REGULARITY OF HOLOMORPHIC CORRESPONDENCES AND APPLICATIONS TO THE MAPPING PROBLEMS

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Abstract. We study the regularity results of holomorphic correspondences. As an application, we combine it with certain recently developed methods to obtain the extension theorem for proper holomorphic mappings between domains with real analytic boundaries in the complex 2-space.

Introduction

This paper is concerned with the boundary regularity problem for holomorphic mappings between domains in complex euclidean spaces. Our main purpose is to present certain results related to the following mapping conjecture in several complex variables:

Conjecture 0.1: Let M_1 and M_2 be two real analytic hypersurfaces of finite D-type in $\mathbf{C^n}$ with $n > 1$. Suppose that $D \subset \mathbf{C^n}$ is a domain with M_1 as part of its real analytic boundary. Also, suppose that f is a holomorphic mapping from D into $\mathbf{C^n}$, that is continuous up to $D \cup M_1$ and maps M_1 into M_2. Then f admits a holomorphic extension across M_1.

Here, we recall that a real analytic hypersurface is of finite D-type [Da] if and only if it does not contain any non-trivial complex analytic variety.

The extensive study of such kinds of problems was initiated by the foundational work of Fefferman in the case of biholomorphic mappings between strongly pseudoconvex domains. Recent related papers include those by Lewy [Le], Pinchuk [Pi1], Webster [We], Diederich- Webster [DW], Bell [Be1] [Be3], Baouendi-Jacobowitz-Treves [BJT], Bedford-Bell [BB], Baouendi - Bell- Rothschild [BBR], Baouendi- Rothschild [BR1] [BR2] [BR3] [BR4], Bell-Catlin [BC1] [BC2], Diederich-Fornaess [DF1] [DF2] [DF3] [DF4], Pinchuk-Tsyganov [PT], Diederich- Fornaess- Ye [DFY], Baouendi-Huang- Rothschild [BHR1] [BHR2], Huang - Pan [HP], Pan [Pan], and the references therein. For detailed surveys of this investigation, we refer the reader to the articles by Forstneric [For], Bedford [Bed], and Bell-Narashimhan [BN].

When M_1 and M_2 are pseudoconvex and when f is proper, a powerful method is to use Condition R introduced by Bell-Ligocka (see [Be2], for example) and the

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solutions to the $\overline{\partial}$-Neumann problems to achieve the smooth extension. Then one applies various versions of the reflection principle to obtain the holomorphic extension ([BJT] [BBR] [BR1] [DF1]). However, because of an example by Barrett [Ba1], it is known that a different approach needs to be employed when the assumption of pseudoconvexity is dropped.

In [We], Webster used a class of invariant varieties, introduced by Segre [Se], to analyze biholomorphisms between non-degenerate algebraic hypersurfaces. Later, it became clear that these invariant objects called Segre surfaces may also be useful to study the above problem even in the non-pseudoconvex case (for example, see [DW], [DF2], [DFY]).

The plausible method based of the use of Segre surfaces might be carried out in two steps. First one proves the holomorphic extension under the assumption that the map extends as a holomorphic correspondence (see the following section for a precise definition of this terminology). Then, one reduces the general situation to the above case. A result along these lines was obtained in [DF2], where it was shown that a biholomorphism between two real analytic domains in $\mathbf{C}^2$, that extends as a correspondence, admits a holomorphic extension up to the boundary. (See also related work in [BB], [Art], [BBR], [DF1] when the map is smooth). This, in particular, implies that any biholomorphic map between two bounded algebraic domains in $\mathbf{C}^2$ extends holomorphically across the boundary [DF2]. The idea used in [DF2] depends strongly on a theorem of Barrett [Ba2], and does not seem adaptable to the proper mapping case.

In this paper, we will establish the smoothness results for holomorphic correspondences coming from general holomorphic mappings (Theorem 1.1$'$). As an immediate application, we obtain a solution of Conjecture 0.1 in case $n = 2$, M_1 and M_2 are alegbraic, and f is proper. One of the other main results indicates that any continuous CR mapping between two variety-free real algebraic hypersurfaces in $\mathbf{C}^2$ is real analytic. This, in particular, gives a complete solution to Conjecture 0.1 in the 2-dimensional algebraic case. Previously, Conjecture 0.1 was only known to hold in case hypersurfaces are strongly pseudoconvex. Our result seems to give the first Schwarz reflection principle for *continuous CR mappings* between a large class of hypersurfaces which may not be pseudoconvex, though much more has been well understood when the map is smooth and proper (see the papers mentioned above). Indeed, it is a wide open question to answer whether Conjecture 1.1 holds when the hypersurfaces are pseudoconvex of finite type ([BC]), or when f is smooth up to $D \cup M_1$ but the hypersurfaces may not be pseudoconvex ([BR2]).

Recently, based on the aforementioned result in [DF2] for biholomorphisms and the machinery developed in [We], Diederich -Pinchuk ([DP]) announced a proof that any biholomorphic map between bounded real analytic domains in $\mathbf{C}^2$ extends as a holomorphic correspondence across the boundary (see also closely related work in [DFY]). This together with the result in [DF2] implies the holomorphic extension for biholomorphisms between real analytic domains in $\mathbf{C}^2$. With the main result in the present paper as our major tool (Theorem 1.1 or Theorem 1.1$'$), and with the help of ideas the author learned from [DFY] and [DP], we will obtain, in the the last section of this paper, a solution of Conjecture 0.1 in case $n = 2$ and f is proper. Indeed, once Theorem 1.1 is known, the rest of the argument of the proof of Theorem 1.3 is a modification of the known methods and ideas for biholomorphic maps as appeared in [DFY], and, in particular, [DP]. However, since [DP] may not be presently available, we give a certain detailed discussion on this matter (§7) for completeness of the proof of Theorem 1.3.

Our method of the proof of Theorem 1.1, which is different from that in [DF2], can be described as follows: We first show that an attached reflection function (see, in particular, [BJT], [BBR]) extends holomorphically across the hypersurface. Here the main idea is to use the weak version of the edge of the wedge theorem (see Proposition 7.1 of [BHR1] or [Hu]). Using this reflection function, we will connect the branching locus of the map with the branching locus of the Segre varieties of the target hypersurface. This then gives us many restrictions on the branching behaviors of the map. Finally the above mentioned results and several technical lemmas (in particular, a modified version of the Baouendi-Rothschild Hopf lemma and a preservation principle) give the proof of Theorem 1.1. The proof of Theorem 1.2 is an easy corollary of Theorem 1.1 with the known results. Once one has Theorem 1.1 as the major tool, the proof of Theorem 1.3 can be achieved with the help of the existing ideas (see [DF1] [DFY] and, in particular, [DP]).

We would like to mention that certain notations in this paper (especially, those in §7) are adapted from [DFY] and, in particular, [DP].

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Added in revision: This is the revised version of the author's previous paper with the same title, in which Theorem 1.1 was proved only in case $n = 1$ or M_1 and M_2 are decoupled. Results related to Theorem 1.1$'$ were also announced by Diederich-Pinchuk in the workshop on Analytic and PDE methods in SCVs at MSRI (November, 1995).

1. Statement of main results and related observations

To state our main results, we need the following preliminary notation (see [BB], [DF2], and, in particular, [DFY], [DP]):

Let M_1 and M_2 be two real analytic hypersurfaces of finite D-type in $\mathbf{C^{n+1}}$ $(n \geq 1)$, and let p be a point in M_1. A CR mapping f from M_1 into M_2 is said to extend as a *holomorphic correspondence* to a neighborhood of p if (a) f is continuous near p; (b) for each component f_j of f, there is a polynomial $\mathcal{P}_j(z;X)$ in X with leading coefficient 1 and other coefficients holomorphic in a small neighborhood U_1 of p, such that $\mathcal{P}_j(z; f_j(z)) = 0$ for $z \in U_1 \cap M_1$. It can be shown that f extends as a holomorphic correspondence at p if and only if there are a small neighborhood U_1 of p, a neighborhood U_2 of M_2, and an analytic sub-variety V of $U_1 \times U_2$ such that (a) $V \supset \Gamma_f = \{(z,w) : w = f(z),\ z \in M_1 \cap U_1\}$ (b) the natural projection of V to U_1 is locally proper.

We also recall that a real analytic hypersurface $M \subset \mathbf{C^{n+1}}$ is called *rigid* if, after a holomorphic change of coordinates, it can be defined by a real analytic function of the form $\rho(z,\overline{z}) = z_{n+1} + \overline{z_{n+1}} + \chi(z', \overline{z'})$, where $z' = (z_1, \cdots, z_n)$.

Theorem 1.1: Let M_1 and M_2 be two real analytic hypersurfaces of finite D-type in $\mathbf{C^{n+1}}$ $(n \geq 1)$. Let $p \in M_1$. Suppose that f is a CR mapping from M_1 into M_2 that extends as a holomorphic correspondence to a neighborhood of p. Then f admits a holomorphic extension near p, if either $n = 1$ or M_1 and M_2 are rigid.

See also Theorem $1.1'$ of Section 5 for a more general version of this result and notice that in whole paper, only in §5, what we call Condition S is used.

Recall that a smooth hypersurface is called algebraic, if it can be defined by a real polynomial. Using certain known results in [We] and [DF2], one can apply Theorem 1.1 to obtain the following:

Theorem 1.2: Let M_1 and M_2 be two algebraic hypersurfaces of finite type in $\mathbf{C^2}$. Let f be a continuous CR mapping from M_1 to M_2. Then f is real analytic on M_1.

Corollary 1.2$'$: Let D and D' be two smoothly bounded domains with real algebraic boundaries in $\mathbf{C^2}$. Let f be a proper holomorphic mapping from D to D'. Then f admits a holomorphic extension to a neighborhood of $\overline{D}$.

We remark that Theorem 1.2 is sharp in a certain sense. In fact, using the inner functions defined on the ball in $\mathbf{C^2}$, one can construct a bounded CR mapping between spheres in $\mathbf{C^2}$ which is even not continuous at any point. It is worthwhile to mention that the map in Theorem 1.2 is not assumed to be the boundary value of some globally defined proper holomorphic map. As is known, it is always a difficult problem to understand when a CR mapping comes from a globally defined proper holomorphic map. Also, we notice that when the map in Theorem 1.2 is assumed to be in the smoothness class C^k with $k >> 1$, then much less restriction to the hypersurfaces is required to obtain the real analyticity ([BHR1]).

Using the two dimensional homeomorphic version of Theorem 1.1 (which has been established in [DF2]), Diederich-Pinchuk recently announced that any biholomorphic map between real analytic bounded domains in $\mathbf{C^2}$ admits a correspondence extension across the boundary [DP] (see also the related work in [DFY]). With Theorem 1.1 at our disposal and with the help of an idea of Diederich-Pinchuk [DP] on the application of the Bishop extension lemma, we will prove in §7 the following result (see [BBR] for the pseudoconvex case, and Theorem 1.3$''$ in §7 for a local version of this result).

Theorem 1.3: Let D and D' be two bounded domains in $\mathbf{C^2}$ with real analytic boundaries. Suppose that f is a proper holomorphic map from D to D', which extends continuously across ∂D. Then f extends holomorphically across ∂D.

In higher dimensions, we will be content in this paper with the following application of Theorem 1.1 (see Theorem 1.1 $'$ for a more general result on this):

Theorem 1.4: Let D and D' be two smoothly bounded algebraic domains in $\mathbf{C^n}$ and let f be a proper holomorphic map from D to D'. If D and D' are rigid then f admits a holomorphic extension to a neighborhood of $\overline{D}$.

The organization of the remainder of this paper is as follows: §2 through §5 is devoted to the proof of Theorem 1.1. In §2, we will study the holomorphic extendibility of a λ-function associated with the map and the target surface. This function was used in various different forms in [Le], [Pi], [BJT], [BBR], and [BR1] to study smooth CR-mappings (see, in particular, the work in [BBR] and [BR1]). In §3, we will present the connections between the branching properties of the map and the invariant varieties (Segre-surfaces). In §4, we will prove a localized Hopf lemma by modifying an argument that appeared in [BR2], which will then be used to prove a preservation principle for our (multiple-valued) map. Section 5 is devoted to the proof of Theorem 1.1, using results established in the previous sections. In §6, we will complete the proofs of Theorem 1.2 and Theorem 1.4. With Theorem 1.1 at our disposal and using some known ideas (in particular, those appeared in [DP]), in the last section (§7), we will present a proof of Theorem 1.3.

2. Basic set-ups and holomorphic extendibility of a reflection function

In this section, we first set up some notation which will be used throughout the paper. Then we study the holomorphic extendibility of a reflection function. We will use some notations, which the author adapts from [DP].

To start with, we let $M_1 \subset \mathbf{C^{n+1}}$, $M_2 \subset \mathbf{C^{n+1}}$ be real analytic hypersurfaces of finite D-type, and let f be a *non-constant* CR mapping from M_1 to M_2. Assume that f extends as a holomorphic correspondence to a neighborhood of $p(\in M_1)$. Our main objective from §2 to §5 is to understand under what circumstances f admits a holomorphic extension near p. Since this is purely a local problem, after holomorphic changes of coordinates and shrinking the size of the hypersurfaces, we can assume that $p = f(p) = 0$; M_1 , M_2 are connected and defined, respectively, by the functions of the following form (see [BJT], for example):

$$\rho_1(z,\overline{z}) = z_{n+1} + \overline{z_{n+1}} + \sum_{j=0}^{\infty} \phi_j(z',\overline{z'}) \left(\frac{z_{n+1}-\overline{z_{n+1}}}{2i}\right)^j, \tag{2.1}$$

$$\rho_2(z,\overline{z}) = z_{n+1} + \overline{z_{n+1}} + \sum_{j=0}^{\infty} \psi_j(z',\overline{z'}) \left(\frac{z_{n+1}-\overline{z_{n+1}}}{2i}\right)^j, \tag{2.2}$$

with $\phi_j(z',0) = \phi_j(0,\overline{z'}) = \psi_j(0,\overline{z'}) = \psi_j(z',0) \equiv 0$ and ϕ_j,ψ_j holomorphic in $z',\overline{z'}$ over certain fixed open neighborhood of $(0,0)$. Here we use $z = (z', z_{n+1})$ with $z' = (z_1,\cdots,z_n)$ for the coordinates in $\mathbf{C^{n+1}}$.

