$ was defined before Proposition 1.10. $\square$

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