TWO DISTINGUISHED SUBSPACES OF PRODUCT BMO AND THE NEHARI–AAK THEORY FOR HANKEL OPERATORS ON THE TORUS

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Abstract. In this paper we show that the theory of Hankel operators in the torus $\mathbb{T}^d$, for $d > 1$, presents striking differences with that on the circle $\mathbb{T}$, starting with bounded Hankel operators with no bounded symbols. Such differences are circumvented here by replacing the space of symbols $L^\infty(\mathbb{T})$ by $\mathrm{BMOr}(\mathbb{T}^d)$, a subspace of product BMO, and the singular numbers of Hankel operators by so-called sigma numbers. This leads to versions of the Nehari–AAK and Kronecker theorems, and provides conditions for the existence of solutions of product Pick problems through finite Pick-type matrices. We give geometric and duality characterizations of BMOr, and of a subspace of it, bmo, closely linked with A_2 weights. This completes some aspects of the theory of BMO in product spaces.

Introduction

This paper deals with the extension of the classical theory of bounded Hankel operators in the circle $\mathbb{T}$ to (big) Hankel operators in the torus $\mathbb{T}^d$, for $d > 1$. Some crucial results in the one-variable theory, involving the notions of L^∞ symbols and the singular numbers of the operators, cannot have, as stated, meaningful extensions to the torus. This difficulty can be overcome by introducing so-called BMOr symbols and sigma numbers of Hankel operators. To explain what changes are to be made in dimension $d > 1$, we recall some basic features of the theory in $\mathbb{T}$.

Each function $\phi \in L^2(\mathbb{T}^d)$ gives rise to a Hankel operator Γ_ϕ, and ϕ is called a symbol for the operator. In the case $d = 1$, these operators are closely related to the space BMO, since, by the Nehari theorem [N], a Hankel operator Γ is bounded if and only if $\Gamma 1 \in \mathrm{BMO}$, and if and only if $\Gamma = \Gamma_\varphi$ with $\varphi \in L^\infty$, while $\phi \in \mathrm{BMO}$ implies Γ_ϕ bounded with $\|\Gamma_\phi\| = \|\phi\|_{\mathrm{BMO}}$. In turn, the Helson–Szegő theorem [HS] relates BMO to the boundedness of the Hilbert transform in $L^2(\mu)$, for μ a given measure on the circle $\mathbb{T}$.

The Nehari theorem gives the distance of a bounded function φ to the space $H^\infty(\mathbb{T})$ as the norm of the Hankel operator Γ_φ, and the theorem of Adamjan, Arov and Krein

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(AAK) refines this by giving its distance to $H^\infty(\mathbb{T}) + R_n$ (where R_n is the space of rational functions with n poles in the disk) as the singular number s_n of the operator, or, equivalently, as the distance of the operator to those Hankel operators of finite rank n [AAK].

From the Beurling characterization of the invariant subspaces of $H^2(\mathbb{T})$ of finite codimension, it follows that a Hankel operator Γ is of finite rank n if and only if $\Gamma = \Gamma_\phi$ with $\phi = \bar{b}h$, where $h \in H^\infty$ and b is a Blaschke product with n zeros at $z_1, \ldots, z_n$. If this is the case, the operator Γ_ϕ is closely related to a model operator in a finite subspace of H^2, so that its norm $\|\Gamma_\phi\|$ equals that of a finite $n \times n$ matrix explicitly given in terms of the z_k's and $\phi(z_k)$'s: the Pick matrix. One of the main applications of the Nehari theorem is that it provides a condition for the existence of solutions of the Pick interpolation problems in terms of the norm of an associated Hankel operator Γ_ϕ of finite rank, thus yielding the classical Pick condition in terms of Pick matrices.

The basic properties of BMO($\mathbb{T}$) can be deduced in a unified way [ACS] through a generalized Bochner theorem, which includes also the results of Nehari and Helson–Szegő. The extension of this theorem to several dimensions led in [CS2] and [CS3] to an extension of the Nehari theorem to $\mathbb{T}^d$, for $d > 1$, in terms of a class of symbols that we called BMOr (for "restricted" BMO). The extension of the Helson–Szegő theorem to several dimensions was given in [CS1], in terms of a subspace of product BMO = BMO($\mathbb{T}^d$) (defined in [ChF1]), that here we call bmo (for "small" BMO).

Section 1 gives some basic properties of these subspaces of product BMO, starting with the continuous proper inclusions

$$L^\infty(\mathbb{T}^d) \subset \mathrm{bmo} \subset \mathrm{BMOr} \subset \mathrm{BMO}(\mathbb{T}^d).$$

The preduals of bmo and BMOr are determined, providing counterparts of the duality result of Chang and Fefferman in product domains [ChF2].

As a corollary of the duality result for BMOr, in Section 2 it is shown that, when $d > 1$, there are bounded Hankel operators without bounded symbols (Theorem 2.1). This indicates that L^∞ symbols are not enough to characterize bounded Hankel operators, and that BMOr is the right class of symbols in product domains [CS3].

For $d > 1$ it is known [Am] that the positivity of the Pick matrix is necessary but not sufficient for the existence of a solution of the Pick problem. Necessary and sufficient conditions involving Pick matrices have been given by Agler for $d = 2$ [Ag], and by Cole, Lewis and Wermer for all $d > 1$ [CLW]. However, their conditions are not verifiable in practice, and the relation with Hankel operators is lost in their approach. In Section 3 we return to the consideration of analogues to the Pick problem with BMOr-norm control initiated in [CS3], and give necessary and sufficient conditions for the existence of solutions of a coordinate-wise Pick problem in terms of either the boundedness of a Hankel operator with symbol specified by the data, or the positiveness of d associated $n \times n$ Pick matrices.

In the case $d > 1$, all singular numbers of a Hankel operator are bounded below by $d^{-1/2}$ times its norm (Theorem B), so that all Hankel operators of finite rank are zero [CS2]. This abrupt change from the one-dimensional case is closely related to the failure of the Beurling characterization of invariant subspaces to hold in the polydisk [AhC], and shows that an AAK theory cannot be meaningful in $\mathbb{T}^d$, for $d > 1$. To recover the main features of the Nehari–AAK theory we need to introduce, not only BMOr symbols, but *sigma numbers* to replace the singular numbers, and a notion of operators of *finite type*, to replace that of finite rank.

In Section 4 we rely on a version of Beurling's characterization in the polydisk given in [CS4] to characterize the symbols of Hankel operators of finite type in terms of tensor products of finite Blaschke products, and to extend the AAK result mentioned above in terms of the sigma numbers of the Hankel operators.

In Section 1 it is shown that, when passing from $\mathbb{T}$ to $\mathbb{T}^d$, for $d > 1$, the different equivalent characterizations of BMO($\mathbb{T}$) give rise to distinct spaces. Similarly, the different characterizations of Carleson measures in $\mathbb{D}$ give rise to different notions in $\mathbb{D}^d$, for $d > 1$. One such characterization is that a measure in $\mathbb{D}$ is Carleson if and only if a canonically associated function is in BMO. In Section 5 we extend this canonical association to $d > 1$, by defining Carleson–Nikolskii measures, and proving that a measure is of this type if and only if a canonically associated function is in BMOr.

In the circle, the norms of Hankel operators of finite rank coincide with the norms of multipliers acting in finite-dimensional model subspaces, which in turn are determined by finite Pick matrices. In Section 6 we prove that the norms of Hankel operators of finite type coincide with those of multipliers acting in corresponding model subspaces, which now are not finite-dimensional but of bi-finite type, like those appearing in Sections 4 and 5. This significantly reduces the number of steps required to verify norm boundedness.

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Basic Notations. The following notations will be used throughout the paper. For $d \geq 1$, $\mathcal{P} = \mathcal{P}(\mathbb{T}^d)$ is the class of trigonometric polynomials; $\hat{f}$ represents the Fourier transform of f;

$$H^2(\mathbb{T}^d) = \{f \in L^2(\mathbb{T}^d) : \hat{f}(n_1, \dots, n_d) = 0 \text{ if } n_k < 0 \text{ for some } k = 1, \dots, d\};$$
$$H^2_{x_k} = \{f \in L^2(\mathbb{T}^d) = \hat{f}(n) = 0 \text{ for } n_k < 0\};$$
$$H^2(\mathbb{T}^d)^\perp = L^2(\mathbb{T}^d) \ominus H^2(\mathbb{T}^d);$$

and the orthogonal projector $P : L^2 \to H^2(\mathbb{T}^d)$ is called the analytic projector.

The d shifts $S_k = S_{x_k}$, in $L^2(\mathbb{T}^d)$, where $k = 1, \dots, d$, are defined by

$$S_k f(x) = S_k f(x_1, \dots, x_d) := \exp(ix_k) f(x).$$

In the case $d = 2$, we write (x, y) for (x_1, x_2) and (m, n) for (n_1, n_2), and consider the subspaces of $H^2(\mathbb{T}^2)$ given by

$$H^2_x = \{f \in L^2(\mathbb{T}^2) : \hat{f}(m, n) = 0 \text{ for } m < 0\}, \quad H^2_{-x} = L^2 \ominus H^2_x,$$

and

$$H^2_y = \{f \in L^2(\mathbb{T}^2) : \hat{f}(m, n) = 0 \text{ for } n < 0\}, \quad H^2_{-y} = L^2 \ominus H^2_y,$$

as well as the projectors

$$P_x : L^2 \to H^2_x, \quad P_{-x} := (I - P_x) : L^2 \to H^2_{-x}$$

and

$$P_y : L^2 \to H^2_y, \quad P_{-y} := (I - P_y).$$

The two shifts S_1 and S_2 in $L^2(\mathbb{T}^2)$ satisfy

$$S_1 f(x, y) = e^{ix} f(x), \quad S_2 f(x, y) = e^{iy} f(y).$$

Observe that

$$H^2(\mathbb{T}^2)^\perp = H^2_{-x} + H^2_{-y} = H^2_{-x} \dot{+} (H^2_{-y} \cap H^2_x) = H^2_{-y} \dot{+} (H^2_{-x} \cap H^2_y).$$

1. Two Distinguished Subspaces of Product BMO

An integrable function in $\mathbb{T}$ is of *bounded mean oscillation* if

$$\frac{1}{|I|} \int_I |f(x) - f_I| \, dx \le C \qquad \text{for all intervals } I, \tag{1.1}$$

where $f_I = |I|^{-1} \int_I f(x) \, dx$. The class BMO of functions of bounded mean oscillation is important in analysis. It is closely related to the Carleson measures and to the A_p weights, as well as to bounded Hankel operators.

A function ϕ is in BMO $=$ BMO$(\mathbb{T})$ if and only if a canonically associated measure μ in $\mathbb{D}$ is Carleson, and a measure μ in $\mathbb{D}$ is Carleson if and only if $\Gamma 1 \in$ BMO for a canonically associated Hankel operator Γ. (See definitions below.) As there are different characterizations for the elements of BMO in $\mathbb{T}$, the same is true for Carleson measures in $\mathbb{D}$.

BMO coincides with the space $L^\infty + HL^\infty$, where H is the Hilbert transform. This characterization follows from Charles Fefferman's famous duality result, asserting that BMO is the (real) dual of the Hardy space H^1. Another way to prove

$$\mathrm{BMO} = L^\infty + HL^\infty \tag{1.2}$$

is through the characterizations of the weights w for which H is bounded in $L^2(w)$ given by the A_2 condition and by the Helson–Szegő theorem [HS].

In passing from $\mathbb{T}$ to $\mathbb{T}^d$, for $d > 1$, the extension of the BMO theory to product domains presents various difficulties [ChF2]. S.-Y. Alice Chang and Robert Fefferman were able to introduce a notion of *product* BMO $=$ BMO$(\mathbb{T}^d)$, dual to the space $H^1(\mathbb{D}^d)$, and for which an analogue of (1.2) is retained [ChF1]. In fact,

$$\begin{aligned} &\phi \in \mathrm{BMO}(\mathbb{T}^d) \iff \\ &\phi = f_1 + H_{x_1} f_2 + \cdots + H_{x_d} f_{d+1} + H_{x_1} H_{x_2} f_{d+2} + \cdots + H_{x_1} H_{x_2} \ldots H_{x_d} f_{2^d} \qquad (1.3)\\ &\qquad \text{for } f_1, \ldots, f_{2^d} \in L^\infty(\mathbb{T}^d), \end{aligned}$$

where H_{x_j} is the Hilbert transform with respect to the variable x_j, for $j = 1, \ldots, d$, and BMO is a complete normed space with respect to

$$\|\phi\|_{\mathrm{BMO}} := \inf\{\max_j \|f_j\|_\infty : \text{ all decompositions (1.3)}\}.$$

But for product BMO the geometric characterizations by mean oscillation and by associated Carleson measures become considerably more complicated (they do not correspond to bounded mean oscillation with respect to rectangles), and, furthermore, the connections with weights and Hankel operators are lost.

In previous work ([CS1], [CS3]), we gave results in product spaces analogous to those linking BMO to weights and to Hankel operators in one variable, in terms of classes of functions that are properly contained in product BMO. In this section we clarify the relation of these classes with product BMO, give some of their basic properties, and characterize their preduals.

Definition 1 (small BMO). A function $\phi \in L^2(\mathbb{T}^d)$, for $d \geq 1$, is in bmo$(\mathbb{T}^d)$ if there exist $f_1, \ldots, f_d, g_1, \ldots, g_d \in L^\infty(\mathbb{T}^d)$ such that

$$\phi = f_1 + H_{x_1} g_1 = \cdots = f_d + H_{x_d} g_d \tag{1.4}$$

and

$$\|\phi\|_{\mathrm{bmo}} := \inf\{\max_{1\leq j\leq d}\{\|f_j\|_\infty, \|g_j\|_\infty\} : \text{ all decompositions (1.4)}\}.$$

Observe that $\|\phi\|_{\mathrm{bmo}} = 0$ if and only if ϕ is constant, and bmo$/\mathbb{C}$ is a complete normed space with respect to $\|\cdot\|_{\mathrm{bmo}}$.