Applying a CR extension result in [BT] or [Tr] and making M_1 sufficiently small, we can further assume that there is a domain D which contains M_1 as part of its smooth boundary such that every CR function defined over M_1 can be extended holomorphically to D. Hence, without loss of generality, we can assume, in all that follows, that f is holomorphic in D and continuous up to $D \cup M_1$.

Applying the implicit function theorem to (2.1), (2.2) and shrinking the size of the hypersurfaces further (if necessary), we see that M_1 and M_2 can also be presented, respectively, by functions of the following form:

$$\widetilde{\rho_1}(z,\overline{z}) = z_{n+1} + \overline{z_{n+1}} + \sum_{j=0}^{\infty} \widetilde{\phi_j}(z',\overline{z'}) z_{n+1}^j = 0, \tag{2.1$'$}$$

$$\widetilde{\rho_2}(w,\overline{w}) = z_{n+1} + \overline{z_{n+1}} + \sum_{j=0}^{\infty} \widetilde{\psi_j}(z',\overline{z'}) z_{n+1}^j = 0, \tag{2.2$'$}$$

where $\widetilde{\phi_j}$ are determined by ϕ_j's, $\widetilde{\phi_j}(z',0) = \widetilde{\phi_j}(0,\overline{z}') = \widetilde{\psi_j}(z',0) = \widetilde{\psi_j}(0,\overline{z'}) = 0$; and

$$\widetilde{\rho_j} = \rho_j h_j. \tag{2.3}$$

Here h_j is a real analytic function near M_1 for $j = 1, 2$ and $h_j(0,0) = 1$. Complexifying (2.3), we then obtain

$$\widetilde{\rho_j}(z,\overline{w}) = \rho_j(z,\overline{w}) h_j(z,\overline{w}). \tag{2.3$'$}$$

Therefore, for z, w close to 0, $\rho_j(z,\overline{w}) = 0$ if and only if $\widetilde{\rho_j}(z,\overline{w}) = 0$.

We now recall the definition of Segre varieties and some related notions, which were first used in [We] (see also [DW] and [DF1]) for the study of the mapping problems.

Since M_2 is completely symmetric to M_1, we will only introduce notations for M_1 and add $'$ for those corresponding to M_2.

We first choose a small neighborhood $\mathcal{U}_3$ of 0 such that $\rho_1(z,w)$, $\widetilde{\rho_1}(z,w)$, $h_1(z,w)$ are holomorphic over $\mathcal{U}_3 \times \mathrm{conj}(\mathcal{U}_3)$, and $h_1(z,w) \neq 0$ in $\mathcal{U}_3 \times \mathrm{conj}(\mathcal{U}_3)$. Here and in what follows, for a subset A in the complex spaces we write $\mathrm{conj}(A) = \{\overline{z} : z \in A\}$. For each point $w \in \mathcal{U}_3$, the Segre variety associated to M_1 and w is defined by $Q_w = \{z \in \mathcal{U}_3 : \rho_1(z, \overline{w}) = 0\}$. By making $\mathcal{U}_3$ small, it can be seen that Q_w is actually a complex submanifold of $\mathcal{U}_3$. It is easy to show that there are two small neighborhoods $\mathcal{U}_2$, $\mathcal{U}_1$ of 0 such that the following holds (see [DF1] for more details on this matter):

(I.a) $\mathcal{U}_3 \supset\supset \mathcal{U}_2 \supset\supset \mathcal{U}_1 \ni 0$.

(I.b) $\mathcal{U}_1 \setminus M_1$ has two connected components, which are in different sides of M_1. Moreover, they are homeomorphic to the ball.

(I.c) For each $w \in \mathcal{U}_1$, $\mathcal{U}_2 \cap Q_w$ is connected.

(I.d) For a sequence $\{w_j\} \subset \mathcal{U}_1$ with $w_j \to w \in \mathcal{U}_1$, the limit set of $\{Q_{w_j}\}$, denoted by $\lim Q_{w_j}$, is Q_w.

In what follows, we call $\{\mathcal{U}_3, \mathcal{U}_2, \mathcal{U}_1\}$ a Segre neighborhood system around 0. It is clear that for any open set containing 0, we can always construct inside this set a Segre neighborhood system around 0. Meanwhile, when $\mathcal{U}_3$ ($\mathcal{U}_2$, respectively) is fixed, we can arrange $\mathcal{U}_2$ ($\mathcal{U}_1$, respectively) as small as we need.

For convenience of the reader, we indicate here how (I.c) and (I.d) can be achieved (much more details for related discussion can be found in work of [DW] and [DF1]).

In the coordinates system where M_1 is defined by (2.1), we can choose a small polydisc $P = P(\delta_1, \cdots, \delta_{n+1}) = \{z : |z_j| < \delta_j(\ j = 1, \cdots, n+1)\} (\subset\subset \mathcal{U}_3)$. Then, applying the implicit function theorem, we can see that for each w with $|w|$ small, $Q_w \cap P$ is contained in the graph of certain function , say $z_{n+1} = g(z', \overline{w})$, over the polydisc $P' = P'(\delta_1, \cdots, \delta_n) = \{z' : |z_j| < \delta_j,\ j = 1, \cdots, n\}$. Here $g(z', \overline{w})$ is holomorphic for $z, w \in \mathcal{U}_3$ and $g(z', 0) = 0$. Now, if we make w close enough to the origin, then $Q_w \cap P$ will be exactly the graph of $g(z', \overline{w})$ over P'. Since it is biholomorphic to P', it is of course connected. Hence, we can simply take $\mathcal{U}_2$ to be the polydisc P chosen in such a manner. Once $\mathcal{U}_2$ is chosen, it is then clear that we can simply take $\mathcal{U}_1$ to be a sufficiently small ball around 0. Next, if $w_j \to w \in \mathcal{U}_1$, then we clearly see that $Q_{w_j} \to Q_w$; for $g(z', \overline{w_j}) \to g(z', \overline{w})$. This gives (I.d).

We next set up the notation for a reflection mapping $\mathcal{R}$.

By examining the proof of Lemma 1.1 of [BJT], it is clear that for each point $p \in M_1$ close to 0, there exists a biholomorphic mapping $\Phi(\cdot; p)$, which depends real analytically on p, such that the following holds:

(i) $\Phi(\cdot; 0) = \mathrm{id}$ and $\Phi(p; p) = 0$.

(ii) $\Phi(\cdot; p)$ (and thus its inverse) is well-defined over some fixed open neighborhood of 0;

(iii) $\Phi(M_1;p)$ is defined near 0 by the following equation:

$$\rho_1(z,\overline{z};p) = z_{n+1} + \overline{z_{n+1}} + \sum_{j=0}^{\infty} \phi_j(z',\overline{z'};p)\left(\frac{z_{n+1}-\overline{z_{n+1}}}{2i}\right)^j,$$

with $\phi_j(0,\overline{z'};p) = \phi_j(z',0;p) \equiv 0$ and ϕ_j holomorphic in $z',\overline{z'}$ over certain fixed open neighborhood of $(0,0)$.

For a small positive ϵ_0, define Φ: $(-\epsilon_0,\epsilon_0)\times M_1 \to \mathbf{C^{n+1}}$ by $\Phi(t,p) = \Phi^{-1}((0',t);p)$. Using the facts that $\Phi(0,p) = p$ and $\Phi(t,0) = (0',t)$, it can be seen that Φ is an analytic diffeomorphism near $(0,0)$.

We now define the reflection mapping $\mathcal{R}$ as follows: For z close to 0, if $\Phi^{-1}(z) = (t,p)$, then $\mathcal{R}(z) = \Phi(-t,p)$. Clearly, $\mathcal{R}$ is a diffeomorphism near the origin. By the way it was constructed, one can verify that $\mathcal{R}|_{M_1\cap O(0)} = \mathrm{id}$, $\mathcal{R}^2(z) = z$ for z close to 0, $\mathcal{R}(D\cap O(0)) \subset D^c$, $\mathcal{R}(D^c\cap O(0)) \subset D$. Here $D^c = U\setminus(D\cup M_1)$ with U a neighborhood of M_1; and in all that follows we write $O(a)$ for a *small* neighborhood of a, whose size may be different in different contexts.

We observe that for a real number t with $|t|$ small, the Segre variety of $\Phi(M_1;p)$ at $(0,t)$ is an open subset of the affine space $\{(z',-t)\}$ near $(0,-t)$. Using the invariant property of Segre varieties under biholomorphic mappings ([We] [DW]) and by arranging the size of $\{\mathcal{U}_3,\mathcal{U}_2,\mathcal{U}_1\}$ suitably, one can see that $\mathcal{R}(z)\in Q_z$ for each $z\in\mathcal{U}_2$.

Since we assumed that f extends as a correspondence to a neighborhood of 0, after shrinking M_1 if necessary, we can assume that for each component f_j of f, there is an *irreducible* Weierstrass polynomial

$$\mathcal{P}_j(z;X) = X^{N_j} + \sum_{0\le l<N_j} a_{j,l}(z)X^l \tag{2.4}$$

with $a_{j,l}$ holomorphic over a neighborhood, say U, of M_1 such that $\mathcal{P}_j(z;f_j(z)) = 0$ for $z\in M_1$. Since f is known holomorphic on D, $\mathcal{P}_j(z;f_j(z)) = 0$ for $z\in D\cup M_1$. In all that follows, we always make (a) $U \supset\supset \mathcal{U}_3\cup D$; (b) $D\supset\supset \mathcal{U}_3\cap D$; (c) $\mathcal{R}$ is a diffeomorphism from $\mathcal{U}_3$; and (d) $\mathcal{R}(D\cap\mathcal{U}_1)\subset D^c\cap\mathcal{U}_2$, $\mathcal{R}(D^c\cap\mathcal{U}_1)\subset D\cap\mathcal{U}_2$.

As usual, we define the branching locus $\mathcal{E}$ of f to be the union of the branching locus of f_j $(j=1,\cdots,n+1)$. That is, a point $z\in U$ is in $\mathcal{E}$ if and only if for some index j and number X, one has $\mathcal{P}_j(z;X) = \frac{\partial\mathcal{P}_j}{\partial X}(z;X) = 0$. Write $E_{M_1} = \mathcal{E}\cap M_1$. Then by the finite type assumption of M_1, one can see that E_{M_1} is a real analytic subset of M_1 with codimension ≥ 2. Obviously, f admits a holomorphic extension across $M_1\setminus E_{M_1}$.

We now introduce the following λ-function associated to the map f and M_2 (see [Le], [Pi1], [BJT], and, in particular, [BBR], for closely related notions):

$$G(f(z),\lambda) = -f_{n+1}(z) - \sum_{j=0}^{\infty}\widetilde{\psi_j}(f^*(z),\lambda)f_{n+1}^j(z). \quad z\in M_1\cup D \tag{2.5}$$

Here $\widetilde{\psi_j}$ are the same as in $(2.2)'$ and we write $f^* = (f_1, \cdots, f_n)$. Then it is clear that $G(f(z), \lambda)$ is holomorphic over $D \times O_\lambda(0)$ and continuous on $(D \cup M_1) \times O_\lambda(0)$. Here and in all that follows, we put a subscript to $O(a)$ to emphasize the coordinates used for the space where $O(a)$ stays. For example, $O_b(a)$ denotes a small open neighborhood of a in the complex b-space, whose size may be different in different contexts.

Our first lemma in this paper is to extend this function across M_1 for small λ.

Lemma 2.1: Under the above notations and assumptions, $G(f(z), \lambda)$ extends holomorphically to $O_z \times O_\lambda$.

The proof we present here is to take the differentiation along the boundary, as did in [Le], [Pi], [BJT], [BBR], [BR1], [Hu], and [BHR1] (in particular, see [BJT], [BBR], and §2.4.2 of [Hu] or Proposition 7.1 of [BHR1]). However, there is an essential difference here. That is, our map is not assumed to be smooth. So, we can only do it almost everywhere. To reach the bad points, we jump into the domain and use the hypothesis to control the rate of blowing-up.

We first observe that when $D \cup M_1$ is pseudo-concave along M_1, then clearly f extends holomorphically across M_1. So, without loss of generality, we will always assume the existence of a non empty pseudoconvex piece in M_1. Furthermore, using the finite type assumption for M_1, we can then see the existence of strongly pseudoconvex points in M_1 (see, for example, [BHR1]). Since it is now clear that strongly pseudoconvex points form a non empty open subset of M_1 and since f extends holomorphically across $M_1 \setminus E_{M_1}$, by our non constant assumption of f and by a result in [BR3], the Jacobian J_f of f is well-defined and non zero on an open dense subset in M_1.