Definition 2 (restricted BMO). A function $\phi \in L^2(\mathbb{T}^d)$, for $d \geq 1$, is in BMOr if there exist $\varphi_0, \varphi_1, \ldots, \varphi_d \in L^\infty(\mathbb{T}^d)$ such that

$$\begin{cases} (I - P_{x_j})\phi = (I - P_{x_j})\varphi_j & \text{for } j = 1, \ldots, d, \quad \text{and} \\ P_{x_1} P_{x_2} \ldots P_{x_d} \phi = P_{x_1} P_{x_2} \ldots P_{x_d} \varphi_0, \end{cases} \tag{1.5}$$

where $P_{x_j} : L^2 \to H^2_{x_j}$ is the analytic projector in x_j, for $j = 1, \ldots, d$. Moreover,

$$\begin{aligned} \|\phi\|_{\mathrm{BMOr}} &:= \inf\{\max_{0 \leq j \leq d} \|\varphi_j\|_\infty : \text{ all decompositions (1.5)}\} \\ &= \max\{\max_{1 \leq j \leq d}\{\inf\{\|\phi - h_{x_j}\|_\infty : h_{x_j} \in H^2_{x_j}\}\}, \inf\{\|\phi - h^\perp\|_\infty : h^\perp \in H^{2\perp}\}\}. \end{aligned}$$

Observe that BMOr is a complete normed space with respect to $\| \cdot \|_{\mathrm{BMOr}}$, and coincides with the space restricted BMO introduced in [CS3].

The two definitions given above are justified by the following results.

Theorem (Helson–Szegő theorem in $\mathbb{T}^d$, for $d \geq 1$). [CS1] *A weight* $0 \leq w \in L^1(\mathbb{T}^d)$ *satisfies*

$$\int_{\mathbb{T}^d} |Hf|^2 w \leq M^2 \int_{\mathbb{T}^d} |f|^2 w \quad \text{for all } f \in \mathcal{P},$$

where $H = H_{x_1} \ldots H_{x_d}$ *is the product Hilbert transform, if and only if* $\phi = \log w \in \mathrm{bmo}(\mathbb{T}^d)$, *with*

$$\phi = u_1 + H_{x_1} v_1 = \cdots = u_d + H_{x_d} v_d$$

for $u_1, \ldots, u_d, v_1, \ldots, v_d$ *real-valued bounded functions in* $\mathbb{T}^d$ *satisfying* $\|u_j\|_\infty \leq C_M$ *and* $\|v_j\|_\infty \leq \pi/2 - \varepsilon_M$ *for* $j = 1, \ldots, d$.

Theorem (Nehari theorem in $\mathbb{T}^d$, for $d \geq 1$). [CS3] *Let* $\Gamma : H^2(\mathbb{T}^d) \cap \mathcal{P} \to H^2(\mathbb{T}^2)^\perp$ *be a Hankel operator.* (The definition of a Hankel operator is given in Section 2.) Γ *is bounded if and only if there exists* $\phi \in \mathrm{BMOr}$ *satisfying* $\Gamma f = (I - P)(\phi f)$, *for all* $f \in H^2(\mathbb{T}^d)$, *where* $P : L^2 \to H^2$ *is the orthogonal projector, and* $\|\phi\|_{\mathrm{BMOr}} \leq \|\Gamma\| \leq \sqrt{d}\|\phi\|_{\mathrm{BMOr}}$.

Definitions 1 and 2 impose constraints on the functions in small and restricted BMO, which follow immediately from the relation between the analytic projector and the Hilbert transforms,

$$P_{x_j} = \frac{1}{2}(I + H_{x_j}) \quad \text{for } j = 1, \ldots, d, \tag{1.6}$$

and can be summarized as follows:

Lemma 1.1. (i) *For* $\phi \in \mathrm{bmo}$ *given by* (1.4), *and for* $j = 1, \ldots, d$, *we have*

$$\begin{aligned} P_{x_j}\phi &= P_{x_j}(f_j - ig_j) \\ (I - P_{x_j})\phi &= (I - P_{x_j})(f_j + ig_j). \end{aligned} \tag{1.7}$$

In particular,

$$P_{x_1} \dots P_{x_d}(f_1 - ig_1) = \dots = P_{x_1} \dots P_{x_d}(f_d - ig_d) \tag{1.8}$$

and

$$(I - P_{x_1}) \dots (I - P_{x_d})(f_1 + ig_1) = \dots = (I - P_{x_1}) \dots (I - P_{x_d})(f_d + ig_d). \tag{1.9}$$

(ii) *For* $\phi \in \mathrm{BMOr}$ *given by* (1.5)*, we have*

$$(I - P_{x_1}) \dots (I - P_{x_d})\varphi_1 = \dots = (I - P_{x_1}) \dots (I - P_{x_d})\varphi_d. \tag{1.10}$$

Lemma 1.1 implies that in order to define functions in bmo or BMOr by d or $d+1$ bounded functions, respectively, those bounded functions have to satisfy the constraints (1.8), (1.9) and (1.10).

The relation between small, restricted and product BMOs is the following:

Proposition 1.2. *The inclusions*

$$L^\infty(\mathbb{T}^d) \subset \mathrm{bmo}(\mathbb{T}^d) \subset \mathrm{BMOr}(\mathbb{T}^d) \subset \mathrm{BMO}(\mathbb{T}^d)$$

are topological, and proper for $d > 1$*. For* $d = 1$ *we have* $\mathrm{bmo}(\mathbb{T}) = \mathrm{BMOr}(\mathbb{T}) = \mathrm{BMO}(\mathbb{T})$*.*

Proof. The topological inclusion $L^\infty \subset \mathrm{bmo}$ is immediate from Definition 1. If $\phi \in \mathrm{bmo}$, by Lemma 1.1(i), for $j = 1, \dots, d$, we have $(I - P_{x_j})\phi = (I - P_{x_j})(f_j + ig_j)$, and $P_{x_1} \dots P_{x_d}\phi = P_{x_1} \dots P_{x_d}(f_1 - ig_1) = \dots = P_{x_1} \dots P_{x_d}(f_d - ig_d)$, with $f_j \pm ig_j \in L^\infty(\mathbb{T}^d)$, which means that condition (1.5) is satisfied and $\phi \in \mathrm{BMOr}$ with $\|\phi\|_{\mathrm{bmo}} \geq \|\phi\|_{\mathrm{BMOr}}$. It follows from (1.5) and (1.6) that $\phi \in \mathrm{BMOr}$ implies $\phi \in \mathrm{BMO}$, with $\|\phi\|_{\mathrm{BMOr}} \geq \|\phi\|_{\mathrm{BMO}}$. To show that the inclusions are proper it is enough to consider $d = 2$.

(a) Example of $\phi \in \mathrm{bmo} \backslash L^\infty$. Fix $v \in L^\infty(\mathbb{T})$ such that $Hv \notin L^\infty(\mathbb{T})$, and let $\phi(x, y) = Hv(x - y)$. Then $\phi = H_x g_1 = H_y g_2$, for $g_1, g_2 \in L^\infty(\mathbb{T}^2)$, defined by $g_1(x, y) = v(x - y)$, $g_2(x, y) = -v(x - y)$, and $\phi \in \mathrm{bmo} \backslash L^\infty(\mathbb{T}^2)$.

(b) Example of $\phi \in \mathrm{BMOr} \backslash \mathrm{bmo}$. Let $\varphi_0(x, y) \equiv 0$, $\varphi_1(x, y) = v(x)h(y)$, $\varphi_2(x, y) = h(x)v(y)$, where $v \in L^\infty(\mathbb{T})$ is as in (a) and $h \in H^\infty(\mathbb{T})$ is not a constant. Define $\phi \in \mathrm{BMOr}(\mathbb{T}^2)$ by condition (1.5), that is, $P_x P_y \phi = P_x P_y \varphi_0$, $(I - P_x)\phi = (I - P_x)\varphi_1$, $(I - P_y)\phi = (I - P_y)\varphi_2$, which can be done since (1.10) is satisfied:

$$(I - P_x)(I - P_y)\varphi_1 = 0 = (I - P_x)(I - P_y)\varphi_2.$$

If ϕ were in $\mathrm{bmo}(\mathbb{T}^2)$, by (1.7), we would have $P_x\phi = P_x(f_1 - ig_1)$ and $P_y\phi = P_y(f_2 - ig_2)$, with $f_1 - ig_1$, $f_2 - ig_2 \in L^\infty(\mathbb{T}^2)$. But, in our case, $P_x\phi = P_x P_y \phi + P_x(I - P_y)\phi = P_x P_y \varphi_0 + P_x(I - P_y)\varphi_2 = h(x)(I - P_y)v(y)$, with $(I - P_y)v(y) \notin L^\infty(\mathbb{T})$ by assumption. Since $P_x\phi = P_x\varphi$ for some $\varphi \in L^\infty(\mathbb{T}^2)$, the function $h(x)(I - P_y))v(y) = P_x\varphi(x, y)$ should be, for all y fixed, a function in $\mathrm{BMO}(\mathbb{T})$ satisfying $|(I-P_y)v(y)|\|h\|_{\mathrm{BMO}(\mathbb{T})} = \|P_x\varphi(\cdot,y)\|_{\mathrm{BMO}(\mathbb{T})} \le c\|\varphi\|_\infty$. Since $\|h\|_{\mathrm{BMO}} \neq 0$ and $(I-P_y)v(y) \notin L^\infty(\mathbb{T})$, this is a contradiction.

(c) Example of $\phi \in \mathrm{BMO}\backslash\mathrm{BMOr}$. Given $\psi_1, \psi_2 \in L^\infty(\mathbb{T}^2)$, take $\phi = H_x\psi_1 + H_y\psi_2 \in \mathrm{BMO}(\mathbb{T}^2)$. In this case, $(I-P_x)\phi = (I-P_x)(i\psi_1 + i\psi_2) - (I-P_x)P_y(2i\psi_2)$, so, for ϕ to be in BMOr, by (1.5), $(I-P_x)P_y\psi_2$ should equal $(I-P_x)\varphi$, for some $\varphi \in L^\infty(\mathbb{T}^2)$. Taking $\psi_2(x,y) = f(x)v(y)$, for $f, v \in L^\infty(\mathbb{T})$, with v as in example (a), this means that, for all y fixed, $|P_yv(y)|\|(I-P_x)f\|_{\mathrm{BMO}} = \|(I-P_x)P_y\varphi(\cdot,y)\|_{\mathrm{BMO}(\mathbb{T})} \le C\|\varphi\|_\infty$, which is a contradiction.

□

In what follows we limit the statements, as well as their proofs, to the case $d = 2$, in order to simplify notations. All results remain valid, with obvious modifications, for $d > 1$.

We will write $\phi \in \mathrm{BMO}_x(\mathbb{T}^2)$ if $\phi(\,\cdot, y) \in \mathrm{BMO}(\mathbb{T})$ for every y. If, in addition, we have $\sup_y \|\phi(\,\cdot, y)\|_{\mathrm{BMO}} \le C$ for some constant C, we say that $\phi \in \mathrm{BMO}_x(\mathbb{T}^2)$ *with uniformly bounded norm.* We define $\mathrm{BMO}_y(\mathbb{T}^2)$ similarly.

Proposition 1.3 (Bounded mean oscillation on rectangles)**.** *The following conditions on a function ϕ are equivalent:*

(a) $\phi \in \mathrm{bmo}(\mathbb{T}^2)$.

(b) *For a constant $C > 0$ we have*

$$\frac{1}{|R|}\iint_R |\phi(x,y) - \phi_R|\,dx\,dy \le C \quad \text{for all } R = I \times J, \tag{1.11}$$

where $I, J \subset \mathbb{T}$ are intervals and

$$\phi_R = \frac{1}{|R|}\iint_R \phi(x,y)\,dx\,dy.$$

(c) $\phi \in \mathrm{BMO}_x(\mathbb{T}^2)$ *with uniformly bounded norm and* $\phi \in \mathrm{BMO}_y(\mathbb{T}^2)$ *with uniformly bounded norm.*

Proof. (b) $\Longrightarrow$ (c). Condition (1.11), of bounded mean oscillation on rectangles, can be written as

$$\frac{1}{|I|}\frac{I}{|J|}\int_I\int_J |\phi(x,y) - \phi_R|\,dx\,dy = \frac{1}{|I|}\int_I F(x,J)\,dx \le C$$

for all intervals I, J. This implies, for almost every $x \in$ I,

$$F(x,J) = \frac{1}{|J|}\int_J |\phi(x,y) - \phi_R|\,dy \le C \quad \text{for all} J, \tag{1.12}$$

which is to say that $\phi \in \mathrm{BMO}_y$ with uniformly bounded norm. Similarly, (1.11) implies $\phi \in \mathrm{BMO}_x$ with uniformly bounded norm.

(c) $\iff$ (a). Obviously, $\phi \in \mathrm{bmo}(\mathbb{T}^2)$ implies $\phi \in \mathrm{BMO}_x(\mathbb{T}^2)$ and $\phi \in \mathrm{BMO}_y(\mathbb{T}^2)$, both with uniformly bounded norm. Conversely, from (c) we have $\phi = f_1 + H_x g_1 = f_2 + H_y g_2$, for f_1 and g_1 bounded functions in x, uniformly in y, and f_2 and g_2 bounded functions in y, uniformly in x. This means that f_1, f_2, g_1 and g_2 are bounded functions of both x and in y, which is (a).

(c) $\Longrightarrow$ (b). By (1.1), the condition that $\phi \in \mathrm{BMO}_x$ with uniformly bounded norm is equivalent to

$$\frac{1}{|I|}\int_I |\phi(x,y) - \phi_I(y)|\,dx \le c \quad \text{for all } I, \text{ uniformly in } y, \tag{1.13}$$

and the condition that $\phi \in \mathrm{BMO}_y$ with uniformly bounded norm is equivalent to

$$\frac{1}{|J|}\int_J |\phi(x,y) - \phi_J(x)|\,dy \le c', \quad \text{for all } J, \text{ uniformly in } x, \tag{1.14}$$

where

$$\phi_I(y) = \frac{1}{|I|}\int_I \phi(x,y)\,dx, \qquad \phi_J(x) = \frac{1}{|J|}\int_J \phi(x,y)\,dy,$$

and c, c' are positive constants. From (1.14) it follows, for almost all $x \in I$, and $y \in J$, that

$$|\phi(x,y) - \phi_J(x)| \le c' \quad \text{for all } J,$$

so that, for $R = I \times J$,

$$\begin{aligned}
|\phi_I(y) - \phi_R| &= \left|\frac{1}{|I|}\int_I \phi(x,y)\,dx - \frac{1}{|I|}\frac{1}{|J|}\int_I\int_J \phi(x,y)\,dx\,dy\right| \\
&= \left|\frac{1}{|I|}\int_I \phi(x,y)\,dx - \frac{1}{|I|}\int_I \phi_J(x)\,dx\right| \\
&\le \frac{1}{|I|}\int_I |\phi(x,y) - \phi_J(x)|\,dx \le c'.
\end{aligned}$$

Then, by (1.13),

$$\begin{aligned}
&\frac{1}{|R|}\iint_R |\phi(x,y) - \phi_R|\,dx\,dy \\
&\qquad = \frac{1}{|J|}\int_J \left(\frac{1}{|I|}\int_I |\phi(x,y) - \phi_R|\,dx\right) dy \\
&\qquad \le \frac{1}{|J|}\int_J \left(\frac{1}{|I|}\int_I |\phi(x,y) - \phi_I(y)|\,dx + \frac{1}{|I|}\int_I |\phi_I(y) - \phi_R|\,dx\right) dy \\
&\qquad \le c + c' = C,
\end{aligned}$$

which is (b). □

The relation between BMOr, bounded Hankel operators in $\mathbb{T}^d$, for $d > 1$, and Carleson measures will be treated in Section 4. Now we consider duality results.

In the one-dimensional case, BMO($\mathbb{T}$) is the dual of the (real) Hardy space

$$H^1(\mathbb{T}) := \{f \in L^1(\mathbb{T}) : Hf \in L^1(\mathbb{T})\},$$

or, equivalently, is the space of functions f such that Pf and $(I-P)f \in L^1(\mathbb{T})$, where $f = P + (I-P)f$ is a canonical decomposition of f given by the analytic projector P. In the two-dimensional case, for each trigonometric polynomial $f \in \mathcal{P}(\mathbb{T}^2)$, we consider three canonical decompositions of f, given in terms of the analytic projectors $P_x : L^2 \to H^2_x$ and $P_y : L^2 \to H^2_y$, as well as of $P_{-x} := (I - P_x)$ and $P_{-y} := (I - P_y)$:

$$f \;=\; P_xf + P_{-x}f = P_yf + P_{-y}f; \tag{1.15}$$

$$f \;=\; P_xf + P_{-x}P_yf + P_{-x}P_{-y}f = P_yf + P_{-y}P_xf + P_{-y}P_{-x}f; \tag{1.16}$$

$$f \;=\; P_xP_yf + P_xP_{-y}f + P_{-x}P_yf + P_{-x}P_{-y}f; \tag{1.17}$$

and norm $\mathcal{P}(\mathbb{T}^2)$ with three different norms, all stronger than the L^1 norm. The completion of $\mathcal{P}(\mathbb{T}^2)$ with respect to these three norms gives rise to Banach spaces, denoted as follows:

(A) The space

$$H^1_x(\mathbb{T}^2) + H^1_y(\mathbb{T}^2) := L^1(\mathbb{T}_y; H^1(\mathbb{T}_x)) + L^1(\mathbb{T}_x; H^1(\mathbb{T}_y)), \tag{1.18}$$

whose elements are functions $f = f(x, y) = f_x(y)$, integrable in y, with values in $H^1(\mathbb{T}_x)$, and $f = f_y(x)$, integrable in x, with values in $H^1(\mathbb{T}_y)$; that is, the closure of $\mathcal{P}(\mathbb{T}^2)$ in the norm

$$[f] := \inf\{\|g\|_{H^1_x} + \|h\|_{H^1_y} : f = g + h\},$$

where

$$\|f\|_{H^1_x} := \|P_xf\|_1 + \|P_{-x}f\|_1 \quad \text{and} \quad \|f\|_{H^1_y} := \|P_yf\|_1 + \|P_{-y}f\|_1$$

correspond to partition (1.15).

(B) The space

$$\mathcal{H}^1(\mathbb{T}^2) := \mathcal{H}^1_x(\mathbb{T}^2) + \mathcal{H}^2_y(\mathbb{T}^2), \tag{1.19}$$

where $\mathcal{H}^1_x$ and $\mathcal{H}^1_y$ are the closures of $\mathcal{P}(\mathbb{T}^2)$ under the norms corresponding to partition (1.16), namely

$$|||f|||_{(x)} := \|P_xf\|_1 + (\|P_{-x}P_yf\|_1 + \|P_{-x}P_{-y}f\|_1),$$
$$|||f|||_{(y)} := \|P_yf\|_1 + (\|P_{-y}P_xf\|_1 + \|P_{-y}P_{-x}f\|_1),$$

and $\mathcal{H}^1$ is normed by

$$|||f||| := \inf\{|||g|||_{(x)} + |||h|||_{(y)} : f = g + h\}. \tag{1.20}$$

Observe that, in particular,

$$|||f||| \geq \|P_{-x}P_{-y}f\|_1 \quad \text{for all } f. \tag{1.21}$$

(C) The space

$$H^1(\mathbb{T}^2) = H^1(\mathbb{T}_x; H^1(\mathbb{T}_y)) = H^1(\mathbb{T}_y; H^1(\mathbb{T}_x)), \tag{1.22}$$

normed by

$$\|f\|_{H^1} := \|P_xP_yf\|_1 + \|P_xP_{-y}f\|_1 + \|P_{-x}P_yf\|_1 + \|P_{-x}P_{-y}f\|_1, \tag{1.23}$$

corresponding to partition (1.17).

Observe that $\mathcal{P}(\mathbb{T}^2)$ can be partitioned in more ways than those in (1.15)–(1.17). For instance, a function f can be written as $f = P_xP_yf + (I - P_xP_y)f$, giving rise to the norm $\|f\| := \|P_xP_yf\|_1 + \|(I - P_xP_y)f\|_1$. Since the Hilbert transform, as well as the analytic projection, is unbounded in $L^1(\mathbb{T})$, this norm $\|\cdot\|$ is not comparable to those above, and in particular to $|||\cdot|||$.

Proposition 1.4. *For each $\varepsilon > 0$, there is an f satisfying*

$$\varepsilon|||f||| \geq \|f\| := \|P_xP_yf\|_1 + \|(I - P_xP_y)f\|_1,$$

where $|||\cdot|||$ is defined in (1.20).

Proof. Consider $f(x,y) = \overline{u(x)}\,v(y) + v(x)\,\overline{u(y)}$, with u an inner function and $v \in L^1$ such that $\|v\|_1 = 1$, while $\|w\|_1 > \varepsilon^{-1}$, for $w = (I - P)v$. Since this f satisfies $P_xP_yf = 0$ and $P_{-x}P_{-y}f = \overline{u(x)}w(y) + w(x)\overline{u(y)}$, we have $\|f\| = \|f\|_1 \geq 2$, and, by (1.21), $|||f||| \geq \|P_{-x}P_{-y}f\|_1$, which, after multiplying by $u(x)u(y)$, is equal to

$$\iint |u(x)w(x) + u(y)w(y)|\,dx\,dy > \int |w(x)|\,dx > \varepsilon^{-1}.$$

To justify the last inequality, it is enough to choose a test function G in the predual of $L^1(\mathbb{T})$, such that $\int u(y)w(y)G(y)\,dy = 0$, since then

$$\begin{aligned}\iint |u(x)w(x) + u(y)w(y)|\,dx\,dy &\geq \sup_F \iint (u(x)w(x) + u(y)w(y))F(x)G(y)\,dx\,dy \\ &= \sup_F \int u(x)w(x)F(x)\,dx = \int |u(x)w(x)|\,dx. \quad \square\end{aligned}$$

Theorem 1.5 (Duality)**.** *The spaces defined in* (1.18), (1.19) *and* (1.22) *are the preduals of the* BMO*s in* $\mathbb{T}^2$. *More precisely:*

(a) $\mathrm{bmo}(\mathbb{T}^2)$ *is the dual of* $H^1_x(\mathbb{T}^2) + H^1_y(\mathbb{T}^2)$.
(b) $\mathrm{BMOr}(\mathbb{T}^2)$ *is the dual of* $\mathcal{H}^1(\mathbb{T}^2) = \mathcal{H}^1_x(\mathbb{T}^2) + \mathcal{H}^1_y(\mathbb{T}^2)$.
(c) $\mathrm{BMO}(\mathbb{T}^2)$ *is the dual of* $H^1(\mathbb{T}^2)$ [ChF1].

Proof. Note that, for any pair of functions f and ϕ in the variables x and y for which the integrals make sense,

$$\int (P_{\pm x} f)(P_{\pm x}\phi)\, dx = \int (P_{\pm y} f)(P_{\pm y}\phi)\, dy = 0,$$

so that

$$\int (P_{\pm x} f)(P_{\mp x}\phi)\, dx = \int (P_{\pm x} f)\phi\, dx = \int f(P_{\mp x}\phi)\, dx, \tag{1.24}$$

and similarly for $P_{\pm y}$.

(a) Let $\phi \in \mathrm{bmo}(\mathbb{T}^2)$ and $f \in H^1_x + H^1_y$. Then, by Lemma 1.1(i) and (1.24), we have $\int f\phi\, dx\, dy = \int g\phi\, dx\, dy + \int h\phi\, dx\, dy$, where

$$\begin{aligned}\int g\phi\, dx\, dy &= \int (P_x g)\phi + \int (P_{-x} g)\phi = \int (P_x g)(P_{-x}\phi) + \int (P_{-x} g)(P_x\phi)\\ &= \int (P_x g)(P_{-x}\psi_1) + \int (P_{-x} g)(P_x\varphi_1)\\ &= \int (P_x g)\psi_1 + \int (P_{-x} g)\varphi_1,\end{aligned}$$

for $\psi_1, \varphi_1 \in L^\infty(\mathbb{T}^2)$. Similarly,

$$\int h\phi = \int (P_y h)\psi_2 + \int (P_{-y} h)\varphi_2,$$

for $\psi_2, \varphi_2 \in L^\infty(\mathbb{T}^2)$. Thus,

$$\left|\int g\phi\right| \le \|P_x g\|_1 \|\psi_1\|_\infty + \|P_{-x} g\|_1 \|\varphi_1\|_\infty$$

and

$$\left|\int h\phi\right| \le \|P_y h\|_1 \|\psi_2\|_\infty + \|P_{-y} h\|_1 \|\varphi_2\|_\infty,$$

which imply

$$\left|\int g\phi\right| \le \|g\|_{H^1_x} \|\phi\|_{\mathrm{bmo}} \quad \text{and} \quad \left|\int h\phi\right| \le \|h\|_{H^1_y} \|\phi\|_{\mathrm{bmo}},$$

and hence

$$\left|\int f\phi\right| \le [f] \|\phi\|_{\mathrm{bmo}}.$$

Conversely, if $l \in (H^1_x + H^1_y)^*$, then for every $f \in \mathcal{P}(\mathbb{T}^2)$, we have $|l(f)| \le C\|f\|_{H^1_x}$, independently of y, and $|l(f)| \le C\|f\|_{H^1_y}$, independently of x. Since $(H^1_x)^* = \mathrm{BMO}_x$ and $(H^1_y)^* = \mathrm{BMO}_y$, this implies, by Proposition 1.3, that l can be given by a function in $\mathrm{bmo}(\mathbb{T}^2)$.

(b) Let $\phi \in \mathrm{BMOr}(\mathbb{T}^2)$ be given by

$$\phi = P_xP_y\varphi_0 + P_xP_{-y}\varphi_2 + P_{-x}\varphi_1 = P_xP_y\varphi_0 + P_{-x}P_y\varphi_1 + P_{-y}\varphi_2$$

for $\varphi_0, \varphi_1, \varphi_2 \in L^\infty(\mathbb{T}^2)$, and let $f \in \mathcal{H}^1_x + \mathcal{H}^1_y$. Then, for $f = g + h$, $g \in \mathcal{H}^1_x$, $h \in \mathcal{H}^1_y$, we have

$$\int g\phi = \int (P_{-x}P_{-y}g)\varphi_0 + \int (P_{-x}P_yg)\varphi_2 + \int (P_xg)\varphi_1$$

and $|\int g\phi| \le |||g|||_{(x)}\|\phi\|_{\mathrm{BMOr}}$. Similarly, $|\int h\phi| \le |||h|||_{(y)}\|\phi\|_{\mathrm{BMOr}}$; hence

$$\left|\int f\phi\right| \le |||f|||\,\|\phi\|_{\mathrm{BMOr}}.$$

Conversely, if l is continuous on $\mathcal{H}^1(\mathbb{T}^2)$, it is continuous on $\mathcal{H}^1_x$ and on $\mathcal{H}^1_y$. In particular, for all $f \in \mathcal{P}(\mathbb{T}^2)$,

$$|l(f)| \le C|||f|||_{(x)} = C\|P_xf\|_1 + C(\|P_yP_{-x}f\|_1 + \|P_{-y}P_{-x}f\|_1).$$

In particular, if $f = P_xf$, we have $|l(f)| \le C\|f\|_1$, and there exists $F \in L^\infty$ such that

$$l(P_xf) = \int F(P_xf)\,dx\,dy = \int (P_{-x}F)f\,dx\,dy,$$

by (1.24). Similarly, if $f = P_{-x}f$,

$$|l(f)| \le C(\|P_yf\|_1 + \|P_{-y}f\|_1) = C\|f\|_{H^1_y},$$

and there exists $G \in \mathrm{BMO}_y$ such that

$$l(P_{-x}f) = \int G(P_{-x}f)\,dx\,dy = \int (P_xG)f\,dx\,dy.$$

Then

$$l(f) = l(P_xf) + l(P_{-x}f) = \int (P_{-x}F + P_xG)f\,dx\,dy \tag{1.25}$$

for $F \in L^\infty$ and $G \in \mathrm{BMO}_y$. Similarly,

$$l(f) = l(P_yf) + l(P_{-y}f) = \int (P_{-y}F' + P_yG')f\,dx\,dy \tag{1.26}$$

for $F' \in L^\infty$ and $G' \in \mathrm{BMO}_x$.

Since, by (1.25) and (1.26), the two functions representing l coincide as functionals on all $f \in \mathcal{P}(\mathbb{T}^2)$, we conclude that

$$P_{-x}F + P_xG = P_{-y}F' + P_yG' = \phi.$$

Now observe that $G \in \mathrm{BMO}_y$ and $G' \in \mathrm{BMO}_x$ imply that $P_xG = P_x\varphi$ and $P_yG' = P_y\varphi'$ for some $\varphi, \varphi' \in L^\infty(\mathbb{T}^2)$. Therefore, the function ϕ satisfies

$$P_xP_y\phi = P_xP_y\varphi = P_xP_y\varphi' \quad \text{and} \quad P_{-x}\phi = P_{-x}F, \quad P_{-y}\phi = P_{-y}F'$$

for $\varphi, \varphi', F, F' \in L^\infty(\mathbb{T}^2)$, which means, by definition, that $\phi \in \mathrm{BMOr}(\mathbb{T}^2)$.