Proof of Lemma 2.1: By the definition of $G(f(z), \lambda)$ and using the assumption that $f(M_1) \subset M_2$, we have

$$\overline{f_{n+1}(z)} = G(f(z), \overline{f^*(z)}) \quad \text{for } z \in M_1. \tag{2.6}$$

Shrinking the size of M_1 if necessary, we can choose a basis $\{\mathcal{L}\}_{j=1}^n$ of $\mathrm{T}^{(1,0)}M_1$, whose coefficients are real analytic in z. As for the smooth mappings case (see [BJT], [BR1]), we apply $\overline{\mathcal{L}}_j$ to (2.6), to obtain

$$\overline{\mathcal{L}}_j \overline{f}_{n+1}(z) = \sum_{l=1}^n \frac{\partial G}{\partial \lambda_l} \overline{\mathcal{L}}_j \overline{f}_l, \quad \text{for } z \notin E_{M_1}.$$

Let $J = \det(\overline{\mathcal{L}}_j \overline{f_l})_{1 \le j,l \le n}$ and let $\mathcal{J}$ be the matrix $(\overline{\mathcal{L}}_j \overline{f_l})_{1 \le j,l \le n}$, which are well-defined on $M_1 \setminus E_{M_1}$. We claim that $J \not\equiv 0$ on $M_1 \setminus E_{M_1}$. To see this, by the above observation, we can assume that there is a strongly pseudoconvex point $p_0 \in M_1$ such that f extends biholomorphically to a neighborhood of p_0 and maps p_0 to a strongly pseudoconvex point q_0 of M_2. Now, seeking a contradiction suppose $J(z) \equiv 0$ near

p_0. Let $\mathcal{J}_j(z) = (\mathcal{L}_j f_1, \cdots, \mathcal{L}_j f_n)$. By passing to a nearby point, we can assume, without loss of generality, that the semi-continuous function $d(z)$, which is defined as the complex dimension of the vector space $V(z)$ spanned by $\{\mathcal{J}_j(z)\}$, takes a local maximal value, say k $(< n)$, at p_0. Here z is close to p_0 but stays in M_1. Moreover, we can assume that $\{\mathcal{J}_1(p_0), \cdots, \mathcal{J}_k(p_0)\}$ forms a basis of $V(p_0)$. Now, by the local maximality of $d(z)$ at p_0 and by the real analyticity of $\mathcal{J}_j(z)'s$ with respect to z, it is clear, from some simple arguments in linear algebra, that near p_0 there exist real analytic functions $a_j(z)$ $(j = 1, \cdots, k)$ with $z \in M_1$ sufficiently close to p_0 such that $\mathcal{J}_{k+1}(z) = \sum_{j=1}^{k} a_j(z)\mathcal{J}_j(z)$ for $z (\in M_1) \approx p_0$. Write $T(z) = \mathcal{L}_{k+1}(z) - \sum_{j=1}^{k} a_j(z)\mathcal{L}_j(z)$. Then $T(z)$ annihilates $f^*(z)$ for $z (\in M_1) \approx p_0$. Since $T(p_0) \neq 0$ and M_1 is strongly pseudoconvex at p_0, it follows that $(d\rho_1, [T, \overline{T}])(p_0) \neq 0$, and thus $\{\mathcal{L}_1, \cdots, \mathcal{L}_n, [T, \overline{T}]\}$ forms a basis of $T_{p_0}M_1$. Write $N(z)$ for the $(1, 0)$-component of $[T, \overline{T}](z)$. Noticing that f is holomorphic near p_0 and $[T, \overline{T}]f^* = T((\overline{T}f^*)) - \overline{T}(Tf^*) = 0$, it is clear that $N(p_0)$ annihilates f^*, too. Observe that $\{\mathcal{L}_1, \cdots, \mathcal{L}_n, N\}(p_0)$ forms a basis of $T^{(1,0)}_{p_0}\mathbf{C}^{\mathbf{n+1}}$. Therefore, making use of the fact $N(p_0)(f^*) = 0$ and by a simple linear algebra argument, it follows that $J_f(p_0)$ is given by some constant times $\overline{J(p_0)}$ and thus equals to 0. This contradicts our choice of p_0.

Now, we let $Z^* = \{z \in M_1 \setminus E_{M_1} : \ J(z) = 0\}$. Then, since $M_1 \setminus E_{M_1}$ has to be connected, we see that $M_1 \setminus Z^*$ is a dense open subset in M_1. In fact, with a little more effort, we will next show that Z^* is contained in a real analytic subset of codimension at least 1. Indeed, notice that $J(z) = J(z, \overline{z}, \overline{Df})$ can be written as a polynomial in $Df = ({f_1}'_{z_1}, \cdots, {f_n}'_{z_{n+1}})$ with coefficients holomorphic in $(z, \overline{z})$ for z in a small neighborhood of M_1. Since (2.4) annihilates f_j, one can easily show that each component of Df can be presented as a rational function of f with coefficients holomorphic near M_1. Thus, by some standard algebra arguments involving symmetric functions, one can conclude that for some irreducible polynomial in X: $\mathcal{P}(z, w; X) = \sum_{j=0}^{N^*} A_j(z, w)X^j$, it holds that $\mathcal{P}(z, \overline{z}; J(z, \overline{z}, \overline{Df(z)})) = 0$ for $z \in M_1 \setminus E_{M_1}$. Here $A_j(z, w)$ are holomorphic for z and $\overline{w}$ in a small neighborhood of M_1. Since $A_j(z, \overline{z}) = 0$ on M_1 if and only if $A_j(z, \overline{z}) = \rho_1(z, \overline{z})h_j(z, \overline{z})$ for some real analytic function h_j near M_1, after doing some cancellation, we may assume that not all $A_j(z, \overline{z})$ vanish on M_1. Now, since $J(z) \neq 0$ on a dense open subset, the zero set of $J(z)$ is obviously contained in the zero set of the coefficient $A_{j^*}(z, \overline{z})$, where $A_{j^*}(z, \overline{z}) \not\equiv 0$ but $A_j(z, \overline{z}) \equiv 0$ for $z \in M_1$ and for $j < j^*$. Hence Z^* is contained in the proper real analytic subset of M_1 consisting of E_{M_1} and the zeros of $A_{j^*}(z, \overline{z})$ in M_1.

Away from Z^*, we now have

$$\left(\frac{\partial G}{\partial \lambda_1}, \cdots, \frac{\partial G}{\partial \lambda_n}\right)^t = \mathcal{J}^{-1}(z, \overline{z}, \overline{Df})(\overline{\mathcal{L}}_1\overline{f}_{n+1}, \cdots, \overline{\mathcal{L}}_n\overline{f}_{n+1})^t. \tag{2.7}$$

Writing (2.7) as n scalar equations, applying $\overline{\mathcal{L}}_j$ to each of them, and keeping doing in this manner, we see, by induction, that for each multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$; there are two holomorphic functions $g_\alpha^{(1)}$ and $g_\alpha^{(2)}$ in

$$(z, \overline{z}, \overline{f}, \cdots, \overline{D^\beta f}, \cdots)$$

(where $\beta \leq |\alpha|$) such that for each $z \in M_1 \setminus Z^*$, one has $g_\alpha^{(2)}(z, \overline{z}, \cdots, \overline{D^\beta f}, \cdots) \neq 0$ and

$$D_\lambda^\alpha G(f, \overline{f^*(z)}) = \frac{g_\alpha^{(1)}(z, \overline{z}, \cdots, \overline{D^\beta f}, \cdots)}{g_\alpha^{(2)}(z, \overline{z}, \cdots, \overline{D^\beta f}, \cdots)}.$$

Here D^β denotes the vector formed by all derivatives of f with order β. We remark that $g_\alpha^{(j)}$ is actually a polynomial in $(\overline{Df}, \cdots, \overline{D^\beta f}, \cdots)$ $(j = 1, 2)$ with coefficients real analytic in z for z close to M_1. By passing to the limit, we see that the function $\frac{g_\alpha^{(1)}}{g_\alpha^{(2)}}$ has a continuous extension to M_1, which we will denote by $h_\alpha(z, \overline{z}, \cdots, \overline{D^\beta f}, \cdots,)$. Notice also that for $\|w\|, \|\lambda\| \ll 1$, there exits a large constant R so that $|D_\lambda^\alpha G(w, \lambda)| \lesssim \alpha! R^{|\alpha|}$.

Denote all the solutions of (2.4) by $\eta_{j,l}(z)$ $(j = 1, \cdots, n+1, \ l = 1, \cdots, N_j)$. We now can easily find a totally real submanifold $M' \subset M_1$ of dimension $n+1$ such that $0 \in M'$ and $M' \cap Z^*$ is contained in a real analytic subset of M' with codimension at least 1 (see, for example, [Hu] or [BHR1]). Choose $\mathcal{H}$: $(\mathbf{C^{n+1}}, 0) \to (\mathbf{C^{n+1}}, 0)$, a germ of biholomorphism which sends a small open neighborhood of 0 in $\mathbf{R}^{n+1}$ into M' and maps the standard wedge W^+ into D, W^- into D^c, respectively. Here the notation D^c is as before.

Consider the following equation in X:

$$\prod_{l_1, \cdots, l_{2n+1}} \left(X - \frac{1}{\alpha!} (D_\lambda^\alpha G(\eta_{1,l_1} \circ \mathcal{H}(z), \cdots, \eta_{n+1,l_n} \circ \mathcal{H}(z), \cdots, \overline{\eta_{n,l_{2n+1}} \circ \mathcal{H}(\overline{z})}) \right)$$

$$= X^N + \sum_{j<N} c_j(z) X^j = 0,$$

where l_j and l_{n+1+j} run from 1 to N_j for $j \leq n+1$, and $N = N_1^2 \cdots N_n^2 N_{n+1}$. $c_j(z)'$s can be seen to be the symmetric functions of some Weierstrass polynomial equations and hence can be seen to be holomorphic near 0. Moreover, one can obtain the Cauchy estimates $|c_j(z)| \lesssim R^{N|\alpha|}$. Now, we notice that

$$\widetilde{h}_\alpha(z) = \frac{1}{\alpha!} h_\alpha(\mathcal{H}(z), \overline{\mathcal{H}(\overline{z})}, \cdots, \overline{D^\beta f(\mathcal{H}(\overline{z}))}, \cdots)$$

is a solution of the above equation for almost all $z \in \mathbf{R}^{n+1}$ near 0. Meanwhile, it clearly extends to a meromorphic function to $W^- \cap O_z(0)$; for f is holomorphic over D. Thus, by the uniqueness of holomorphic functions [Pi2], it follows that

$$(\widetilde{h}_\alpha(z))^N + \sum_{j<N} c_j(z) (\widetilde{h}_\alpha(z))^j = 0,$$

for all

$$z \in W^- \cap O_z(0) \setminus \{\text{the singular set of } \widetilde{h_\alpha} \text{ in } W^- \cap O_z(0)\}.$$

In particular, we see that $\widetilde{h}_\alpha(z)$ is bounded. Using the Riemann extension theorem, we conclude that $\widetilde{h_\alpha}$ extends holomorphically to $W^- \cap O_z(0)$. Moreover, we have $|\widetilde{h}_\alpha(z)| \lesssim R^{N|\alpha|}$ for $z \in W^- \cap O_z(0)$; for its coefficients have the same sort of estimates.

Next, fix a small open subset U containing 0. Let

$$\phi_\alpha^-(z) = \frac{1}{\alpha!}\left(D_\lambda^\alpha \sum_\beta \widetilde{h}_\beta(z)\left(\lambda - \overline{f^* \circ \mathcal{H}(\overline{z})}\right)^\beta\right)_{\lambda=0}$$

for $z \in W^- \cap U$ and let $\phi_\alpha^+(z) = \frac{1}{\alpha!}D_\lambda^\alpha G(f \circ \mathcal{H}(z), \lambda)|_{\lambda=0}$ for $z \in W^+ \cap U$. Then it can be seen that $|\phi_\alpha^+(z)|,\ |\phi_\alpha^-(z)| \lesssim R'^{|\alpha|}$ for some large constant R', where z stays in their defining regions, respectively. Notice that ϕ_α^+ matches up with ϕ_α^- over $U \cap \mathbf{R}^{n+1} \setminus \{$ a thin real analytic subset$\}$. The classic edge of the wedge theorem then indicates that ϕ_α^+ extends to a holomorphic function ϕ_α defined over some sufficiently small neighborhood U' of 0, whose size depends only on the size of the wedges where $\phi_\alpha^\pm$ are defined, and therefore is independent of α (see [Vad] or [BHR1], for example). Moreover, U' can be filled in by analytic disks with boundary staying in the closure of $(W^+ \cup W^-)$ ([Val]). So, the maximal principle tells that ϕ_α has the same kind of Cauchy estimates as ϕ_α^+ and ϕ_α^- do. Now, $(\sum_\alpha \phi_\alpha \lambda^\alpha) \circ \mathcal{H}^{-1}$ clearly gives the holomorphic extension for $G(f(z), \lambda)$ to $O_z(0) \times O_\lambda(0)$. The proof of Lemma 2.1 is now complete. □

Remark 2.2: (a) As an immediate application of Lemma 2.1, by letting $\lambda = 0$ in (2.4), we conclude that $f_{n+1}(z)$ admits a holomorphic extension near 0. Thus, $\mathcal{P}_{n+1}(z; X) = X - f_{n+1}(z)$.

(b) From the proof of Lemma 2.1, one can see that the same result holds whenever f extends to one side and $M_1 \setminus \mathcal{E}$ contains a strongly pseudoconvex point where J_f is not 0. In particular, when f extends to one side, this can be applied to the case in which M_1 and M_2 are not Levi-flat in $\mathbf{C}^2$.