(c) Let $\phi \in \mathrm{BMO}(\mathbb{T}^2) = L^\infty + H_x L^\infty + H_y L^\infty + H_x H_y L^\infty$ (see [ChF1]) and $f \in H^1$. Writing

$$\phi = P_x P_y \psi_1 + P_x P_{-y} \psi_2 + P_{-x} P_y \psi_3 + P_{-x} P_{-y} \psi_4,$$

for $\psi_1, \psi_2, \psi_3, \psi_4 \in L^\infty$, we get

$$\begin{aligned}\left|\int f\phi\right| &= \left|\int (P_x P_y f)(P_{-x} P_{-y}\psi_4) + \int (P_x P_{-y} f)(P_{-x} P_y \psi_3)\right.\\ &\qquad\qquad \left. + \int (P_{-x} P_y f)(P_x P_{-y}\psi_2) + \int (P_{-x} P_{-y} f)(P_x P_y \psi_1)\right|\\ &\leq \|P_x P_y f\|_1 \|\psi_4\|_\infty + \|P_x P_{-y} f\|_1 \|\psi_3\|_\infty\\ &\qquad\qquad + \|P_{-x} P_y f\|_1 \|\psi_2\|_\infty + \|P_{-x} P_{-y} f\|_1 \|\psi_1\|_\infty,\end{aligned}$$

which implies $\left|\int f\phi\right| \leq \|f\|_{H^1} \|\phi\|_{\mathrm{BMO}}$.

Conversely, if $l \in (H^1)^*$, the usual duality argument shows that l is given by a function in $L^\infty + H_x L^\infty + H_y L^\infty + H_x H_y L^\infty = \mathrm{BMO}(\mathbb{T}^2)$, as in [ChF1]. $\square$

A more detailed study of bmo and BMOr in $\mathbb{T}^d$, for $d > 1$, including their atomic decompositions and their associated Carleson measures, will be the object of a future paper.

2. Big Hankel Operators and Their BMOr Symbols

We consider operators $\Gamma : \mathcal{P} \cap H^2(\mathbb{T}^d) \to H^2(\mathbb{T}^d)^\perp$, for $d \geq 1$. Such operators Γ are called bounded if $\sup \|\Gamma f\|_2 / \|f\|_2 =: \|\Gamma\| < \infty$. A bounded Γ has a unique bounded extension, $\Gamma : H^2 \to H^{2\perp}$. It is easy to check that, for every $\Gamma : \mathcal{P} \cap H^2 \to H^{2\perp}$, the following conditions are equivalent:

(a) $\langle \Gamma S_k f,\, g\rangle = \langle \Gamma f, S_{-k} g\rangle$ for $k = 1, \ldots, d$ and all $f \in \mathcal{P} \cap H^2$ and $g \in H^{2\perp}$;
(b) $\Gamma S_k f = (I - P) S_k \Gamma f$ for $k = 1, \ldots, d$, where $P : L^2 \to H^2$ is the orthoprojector;
(c) There exists $\phi \in L^2(\mathbb{T}^2) \ni \Gamma = \Gamma_\phi$, that is, $\Gamma f = (I - P)\phi f$ for all $f \in H^2(\mathbb{T}^2)$;
(d) $\Gamma = \Gamma_{\phi_-}$ for $\phi_- = \Gamma 1 \in H^{2\perp}$.

If (a)–(d) are verified, Γ is called a *big Hankel operator*, and ϕ as in (c) is called a *symbol* of Γ. Since

$$\Gamma_\phi = \Gamma_\psi \iff \phi - \psi = h \in H^2,$$

we see that if ϕ is a symbol for Γ so are all $\phi + h$, for $h \in H^2$. Moreover, among all symbols, there is a unique one in $H^{2\perp}$, which is $\Gamma 1$. In what follows, (big) Hankel operators will be referred to as *Hankel.*

If $\varphi \in L^\infty$, then Γ_φ is a bounded operator, with $\|\Gamma_\varphi \leq \|\varphi\|_\infty$. In the one-dimensional case, the Nehari Theorem gives the converse: A Hankel operator Γ is bounded if and only if $\exists\varphi \in L^\infty$ with $\Gamma_\varphi = \Gamma$, if and only if $\exists\varphi \in L^\infty$ with $\Gamma_\varphi = \Gamma$ and $\|\varphi\|_\infty = \|\Gamma\|$, and if and only if $\Gamma 1 \in \mathrm{BMO}$. Also, $\|\Gamma_\varphi\| = \mathrm{dist}_{L^\infty}(\varphi, H^\infty)$. Since $\phi \in \mathrm{BMO}(\mathbb{T})$ implies $\phi = \varphi + h$, for $\varphi \in L^\infty$ and, $h \in H^2$, we have

$$\phi \in \mathrm{BMO} \implies \Gamma_\phi = \Gamma_\varphi \text{ is bounded, and } \|\Gamma_\phi\| \leq \|\phi\|_{\mathrm{BMO}}.$$

Thus, in the one-dimensional case, $\mathrm{BMO}(\mathbb{T})$ appears as an essential feature both in the weighted norm inequalities for the Hilbert transform, and in the boundedness of the Hankel operators. In [ACS] it was shown that the basic properties of $\mathrm{BMO}(\mathbb{T})$ can be deduced in a unified way from a Generalized Bochner Theorem (GBT) that is equivalent to the Nehari theorem in $H^2(\mathbb{T};\mu)$, and which unifies the results of Nehari and Helson–Szegő. An abstract version of this GBT led to a version of the Nehari theorem in $\mathbb{T}^d$ in terms of BMOr, and to an extension of the Helson–Szegő theorem in terms of $\mathrm{bmo}(\mathbb{T}^d)$. Since $\mathrm{bmo} \neq L^\infty$ and $\mathrm{BMOr} \neq \mathrm{BMO}$ for $d > 1$, this underlines the importance of these two subspaces of product BMO.

Here we will base our considerations on the two-dimensional version of Nehari theorem:

Theorem A. [CS1], [CS2] *For every (big) Hankel operator $\Gamma : \mathcal{P} \cap H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp$, the following conditions are equivalent:*

(a) *Γ is bounded.*

(b) *There exist $\varphi_1, \varphi_2 \in L^\infty(\mathbb{T}^2)$, with $\max\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\} \leq \|\Gamma\|$, and such that*

$$P_{-x}\Gamma = P_{-x}\Gamma_{\varphi_1}, \quad P_{-y}\Gamma = P_{-y}\Gamma_{\varphi_2}. \tag{2.1}$$

(c) *There exists $\phi \in \mathrm{BMOr} \cap H^{2\perp}$ with $\Gamma = \Gamma_\phi$ and $\|\phi\|_{\mathrm{BMOr}} \leq \|\Gamma\| \leq \sqrt{2}\|\phi\|_{\mathrm{BMOr}}$.*

(d) *$\Gamma 1 \in \mathrm{BMOr}$.*

Remark that (2.1) implies, for φ_1 and φ_2 as in (b), that

$$\Gamma = P_{-x}\Gamma_{\varphi_1} + P_x P_{-y}\Gamma_{\varphi_2} = P_{-y}\Gamma_{\varphi_2} + P_{-x}P_y\Gamma_{\varphi_1}. \tag{2.2}$$

Corollary. *For every $\phi \in \mathrm{BMOr}$, Γ_ϕ is bounded and*

$$\|\Gamma_\phi\| = \max\{\mathrm{dist}_{\mathrm{BMOr}}(\phi, \mathrm{BMOr} \cap H^2_x), \mathrm{dist}_{\mathrm{BMOr}}(\phi, \mathrm{BMOr} \cap H^2_y)\}. \tag{2.3}$$

From Theorem A and the fact that $\phi \in \mathrm{BMOr}$ implies $\phi_- = (I - P)\phi \in \mathrm{BMOr} \cap H^{2\perp}$, with $\|\phi_-\|_{\mathrm{BMOr}} \leq \|\phi\|_{\mathrm{BMOr}}$, it follows that, for all $\phi \in \mathrm{BMOr}$, the operator Γ_ϕ is bounded and satisfies $\|\Gamma_\phi\| \leq \sqrt{2}\|\phi\|_{\mathrm{BMOr}}$.

Then, $\phi \mapsto \Gamma_\phi$ is a surjective map from BMOr onto the space $\mathcal{G}$ of the bounded Hankel operators, whose restriction to $\mathrm{BMOr} \cap H^{2\perp}$ is a bijection. The symbols $\phi \in \mathrm{BMOr}$, a proper subspace of product BMO, are thus enough for the theory of big Hankel operators. The duality theorem for BMOr leads to the following theorem, which highlights that the symbols in L^∞ are not enough, so that equivalence (c) in Theorem A can be considered sharp.

The map $\varphi \mapsto \Gamma_\varphi$ from $L^\infty(\mathbb{T}^2)$ to the space $\mathcal{G}$ has kernel $H^\infty(\mathbb{T}^2)$ and induces an injective map from L^∞/H^∞ into $\mathcal{G}$. If this map were also surjective, by the Banach open mapping theorem, there would be a constant $K > 0$ such that, for each $\Gamma \in \mathcal{G}$, there would be a $\varphi \in L^\infty$ with

$$\Gamma_\varphi = \Gamma \quad \text{ and } \quad \|\varphi\|_\infty \le K\|\Gamma\|. \tag{2.4}$$

If (2.4) held, the space $L^\infty + PL^\infty$, the "natural" extension of BMO($\mathbb{T}$) to $d > 1$, would indeed coincide with BMOr. We see that this is not the case by showing that the map $\varphi \mapsto \Gamma$ is not surjective from L^∞/H^∞ to $\mathcal{G}$.

Theorem 2.1. *There are bounded big Hankel operators from $H^2(\mathbb{T}^2)$ to $H^2(\mathbb{T}^2)^\perp$ that have no bounded symbol.*

Proof. If the map $\varphi \mapsto \Gamma$ from L^∞/H^∞ to $\mathcal{G}$ were surjective, there would exist a $K > 0$ for which (2.4) would be satisfied. If a pair $\varphi_1, \varphi_2 \in L^\infty(\mathbb{T}^2)$, with $\|\varphi_1\|_\infty \le 1$ and $\|\varphi_2\|_\infty \le 1$, coincide as functionals on $H^2(\mathbb{T}^2)$, by (2.1) it defines a bounded Hankel operator Γ, with $\|\Gamma\| \le \sqrt{2}$, so that there would be a $\varphi \in L^\infty$ with $\Gamma = \Gamma_\varphi$ and $\|\varphi\|_\infty \le \sqrt{2}K$. Now given any trigonometric polynomials $p_0 \in H^2_{-x,-y} := H^2_{-x} \cap H^2_{-y}$, $p_1 \in H^2_{-x,y} := H^2_{-x} \cap H^2_y$, and $p_2 \in H^2_{x,-y} := H^2_x \cap H^2_{-y}$, we have by (2.2):

$$\begin{aligned}\langle \Gamma 1,\, p_0 + p_1 + p_2\rangle &= \langle P_{-x}\Gamma_{\varphi_1} 1,\, p_0 + p_1\rangle + \langle P_{-y}P_x\Gamma_{\varphi_2} 1, p_2\rangle \\ &= \langle \Gamma_{\varphi_1} 1,\, p_0 + p_1\rangle + \langle \Gamma_{\varphi_2} 1, p_2\rangle = \int \varphi_1(\bar{p}_0 + \bar{p}_1) + \int \varphi_2 \bar{p}_2.\end{aligned}$$

On the other hand, if (2.4) holds, we have

$$\begin{aligned}|\langle \Gamma 1, p_0 + p_1 + p_2\rangle| = |\langle \Gamma_\varphi 1, p_0 + p_1 + p_2\rangle| &= |\int \varphi(\bar{p}_0 + \bar{p}_1 + \bar{p}_2)| \\ &\le \|\varphi\|_\infty \|p_0 + p_1 + p_2\|_1 \le \sqrt{2}K\|p_0 + p_1 + p_2\|_1,\end{aligned}$$

so that

$$\left|\int \varphi_1(\bar{p}_0 + \bar{p}_1) + \int \varphi_2 \bar{p}_2\right| \le \sqrt{2}K\|p_0 + p_1 + p_2\|_1. \tag{2.5}$$

We will now show that (2.5) leads to a contradiction. In fact, to give any pair φ_1, φ_2 as above is the same as to give a $\phi_- \in \text{BMOr} \cap H^{2\perp}$, with $P_{-x}\phi_- = P_{-x}\varphi_1$, $P_{-y}\phi_- = P_{-y}\varphi_2$, and $P_xP_y\phi_- = 0$. Then (2.5) can be rewritten as

$$\left|\int \phi_-(\bar{p}_0 + \bar{p}_1 + \bar{p}_2)\right| \le \sqrt{2}K\|p_0 + p_1 + p_2\|_1 \tag{2.6}$$

for all trigonometric polynomials $p_0 + p_1 + p_2 \in H^2_{-x,-y} + H^2_{-x,y} + H^2_{x,-y} = H^{2\perp}$.

Now, any $\phi \in \mathrm{BMOr}$ can be written as $\phi = P_x P_y \phi + (I - P_x P_y)\phi = P_x P_y \varphi_0 + \phi_-$ for some $\varphi_0 \in L^\infty$ and $\phi_- \in \mathrm{BMOr} \cap H^{2\perp}$. Thus, for every trigonometric polynomial $f = \bar{p} + \bar{p}_0 + \bar{p}_1 + \bar{p}_2$, with $p \in H^2(\mathbb{T}^2)$, and $\phi \in \mathrm{BMOr}$, (2.6) yields

$$\Big|\int \phi f\Big| = \Big|\int \varphi_0 \bar{p} + \int \phi_-(\bar{p}_0 + \bar{p}_1 + \bar{p}_2)\Big| \leq \|\varphi_0\|_\infty \|p\|_1 + \sqrt{2}K\|p_0 + p_1 + p_2\|_1$$
$$\leq (\|\phi\|_{\mathrm{BMOr}} + \sqrt{2}K)(\|P_x P_y f\|_1 + \|(I - P_x P_y)f\|_1), \tag{2.7}$$

where $p = P_x P_y f$ and $p_0 + p_1 + p_2 = (I - P_x P_y)f$.

But since the Hilbert transforms, as well as the analytic projections, are unbounded in L^1, the norms $\|P_x P_y f\|_1 + \|(I - P_x P_y)f\|_1$ and $|||f|||$ are not comparable (see Proposition 1.4), and there exists for every $\varepsilon > 0$ an $f \in \mathcal{P}(\mathbb{T}^2)$ such that

$$\|P_x P_y f\|_1 + \|(I - P_x P_y)f\|_1 < \frac{\varepsilon}{1 + \sqrt{2}K}|||f|||.$$

By Hahn–Banach and the duality of BMOr, there exists $\phi \in \mathrm{BMOr}$ such that $\|\phi\|_{\mathrm{BMOr}} \leq 1$ and $\int \phi f = |||f|||$, so that (2.7) implies $\int \phi f < \varepsilon \int \phi f$, which is a contradiction. □

An important open question is whether for every $\varphi \in L^\infty(\mathbb{T}^2)$ there is another $\psi \in L^\infty(\mathbb{T}^2)$ such that $\Gamma_\psi = \Gamma_\varphi$ and $\|\psi\|_\infty \leq K\|\Gamma_\varphi\|$, where K is a universal constant and Γ_φ is the big Hankel operator defined by $\mathbb{T}_\varphi f = (I - P)\varphi f$. Some geometric properties of $H^\infty(\mathbb{T}^2)$ and BMOr make highly improbable a positive answer to this question, which will be considered elsewhere.