(d) In the following sections, we always assume that $G(f(z), \lambda)$ is holomorphic over the closure of $\mathcal{U}_3 \times O_\lambda(0)$

3. Branches of $\mathcal{F}$ and Segre varieties

In this section, we start to study the branching property of the polynomial equations defining f. The argument will be based on the extensive use of Segre varieties and Lemma 2.1. Before proceeding, we set up the following notations:

Let $\mathcal{E}$ still be as before. For each $z \in \mathcal{U}_3$, we write $\mathcal{F}(z)$ for the multiple-valued map formed by extending f from $\mathcal{U}_3 \cap D$. More precisely, when $z \notin \mathcal{E}$, $\widetilde{f}(z) \in \mathcal{F}(z)$ if there exists a path $\gamma \subset \mathcal{U}_3 \backslash \mathcal{E}$ with $\gamma(0) \in D$, $\gamma(1) = z$, and a continuous section $\mathcal{G}$ from $[0,1]$ to the sheaf of germs of holomorphic mappings to $\mathbf{C^{n+1}}$, denoted by $(\mathcal{O}^{n+1}, \pi, \mathbf{C^{n+1}})$, such that $\mathcal{G}(0)$ coincides with the germ of f at $\gamma(0)$ and $g(\gamma(1)) = \widetilde{f}(z)$, where g is a representation of $\mathcal{G}(\gamma(1))$, i.e, the germ of g at $\gamma(1)$ is $\mathcal{G}(\gamma(1))$. When $z \in \mathcal{U}_3 \cap \mathcal{E}$, we let $\mathcal{F}(z)$ be the set of the limit points of all possible sequences $\{\mathcal{F}(z_j)\}$ with $z_j \notin \mathcal{E}$ and $z_j \to z$. For each $z \in \mathcal{U}_2$, we write $\mathcal{F}(Q_z) = \cup_{w \in Q_z \cap \mathcal{U}_2} \mathcal{F}(w)$. We also notice that for each point in $\mathcal{F}(z)$, its $m-$th component is a solution of the equation $\mathcal{P}_m(z;X) = 0$ ((2.4)). From this, it follows that when z is close to 0, then $\mathcal{F}(z)$ is close to the origin, too. Moreover, Lemma 2.1 now indicates that $G(\widetilde{f}(z), \lambda)$ is independent of the choices of $\widetilde{f}(z) \in \mathcal{F}(z)$. So, in what follows, we use the notation $G(\mathcal{F}, \lambda)$ to replace $G(f(z), \lambda)$.

For $w \in \mathcal{U}_2$, we denote by A_w the set $\{\omega \in \mathcal{U}_2 : Q_w \cap \mathcal{U}_2 = Q_\omega \cap \mathcal{U}_2\}$. We mention that A_w is invariantly attached to the hypersurafce M_1, and will play an important role in our proof of Theorem 1.1. By our finite D-type assumption for M_1 and by shrinking $\mathcal{U}_3$, we can make $A_0 = \{0\}$ and A_z finite for $z \in \mathcal{U}_2$ (see [DW], [BJT], and [DF1]). Similarly, by making $\mathcal{U}_3'$ small, we can assume that $A_0' = \{0\}$ and $\#A_w' < \infty$ for $w \in \mathcal{U}_2'$. For more properties of A_w, we refer the reader to ([DF1], Lemma 1).

Before proceeding to the proof of the first lemma in this section, we mention that by shrinking $\mathcal{U}_2$ if necessary, we can assume that $\mathcal{F}(\mathcal{U}_2) \subset \mathcal{U}_1'$. Meanwhile, it is clear that $\mathcal{F}(0) = 0$. We also retain notations which have been set up so far.

Lemma 3.0 (i) $\#\mathcal{F}(z)$ is constant for $z \in \mathcal{U}_3 \setminus \mathcal{E}$.
(ii) The limit set of $\mathcal{F}(z_j)$ is $\mathcal{F}(z_0)$ if $z_j \to z_0 (\in \mathcal{U}_3)$.
(iii) In case $n = 1$, $\mathcal{F}(z) = \{(X, f_2(z)) : \mathcal{P}_1(z;X) = 0\}$.
(iv) For $w_1, w_2 \in \mathcal{U}_1$, if the variety $Q_{w_1} \cap Q_{w_2} \cap \mathcal{U}_2$ contains a point of complex dimension n, then $Q_{w_1} \cap \mathcal{U}_2 = Q_{w_2} \cap \mathcal{U}_2$. Thus $A_{w_1} = A_{w_2}$.

Proof of Lemma 3.0: (i) is an immediate consequence of what is called the monodromy theorem. (iv) is a consequence of (I.c) and the fact that two connected closed complex submanifolds of a domain coincide if they have an open subset in common. (iii) follows easily from the irreducibility assumption of $\mathcal{P}_1$ and the following standard argument: For $z \notin \mathcal{E}$, write $\mathcal{F}(z) = \{(g_j(z), f_2(z))\}_{j=1}^k$. Then (i) indicates that k is constant. Now, the Riemann extension theorem tells that $\widetilde{\mathcal{P}_1} = \prod_{j=1}^k (X - g_j(z))$ is a Weiestrass polynomial in X. It, of course, divides $\mathcal{P}_1$ by the previous observation. So, from the irreducibility of $\mathcal{P}_1$, it follows that $\mathcal{P}_1 = \widetilde{\mathcal{P}}_1$, and this clearly completes the proof of (iii). So, it now suffices for us to give the proof of (ii) to complete the proof of Lemma 3.0.

When $z_0 \notin \mathcal{E}$, then $\mathcal{F}(z)$ can be stratified into k holomorphic branches near z_0 with k =the generic counting number of $\mathcal{F}(z)$. That is, we can write $\mathcal{F}(z) = \{g_1(z), \cdots, g_k(z)\}$ with g_j's holomorphic near z_0. Now, when z_j is close enough to z_0, then $\mathcal{F}(z_j) = \{g_1(z_j), \cdots, g_k(z_j)\}$. From this, it is easy to see that $\lim \mathcal{F}(z_j) = \mathcal{F}(z_0)$.

When $z_0 \in \mathcal{E}$, by passing to a nearby point, we can assume, without loss of generality, that all z_j's are not in $\mathcal{E}$. Now, for another sequence $\{w_j\}(\subset \mathcal{U}_3 \setminus \mathcal{E})$ with $w_j \to z_0$, we first find a family of curves $\{\gamma_j\}$ such that $\|\gamma_j - z_0\| \to 0$ and $\gamma_j(0) = z_j$, $\gamma_j(1) = w_j$. Still write $\mathcal{F}(z) = \{g_1(z), \cdots, g_k(z)\}$ for $z \notin \mathcal{E}$. Then for each l, there is a continuous map $I_l(t)$ from $[0,1]$, which maps t to a point in $\mathcal{F}(\gamma_j(t))$ such that $I_l(0) = g_l(z_j)$ and $I_l(1) = g_{\sigma_{z_j}(l)}(w_j)$, where σ_{z_j} is a permutation of $\{1, \cdots, k\}$. Moreover, it is clear that the m-th component of $I_l(t)$ is a solution of $\mathcal{P}_m(\gamma_j(t); X) = 0$, as observed above. Now, by Lemma 2.5 of [Mal] and our choices of γ_j, it follows that $\|g_l(z_j) - g_{\sigma_{z_j}(l)}(w_j)\| \to 0$. Therefore, we see that $\lim \mathcal{F}(z_j) = \lim \mathcal{F}(w_j)$, which must be $\mathcal{F}(z_0)$ by the definition. $\square$

Lemma 3.1. After shrinking $\mathcal{U}_1$, if necessary, then for each $z \in \mathcal{U}_1$, it holds that $\mathcal{F}(z) \subset A'_{\widetilde{f(z)}}$ and $\mathcal{F}(Q_z) \subset Q'_{\widetilde{f}(z)}$ for any $\widetilde{f}(z) \in \mathcal{F}(z)$.

Proof of Lemma 3.1: We would like to mention that by a simple unique continuation argument and by using the invariant property of Segre varieties, one can easily show that for each nice branch $\widetilde{f}$ with $\widetilde{f}|_D = f$, it holds that $\widetilde{f}(Q_z) \subset Q_{\widetilde{f(z)}}$ (see also [DF2], for example). The main idea of the proof of the lemma is to use the fact that $\widetilde{\rho_2}(\mathcal{F}(z), \widetilde{f}(\omega))$ is single-valued for each fixed ω, by the above established λ-function.

We let $z \in \mathcal{U}_1 \cap D$, and assume that $z,\ z^* \notin \mathcal{E}$. Here, we use z^* to denote the reflection point of z, i.e, $z^* = \mathcal{R}(z)$. We also choose a simply connected smooth curve $\gamma : [0,1] \to \mathcal{U}_3$ with $\gamma \subset \mathcal{R}(\mathcal{U}_1 \setminus D) \setminus \mathcal{E}$ and $\gamma(1) = z^*$, $\gamma \cap M_1 = \{\gamma(0)\}$. Here, when there is no confusion arising, we also use the letter γ to denote its image set. Moreover, we assume that γ intersects M_1 transversally at $\gamma(0)$. Let $\widetilde{\gamma} = \mathcal{R}(\gamma) \cup \gamma$. Then $\widetilde{\gamma}$ is still a simply connected curve in $\mathcal{U}_2$ by the discussions in §2. Thickening $\widetilde{\gamma}$ suitably, we can then obtain a simply connected domain, which we will denote by $O(\subset \mathcal{U}_2)$. Now, we can well define a holomorphic map $\hat{f}$ from O, which coincides with f on $O \cap D$. Let $M_{1c} = \{(z,\omega) \in \mathcal{U}_3 \times \mathrm{conj}(\mathcal{U}_3) : \widetilde{\rho_1}(z,\omega) = 0\}$ and consider $M^*_{1c} = M_{1c} \cap \{O \times \mathrm{conj}(O)\}$. Now, the following function is well-defined and holomorphic over M^*_{1c}:

$$\Xi(z,\omega) = \overline{f_{n+1}(\overline{\omega})} - G(\mathcal{F}(z), \overline{\hat{f}(\overline{\omega})}) \left(= \widetilde{\rho_2}(\mathcal{F}(z), \overline{\hat{f}(\overline{\omega})})\right)$$

which vanishes on $M^*_1 = \{(a, \overline{a}) : a \in M_1 \cap \mathcal{U}_3\}$. Notice that M^*_1 is a totally real subset in M^*_{1c} with maximal dimension, we conclude from the uniqueness property of holomorphic functions [Pi2] that $\Xi(z,\omega) \equiv 0$ in the union, denoted by M^{**}_{1c}, of the connected components of M^*_{1c} which have non-empty intersections with M^*_1. By our choice of $\widetilde{\gamma}$, we see that $(z, \overline{z^*})$, $(z^*, \overline{z}) \in M^{**}_{1c}$; for $(\gamma(t), \overline{\mathcal{R}(\gamma(t))}) \in M^*_{1c}$ $(\mathcal{R}(\gamma(t)), \overline{\gamma(t)}) \in M^*_{1c}$ $(t \in [0,1])$ connect them to M^*_1, respectively. Therefore, $(z, \mathrm{Conj}(Q_z \cap O(z^*))) \subset M^*_{1c}$, $(z^*, \mathrm{Conj}(Q_{z^*} \cap O(z))) \subset M^*_{1c}$ by the definition of Segre varieties. This implies that $G(\mathcal{F}(z), \overline{\hat{f}(\omega)}) = \overline{f_{n+1}(\omega)}$ for $\omega \in Q_z \cap O(z^*)$; and

$G(\mathcal{F}(z^*), \overline{\hat{f}(\omega)}) = \overline{f_{n+1}(\omega)}$ for $\omega \in Q_{z^*} \cap O(z)$. Equivalently, we have $\widetilde{\rho}_2(\mathcal{F}(z), \overline{\hat{f}(\omega)}) = 0$ for $\omega \in Q_z \cap O(z^*)$; and $\widetilde{\rho}_2(\mathcal{F}(z^*), \overline{\hat{f}(\omega)}) = 0$ for $\omega \in Q_{z^*} \cap O(z)$.

From a basic fact of Segre varieties: $w \in Q_z$ if and only if $z \in Q_w$; it now follows that $\hat{f}(Q_z \cap O(z^*)) \subset \cap Q'_{\widetilde{f(z)}}$ with $\widetilde{f}(z) \in \mathcal{F}(z)$; and $\hat{f}(Q_{z^*} \cap O(z)) \subset \cap Q'_{\widetilde{f}(z^*)}$ with $\widetilde{f}(z^*) \in \mathcal{F}(z^*)$.

By slightly perturbing z if necessary, we assume momentarily that $J_{\hat{f}} \neq 0$ near z^*. Now, since $\hat{f}(Q_z \cap O(z^*)) \subset Q'_{\widetilde{f}(z)}$ for any $\widetilde{f}(z) \in \mathcal{F}(z)$, and since each of them is a connected complex submanifold of dimension n near $\hat{f}(z^*)$, all these submanifolds therefore coincide near $\hat{f}(z^*)$. Hence, it follows from Lemma 3.0 (iv) that all $Q'_{\widetilde{f}(z)} \cap \mathcal{U}_2$ are the same. So, $\mathcal{F}(z) \subset A'_{\widetilde{f}(z)}$ for any given $\widetilde{f}(z) \in \mathcal{F}(z)$.

Assume also that $J_f(z) \neq 0$. In a similar manner, we then also see that $\mathcal{F}(z^*) \subset A'_{\widetilde{f}(z^*)}$ for any given $\widetilde{f}(z^*) \in \mathcal{F}(z^*)$.

Next, we note that $\widetilde{\rho}_2(\mathcal{F}(z), \overline{\hat{f}}(\omega)) = 0$ if and only if $\rho_2(\mathcal{F}(z), \overline{\hat{f}}(z)) = 0$. Let $q \in \mathcal{F}(z)$. Then, by what we just obtained and by the reality of ρ_2, we have $\rho_2(\hat{f}(\omega), \overline{q}) = 0$ and therefore, $\widetilde{\rho_2}(\hat{f}(\omega), \overline{q}) = 0$ for $\omega \in Q_z \cap O(z^*)$. Now, since Lemma 2.1 indicates that $\widetilde{\rho_2}(\mathcal{F}(\omega), \overline{q})$ is well defined and holomorphic for $\omega \in \mathcal{U}_3$, we conclude that $\widetilde{\rho_2}(\mathcal{F}(\omega), \overline{q}) = 0$ for $\omega \in \mathcal{U}_2 \cap Q_z$. This implies that $\mathcal{F}(Q_z) \subset Q'_q$.

In a similar manner, we can also show that $\mathcal{F}(Q_{z^*}) \subset Q'_{q^*}$ for any $q^* \in \mathcal{F}(z^*)$.

Now, we will complete the proof of Lemma 3.1 by passing to the limit.

Let z close to 0 be such that either $z \in \mathcal{E} \cup \mathcal{R}(\mathcal{E}) \cup M_1$ or $J_f(z) = 0$, or $J_f(z^*) = 0$. By what we did above, we can find a sequence $\{z_j\}$ with $z_j \to z$ so that $\mathcal{F}(z_j) \subset A'_{\widetilde{f}(z_j)}$ for any $\widetilde{f}(z_j) \in \mathcal{F}(z_j)$. In the other words, $Q'_{\widetilde{f}(z_j)} = Q'_{\hat{f}(z_j)}$ for any $\widetilde{f}(z_j), \hat{f}(z_j) \in \mathcal{F}(z_j)$. By Lemma (3.0) (ii) and (I.d), it then follows that $Q'_{\widetilde{f}(z)} = Q'_{\hat{f}(z)}$ for any $\widetilde{f}(z), \hat{f}(z) \in \mathcal{F}(z)$, i.e, $\mathcal{F}(z) \in Q'_{\widetilde{f}(z)}$ for any $\widetilde{f}(z) \in \mathcal{F}(z)$. Similarly, we also have $\mathcal{F}(Q_z) \subset Q'_{\widetilde{f}(z)}$ for any $\widetilde{f}(z) \in \mathcal{F}(z)$.

Hence, after shrinking $\mathcal{U}_1$ one more time if necessary, we see the proof of Lemma 3.1.

□

Remark 3.2 (a) By Lemma 3.1, we can now well-define $Q'_{\mathcal{F}(z)}$ to be Q'_q and $A'_{\mathcal{F}(z)} = A'_q$ for some $q \in \mathcal{F}(z)$. Then Lemma 3.1 can be written as $\mathcal{F}(z) \subset A'_{\mathcal{F}(z)}$ and $\mathcal{F}(Q_z) \subset Q'_{\mathcal{F}(z)}$.

(b) As an application of Lemma 3.1, we conclude that when A'_w is just a single point for $w \approx 0$, then f extends holomorphically near 0. This is the case when the target point is a non-degenerate point or has some special bi-type property. Define $\mathcal{A}'$ by sending each point w to A'_w. One can see that if f does not allow holomorphic extension, then $\mathcal{A}'$ branches at 0 (we will make this more precise in §5). It is this fact that links the branching points of $\mathcal{F}$ with the singular points of $\mathcal{A}'$, which will be the key observation for the proof of Theorem 1.1.

More specifically, let $D_1 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 < 1\}$ and $D_2 = \{(z_1, z_2) : |z_1|^4 + |z_2|^2 < 1\}$. Let $\mathcal{G} = (\sqrt{z_1}, z_2)$ be the multiple-valued map from D_1 to D_2. Then the branching locus of $\mathcal{G}$ is given by $Z = \{(0, z_2)\}$, and $\mathcal{G}(Z) = \{(0, z_2)\}$. We observe that $\mathcal{G}(Z)$ is exactly the branching locus of the $\mathcal{A}'$-map of ∂D_2.

Lemma 3.3: (i) Let M_2 be as given in (2.2). Let $\Omega^{\pm}$ be defined by $\pm\rho_2 < 0$ respectively. Write $\hat{n}^{\pm} = \{b = (0', b_{n+1}) : \pm b_{n+1} < 0\}$ with $|b_{n+1}|$ small. Then $A'_b \subset \Omega^{\pm}$ when $b \in \hat{n}^{\pm}$, respectively.
(ii) If for some $q \in \mathcal{F}(z)$, it holds that $q \in M_2$, then $\mathcal{F}(z) \subset M_2$. Hence, $\mathcal{F}(M_1 \cap \mathcal{U}_1) \subset M_2$.
(iii) $f_{n+1}(z) = z_{n+1}^k g(z)$ for some holomorphic function $g(z)$ defined near 0.
(iv) $\{\mathcal{F}^{-1}(0)\} \cap \mathcal{U}_2 = \{0\}$ after shrinking $\mathcal{U}_2$, i.e , $\widetilde{f}(z) \neq 0$ for any $z (\in \mathcal{U}_2) \neq 0$ and $\widetilde{f}(z) \in \mathcal{F}(z)$. Moreover $\mathcal{F}^{-1}(w) \cap \mathcal{U}_2$ is a finite set for any $w \approx 0$; and for any analytic variety E passing through $0 \in \mathcal{U}_1$, $\mathcal{F}(E)$ also gives the germ of an analytic variety at $0 \in \mathcal{U}'_1$ with $\dim_0 E = \dim_0(\mathcal{F}(E))$.
(v) After shrinking $\mathcal{U}_1$ if necessary, then for any $z \in \mathcal{U}_1$, either $\mathcal{F}(z) \subset \Omega^+$ or $\mathcal{F}(z) \subset \Omega^-$, or $\mathcal{F}(z) \subset M_2$.

Proof of Lemma 3.3: (i): Let $\eta = (\eta', \eta_{n+1}) \in A'_b$, i.e, $Q'_\eta \cap \mathcal{U}'_2 = Q'_b \cap \mathcal{U}'_2$. We see that

$$\{(w', w_{n+1}) \in \mathcal{U}'_2 : w_{n+1} + \overline{\eta_{n+1}} + \rho_2^*(w', \overline{\eta}', \frac{w_{n+1} - \overline{\eta_{n+1}}}{2i}) = 0\}$$

$$= \{(w', w_{n+1}) \in \mathcal{U}'_2 : w_{n+1} + \overline{b_{n+1}} + \rho_2^*(w', \overline{b}', \frac{w_{n+1} - \overline{b_{n+1}}}{2i}) = 0\}.,$$

where $\rho^* = \sum_{j \geq 0} \psi_j(w', \overline{w'})(\mathrm{Im} w_{n+1})^j$. Letting $w' = 0$, we see that $\eta_{n+1} = b_{n+1}$. Meanwhile, since $\{(w', -\overline{\eta_{n+1}})\} \cap \mathcal{U}'_2 = Q'_\eta$, it follows that $\rho_2^*(w', \overline{\eta'}, i\overline{\eta_{n+1}}) = 0$ for all w' with $(w', b_{n+1}) \in \mathcal{U}'_2$. We see, in particular, that $\rho_2^*(\eta', \overline{\eta'}, i\overline{\eta_{n+1}}) = 0$. Therefore, $\psi_0(\eta', \overline{\eta'}) = -\sum_{j>0}^{\infty} \psi_j(\eta', \overline{\eta'})(i\overline{\eta_{n+1}})^j = o(|\eta_{n+1}|)$. Now, $\rho_2(\eta, \overline{\eta}) = \eta_{n+1} + \overline{\eta_{n+1}} + \rho_2^*(\eta', \overline{\eta'}, \frac{\eta_{n+1} - \overline{\eta_{n+1}}}{2i}) = \eta_{n+1} + \overline{\eta_{n+1}} + \psi_0(\eta', \overline{\eta'}) + o(|\eta_{n+1}|) = 2\eta_{n+1} + o(\eta_{n+1})$, which is positive when $b_{n+1} = \eta_{n+1} > 0$ and small; and negative when $b_{n+1} < 0$. This gives the proof of part (i).

(ii). Let $q \in M_2 \cap \mathcal{F}(z)$. Then $q \in Q'_q = Q'_{\widetilde{f}(z)}$ for each $\widetilde{f}(z) \in \mathcal{F}(z)$. Thus $\widetilde{f}(z) \in Q'_q = Q'_{\widetilde{f}(z)}$, by Lemma 3.1. So, $\widetilde{f(z)} \in M_2$ (see [DF1]).

(iii). By Lemma 3.1, it follows that $\mathcal{F}(Q_0) \subset Q'_{\mathcal{F}(0)} = Q'_0$. Thus, we see that $f_{n+1}(z', 0) \equiv 0$. Since f_{n+1} is holomorphic near 0, we conclude that $f_{n+1}(z', z_{n+1}) = z_{n+1}^k g(z)$ for some holomorphic function g near 0 and some positive integer k.

(iv). The following argument will be based on some facts on local analytic covers and the elimination theorem (see [Chi], [Wh], or [GR]).

We start by setting up some notations:

Let $\widetilde{V} = \{(z,w) \approx (0,0) : w_{n+1} = f_{n+1}(z), \mathcal{P}_j(z,w_j) = 0, j = 1,\cdots,n\}$ and let V be its irreducible component at $(0,0)$ which contains $\Gamma_f = \{(z,w) : w = f(z), z(\approx 0) \in D\}$. Let π and π' be the natural projections from V to the first and second copies of $\mathbf{C^{n+1}}$, respectively. Since $\mathcal{P}_j(0;X) = 0$ has only zero solutions, i.e, $\pi^{-1}(0) = 0$, π is locally proper. Thus, we can choose a small neighborhood U of 0 such that (a) $\pi^{-1}(U)$ is irreducible; (b) π is proper from $\pi^{-1}(U)$; and (c) π is a sheeted covering map from $\pi^{-1}(U) \setminus \pi^{-1}(\mathcal{E})$ to $U \setminus \mathcal{E}$ (see, for example , Theorem 1 of [Chi], pp 122). In the following discussion, we make $\mathcal{U}_2 \supset U \supset \mathcal{U}_1$ and restrict π to $\pi^{-1}(U)$.

Now, by using some standard arguments as appeared in the proof of Lemma 3.0 (iii) and using the irreducibility of $\pi^{-1}(U)$, one can see that $\mathcal{F}(z) = \pi' \circ \pi^{-1}(z)$, for $z \in \mathcal{U}_2$. (Otherwise, we can form a proper analytic subvariety of V (of the same dimension), which is defined by $\{(z,w) : \ w \in \mathcal{F}(z)\}$).

On the other hand, it obviously holds that $\mathcal{F}^{-1}(0) \cap U = \pi(\pi'^{-1}(0) \cap \pi^{-1}(U))$. Thus, if 0 is an accumulation point of $Y = \mathcal{F}^{-1}(0)$, then $Y \cap U$ is an analytic variety of U (by the elimination theorem) and it must have positive dimension at 0. We will assume this and seek a contradiction.

Then, $Y \cap U$ contains some holomorphic curve $Y^* \subset \mathcal{U}_1$ parametrized by $z' = \phi(t), z_{n+1} = \psi(t)$, ($t \in \Delta$, the unit disk in $\mathbf{C^1}$) with $\phi(0) = \psi(0) = 0$, $\|d\phi\| + \|d\psi\| \neq 0$ for $t \neq 0$. For each $z \in Y \cap \mathcal{U}_1$, $\mathcal{F}(Q_z) \subset Q'_0$, by Lemma 3.1. We claim that $\cup Q_z$ with $z \in Y^*$ fills in an open subset in $\mathbf{C^{n+1}}$. This then gives us a contradiction; for we assumed that $J_f \not\equiv 0$ and Q'_0 is a complex hypersurface.

To see the size of $U^* = \cup Q_z$ with $z \in Y^*$, we note that M_1 is also defined by $\overline{\widetilde{\rho_1}(z,\overline{z})} = 0$. Therefore, it can be seen that Q_z can be parametrized by

$$z_{n+1} = -\overline{\psi(t)} - \sum_{j=0}^{\infty} \overline{\widetilde{\phi_j}}(\overline{\phi(t)}, z')\overline{\psi(t)}^j \equiv g(z', \overline{t}).$$

Here and in what follows, we use the notation $\overline{h}(z)$ for the function $\overline{h(\overline{z})}$. So, U^* can be parametrized by the map

$$\mathcal{T}(z', \overline{t}) : \ \mathbf{C^n} \times \Delta \to \mathbf{C}^{n+1}$$

$\mathcal{T}(z',\overline{t}) = (z', -\overline{\psi(t)} - \sum_{j=0}^{\infty} \overline{\widetilde{\phi_j}}(\overline{\phi(t)}, z')\overline{\psi(t)}^j)$. To see that $\mathcal{T}(z,t)$ is a biholomorphism at certain point $(z',\overline{t})(\approx (0,0))$, it suffices to show that $\frac{\partial g}{\partial \overline{t}} \neq 0$ at $(z',\overline{t})$. Indeed, this can be argued as follows: When $\psi(t) \not\equiv 0$, we can simply choose $(0',t)$ for some t with $\psi'(t) \neq 0$; when $\psi \equiv 0$, if $\frac{\partial g}{\partial \overline{t}} \equiv 0$, then for any given z' with $|z'|$ small, $\overline{\widetilde{\phi_0}}(\overline{\phi(t)}, z')$ is independent of t. Hence it has to be 0; for $\phi(0) = 0$ and thus $\overline{\widetilde{\phi_0}}(\phi(0), z') = 0$. Letting $z' = \phi(t)$, it follows that $\overline{\widetilde{\phi_0}}(\overline{\phi(t)}, \phi(t)) \equiv 0$. This contradicts the finite type assumption on M_1; for it implies that $Y^* = (\phi(t), 0)$ stays inside M_1. So, by making $\mathcal{U}_2$ small, it holds that $\mathcal{F}^{-1}(0) \cap \mathcal{U}_2 = \{0\}$.

Therefore, we conclude that 0 is an isolated point of ${\pi'}^{-1}(0)$. Hence, π' is locally finite near 0. Moreover, $\mathcal{F}^{-1}$, when restricted as a map from $O(0)$ to $\mathcal{U}_2$, maps a small neighborhood $0 \in \mathcal{U}_1'$ into a small neighborhood of $0 \in \mathcal{U}_1$.

Now, shrinking $\mathcal{U}_2$ if necessary, we assume that $q \approx 0$, $\mathcal{F}^{-1}(q) \cap \mathcal{U}_2 = \pi({\pi'}^{-1}(q))$ is finite. By the elimination theorem ([Chi], Theorem 1 of pp122), there exists an open neighborhood U' of 0 such that π' is proper from ${\pi'}^{-1}(U')$ to U'; and away from a proper analytic set, π' gives a sheeted covering map. In particular, π' is an open mapping (see [GR], Lemma 6 pp 102, and notice the openness follows from the onto part of that lemma).