3. Interpolation Problems in the Polydisk, Hankel Operators and Pick Matrices

A basic interpolation problem in $\mathbb{D}^d$, for $d \geq 1$, is the *Pick problem*: Given $z_1, \ldots, z_n \in \mathbb{D}^d$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, find a necessary and sufficient condition for the existence of an analytic function F on $\mathbb{D}^d$ satisfying $F(z_k) = \lambda_k$, for $k = 1, \ldots, n$, with $\|F\|_\infty \leq 1$. This problem can be reformulated in a way that is slightly more general only for $d > 1$, as follows: Given $z_1, \ldots, z_n \in \mathbb{D}^d$ and $G \in H^\infty(\mathbb{D}^d)$, find an analytic F satisfying $F(z_k) = G(z_k)$ for $k = 1, \ldots, n$, and $\|F\|_\infty \leq 1$.

In the case of $d = 1$, the problem was solved by G. Pick in 1916, in terms of the positivity of an associated $n \times n$ matrix given by the data. Another solution has been given in terms of the boundedness of an associated Hankel operator given by the data.

Theorem 3.1 (Pick). *Given $z_1, \ldots, z_n \in \mathbb{D}$ and $G \in H^\infty(\mathbb{D})$, the following assertions are equivalent:*

(i) *The Pick matrix*

$$((1 - G(z_j)\overline{G(z_k)})(1 - z_j\overline{z_k})^{-1})_{j,k=1,\dots,n} \tag{3.1}$$

is positive definite.

(ii) *The Hankel operator Γ_ϕ with symbol $\phi = \bar{b}G$, where b is the Blaschke product with simple zeros at $z_1, \dots, z_n$, is bounded, and $\|\Gamma_\phi\| \leq 1$.*

(iii) *The Pick problem has a solution.*

The equivalence of (i) and (iii) was proved in [P], and that of (ii) and (iii) can be obtained as a corollary of the Nehari theorem (see, for instance, [Ni]).

For $d > 1$ it is known [Am] that the positivity of the Pick matrix analogous to (3.1) is necessary but not sufficient for the existence of a solution to the Pick problem. Necessary and sufficient conditions involving Pick matrices have been given by Agler for $d = 2$ [Ag], and by Cole, Lewis and Wermer for all $d > 1$ [CLW]. However, their conditions are not verifiable in practice. Moreover, in their approach the relation with Hankel operators is lost.

As the Nehari theorem for $d > 1$ can be recovered by replacing the L^∞ norm by the BMOr norm, considering the Pick problem with BMOr-norm control allowed us in [CS3] to retain the relation with Hankel operators (within a constant $\sqrt{d}$), but not a Pick condition.

The following result, which also reduces to the Pick theorem when $d = 1$, gives necessary and sufficient conditions for the existence of solutions of a coordinate-wise Pick problem in terms of either the boundedness of a Hankel operator with symbol specified by the data, or the positivity of d associated $n \times n$ Pick matrices. We state it here only for $d = 2$, but it holds for all $d > 1$, with obvious changes.

Theorem 3.2. *Given $(z_1, w_1), \dots, (z_n, w_n) \in \mathbb{D}^2$ and $G \in H^\infty(\mathbb{D}^2)$, let $b = b_1 \otimes b_2$, where b_1 and b_2 are finite one-dimensional Blaschke products with simple zeros at $z_1, \dots, z_n$ and $w_1, \dots, w_n$, respectively. The following assertions are equivalent (up to a constant $\sqrt{2}$):*

(i) *The Pick matrices*

$$((1 - G(z_j, y)\overline{G(z_k, y)})(1 - z_j\overline{z_k})^{-1})_{j,k=1,\dots,n} \tag{3.2}$$

and

$$((1 - G(x, w_j)\overline{G(x, w_k)})(1 - w_j\overline{w_k})^{-1})_{j,k=1,\dots,n} \tag{3.3}$$

are positive definite for every $y \in \mathbb{T}$ and every $x \in \mathbb{T}$, respectively.

(ii) *The Hankel operator Γ_ϕ, for $\phi = \bar{b}G$, is bounded, with $\|\Gamma_\phi\| \leq 1$.*

(iii) *There is a function $F \in H^2(\mathbb{D}^2)$ satisfying $F(z_k, w) = G(z_k, w)$ and $F(z, w_k) = G(z, w_k)$, for $k = 1, \dots, n$, with $\|\bar{b}F\|_{\mathrm{BMOr}} \leq 1$.*

(iv) *There exist two bounded functions on* $\mathbb{D}^2$, F_1 *analytic in* z *and* F_2 *analytic in* w, *satisfying* $F_1(z_k, y) = G(z_k, y)$ *and* $F_2(x, w_k) = G(x, w_k)$, *for* $k = 1, \ldots, n$, *with* $\|F_1\|_\infty \leq 1$ *and* $\|F_2\|_\infty \leq 1$.

More precisely, (ii) *implies* (iii) *and* (iv), *and either* (iii) *or* (iv) *imply* (ii) *with* $\|\Gamma_\phi\| \leq \sqrt{2}$, *while* (i) *is equivalent to* (iv). (Compare [BH].)

Remark. Observe that the loss of the L^∞ norm in condition (iii) is compensated by the strengthening of the interpolation condition to each variable independently.

Proof. (ii) $\Longrightarrow$ (iii). If $\|\Gamma_\phi\| \leq 1$, by Theorem A there exists $\psi \in$ BMOr with $\|\psi\|_{\mathrm{BMOr}} \leq 1$, such that $\Gamma_\phi = \Gamma_\psi$. Therefore $\psi - \phi = \psi - \bar{b}G \in H^2(\mathbb{D}^2)$. Setting $F = b\psi = G + b_1 b_2 h \in H^2(\mathbb{D}^2)$ it is immediate that F satisfies all the conditions of (iii).

(iii) $\Longrightarrow$ (ii). If F satisfies the interpolation conditions (iii), then

$$F - G = b_1(z)h_1(z, w) = b_2(w)h_2(z, w)$$

for $h_1, h_2 \in H^2(\mathbb{D}^2)$. This implies that there is an $h \in H^2(\mathbb{D}^2)$ such that $F - G = b_1(z)b_2(w)h(z, w)$. Thus, setting $\psi = \bar{b}F$, we have $\|\psi\|_{\mathrm{BMOr}} \leq 1$, and, by Theorem A, $\|\Gamma_\psi\| \leq \sqrt{2}$. But $\Gamma_\psi = \Gamma_\phi$, since $\psi - \phi = \bar{b}(F - G) = h$, so $\|\Gamma_\phi\| \leq \sqrt{2}$.

(ii) $\Longrightarrow$ (iv). By Theorem A(b), the condition $\|\Gamma_\phi\| \leq 1$ implies that there exist $\varphi_1, \varphi_2 \in L^\infty(\mathbb{T}^2)$ with $\|\varphi_j\|_\infty \leq 1$ for $j = 1, 2$, such that $\phi = \bar{b}G = \phi_1 + h_x = \phi_2 + h_y$ for $h_x \in H^2_x$ and $h_y \in H^2_y$. The functions $F_1 = b_1\varphi_1$ and $F_2 = b_2\varphi_2$ satisfy $\|F_j\|_\infty \leq 1$ for $j = 1, 2$, as well as $F_1(z_k, w) = G(z_k, w)$ and $F_2(z, w_k) = G(z, w_k)$. Moreover, $F_1 = \overline{b_2(w)}G(z, w) + b_1(z)h_x(z, w)$ is analytic in z, and $F_2 = \overline{b_1(z)}G(z, w) + b_2(w)h_y(z, w)$ is analytic in w.

(iv) $\Longrightarrow$ (ii). By the interpolation conditions satisfied by F_1 and F_2, for each $y \in \mathbb{T}$ we have $G(z, y) - F_1(z, y) = b_1(z)h_x(z, y)$ for $h_x \in H^2_x$, and, for each $x \in \mathbb{T}$, we have $G(x, w) - F_2(x, w) = b_2(w)h_y(x, w)$ for $h_y \in H^2_y$. Hence, setting $\varphi_j = \bar{b}F_j$ for $j = 1, 2$, we have $\|\varphi_j\|_\infty \leq 1$, and

$$\phi - \varphi_1 = \bar{b}(G - F_1) = \overline{b_2}h_x \in H^2_x,$$
$$\phi - \varphi_2 = \bar{b}(G - F_2) = \overline{b_1}h_y \in H^2_y.$$

Again by Theorem A, this implies $\|\Gamma_\phi\| \leq \sqrt{2}$.

(i) $\Longleftrightarrow$ (iv). Apply Pick's Theorem to the one-variable functions $G(z, \cdot)$ and $G(\cdot, w)$ separately. Then the two solutions $F_1(z, \cdot)$ and $F_2(\cdot, w)$ satisfy $\|F_1(\cdot, w)\|_\infty \leq 1$ for all $w \in \mathbb{D}$, and $\|F_2(z, \cdot)\|_\infty \leq 1$ for all $z \in \mathbb{D}$, so that $\|F_1\|_\infty \leq 1$ and $\|F_2\|_\infty \leq 1$. Conversely, observe that (iv) implies the analogues of (3.2) and (3.3) with $w \in \mathbb{D}$ instead of $y \in \mathbb{T}$, and $z \in \mathbb{D}$ instead of $x \in \mathbb{T}$, respectively, which is equivalent to (3.2) and (3.3) by the analyticity of G in both variables. □

Given $(z_1, w_1), \dots, (z_n, w_n) \in \mathbb{D}^2$ and $\lambda_1, \dots, \lambda_n$ in $\mathbb{C}$, let b_1 and b_2 be the corresponding one-dimensional Blaschke products. Writing $\lambda_k = \lambda'_k \lambda''_k$, for $k = 1, \dots, n$, set

$$G_1(z) = \sum_{k=1}^{n} \frac{b_1(z)}{b'_1(z_k)} \frac{\lambda'_k}{z - z_k} \qquad \text{and} \qquad G_2(w) = \sum_{k=1}^{n} \frac{b_2(w)}{b'_2(w_k)} \frac{\lambda''_k}{w - w_k},$$

so that $G_1(z_k) = \lambda'_k$, $G_2(w_k) = \lambda''_k$, and $G(z_k, w_k) = \lambda_k$ for $G(z, w) = G_1(z)G_2(w)$. For such G_1 and G_2, or any others satisfying the interpolating conditions, we have:

Corollary 3.3. *Given $(z_1, w_1), \dots, (z_n, w_n) \in \mathbb{D}^2$ and $\lambda_1, \dots, \lambda_n$ in $\mathbb{C}$, there exists a function $F \in H^2(\mathbb{D}^2)$ satisfying $F(z_k, w) = G(z_k, w)$ and $F(z, w_k) = G(z, w_k)$, for $k = 1, \dots, n$, with $\|\bar{b}F\|_{\mathrm{BMOr}} \le 1$, as well as two bounded functions on $\mathbb{D}^2$, F_1 analytic in z and F_2 analytic in w, satisfying $F_1(z_k, y) = G(z_k, y)$ and $F_2(x, w_k) = G(x, w_k)$, for $k = 1, \dots, n$, with $\|F_1\|_\infty \le 1$ and $\|F_2\|_\infty \le 1$, whenever the two numerical $n \times n$ matrices*

$$((1 - \|G_2\|_\infty^2 G_1(z_j)\overline{G_1(z_k)})(1 - z_j\overline{z_k})^{-1})_{j,k=1,\dots,n}$$

and

$$((1 - \|G_1\|_\infty^2 G_2(w_j)\overline{G_2(w_k)})(1 - w_j\overline{w_k})^{-1})_{j,k=1,\dots,n}$$

are positive definite.

4. Hankel Operators of Finite Type and Versions of the Kronecker and AAK Theorems

In the one-dimensional case, once the relation between the bounded Hankel operators and their symbols was established, it was important to characterize the symbols of operators of finite rank. The characterization is given by the Kronecker theorem: A bounded Hankel operator Γ is of finite rank if and only if $\Gamma = \Gamma_\varphi$ for $\varphi = \bar{b}h$, where b is a finite Blaschke product and $h \in H^\infty$, so $\bar{b}h \in H^\infty + R_n$ (where R_n is the class of rational functions with n poles in the disk). Since the range of Γ is finite-dimensional if and only if its kernel has finite codimension, and since this kernel is a subspace of $H^2(\mathbb{T})$ invariant under the shift S, the Kronecker theorem can be deduced from the Beurling theorem, asserting that a subspace $\mathfrak{I} \subset H^2(\mathbb{T})$ is invariant if and only if $\mathfrak{I} = \theta H^2(\mathbb{T})$, where θ is an inner function with $|\theta| \equiv 1$, and that an invariant subspace $\mathbb{T}$ is of finite codimension if and only if $\theta = b$, a finite Blaschke product. The S^*-invariant subspaces, called the model spaces, are of the form $K_\theta = H^2(\mathbb{T}) \ominus \theta H^2(\mathbb{T})$, and K_θ is finite-dimensional if and only if $\theta = b$.

Recall that for an operator T and for $n \in \mathbb{N}$, the singular numbers of T are defined as

$$s_n(T) := \inf\{\|T - T_n\| : T_n \text{ of finite rank } \le n\}, \tag{4.1}$$

which is equivalent to

$$s_n(T) = \inf\{\|T\big|_E\| : E \text{ of codimension } \leq n\}.$$

Here $s_0(T) = \|T\| \geq s_1 \geq \cdots \geq s_n \geq \cdots$, and T is of finite rank if there is an $m \in \mathbb{N}$ such that $s_n(T) = 0$ for $n > m$.

A theorem of Adamjan–Arov–Krein [AAK] asserts that for every Hankel operator $\Gamma : H^2(\mathbb{T}) \to H^2(\mathbb{T})^\perp$ we have, for $n \in \mathbb{N}$,

$$s_n(\Gamma) = \inf\{\|\Gamma - \Gamma_n\| : \Gamma_n \text{ Hankel and of finite rank} \leq n\}. \tag{4.2}$$

This, combined with the Kronecker theorem, gives, for all $\varphi \in L^\infty(\mathbb{T})$,

$$s_n(\Gamma_\varphi) = \operatorname{dist}(\varphi,\ H^\infty + R_n).$$

An equivalent form of (4.2) was given by S. Treil [T1] as

$$s_n(\Gamma) = \inf\{\|\Gamma|_{\mathfrak{I}}\| : \mathfrak{I} \text{ invariant under } S \text{ and codim } \mathfrak{I} \leq n\}, \tag{4.3}$$

where, by Beurling's theorem, the subspace $\mathfrak{I}$ is of the form $bH^2(\mathbb{T})$, for b a Blaschke product with n factors.