In the following discussion, we make $\mathcal{F}(U) \subset U'$, and we restrict π' to ${\pi'}^{-1}(U')$. By the above discussion, we also observe the fact: $\pi^{-1}(U) \subset {\pi'}^{-1}(U')$.

Using again the elimination theorem and noting that π, π' are local analytic covering maps, it follows that for any analytic variety $E \subset U$, $\mathcal{F}(E) \cap \pi'(\pi^{-1}(U)) = \pi'(\pi^{-1}(E) \cap \pi'(\pi^{-1}(U)))$ is an analytic variety in $\pi'(\pi^{-1}(U))$ with the same dimension at the origin (see for example, Theorem 11 E pp 68 of [Wh]; or Theorem 1, pp122, of [Chi]). This completes the proof of (iv).

(v) Since π is proper, it is also closed. Also, both are open mappings; for they are local analytic covering mappings, too. (See [GR], Lemma 6, pp102)

For any closed subset B of U', we first notice that

$$\mathcal{F}(B) \cap U = \pi\left({\pi'}^{-1}(B) \cap \pi^{-1}(U)\right).$$

Since ${\pi'}^{-1}(B)$ is closed in ${\pi'}^{-1}(B) \cap {\pi'}^{-1}(U')$, it is closed in ${\pi'}^{-1}(B) \cap \pi^{-1}(U)$ by the above arrangement. By using the closeness of π, it follows that $\mathcal{F}^{-1}(B)$ is closed in U. In particular, we see that $\widetilde{M_1} \cap U = \mathcal{F}^{-1}(M_2 \cap U')$ is closed in U. Clearly, $\mathcal{F}(\widetilde{M_1}) \subset M_2$ by part (ii) of this lemma.

Now, let $U^0 = U \setminus \widetilde{M_1}$. Then it is open. We notice that ∂U^0 is contained in $\partial U \cup \widetilde{M_1}$. Since $\mathcal{F} = \pi' \circ \pi^{-1}$, the openness of π' implies the openness of $\mathcal{F}$ from U as a multiple-valued map. Therefore it sends interior points to interior points. On the other hand, using the weak continuity of $\mathcal{F}$ (Lemma 3.0 (ii)), one sees that $\mathcal{F}$ maps the closure of U^0 to the closure of $\mathcal{F}(U^0)$. Hence, a simple topological argument shows that the boundary of $\mathcal{F}(U^0)$ is contained in $\mathcal{F}(\partial U) \cup M_2$. Since $\mathcal{F}^{-1}(0) \cap \mathcal{U}_2 = \{0\}$, it follows easily that 0 is not contained in the closure of $\mathcal{F}(\partial U)$. Hence, we see that there is a small ball, denoted by B_0, centered at the origin such that $B_0 \cap \mathcal{F}(\partial U) = \emptyset$. Let U^* be a small neighborhood of 0 such that $\mathcal{F}(U^*) \subset\subset B_0$. If $U^0 \cap U^* = \emptyset$, then Part (ii) of this lemma shows that $\mathcal{F}(U^*) \subset \mathcal{F}(\widetilde{M_1}) \subset M_2$ and thus we are done. So, without loss of generality, we assume that $U^* \cap U^0 \neq \emptyset$. Let D^* be a connected component of U^0 with $U^* \cap U^0 \neq \emptyset$. Note that $\partial \mathcal{F}(D^*) \cap B_0 = \emptyset$. Then a simple topological argument indicates that either $\mathcal{F}(D^*) \supset B_0 \cap \Omega^+$ or $\mathcal{F}(D^*) \supset \Omega^- \cap B_0$ (see, for example, [BHR2]). Without loss of generality, we assume the first case. Then, there exists a point $p_0 \in D^*$ and certain $\widetilde{f}(p_0) \in \mathcal{F}(p_0)$ such that $\widetilde{f}(p_0) \in \hat{n}^+$ and $\widetilde{f}(p_0)$ is close to 0. By the first part of this lemma and Lemma 3.1, we see that $\mathcal{F}(p_0) \subset \Omega^+$. Now, for any $z \in D^* \setminus \mathcal{E}$, let γ be a curve in D^* with $\gamma(0) = p_0$, $\gamma(1) = z$, and $\gamma((0,1)) \cap \mathcal{E} = \emptyset$. Then for any $\widetilde{f}(z) \in \mathcal{F}(z)$, there is a branch $\hat{f}$ of $\mathcal{F}$, which is continuous on γ and $\hat{f}(z)$ coincides with $\widetilde{f}(z)$. Now, since $\hat{f} \circ \gamma$ does not meet M_2 by our choice of U^0, and $\hat{f} \circ \gamma(t) \in \Omega^+$ for t close enough to 0 (this can be seen by the fact that all limit points of $\hat{f}(\gamma(t))$ $(t \to 0)$ are in $\mathcal{F}(p_0)$), we conclude that $\hat{f}(z)$ and thus $\widetilde{f}(z)$ have to be in Ω^+. For $z \in \mathcal{E} \cap D^*$, by passing to a limit and noting $\mathcal{F}(z) \cap M_2 = \emptyset$, we can also see that $\mathcal{F}(z) \subset \Omega^+$. Thus the proof of the last part of Lemma 3.3 is complete if we make $\mathcal{U}_1$ small. □

Remark 3.4: The following observations will be used in the later discussions:

(a) From the proof of Lemma 3.3 (iv), we also notice the following property of $\mathcal{F}$: There exists an open neighborhood $\mathcal{U}_0'$ of 0 such that $\mathcal{U}_0 = \mathcal{F}^{-1}(\mathcal{U}_0') \cap \mathcal{U}_2$ is an open subset of $\mathcal{U}_1$. Moreover, $\mathcal{F}|_{\mathcal{U}_1 \to \mathcal{F}(\mathcal{U}_1)}$ and $\mathcal{F}^{-1}|_{\mathcal{U}_0' \to \mathcal{U}_0}$ are multiple-valued open mappings.

(b) Let $(V, \pi, \pi(V))$ be the m-sheeted analytic covering space, as in Lemma 3.3 (iv). Let $z_0 \in \mathcal{E} \cap \pi(V)$ be close to the origin. Then, it is easy to see that in case $\#\pi^{-1}(z_0) < m$, then there are two distinct sequences $\xi_j, \eta_j \in \pi^{-1}(z_j) \subset V$ such that $z_j \to z_0$, $\xi_j, \eta_j \to w_0 \in \pi^{-1}(z_0)$.

(c) Still let V as above. Write (Y, σ, V) for the standard normalization of V (see [Wh], Chapter 8). We remark that after making V small, $\sigma^{-1}(0)$ is a single point. Write $\mathcal{E}_0 = \pi \circ \sigma(S)$, where

$$S = \{x \in Y : \text{ either x is singular or x is smooth but } \mathrm{d}_x\sigma \text{ is singular}\}.$$

Write $S_0 = (\pi \circ \sigma)^{-1}(\pi \circ \sigma)(S)$. Then $\pi \circ \sigma$ is a local biholomorphic mapping from $Y \setminus S_0$ to its image and gives a finitely sheeted covering mapping away from the singular set. (see [Wh], Chapter 8 and [GR], pp 108). Therefore, if V is singular at 0, i.e, f is not holomorphic at 0, then $\mathcal{E}_0 = \pi \circ \sigma(S)$ is an analytic variety of codimension 1 at 0. (This follows from the fact that any finitely sheeted analytic covering space with 'genuine' singular set of codimension ≥ 2 is trivial) (see the following observation in Remark 3.4 (d)). Now, for each $z \in \mathcal{E}_0 \cap O(0) (\subset \mathcal{E})$ and any small neighborhood U_z of z, the basic property for the normal space ([Wh], in particular Lemma 2F pp 254) indicates that V can not be smooth at $\pi^{-1}(z)$. Therefore, $\#\pi^{-1}(z) < m$.

(d) More generally, let $P(z, X) = X^N + \sum_{j<N} a(z) X^j = 0$ be an irreducible polynomial equation in X for z near 0. Let Z^* be its branching locus. A point z_0 is called a genuine branching point of $P = 0$ if $P(z, X)$ can not be factorized into linear functions in X in the Noetherian ring $\mathcal{O}_{z_0}$. Write Z for the collection of all genuine branching locus of P. The normalization argument as in (c) shows that Z is an analytic variety near the origin. Moreover, it can be seen that the following monodromy

theorem holds: Let γ_1, γ_2 be loops in $O(0) \setminus Z$, which are based at $p_0 \in O(0) \setminus Z^*$ and are homotopic to each other in $O(0) \setminus Z$, relative to the base point. Then, any analytic continuation of the roots of P along γ_1, γ_2, starting with the same initial value, will end up with the same value when coming back to p_0. From this fact, if follows that in case $Z \neq \emptyset$, Z must be of codimension 1 everywhere.

4. A local Hopf lemma and a preservation principle of $\mathcal{F}$

Before proceeding further, we need to strengthen Lemma 3.3 (iii) to the following version:

Lemma 4.1: $f_{n+1}(z) = z_{n+1} g(z)$ with $g(0) \neq 0$.

This sort of the Hopf lemma was established in [BR2] in the case when the map is assumed to be smooth. Since we now have a nice control of the branches of $\mathcal{F}$ and we know that f_{n+1} is holomorphic, Lemma 4.1 can be proved by using the same approach and ideas as in [BR2]. Because our emphasis of the present paper is the two dimensional case, for completeness, we will present a detailed proof of Lemma 4.1 when $n = 1$ and leave the easy modification in general dimensions to Remark 4.1′.

We also mention that when f is a CR-homeomorphism, the proof of Lemma 4.1 readily follows from Lemma 3.3 (iii). To see this, write $f^{-1} = (\cdots, f^{-1}_{n+1})$. Then the same argument as above shows that $f^{-1}_{n+1}(w) = w^m g^*(w)$ for some $m \geq 1$. Thus $z_{n+1} = {z_{n+1}}^{km} O(1)$. So $k = m = 1$.

Proof of Lemma 4.1: We will use the approach appeared in [BR2]. We assume $n = 1$. Seeking a contradiction, we suppose that $f_2(z) = z_2^k g(z)$ with $k \geq 2$.

We start with the function $G(\mathcal{F}, \lambda)$, which has the following property:

$$\overline{f_2(\omega)} = G(\mathcal{F}(z), \overline{\widetilde{f}_1(\omega)})$$

for any $z \in Q_\omega$ and $(\widetilde{f_1}, f_2) \in \mathcal{F}$. Write the defining equation of M_1 in the following form (see (2.1)): $t = \phi(z_1, \overline{z_1}, s)$ with $t = 2\mathrm{Re} z_2$, $s = \mathrm{Im} z_2$, and $\phi = \rho_1 - t$. Clearly, $\phi(0, \overline{z_1}, s) = \phi(z_1, 0, s) \equiv 0$. Then $z \in Q_\omega$ reads as

$$z_2 + \overline{\omega_2} = \phi(z_1, \overline{\omega_1}, \frac{z_2 - \overline{\omega_2}}{2i}).$$

We let $\tau = -is + \frac{1}{2}\phi(z_1, \xi, s)$ and $R(z_1, \xi, \tau) = is + \frac{1}{2}\phi(z_1, \xi, s)$. As in [BR2], using the implicit function theorem, we can find a holomorphic function $\mu(z_1, \xi)$ near 0 such that $R(z_1, \xi, \mu(z_1, \xi)) \equiv 0$ and $\mu(z_1, 0) = \mu(0, \xi) \equiv 0$. It is easy to verify that $\mu(z_1, \xi) \not\equiv 0$ and thus $\frac{\partial \mu}{\partial z_1}(z_1, \xi) \not\equiv 0$; for, otherwise, it implies that $\phi_0 \equiv 0$ and thus contradicts the finite type assumption of M_1. Now, one can directly verify that $(z_1, R(z_1, \xi, \tau)) \in Q_{\overline{(\xi, \tau)}}$. So, we have $\overline{(\xi, \tau)} \in Q_{(z_1, R(z_1, \xi, \tau))}$ and thus

$$\overline{f_2(z_1, R(z_1, \xi, \tau))} = G(\mathcal{F}(\overline{\xi}, \overline{\tau}), \overline{\widetilde{f}_1(z_1, R(z_1, \xi, \tau))}).$$

As in the previous section, we write $\overline{\widetilde{f}_1}(z) = \overline{\widetilde{f}_1(\overline{z})}$ and use the same notation for $\overline{f_2}(z)$. Then

$$\overline{f_2}(\overline{z_1}, \overline{R(z_1,\xi,\tau)}) = G(\mathcal{F}(\overline{\xi},\overline{\tau}), \overline{\widetilde{f}_1}(\overline{z_1}, \overline{R(z_1,\xi,\tau)}). \tag{4.1}$$

We still let $\mathcal{E}$ be the branching locus of $\mathcal{F}$, i.e, $\mathcal{E} = \{z \in \mathcal{U}_3 : \mathcal{P}_1(z,X) = \frac{\partial \mathcal{P}_1}{\partial X}(z,X) = 0$ for some $X\}$; and write $\mathcal{F}(z) = (\mathcal{F}_1(z), f_2(z))$.

We will assume momentarily the fact that 0 is an isolated point of $\mathcal{E}\cap Q_0 = \mathcal{E}\cap\{z : z_2 = 0\}$, which will be proved in Lemma 5.1 (a) of §5. (In n-dimensions, we observe the last statement of Lemma 5.1 (a) holds without Condition S).

Since $\{\mathcal{F}^{-1}(w)\}$ is finite for $w \approx 0$ (Lemma 3.3 (iv)), we see that no branch of $\mathcal{F}$ is constant on an open subset of Q_0. We also notice, by Lemma 3.0 (iii), that $\mathcal{F}_1(z_1,0)$ consists of exactly the solutions of $\mathcal{P}_1((z_1,0);X) = 0$. Using the Puiseaux expansion, we can write $\mathcal{F}_1(z_1,0) = g(z_1^{1/k})$, where g is a holomorphic function with $g(z_1) \neq 0$ for each $z_1(\neq 0)$ sufficiently close to 0.

Now, for each non zero z_1 close to the origin, there is a small neighborhood $O((z_1,0))$ near $(z_1,0)$ such that we can stratify $\mathcal{F}$ into several non-constant holomorphic branches. Next, we choose ξ sufficiently small and then let τ be sufficiently close to $\mu(z_1,\xi)$. Then $R(z_1,\xi,\tau) \approx 0$, too. After letting $\widetilde{f}$ be a holomorphic branch of $\mathcal{F}$ over $O((z_1,0))$, we then can take the derivative with respect to $\overline{z}_1$ in (4.1), to obtain

$$\frac{\partial \overline{f_2}}{\partial z_1} + \frac{\partial \overline{f_2}}{\partial z_2}\overline{R'_{z_1}(z_1,\xi,\tau)} = \frac{\partial G}{\partial \lambda}(\frac{\partial \overline{\widetilde{f}_1}}{\partial z_1} + \frac{\partial \overline{\widetilde{f}_1}}{\partial z_2}\overline{R'_{z_1}(z_1,\xi,\tau)}). \tag{4.2}$$

Let $\tau = \mu(z_1,\xi)$ in (4.2) and notice that $f_2 = z_2^k g$ with $k \geq 2$. We obtain

$$\frac{\partial G}{\partial \lambda}(\frac{\partial \overline{\widetilde{f}_1}}{\partial z_1}(\overline{z_1},0) + \frac{\partial \overline{\widetilde{f}_1}}{\partial z_2}(z_1,0)\overline{R'_{z_1}(z_1,\xi,\mu(z_1,\xi))}) = 0. \tag{4.3}$$

By the above discussion, we see that $\frac{\partial \overline{\widetilde{f}_1}}{\partial z_1}(\overline{z_1},0) \neq 0$ for $z_1 \neq 0$. So, for each z_1 with $|z_1|$ small, when ξ is chosen so that $|\xi|$ is sufficiently smaller than $|z_1|$, using the fact that $R'_{z_1}(z_1,0,\tau) = 0$, we see that $R'_{z_1}(z_1,\xi,\mu(z_1,\xi)) \approx 0$. Moreover,

$$\frac{\partial \overline{\widetilde{f}_1}}{\partial z_1}(\overline{z_1},0) + \frac{\partial \overline{\widetilde{f}_1}}{\partial z_2}(z_1,0)\overline{R'_{z_1}(z_1,\xi,\mu(z_1,\xi))} \neq 0.$$

Hence it follows from (4.3), that

$$\frac{\partial G}{\partial \lambda}(\mathcal{F}(\overline{\xi}, \overline{\mu(z_1,\xi)}), \overline{\widetilde{f}_1}(\overline{z_1},0)) \equiv 0.$$

On the other hand, similar to an argument in [BR2], letting $\tau = \mu(z_1,\xi)$ in (4.1), we have

$$G(\mathcal{F}(\overline{\xi}, \overline{\mu(z_1,\xi)}), \overline{\widetilde{f}_1}(\overline{z_1},0)) \equiv 0.$$

Now, let $q_1 = G'_\lambda(w_1, 0, \lambda)$ and $q_0 = G(w_1, 0, \lambda)$. Notice that $G(w_1, w_2, \lambda) = -w_2 + q_0 - \sum_{j>0} \widetilde{\psi}_j(w_1, \lambda)w_2^j$ and $q_1 = (q_0)'_\lambda$. We see that the equations $G(w_1, w_2, \lambda) = 0$, $G'_\lambda(w_1, w_2, \lambda) = 0$ can be solved as $w_2 = q_0 h^*(q_0, w_1, \lambda))$ and $q_1 = q_0\psi^*(w_1, \lambda, q_0)$ for certain holomorphic functions h^*, ψ^*. In the following discussions, we always let $\tau = \mu(z_1, \xi)$.

Let $w_1(z_1, \xi) = \hat{f}_1(\overline{\xi}, \overline{\tau})$ with $\hat{f}_1(\overline{\xi}, \overline{\tau}) \in \mathcal{F}(\overline{\xi}, \overline{\tau})$, and let $\lambda(z_1) = \overline{\widetilde{f}_1}(\overline{z_1}, 0)$ in the above formulas, we obtain the following equality

$$q_1(\hat{f}_1(\overline{\xi}, \overline{\tau}), \overline{\widetilde{f}_1}(\overline{z_1}, 0)) = q_0(\hat{f}_1(\overline{\xi}, \overline{\tau}), \overline{\widetilde{f}_1}(\overline{z_1}, 0)) \times \psi^*(\hat{f}_1(\overline{\xi}, \overline{\tau}), \overline{\widetilde{f}_1}(\overline{z_1}, 0), q_0). \tag{4.4}$$

Here for clarity, we summarize the situations in which (4.4) makes sense and holds:

(a) z_1 is an arbitrarily given non zero complex number near the origin. (b) $\widetilde{f}_1$ is some holomorphic branch of $\mathcal{F}$ over some small neighborhood $O((z_1, 0))$ of $(z_1, 0)$, whose size depends on z_1. (c) ξ is taken in a small neighborhood $O(z_1, \widetilde{f}_1)$ of 0, whose size depends on the choice of $O((z_1, 0))$ and the choice of $\widetilde{f}_1$. (d) $\hat{f}(\overline{\xi}, \overline{\tau})$ is any point in $\mathcal{F}_1(\overline{\xi}, \overline{\tau})$.

Since $f_2 \not\equiv 0$ and $\frac{\partial \mu}{\partial z_1} \not\equiv 0$, $Z^* = \{(z_1, \xi) : \ f_2(\overline{\xi}, \overline{\mu(z_1, \xi)}) = 0\}$ is an analytic variety of dimension 1. So, we can assume that Z^* has only one component of the form $\{(a, \xi)\}$ near the origin, i.e, the one with $a = 0$. We therefore see that $f_2(\overline{\xi}, \overline{\mu(z_1, \xi)})$, when regarded as a function in $\overline{\xi}$, is not constant for each fixed $z_1 \neq 0$; for if it were constant, then it would be identically 0. Meanwhile, noticing that $(\hat{f}_1(\overline{\xi}, \overline{\tau}), f_2(\overline{\xi}, \overline{\tau})) \in Q'_{(\widetilde{f}_1(z_1,0),0)}$, we see that

$$f_2(\overline{\xi}, \overline{\mu(z_1, \xi)}) = -\widetilde{\psi_0}(\hat{f}_1(\overline{\xi}, \overline{\mu(z_1, \xi)}), \overline{\widetilde{f}_1}(\overline{z_1}, 0)). \tag{4.5}$$

Let $Z^{**} \subset \mathbf{C}^2$ be the zero set of $q_1(w_1, \lambda) - q_0(w_1, \lambda)\psi^*(w_1, \lambda, q_0(w_1, \lambda))$. If it is not of dimension 2, then it has only finitely many irreducible one dimensional components near the origin. Thus, we can assume that for any $\lambda (\neq 0)$ with $|\lambda|$ small, $Z^{**} \cap \{(w_1, \lambda)\}$ is of dimension 0. Returning to (4.4), one sees that for each $z_1 \neq 0$ close to the origin, $\hat{f}_1(\overline{\xi}, \overline{\mu(z_1, \xi)})$ takes only finitely many values for $\xi \in O(z_1, \widetilde{f})$. This obviously contradicts (4.5); for $f_2(\overline{\xi}, \overline{\mu}(z_1, \xi))$ can not take only finitely many values when $\xi \in O(z_1, \widetilde{f}_1)$ by the above argument.

Therefore, we showed that $(q_0)'_\lambda = q_1(w_1, \lambda) \equiv q_0(w_1, \lambda)\psi^*(w_1, \lambda, q_0(w_1, \lambda))$. Now, applying the same argument as in [BR2], we can see, by the basic theory of ODE and by the initial condition $q_0(w_1, 0) = G(w_1, 0, 0) = 0$, that $q_0 \equiv 0$. This contradicts the finite type assumption of M_1. □

Remark 4.1′: The above proof can be adapted to the general dimensions as in [BR2], almost without any change. However, we would like to include here some details just for convenience of the reader:

We first replace z_1 by $z' \in \mathbf{C}^n$ and define, in the same manner, the functions τ, μ, R. We then have, by Lemma 3.1, the following

$$\overline{f_{n+1}(z', R(z', \xi, \tau))} = G(\mathcal{F}(\overline{\xi}, \overline{\tau}), \overline{\widetilde{f^*}(z', R(z', \xi, \tau))});$$

for $\overline{(\xi, \tau)} \in Q_{(z', R(z', \xi, \tau))}$ with $\xi \in \mathbf{C^n}$.

Let $\mathcal{E}' = \mathcal{E}_0 \cap Q_0$. By Lemma 5.1 (a), it is a proper analytic variety of Q_0. In the same manner, we let $z' \in Q_0 \setminus \mathcal{E}'$ be close to 0 and choose a small neighborhood $O((z', 0))$ of $(z', 0)$ such that $\mathcal{F}$ can be stratified into several (non-constant) holomorphic branches over $O((z', 0))$. Arguing in the same way, we have for each $l \leq n$,

$$\frac{\partial \overline{f_{n+1}}}{\partial z_l} + \frac{\partial \overline{f_{n+1}}}{\partial z_{n+1}} \overline{R'_{z_l}(z', \xi, \tau)} = \sum_{j=1,\cdots,n} \frac{\partial G}{\partial \lambda_j} \left(\frac{\partial \overline{\widetilde{f_j}}}{\partial z_l} + \frac{\partial \overline{\widetilde{f_j}}}{\partial z_{n+1}} \overline{R'_{z_l}(z', \xi, \tau)} \right).$$

Let $\tau = \mu(z', \xi)$. Assume that $f_{n+1} = z_{n+1}^k g$ with $k > 1$. One then has, for ξ with $\|\xi\|$ sufficiently small, that

$$\sum_{j=1,\cdots,n} \frac{\partial G}{\partial \lambda_j} \left(\frac{\partial \overline{\widetilde{f_j}}}{\partial z_l}(\overline{z'}, 0) + \frac{\partial \overline{\widetilde{f_j}}}{\partial z_{n+1}}(z', 0) \overline{R'_{z_l}(z', \xi, \tau)} \right) = 0.$$

Since $\mathcal{F}^{-1}$ is a finite-to-finite map, $f^*(z', 0)$ is a finite-to-one map from $O(z', 0) \cap Q_0$ too. Moving to a nearby point if necessary, we can assume that

$$\det\left(\frac{\partial \overline{\widetilde{f_j}}}{\partial z_l}(\overline{z'}, 0) \right)_{1 \leq j, l \leq n} \neq 0.$$

Reasoning as in the proof of the two dimensional case and making $|\xi|$ sufficiently small, we have for each $l < n+1$

$$\frac{\partial G}{\partial \lambda_l}(\mathcal{F}(\overline{\xi}, \overline{\mu(z', \xi)}), \overline{\widetilde{f}^*}(\overline{z'}, 0)) \equiv 0,$$

and

$$G(\mathcal{F}(\overline{\xi}, \overline{\mu(z', \xi)}), \overline{\widetilde{f}^*}(\overline{z'}, 0)) \equiv 0.$$

Here, $\|\xi\|$ is sufficiently small. Let $q_l = G'_{\lambda_l}(w', 0, \lambda)$ and $q_0 = G(w', 0, \lambda)$. Similarly, we have

$$q_l(w', \lambda) = q_0(w', \lambda) h_l^*(w', \lambda, q_0(w', \lambda)),$$

where $w' = \hat{f}^*(\overline{\xi}, \overline{\tau})$, $\tau = \mu(z', \xi)$, $\lambda(z') = \overline{\widetilde{f}^*}(\overline{z'}, 0)$, and h_l^* is holomorphic in its variables. (See the notation introduced in the proof of Lemma 4.1).

Now, a similar argument as in the two dimensional case (involving the use of the n-dimensional version of (4.5)) indicates that

$$q_l(w', \lambda) \equiv q_0(w', \lambda) h_l^*(w', \lambda, q_0) \tag{4.6}$$

for $(w', \lambda) \approx 0$ and each $l < n+1$. Indeed, this can also be seen as follows: Let $z'_0 \in \mathbf{C^n}$ be chosen as above such that $\lambda(z') = \overline{\widetilde{f}^*}(\overline{z'}, 0)$ is an open mapping from $O((z'_0, 0)) \cap Q_0$ to $\mathbf{C^n}$. Fix each $z' \in O((z'_0, 0)) \cap Q_0$ and consider the map H which sends each $\xi(\approx 0) \in \mathbf{C}^n$ to $(\hat{f}^*(\overline{\xi}, \overline{\mu(z',\xi)}), f_{n+1}(\overline{\xi}, \overline{\mu(z',\xi)}))$. As argued before, we see that $H(\xi) \in Q'_{(\widetilde{f}^*(z',0),0)}$ and is a finite to one map, too. Now, let π^* be the natural projection from $\mathbf{C^{n+1}}$ to its first n-copies of $\mathbf{C}$. Then, it follows easily that $\pi^* \circ H$ is an open mapping. Hence, from the uniqueness theorem of holomorphic functions, we conclude that (4.6) holds identically. Finally, the same argument as in [BR2] gives a contradiction.

Lemma 4.2: After shrinking $\mathcal{U}_1$, it holds that $\mathcal{F}(\mathcal{U}_1 \cap D) \subset \Omega$ and $\mathcal{F}(\mathcal{U}_1 \cap D^c) \subset \Omega^c$. Here, Ω and Ω^c stay in different sides of M_2.