Through an abstract version of the AAK theorem, in [CS1] it was shown that the situation is radically different for (big) Hankel operators:

Theorem B. *For every bounded $\Gamma : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp$ and for all $n \in \mathbb{N}$, we have*

$$s_n(\Gamma) \geq 1/\sqrt{2}\|\Gamma\|.$$

This theorem implies that all (big) Hankel operators of finite rank are zero, and no satisfactory extension of the AAK theorem can be expected in terms of their singular numbers. This is linked to the fact that, by a theorem of Ahern and Clark [AhC], the subspaces of the form $bH^2(\mathbb{T}^2)$ for $b = b_1 \otimes b_2$, with b_1 and b_2 finite one-dimensional Blaschke products, are not of finite codimension in $H^2(\mathbb{T}^2)$. However, as shown in [CS4], for these subspaces $bH^2(\mathbb{T}^2)$, it is still true that

$$\{f_{y_0}(x) = f(x, y_0) : f(x, y) \in bH^2(\mathbb{T}^2),\ y_0 \in \mathbb{T} \text{ fixed}\}$$

and

$$\{f_{x_0}(y) = f(x_0, y) : f(x, y) \in bH^2(\mathbb{T}^2),\ x_0 \in \mathbb{T}\text{fixed}\}$$

are finite-codimensional subspaces of $H^2(\mathbb{T})$, leading to a notion of subspaces of finite bi-codimension.

A decomposable subspace V of $H^2(\mathbb{T}^2)$, with $V = V_1 \otimes V_2$, where $V_1 \subset H^2(\mathbb{T})$ and $V_2 \subset H^2(\mathbb{T})$, is called of *finite bi-codimension* (m, n) if and only if $\operatorname{codim} V_1 = m$ and $\operatorname{codim} V_2 = n$. The orthogonal complement of such V is

$$V^\perp = H^2(\mathbb{T}^2) \ominus V = V_1^\perp \otimes H^2(\mathbb{T}) + H^2(\mathbb{T}) \otimes V_2^\perp,$$

with $V_k^\perp = H^2(\mathbb{T}) \ominus V_k$ for $k = 1, 2$. Since orthogonal complements of this form will appear again in Section 5 and in other contexts, we give them a name. Given a subspace $\mathfrak{L} \subset L^2(\mathbb{T})$ and two positive integers m and n, a subspace $W \subset L^2(\mathbb{T}^2)$ is said to be of *bi-finite type* $(\mathfrak{L}; m, n)$ if and only if there exist two subspaces W_1 and W_2 of $H^2(\mathbb{T})$, with $\dim W_1 = m$ and $\dim W_2 = n$, such that $W = W_1 \otimes \mathfrak{L} + \mathfrak{L} \otimes W_2$. With this notation, the orthogonal complement $V^\perp$ of a subspace $V \subset H^2(\mathbb{T}^2)$ of finite bi-codimension (m, n) is a subspace of bi-finite type $(H^2(\mathbb{T}); m, n)$.

For these notions we have the following analogue to the Beurling theorem for invariant subspaces of finite codimension in the disk:

Theorem C. [CS4] *For a subspace $\mathfrak{I} \subset H^2(\mathbb{T}^2)$, invariant under both shifts, S_1 and S_2, of $H^2(\mathbb{T}^2)$, the following conditions are equivalent:*

(a) *$\mathfrak{I}$ is of finite bi-codimension (m, n), that is, $\mathfrak{I} = V_1 \otimes V_2$;*
(b) *$\mathfrak{I}^\perp = H^2(\mathbb{T}^2) \ominus \mathfrak{I}$ is of bi-finite type $(H^2(\mathbb{T}); m, n)$, that is, $\mathfrak{I}^\perp = W_1 \otimes H^2(\mathbb{T}) + H^2(T) \otimes W_2$ with $\dim W_1 = m$ and $\dim W_2 = n$.*
(c) *$\mathfrak{I} = bH^2(\mathbb{T}^2) = b_1 H^2(\mathbb{T}) \otimes b_2 H^2(\mathbb{T})$, for $b = b_1 \otimes b_2$, that is, $b(x, y) = b_1(x) b_2(y)$, where b_1 and b_2 are one-dimensional Blaschke products with m and n factors, respectively.*

Furthermore, $W_k = V_k^\perp = H^2(\mathbb{T}) \ominus b_k H^2(\mathbb{T})$ for $k = 1, 2$.

A Hankel operator $\Gamma : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp$ is called of *finite type* $(m, n) \in \mathbb{N}^2$ if the kernel of Γ is of finite bi-codimension (m, n).

Theorem 4.1. (Kronecker-type characterization of Hankel operators of finite type) *Let $\Gamma : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp$ be a bounded Hankel operator. The following conditions are equivalent:*

(a) *Γ is of finite type (m, n).*
(b) *$\Gamma = \Gamma_\phi$ for $\phi \in \mathrm{BMOr} \cap H^{2\perp}$, and $\phi = \bar{b}h$, where $b = b_1 \otimes b_2$ for b_1 and b_2 one-dimensional Blaschke products with m and n factors, respectively, and $h \in H^2(\mathbb{T}^2)$. Moreover, $h = bh_x + \varphi_1 = bh_y + \varphi_2$ for $\varphi_1, \varphi_2 \in L^\infty(\mathbb{T}^2)$, $h_x \in H^2_x(\mathbb{T}^2)$, $h_y \in H^2_y(\mathbb{T}^2)$, so that*

$$\phi = h_x + \bar{b}\varphi_1 = h_y + \bar{b}\varphi_2.$$

Proof. The kernel K of Γ is invariant under both shifts S_1 and S_2, since Γ is Hankel and, for $f \in K$, $\langle \Gamma S_k f, g\rangle = \langle \Gamma f,\, S_k^{-1} g\rangle = 0$, for $k = 1, 2$. Then, by Theorem C, there is $b = b_1 \otimes b_2$ such that $K = bH^2(\mathbb{T}^2)$. For $\phi \in \mathrm{BMOr}$ the $H^{2\perp}$ symbol of Γ, we have $\langle \Gamma f, g\rangle = \int f\bar{g}\phi$, and thus $\int bf\bar{g}\phi = 0$ for all $f \in H^2$ and $g \in H^{2\perp}$, that is, $b\phi = h \in H^2$ and $\phi = \bar{b}h$, and the converse holds. Moreover, since $\bar{b}h = \phi \in \mathrm{BMOr}$, the function h must satisfy $h = bh_x + \varphi_1 = bh_y + \varphi_2$ for $\varphi_1, \varphi_2 \in L^\infty(\mathbb{T}^2)$, $h_x \in H^2_x$, $h_y \in H^2_y$, and the conclusion follows. □

Theorem C suggests, in order to develop a version of the AAK theorem, to replace the ordinary singular numbers of Hankel operators by some σ-numbers defined in analogy with (4.3). For $\Gamma : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp$ a Hankel operator and $(m,n) \in \mathbb{N}^2$, let

$$\begin{aligned}(4.4)\qquad \sigma_{mn}(\Gamma) := \inf\{\|\Gamma|_{\mathcal{J}}\| : \mathcal{J} \subset H^2(\mathbb{T}^2) &\text{ invariant under } S_1 \text{ and } S_2\\ &\text{and of finite bi-codimension } (m,n)\}.\end{aligned}$$

Equivalently, for $\phi \in \mathrm{BMOr}$,

(4.5)

$$\begin{aligned}\sigma_{mn}(\Gamma_\phi) = \inf\{\|\Gamma_{\bar b\phi}\| : b = b_1 \otimes b_2, &\text{ with } b_1 \text{ and } b_2 \text{ one-dimensional Blaschke products,}\\ &\text{having at most } m \text{ and } n \text{ factors, respectively}\}\end{aligned}$$

since, by Theorem C, the subspaces $\mathcal{J}$ in (4.4) can be written as $\mathcal{J} = bH^2(\mathbb{T}^2)$ with $\|bh\|_2 = \|h\|_2$, so that

$$\|\Gamma_\phi|_{\mathcal{J}}\| = \sup_h \frac{\|\Gamma_\phi bh\|_2}{\|bh\|_2} = \sup_h \frac{\|\Gamma_{b\phi} h\|_2}{\|h\|_2} = \|\Gamma_{b\phi}\|.$$

It is easy to check that the infima are attained in (4.4) and (4.5). Clearly, we have again

$$\begin{aligned}\sigma_{00}(\Gamma) &= \|\Gamma\|,\\ \sigma_{mn}(\Gamma) &\geq \sigma_{(m+1)n}(\Gamma),\\ \sigma_{mn}(\Gamma) &\geq \sigma_{m(n+1)}(\Gamma),\\ \sigma_{mn}(\Gamma) &\leq \|\Gamma\|\end{aligned}$$

for all $m, n \in \mathbb{N}$.

Corollary 4.2. *For every $(m,n) \in \mathbb{N}^2$ there exists a non-zero Hankel operator*

$$\Gamma : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp$$

of finite type (m,n), such that

$$\sigma_{pq}(\Gamma) = 0 \textit{ for } p > m, \quad q > n.$$

Proof. Take $h(x,y) = b_1(x)h_1(y) + b_2(y)h_2(y)$ for b_1 and b_2 one-dimensional Blaschke products with at most m and n factors, respectively, and $h_1, h_2 \in H^\infty(\mathbb{T})$. Further take $\phi = \bar b h$, for $b = b_1 \otimes b_2$. Then $h \in H^\infty(\mathbb{T}^2)$, and

$$\phi = \overline{b_1(x)}h_2(x) + \overline{b_2(y)}h_1(y) \in \mathrm{BMOr} \cap H^2(\mathbb{T}^2)^\perp,$$

since

$$P_{-x}\phi = P_{-x}\overline{b_1(x)}h_2(x), \qquad P_{-y}\phi = P_{-y}\overline{b_2(x)}h_1(y),$$

with $\bar b_1 h_2$ and $\bar b_2 h_1$ in $L^\infty(\mathbb{T}^2)$, and $P_x P_y \phi = 0$ for the right choice of h_1 and h_2. By Theorem (4.1), the Hankel operator $\Gamma = \Gamma_\phi$ satisfies the conclusion. $\square$

From Theorem B follows that there are no nonzero compact big Hankel operators, that is, Hankel operators whose sequence of singular numbers tend to zero. Since this corollary says that there are big Hankel operators $\Gamma \neq 0$ with $\sigma_{mn}(\Gamma) \to 0$ as $m, n \to \infty$, it is interesting to study the class of such operators, and this will be done elsewhere.

For bounded Hankel operators in the one-dimensional case the AAK theorem asserts that $s_n(\Gamma_\varphi) = \mathrm{dist}_{L^\infty}(\varphi, H^\infty + R_n)$.

This precise statement does not hold for all bounded (big) Hankel operators $\Gamma = \Gamma_{\phi'}$ given by a symbol $\phi \in$ BMOr, but we still have a substitute by replacing the distance

$$\begin{aligned}\mathrm{dist}(\phi,\ H^\infty + R_n) = \inf\{\|\phi - b^{-1}h\|_\infty : h \in H^\infty = L^\infty \cap H^2,\\ b = b_1 \otimes \cdots \otimes b_d,\ \text{with } b_k \text{ a one-dimensional Blaschke product}\\ \text{of at most } n_k \text{ factors, for } k = 1, \ldots, d\}\\ = \inf\{\|b\phi - h\|_\infty : h \in H^\infty = L^\infty \cap H^2, b\}\end{aligned}$$

by

$$\begin{aligned}\delta(\phi,\ \mathrm{BMOAr} + R_n) := \inf\{\|b\phi - h\|_{\mathrm{BMOr}} : h \in \mathrm{BMOAr} = \mathrm{BMOr} \cap H^2,\\ b = b_1 \otimes \cdots \otimes b_d, \text{with } b_k \text{ a one-dimensional Blaschke product}\\ \text{of at most } n_k \text{ factors, for } k = 1, \ldots, d\}.\end{aligned}$$

Observe that BMOAr = BMOr $\cap H^2$ = BMO $\cap H^2$ = BMOA.

Theorem 4.3. *For every $\phi \in \mathrm{BMOr}(\mathbb{T}^d)$ and $n \in \mathbb{N}^d$, with $d > 1$, we have*

$$1/\sqrt{d}\sigma_n(\Gamma_\phi) \leq \delta(\phi,\ \mathrm{BMOAr} + R_n) \leq \sigma_n(\Gamma_\phi),$$

where Γ_ϕ is the Hankel operator with symbol ϕ.

Proof. By (4.5), for every $\varepsilon > 0$, there are b_1 and b_2, and $b = b_1 \otimes b_2$, such that

$$\sigma_{mn}(\Gamma_\phi) \leq \|\Gamma_{b\phi}\| \leq \sigma_{mn}(\Gamma_\phi) + \varepsilon.$$

The operator $\Gamma_{b\phi} : H^2 \to H^{2\perp}$ is also Hankel, and, by Theorem A, $\Gamma_{b\phi} = \Gamma_\psi$ for some $\psi \in$ BMOr, with $\psi = b\phi - h$, $h \in H^2$, and $\|\Gamma_{b\phi}\| \leq \sqrt{2}\|\psi\|_{\mathrm{BMOr}}$, $\|\psi\|_{\mathrm{BMOr}} \leq \sigma_{mn}(\Gamma_\phi) + \varepsilon$. Hence, $1/\sqrt{2}\sigma_{mn}(\Gamma_\phi) \leq \|b\phi - h\|_{\mathrm{BMOr}} \leq \sigma_{mn}(\Gamma_\phi) + \varepsilon$, for all $\varepsilon > 0$, which is the conclusion. □

5. Carleson Measures, Model Subspaces of Finite Type, and BMOr Symbols

In the one-dimensional case, there is a close relation linking Hankel operators, BMO functions and Carleson measures. Carleson measures in the disk are those positive measures μ satisfying the Carleson imbedding condition

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq C^2 \int_{\mathbb{T}} |f(t)|^2 \, dt, \quad \text{for all } f \in H^2(\mathbb{T}), \tag{5.1}$$

where $f(z)$ stands for the analytic extension of f to $\mathbb{D}$.

Carleson characterized those measures as satisfying the *tent condition* for intervals, that is, $\mu(S(I)) \leq C|I|$ for every interval I, where $S(I)$ is a tent in $\mathbb{D}$ with base I. Moreover, the H^2-imbedding condition (5.1) is equivalent to the H^p-imbedding condition being valid for all p such that $1 \leq p < \infty$.

Following Nikolskii and Treil [Ni], [T2], condition (5.1) can be expressed in terms of projectors on one-dimensional model subspaces

$$K_z = K_{b_z} = H^2 \ominus b_z H^2$$

defined by single-factor Blaschke products

$$b_z(\zeta) = \frac{|z|}{z} \frac{z - \zeta}{1 - \zeta \bar{z}}.$$

It is well known that such a subspace K_z is spanned by the normalized function

$$\phi_z(\xi) = \frac{(1 - |z|^2)^{1/2}}{1 - \bar{z}\xi} \quad \text{for } z \in \mathbb{D} \text{ and } \xi \in \mathbb{T}, \tag{5.2}$$

which has the reproducing property

$$\langle f, \phi_z \rangle = (1 - |z|^2)^{1/2} f(z) \quad \text{for all } f \in H^2. \tag{5.3}$$

Thus, for $P_z : H^2(\mathbb{T}) \to K_z$ the orthogonal projector, the identity

$$\|P_z f\|_2^2 = (1 - |z|^2)|f(z)|^2 \tag{5.4}$$

holds for all $f \in H^2(\mathbb{T})$. The Carleson imbedding condition (5.1) can, therefore, be rewritten as

$$\int_{\mathbb{D}} \|P_z f\|_2^2 \, d\nu(z) \leq C^2 \|f\|_2^2, \quad \text{for all } f \in H^2(\mathbb{T}), \tag{5.5}$$

with $d\nu(z) = (1 - |z|^2)^{-1} \, d\mu(z)$.

Moreover, for $P : L^2 \to H^2$ the analytic projector, we have

$$P_z f = b_z (I - P) \bar{b}_z f = b_z \Gamma_{\bar{b}_z} f \tag{5.6}$$

and

$$\|P_z f\|_2^2 = \|\Gamma_{\bar{b}_z} f\|_2^2. \tag{5.7}$$

From (5.7) it can be deduced (see the development leading to (5.10) below) that $\mu \geq 0$ is Carleson if and only if a canonically associated (vector-valued) Hankel operator Γ is bounded, and (through the Nehari theorem) if and only if its antianalytic symbol $\Gamma 1 \in \mathrm{BMO}$.

In Section 1 we observed that the different definitions of BMO, which coincide for $d = 1$, give rise to different classes in $\mathbb{T}^d$, for $d > 1$. In fact, Chang and Fefferman defined product BMO to circumvent Carleson's counterexample showing that the class of measures in $\mathbb{D}^2$ characterized by the tent condition on rectangles $R = I \times J$ does not necessarily satisfy the H^1-imbedding condition. Enlarging the class of tents in their definition of product Carleson measures, Chang and Fefferman proved that a function is in product BMO if and only if a canonically associated measure is product Carleson. Here we adopt in $\mathbb{D}^d$ the Nikolskii formulation (5.5) and show that a measure is Carleson–Nikolskii if and only if a canonically associated function is BMOr.

In $\mathbb{T}^2$ the one-dimensional subspace K_{b_z}, where b_z is a Blaschke factor, is replaced by $K_{b_z \otimes b_\zeta} =: K_{z\zeta}$, where b_z and b_ζ are one-variable Blaschke factors, and $K_{z\zeta}$ is not a one-dimensional subspace of $H^2(\mathbb{T}^2)$. But now, according to Theorem C in Section 4, $K_{z\zeta}$ is of bi-finite type $(H^2(\mathbb{T}); 1, 1)$, and its elements are of the form $A(y)\phi_z(x) + B(x)\phi_\zeta(y)$, for $A(y)$ and $B(x)$ varying in $H^2(\mathbb{T})$. For the orthogonal projector $P_{z\zeta}$ from $H^2(\mathbb{T}^2)$ onto $K_{z\zeta}$ we have the following result.

Lemma 5.1 (Lemma on the projection). *If $P_{z\zeta} : H^2(\mathbb{T}^2) \to K_{z\zeta}$ is the orthogonal projector, we have, for all $f \in H^2(\mathbb{T}^2)$,*

$$P_{z\zeta} f(x,y) = c_z f(z,y)\phi_z(x) + c_\zeta f(x,\zeta)\phi_\zeta(y) - c_z c_\zeta f(z,\zeta)\phi_z(x)\phi_\zeta(y), \tag{5.8}$$

where $c_\nu = (1 - |\nu|^2)^{1/2}$, $\nu \in \mathbb{D}$. Equivalently,

$$P_{z\zeta} f(x,y) = \langle f, \phi_z\rangle_{L^2(\mathbb{T}_x)}\phi_z(x) + \langle f, \phi_\zeta\rangle_{L^2(\mathbb{T}_y)}\phi_\zeta(y) - \langle f, \phi_z \otimes \phi_\zeta\rangle_{L^2(\mathbb{T}^2)}\phi_z(x)\phi_\zeta(y).$$

Proof. Denoting the right-hand side of (5.8) by $g(x,y)$, for $g \in K_{z\zeta}$, it remains to check that, for arbitrary $A, B \in H^2(\mathbb{T})$, we have

$$\langle g(x,y),\, A(y)\phi_z(x)\rangle = \langle f(x,y),\, A(y)\phi_z(x)\rangle,$$
$$\langle g(x,y),\, B(x)\phi_\zeta(y)\rangle = \langle f(x,y),\, B(x)\phi_\zeta(y)\rangle.$$

Since, by (5.3), for every $F \in H^2(\mathbb{T}^2)$ we have

$$\langle F(x,y),\, \phi_z(x)\rangle = c_z F(z,y) \quad \text{and} \quad \langle F(x,y),\, \phi_\zeta(y)\rangle = c_\zeta F(x,\zeta), \tag{5.9}$$

and, by (5.2), $\int |\phi_z(x)|^2\,dx = \int |\phi_\zeta(y)|^2\,dy = 1$, we obtain, as desired,

$$\begin{aligned}\langle g(x,y),\, A(y)\phi_z(x)\rangle &= c_z \int f(z,y)\overline{A(y)}\,dy + c_\zeta \iint f(x,\zeta)\phi_\zeta(y)\overline{A(y)\phi_z(x)}\,dx\,dy \\ &\qquad - c_z c_\zeta f(z,\zeta) \iint \phi_z(x)\phi_\zeta(y)\overline{A(y)\phi_z(x)}\,dx\,dy \\ &= c_z \int f(z,y)\overline{A(y)}\,dy + c_\zeta c_z f(z,\zeta) c_\zeta \overline{A(\zeta)} - c_z c_\zeta f(z,\zeta) c_\zeta \overline{A(\zeta)} \\ &= \langle f(x,y),\, A(y)\phi_z(x)\rangle,\end{aligned}$$

and similarly for the other term. □

Lemma 5.2. *For $P_{z\zeta} : H^2(\mathbb{T}^2) \to K_{z\zeta}$, where $(z,\zeta) \in \mathbb{D}^2$, and $f \in H^2(\mathbb{T}^2)$, we have*

$$\begin{aligned}\|P_{z\zeta} f\|_2^2 (1-|z|^2)^{-1}(1-|\zeta|^2)^{-1} &= (1-|\zeta|^2)^{-1} \int_{\mathbb{T}} |f(z,y)|^2\,dy \\ &\qquad + (1-|z|^2)^{-1} \int_{\mathbb{T}} |f(x,\zeta)|^2\,dx - |f(z,\zeta)|^2,\end{aligned}$$

where $f(z,y)$, $f(x,\zeta)$ and $f(z,\zeta)$ are the analytic extensions of f to $z \in \mathbb{D}$, $\zeta \in \mathbb{D}$, and $(z,\zeta) \in \mathbb{D}^2$, respectively.

Proof. From the expression of $P_{z\zeta} f$ given in (5.8) it follows, using (5.2), (5.3), (5.4) and (5.9), that

$$\begin{aligned}|P_{z\zeta} f|^2 &= c_z^2 |f(z,y)|^2 |\phi_z(x)|^2 + c_\zeta^2 |f(x,\zeta)|^2 |\phi_\zeta(y)|^2 + c_z^2 c_\zeta^2 |f(z,\zeta)|^2 |\phi_z(x)|^2 |\phi_\zeta(y)|^2 \\ &\quad + 2\,\mathrm{Re}\Big(c_z c_\zeta f(z,y)\overline{f(x,\zeta)}\phi_z(x)\overline{\phi_\zeta(y)} + c_z^2 c_\zeta \overline{f(z,\zeta)} f(z,y) |\phi_z(x)|^2 \phi_\zeta(y) \\ &\qquad - c_z c_\zeta^2 f(z,\zeta)\overline{f(x,\zeta)}\phi_z(x)|\overline{\phi_\zeta(y)}|^2 \Big).\end{aligned}$$

Then,

$$\begin{aligned}\|P_{z\zeta} f\|_2^2 &= \iint |P_{z\zeta} f(x,y)|^2\,dx\,dy \\ &= c_z^2 \int |f(z,y)|^2\,dy + c_\zeta^2 \int |f(x,\zeta)|^2\,dx + c_z^2 c_\zeta^2 |f(z,\zeta)|^2 - 2\,\mathrm{Re}(c_z^2 c_\zeta^2 |f(z,\zeta)|^2),\end{aligned}$$

so

$$(c_z^2 c_\zeta^2)^{-1} \|P_{z\zeta} f\|_2^2 = c_\zeta^{-2} \int |f(z,y)|^2\,dy + c_z^{-2} \int |f(x,\zeta)|^2\,dx - |f(z,\zeta)|^2,$$

which is the conclusion. □

Following Nikolskii's approach, we say that a measure $\mu \geq 0$ defined in $\mathbb{D}^2$ is *Carleson–Nikolskii* if

$$d\nu(z,\zeta) = (1-|z|^2)^{-1}(1-|\zeta|^2)^{-1} d\mu(z,\zeta)$$

satisfies

$$\iint_{\mathbb{D}^2} \|P_{z\zeta} f\|_2^2 \, d\nu(z,\zeta) \le C^2 \|f\|_2^2 \quad \text{for all } f \in H^2(\mathbb{T}^2).$$

Formula (5.6) is still valid in $H^2(\mathbb{T}^2)$, that is, for all $f \in H^2(\mathbb{T}^2)$ we have

$$P_{z\zeta} f = (b_z \otimes b_\zeta)(I-P)(\bar{b}_z \otimes \bar{b}_\zeta) f = (b_z \otimes b_\zeta)\Gamma_{\bar{b}_z \otimes \bar{b}_\zeta} f.$$

Thus, again we have

$$\|P_{z\zeta} f\|_2^2 = \|\Gamma_{\bar{b}_z \otimes \bar{b}_\zeta} f\|_2^2.$$

Therefore, a measure μ is Carleson–Nikolskii if and only if

$$\iint_{\mathbb{D}^2} \|\Gamma_{\bar{b}_z \otimes \bar{b}_\zeta} f\|_2^2 \, d\nu \le C^2 \|f\|_2^2 \quad \text{for all } f \in H^2(\mathbb{T}^2).$$

Let $\Gamma : H^2(\mathbb{T}^2) \to L^2(\mathbb{D}^2, \nu; H^2(\mathbb{T}^2)^\perp)$ be the operator assigning to each $f \in H^2(\mathbb{T}^2)$ the function

$$(z,\zeta) \mapsto \Gamma_{\bar{b}_z \otimes \bar{b}_\zeta} f \in H^2(\mathbb{T}^2)^\perp,$$

so that μ is of Carleson type if and only if Γ is bounded in $L^2(\mathbb{D}^2, \nu)$, with $\|\Gamma\| \le C$. By Fubini's theorem, the space $L^2(\mathbb{D}^2, \nu; H^2(\mathbb{T}^2)^\perp)$ of square integrable functions in the bidisk, with values in $H^2(\mathbb{T}^2)^\perp$, is isometrically isomorphic to the space $H^2(\mathbb{T}^2)^\perp(L^2(\mathbb{D}^2, \nu))$ of antianalytic functions with values in $L^2(\mathbb{D}^2, \nu)$. Under this isomorphism the operator Γ corresponds to the operator

$$\vec{\Gamma} : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp(L^2(\mathbb{D}^2, \nu)). \tag{5.10}$$

This $\vec{\Gamma}$ is a (vector-valued) big Hankel operator, since, for $k = 1, 2$, we have

$$\vec{\Gamma} S_k f = F(x,y;z,\zeta) = \Gamma_{\bar{b}_z \otimes \bar{b}_\zeta}(S_k f)(x,y) = (I-P) S_k \Gamma_{\bar{b}_z \otimes \bar{b}_\zeta} f = (I-P) S_k \vec{\Gamma} f.$$

The operator $\vec{\Gamma}$ is called *the Hankel operator canonically associated to* μ. Theorem A (which can be used since its proof through abstract liftings extends to Hankel operators from the scalar spaces $H^2(\mathbb{T}^2)$ to a vector-valued $H^2(\mathbb{T}^2)^\perp(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space) applied to $\vec{\Gamma}$ yields:

Theorem 5.3. *A measure $\mu \ge 0$ in $\mathbb{D}^2$ is of Carleson type, with constant C, if and only if the canonically associated operator $\vec{\Gamma}$ is bounded with norm $\|\vec{\Gamma}\| = C$, and if and only if $\vec{\Gamma} 1 \in \mathrm{BMOr}(\mathbb{T}^2; L^2(\mathbb{D}^2, \nu))$, with norm $\cong C$.*

Thus the connection between measures satisfying the Carleson imbedding condition, Hankel operators and $\mathrm{BMO}(\mathbb{T})$, is recovered in $\mathbb{T}^2$ in terms of BMOr.

6. Estimates for the Norm of the Hankel Operators of Finite Type

Let us recall some basic properties of the finite-dimensional model subspaces $K_b \subset H^2(\mathbb{T})$, where b is a finite Blaschke product, which include as a special case the properties of the K_z considered in Section 5. For a finite Blaschke product b with simple zeros in $\mathbb{D}$, we again denote by P_b the orthogonal projector from $H^2(\mathbb{T})$ onto $K_b = H^2(\mathbb{T}) \ominus bH^2(\mathbb{T})$, and define the model operator $T_b : K_b \to K_b$ by $T_b := P_b S|K_b$, so that $T_b^* = S^*|K_b$. Similarly, for each $G \in H^\infty(\mathbb{T})$, $G(T_b)$ is defined by $G(T_b)f := P_b G f$.

If ϕ_z is given by (5.2), then, for each $z \in \mathbb{D}$, ϕ_z is an eigenfunction of S^*, and if $z_1, \ldots, z_m \in \mathbb{D}$ are the zeros of b, then $\{\phi_{z_1}, \ldots, \phi_{z_m}\}$ is a basis of K_b composed of eigenfunctions of T_b^*. Similarly, K_b has a basis $\{\psi_{z_1}, \ldots, \psi_{z_m}\}$, of eigenfunctions of T_b, where

$$\psi_z(\xi) = b(\xi - z)^{-1}, \quad T_b \psi_z = z\psi_z. \tag{6.1}$$

Thus T_b and T_b^* are *multiplier operators*, that is, they are given by diagonal finite matrices in the corresponding bases, so that, for each $G \in H^\infty(\mathbb{T})$, the condition

$$\|G(T_b)\| \leq 1$$

is equivalent to the positive definiteness of the associated Pick matrix

$$((1 - G(z_j)\overline{G(z_k)})(1 - z_j \overline{z_k})^{-1})_{j,k=1,\ldots,m}.$$

The Kronecker theorem characterizes the symbols of the Hankel operators $\Gamma : H^2(\mathbb{T}) \to H^2(\mathbb{T})^\perp$ of finite rank n as those of the form $\bar{b}G$, for b a Blaschke product with n factors and $G \in H^\infty(\mathbb{T})$. Since the projector P_b is related to the analytic projector $P : L^2 \to H^2$ by

$$P_b f = b(I - P)\bar{b} f = b\Gamma_{\bar{b}} f, \tag{6.2}$$

we derive the identities

$$|\Gamma_{\bar{b}G} f| = |\Gamma_{\bar{b}} G f| = |P_b G f| = |G(T_b f)|,$$

and thus

$$\|\Gamma_{\bar{b}G}\| = \|G(T_b)\|.$$

This means that, in the circle, *the norm of a Hankel operator of finite rank is equal to the norm of an associated multiplier operator* acting in finite-dimensional K_b, which in turn is determined by a finite Pick matrix.

The same result holds for Hankel operators of finite type in the torus (see Theorem 6.2 below), but the association with the multiplier operators acting in K_b is not so simple. This is due to the fact that here K_b, for b the tensor product of d Blaschke products, is not finite-dimensional but of multiple-finite type. As before, we present here the case $d = 2$.

In $\mathbb{T}^2$, if we restrict ourselves to the case when b_1 and b_2 have the same number of zeros, at $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$, respectively, and when $G = G_1 \otimes G_2$, with $G_1, G_2 \in H^\infty(\mathbb{T})$, we have the following equivalences, in terms of

$$K_{b_1 b_2} = H^2(\mathbb{T}^2) \ominus (b_1 \otimes b_2) H^2(\mathbb{T}^2) \tag{6.3}$$

and

$$K'_{b_1, b_2} = \overline{H^2(\mathbb{T}^2)^\perp} \ominus [(b_1 \otimes b_2) H^2(\mathbb{T}^2) \oplus b_1(H^2_x \cap H^2_{-y}) \oplus b_2(H^2_{-x} \cap H^2_y)] \tag{6.4}$$

subspaces of finite type $(H^2(\mathbb{T}); n, n)$ and $(L^2(\mathbb{T}); n, n)$, respectively, as follows from Theorem C and from Theorem 2 in [CS4] (see Section 4).

Proposition 6.1. *Given $\phi = (\bar{b}_1 \otimes \bar{b}_2)(G_1 \otimes G_2)$, for b_1, b_2, G_1 and G_2 as above, and $K_{b_1 b_2}$ and $K'_{b_1 b_2}$ defined by* (6.3) *and* (6.4)*, the following conditions are equivalent:*

(a) $\|\Gamma_\phi\| \leq 1$, *that is,* $\|\Gamma_\phi f\|_{L^2} \leq \|f\|_{L^2}$, *for all* $f \in H^2(\mathbb{T}^2)$.
(b) $\|\Gamma_\phi | K_{b_1 b_2}\| \leq 1$, *that is,* $\|\Gamma_\phi e\|_{L^2} \leq \|e\|_{L^2}$, *for all* $e \in K_{b_1 b_2}$.
(c) *For all* $e \in K_{b_1 b_2}$ *and* $e' \in K'_{b_1 b_2}$, *the inequality*

$$\left| \iint e \bar{e} \phi \, dx \, dy \right| \leq \|e\|_{L^2} \|e'\|_{L^2}$$

holds.

Proof. (a) $\iff$ (b). Every $f \in H^2(\mathbb{T}^2)$ can be written as an orthogonal sum

$$f(x,y) = b_1(x) b_2(y) h(x,y) + e(x,y), \qquad \text{with } h \in H^2(\mathbb{T}^2) \text{ and } e \in K_{b_1 b_2},$$

and

$$\begin{aligned} \Gamma_\phi (b_1 \otimes b_2) h &= (I - P)(b_1(x) b_2(y) h(x,y) \overline{b_1(x) b_2(y)} G_1(x) G_2(y)) \\ &= (I - P)(h(x,y) G_1(x) G_2(y)) = 0, \end{aligned}$$

since $h G_1 G_2 \in H^2(\mathbb{T}^2)$. Thus, $\Gamma_\phi f = \Gamma_\phi e$, $\forall f \in H^2(\mathbb{T}^2)$ and $\|\Gamma_\phi f\|_2 = \|\Gamma_\phi e\|_2 \leq \|e\|_2 \leq \|f\|_2$, so (b) implies (a). The converse follows from $K_{b_1 b_2} \subset H^2(\mathbb{T}^2)$.

(a) $\iff$ (c). Similar proof, observing the equivalence of (a) with

$$\left| \iint f(x,y) \overline{g(x,y)} \phi(x,y) \, dx \, dy \right| \leq \|f\|_2 \|g\|_2$$

for all $f \in H^2(\mathbb{T}^2)$ and $g \in \overline{H^2(\mathbb{T}^2)^\perp}$, and writing in terms of the decomposition of $H^2(\mathbb{T}^2)^\perp$ in the direct sum of $K'_{b_1 b_2}$ and its orthogonal complement. $\square$

Proposition 6.1 says that $\|\Gamma_\phi\| = \|\Gamma_\phi | K_{b_1 b_2}\|$, and we will prove that $\|\Gamma_\phi | K_{b_1 b_2}\|$ coincides with the norm of a multiplier operator in $K_{b_1 b_2}$ (see Theorem 6.2 below), thus generalizing the one-dimensional results.

The systems of eigenfunctions $\{\psi_{z_1}, \dots, \psi_{z_n}\}$ and $\{\psi_{w_1}, \dots, \psi_{w_n}\}$, where $\{z_1, \dots, z_n\}$ and $\{w_1, \dots, w_n\}$ are the zeros of b_1 and b_2, are bases for K_{b_1} and K_{b_2}, respectively. Through Theorem C of Section 4, this allows to write the elements $e \in K_{b_1 b_2}$ as

$$e(x,y) = \sum_{i=1}^{n} A_i(y)\psi_{z_i}(x) + \sum_{j=1}^{n} B_j(x)\psi_{w_j}(y) \tag{6.5}$$

where, for $i, j = 1, \dots, n$, we have A_i, $B_j \in H^2(\mathbb{T})$. In what follows we write, for simplicity,

$$\psi_{z_i}(x) = \xi_i(x) \text{ and } \psi_{w_j}(y) = \eta_j(y) \quad \text{for } i, j = 1, \dots, n,$$

and remark that the $\xi_i' s$ are eigenfunctions of the model operator T_{b_1}, for which, by (6.1), $T_{b_1}\xi_i = z_i\xi_i$, and the $\eta_j' s$ are eigenfunctions of T_{b_2}, with $T_{b_2}\eta_j = w_j\eta_j$. For their part, each $A_i(y)$, $B_j(x)$ can be written as

$$\begin{aligned} A_i(y) &= b_2(y)h_i''(y) + \sum_k c_{ik}\eta_k(y), \\ B_j(x) &= b_1(x)h_j'(x) + \sum_l d_{jl}\xi e(x) \end{aligned} \tag{6.6}$$

with $h_i'', h_j' \in H^2(\mathbb{T})$, for $i, j = 1, \dots, n$. Moreover

$$\Gamma_\phi e = (I - P)\phi e = (P_{-y} + P_{-x}P_y)\phi e = (P_{-x} + P_x P_{-y})\phi e,$$

and, by (6.2) and the definitions of $\mathbb{T}_b$ and $G(\mathbb{T}_b)$, we get

$$\begin{aligned} &P_{-x}(\bar{b}_1 G_1 \xi_i)(x) = \bar{b}_1 T_{b_1} G_1 \xi_i(x) = \bar{b}_1 G_1(z_i)\xi_i(x), \\ &P_{-y}(\bar{b}_2 G_2 \eta_j)(y) = \bar{b}_2 G_2(w_j)\eta_j(y). \end{aligned}$$

Since $P_x \bar{b}_1 G_1 \xi_i = \bar{b}_1 G_1 \xi_i - P_{-x}\bar{b}_1 G_1 \xi_i$, we have

$$\begin{aligned} &P_x \overline{b_1(x)} G_1(x)\xi_i(x) = \overline{b_1(x)}(G_1(x) - G_1(z_i))\xi_i(x), \\ &P_y \overline{b_2(y)} G_2(y)\eta_j(y) = \overline{b_2(y)}(G_2(y) - G_2(w_j))\eta_j(y). \end{aligned}$$

Now for every $e \in K_{b_1 b_2}$, we can write $\Gamma_\phi e$ in terms of functions expressible by b_1, b_2, G_1 and G_2. By (6.5) and (6.6), every $e \in K_{b_1 b_1}$ has the expression

$$e(x,y) =_i \sum_{i,j=1}^{n} c_{ij}\xi_i(x)\eta_j(y) + \sum_{i=1}^{n} b_2(y)h_i''(y)\xi_i(x) + \sum_{j=1}^{n} b_1(x)h_j'(x)_j(y), \tag{6.7}$$

where, for $i, j = 1, \ldots, n$, we have $c_{ij} \in \mathbb{C}$ and $h'_j, h''_j \in H^2(\mathbb{T})$ are one-variable functions. From all the above and the fact that $e \in K_{b_1 b_2}$, we have

(6.8)
$$\Gamma_\phi e = \overline{b_1(x) b_2(x)} \Big(\sum_{i,j=1}^{n} \big(G_1(x)G_2(w_j) + G_1(z_i)G_2(y) - G_1(z_i)G_2(w_j) \big) \cdot \xi_i(x)\eta_j(y)c_{ij} + \sum_{i=1}^{n} G_1(z_i)G_2(y)\xi_i(x)b_2(y)h''_i(y) + \sum_{j=1}^{n} G_1(x)G_2(w_j)b_1(x)\eta_j(y)h'_j(x) \Big).$$

Remark. Expressions (6.7)–(6.8) allow us to check that the three equivalent conditions of Proposition 6.1 are also equivalent to the positive-definiteness of a finite Pick matrix, defined in terms of b_1, b_2, G_1 and G_2, whose elements are bounded operators acting in $H^2(\mathbb{T})$, $L^2(\mathbb{T})$ or from $H^2(\mathbb{T})$ to $L^2(\mathbb{T})$.

Expression (6.7) shows that $K_{b_1 b_2} = K^0 \oplus K^1 \oplus K^2$, where K^0 is the direct sum of the n^2 one-dimensional spaces $\mathbb{C}\xi_i(x)\eta_j(y)$, K^1 is the direct sum of the n subspaces $b_2(y)\xi_i(x)H^2(\mathbb{T})$, and K^2 is the direct sum of the n subspaces $b_1(x)\eta_j(y)H^2(\mathbb{T})$.

Whenever $F^1, F^2, F^3, F^4 \in H^\infty(\mathbb{T})$, it is clear that, for $i, j = 1, \ldots, n$, we have
$$(F^1(x) + F^2(y))\xi_i(x)\eta_j(y) \in K_{b_1 b_2},$$
$$F^3(y)\xi_i(x)b_2(y)H^2(\mathbb{T}) \subset K_{b_1 b_2},$$
$$F^4(x)\eta_j(y)b_1(x)H^2(\mathbb{T}) \subset K_{b_1 b_2}.$$

Accordingly, we say that an operator $T : K_{b_1 b_2} \to K_{b_1 b_2}$ is a multiplier in $K_{b_1 b_2}$ if, for $i, j = 1, \ldots, n$,

$$T\xi_i(x)\eta_j(y) = (F^1_{ij}(x) + F^2_{ij}(y))\xi_i(x)\eta_j(y), \tag{6.9}$$
$$T\xi_i(x)b_2(y)h''(y) = F^3_i(y)\xi_i(x)b_2(y)h''(y) \quad \text{for } h'' \in H^\infty(\mathbb{T}), \tag{6.10}$$
$$T\eta_j(y)b_1(x)h'(x) = F^4_j(x)\xi_1(x)\eta_j(y)h'(x) \quad \text{for } h' \in H^\infty(\mathbb{T}). \tag{6.11}$$

The development above implies the following result:

Theorem 6.2. *Given two one-dimensional Blaschke products b_1 and b_2, with simple zeros at $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$, respectively, and given $G_1, G_2 \in H^\infty(\mathbb{T})$, let $\phi = (\bar{b}_1 \otimes \bar{b}_2)(G_1 \otimes G_2)$. If Γ_ϕ is the Hankel operator defined by symbol ϕ, then $\|\Gamma^\phi\| = \|\Gamma_\phi\|$, where Γ^ϕ is the multiplier in $K_{b_1 b_2}$ (in the sense of* (6.9)–(6.11)*) defined by*
$$F^1_{ij}(x) + F^2_{ij}(y) = G_1(x)G_2(w_j) + G_1(z_i)G_2(y) - G_1(z_i)G_2(w_j),$$
$$F^3_i(y) = G_1(z_i)G_2(y)$$
$$F^4_j(x) = G_1(x)G_2(w_j)$$
for $i, j = 1, \ldots, n$.

Theorem 6.2 has a valid formulation in $\mathbb{T}^d$, for $d \geq 1$. For $d = 1$ it reduces to the Pick formula.

Theorem 6.2 allows us to write the boundedness condition $\|\Gamma_\phi\| \leq 1$ as a formula of Pick matrix type, but more complicated than in the one-dimensional case, and we will not go into the details here. Still, remark that the verification of boundedness of the norm of Γ^ϕ is not as involved as that for the restriction of Γ_ϕ to the model subspace $K_{b_1 b_2}$ (condition (b) of Proposition 6.1), since it is done through the defining properties (6.9)–(6.11) of multipliers.

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