Proof of Lemma 4.2: We will still use the $\lambda-$function $G(\mathcal{F}, \lambda)$ introduced in Lemma 2.1. Let

$$\rho^*(z, \overline{w}) = 1/N\mathrm{Re}\left(\sum_{\widetilde{f(w)} \in \mathcal{F}(z)} \left(\overline{f_{n+1}(w)} - G(\mathcal{F}(z), \overline{\widetilde{f(w)}})\right)\right),$$

where N is the generic counting number of $\mathcal{F}(z)$. Then by Lemma 2.1 and the Riemann extension theorem, we see that $\rho^*(z, \overline{w})$ is holomorphic in $(z, \overline{w})$. In particular, $\rho^*(z, \overline{z})$ is real analytic near 0. We claim that $\rho^*(z, \overline{z})$ is < 0, when $\mathcal{F}(z) \subset \Omega^+$; > 0 when $\mathcal{F}(z) \subset \Omega^-$; and $= 0$ when $\mathcal{F}(z) \subset M_2$ (see Lemma 3.3 (a) for related notations). In fact, for each fixed $\widetilde{f}(z) \in \mathcal{F}(z)$, we have

$$\mathrm{Re}\left(\overline{f_{n+1}(z)} - G(\widetilde{f}, \overline{\widetilde{f}(z)})\right) = \mathrm{Re}\widetilde{\rho_2}\left(\widetilde{f}, \overline{\widetilde{f}(z)}\right)$$

$$= \rho_2\left(\widetilde{f}, \overline{\widetilde{f}(z)}\right)\mathrm{Re}h_2(\widetilde{f}(z), \overline{\widetilde{f}(z)})) = \rho_2(\widetilde{f}, \overline{\widetilde{f}(z)}))(1 + o(\|\widetilde{f(z)}\|).$$

Thus, the claim follows from Lemma 3.3 (v).

Next, applying Lemma 4.1, one sees that

$$\rho^* = 2\mathrm{Re}(f_{n+1}(z)) + o(|z|) = \mathrm{Re}(g(0)z_{n+1}) + o(|z|)$$

with $g(0) \neq 0$. Thus, $\mathrm{d}_0\rho^* \neq 0$. Therefore, ρ^* serves as a real analytic defining function of M_1 near 0. Without loss of generality, let us assume that $\rho^*(z, \overline{z}) < 0$ for $z \in D$. Then $\rho(z, \overline{z}) > 0$ for $z \in D^c$. Now, by making $\mathcal{U}_1$ small, the above argument shows that $\mathcal{F}(D \cap \mathcal{U}_1) \subset \Omega^+$ and $\mathcal{F}(D^c \cap \mathcal{U}_1) \subset \Omega^-$. So, we can let the Ω in Lemma 4.2 be Ω^+. Of course, Ω^c is then Ω^-. $\square$

Remark 4.3 (a). In what follows, we will assume, without loss of generality, that $\Omega = \Omega^+$. Then we observe that the above results give the fact $g(0) > 0$. To see this, we write $f_{n+1}(z) = \alpha z_{n+1} + o(\|z\|)$ with $\alpha = \alpha_1 + \sqrt{-1}\alpha_2 \neq 0$. Letting $t = x + \sqrt{-1}kx$, then $\rho^*(t) = (\alpha_1 - \alpha_2 k)x + o(|x|)$. Notice that $\rho^*(x) < 0$ for any given real number k and $x(\approx 0) < 0$. We conclude that $(\alpha_1 - \alpha_2 k) > 0$. Thus, we see that $\alpha_2 = 0$ and $\alpha_1 > 0$.

(b). Another immediate application of Lemma 4.2 is that for certain small neighborhoods U^* and $U^{*\prime}$ of 0, f is proper from $U^* \cap D$ to $U^{*\prime} \cap \Omega$. In fact, let B be a sufficiently small ball centered at 0. By certain well-known construction (see [BC], for example) and Lemma 3.3 (iv), we can simply take U^* to be the connected component of the set $\{z \in B \cap D : \ f(z) \notin \partial B \cap D\}$ such that $f(U^*)$ contains $\Omega \cap O(0)$ with $O(0)$ sufficiently small.

5. Completion of the proof of Theorem 1.1

We now proceed to the proof of our main technical theorem. For simplicity, we retain all the notation which we have set up in the previous sections.

As before, denote by $\mathcal{A}$ the map, which sends: $w \in \mathcal{U}_1$ to the finite set A_w. w_0 is called a *separable* point of $\mathcal{A}$ if $A_{w_0} = \{w_j\}_{j=1}^{N^*}$ satisfies the following property: There exist open neighborhoods $O(w_j)$ of w_j $(j = 1, \cdots, N^*)$ such that $\mathcal{A}(w) \cap O_{w_j} = \{w\}$ for any $w \in O_{w_j}$. We write $\mathcal{W} = \{w \in \mathcal{U}_1 : \ w \text{ is not a separable point of } \mathcal{A}\}$. We say that M_1 satisfies *Condition S* at 0, if a holomorphic change of coordinates can be chosen so that M_1 is still defined by an equation of the form as in (2.1) and (2.1)$'$; moreover, the following holds:

(a) $\mathcal{W}$ is contained in an analytic variety E spread over $(z_2, \cdots, z_{n+1})$-space, i.e, E is defined by an equation of the form: $z_1^{N^*} + \sum_{j=0}^{N^*-1} a_j(z_2, \cdots, z_{n+1}) z_1^j = 0$ with $a_j(z_2, \cdots, z_{n+1})$ holomorphic near the origin.

(b) For each small $t > 0$, there is a sufficiently small positive number $\epsilon(t)$ such that for each $b^* = (b_2, \cdots, b_n)$ with $|b^*| < \epsilon(t)$, the set $E \cap \{(z_1, b^*, -t) : z_1 \in \mathbf{C}^1\}$ is contained in some connected component of $D \cap \{(z_1, b^*, -t) : z_1 \in \mathbf{C}^1\}$.

Similarly, we can define $\mathcal{A}'$ and $\mathcal{W}'$ and speak of *Condition S* for M_2. In this case, we use Ω to replace the role of D (see Remark 4.3(a)).

We will see in the following lemma that $\mathcal{W}$ and $\mathcal{W}'$ can be used to control the branching locus of $\mathcal{F}$.

Lemma 5.1: (a) Suppose that M_1 and M_2 satisfy Condition S at the origin. Then $\mathcal{F}(\mathcal{E}_0 \cap O(0)) \subset \mathcal{W}'$, $\mathcal{E}_0 \cap O(0) \subset E$, and $\mathcal{E}_0 \cap O(0) \cap \{(z_1, \cdots, z_n, 0)\}$ is a proper variety near the origin of the $\{(z_1, \cdots, z_n, 0)\}$-space. Here E is chosen as in the definition of Condition S, and $\mathcal{E}_0$ is as defined in Remark 3.4 (c). (When $n = 1$, we always have $\mathcal{F}(\mathcal{E}) \subset \mathcal{W}'$ and thus 0 is an isolated point of $\mathcal{E} \cap Q_0$). (b) M_1 and M_2 satisfy Condition S, when $n = 1$ or when M_1 and M_2 are rigid at the origin.

Proof of Lemma 5.1: We first prove Part (a) of the lemma:

Let $z \in \mathcal{E}_0$ be sufficiently close to 0. Then, by the definition of $\mathcal{E}_0$ and Remarks 3.4 (b) (c), it follows that there is a sequence $z_j \to z$ such that we can find two sequences $\{\eta_j\}$ and $\{\xi_j\}$ with $\xi_j, \eta_j \in \mathcal{F}(z_j)$ $\eta_j \neq \xi_j$, but $\xi_j, \eta_j \to w_0$ for some w_0. By Lemma 3.1, it then follows that $\xi_j \in A'_{\eta_j}$. Hence, w_0 is not a separable point of $\mathcal{A}'$. Again by

Lemma 3.1, we conclude that $\mathcal{F}(z) \subset \mathcal{W}'$. *Notice that for this part, no Condition S is necessary.*

We next prove that $\mathcal{E}_0 \cap O(0) \subset E$. To this aim, we assume, without loss of generality, that $\mathcal{E}_0 \neq \emptyset$. Let $\cup_j \mathcal{E}_j$ be the union of the irreducible components of $\mathcal{E}_0$ near 0. We recall by Remark 3.4 (d), that each $\mathcal{E}_j$ then must be of codimension 1 at 0. Now, since $\mathcal{F}(\mathcal{E}_j) \subset \mathcal{W}'$ is also an analytic variety of codimension 1 at 0 (Lemma 3.3 (iv)), the assumption of Condition S for M_2 indicates that $\mathcal{F}(\mathcal{E}_j) \cap \Omega \neq \emptyset$ near 0. Therefore, it follows from Lemma 4.2, that $\mathcal{E}_j \cap D$ is not empty in any neighborhood of 0. Now, pick any point $z \in \mathcal{E}_j \cap D$, which is sufficiently close to 0. Then, as above, we can find a sequence $z_j \to z$ and $\{\eta_j\}$, $\{\xi_j\}$ with $\xi_j, \eta_j \in \mathcal{F}(z_j)$ $\eta_j \neq \xi_j$, but $\xi_j, \eta_j \to w_0 \in \Omega$. By making z sufficiently close to 0, we can assume by Remark (4.3)(b), that $f(z_0) = w_0$ for some z_0 close to 0. Moreover, f is proper from a small neighborhood U of z_0 and $\overline{U} \cap f^{-1}(w_0) = \{z_0\}$. Since f is single valued, there are two distinct points a_j, $b_j \in U$ such that $f(a_j) = \xi_j$ and $f(b_j) = \eta_j$. Now, the following Claim 5.2 indicates that $a_j \in A_{b_j}$. Since a_j, b_j clearly converge to z_0 as $j \to \infty$, we conclude that $z_0 \in \mathcal{W}$. Again by Lemma 3.1 and Claim 5.2, we see that $z_0, z \in \mathcal{W}$. By the arbitrariness of the choice of z and the assumption of Condition S, this fact implies that $\mathcal{E}_j$ has an open subset contained in $\mathcal{W}$ and thus in E. From the uniqueness of analytic varieties, it follows that $\mathcal{E}_j \subset E$ for any j.

Claim 5.2: Let M_1, M_2, and f be as before. We have that $b \in A_a$ if $Q'_{\mathcal{F}(a)} = Q'_{\mathcal{F}(b)}$, where $a, b \in \mathbf{C}^{n+1}$ are sufficiently close to 0. (For this claim, no Condition S is required).

Proof of Claim 5.2: Let a and b be as in the claim. We then need to show that $Q_a \cap \mathcal{U}_2 = Q_b \cap \mathcal{U}_2$. To this aim, we consider the function $\widetilde{\rho_2}(\mathcal{F}(z), \overline{\widetilde{f}(\omega)})$, for any given $\widetilde{f}(\omega) \in \mathcal{F}(\omega)$. By Lemma 2.1, $\widetilde{\rho_2}(\mathcal{F}(z), \overline{\widetilde{f}(\omega)})$ is well defined and holomorphic in (z, λ) with $\lambda = \overline{\widetilde{f}(\omega)}$ for $(z, \lambda) \in O_z(0) \times O_\lambda(0)$. Meanwhile $\widetilde{\rho_2}(\mathcal{F}(z), \overline{\widetilde{f}(\omega)}) = cz_{n+1} + c\overline{\omega}_{n+1} + O(|z||\lambda| + o(\omega))$ for some $c > 0$, by Remark 4.3 (a). Thus, for each ω sufficiently close to 0, since $|\widetilde{f}(\omega)| \approx 0$, the implicit function theorem indicates that $\widetilde{\rho_2}(\mathcal{F}(z), \overline{\widetilde{f}(\omega)}) = 0$ defines a connected complex hypersurface W_ω in some small neighborhood $U^\#$ of 0, whose size is independent of the parameter ω with $|\omega| << 1$. Notice, by Lemma 3.1, that $\mathcal{F}(Q_\omega) \subset Q'_{\widetilde{f}(\omega)}$ near 0. We see that $Q_\omega \cap U^\# \subset W_\omega$. Thus it follows that $W_\omega = Q_\omega$ in $U^\#$ when $|\omega| << 1$; for Q_ω is also a connected complex submanifold of codimension 1 in $\mathcal{U}_2$.

Now, if $Q'_{\mathcal{F}(b)} = Q'_{\mathcal{F}(a)}$, we then conclude that

$$\widetilde{\rho_2}(\mathcal{F}(z), \overline{\mathcal{F}(b)}) = 0$$

and $\widetilde{\rho}_2(\mathcal{F}(z), \overline{\mathcal{F}(a)}) = 0$ define the same variety. Hence, when $a, b \approx 0$, we conclude, by Lemma 3.0 (iv), that $Q_a = Q_b$; for both of them have a small open subset of

$W_a \cap U^{\#} = W_b \cap U^{\#}$ in common. This gives the proof of Claim 5.2. (We mention that for the proof Claim 5.2, no assumption of Condition S is required).

The last statement in (a) follows clearly from the above facts and the assumption of Condition S for the hypersurfaces. More precisely, we notice that $\mathcal{F}(\mathcal{E}_0 \cap O(0)) \subset \mathcal{W}'$ and $\mathcal{F}(Q_0) \subset Q'_0$. From Lemma 3.3 (iv) and the fact that $\mathcal{W}' \cap \{(z_1, \cdots, z_n, 0)\}$ is contained in a proper subvariety (of codimension at least 1) in Q'_0, it follows that $\mathcal{E}_0 \cap Q_0$ is also of codimension at least 1 in Q_0. *Actually, using the same proof as in Part (b) for the rigid case, and using a simple projection lemma, we notice that this part does not require Condition S neither.* Also, we notice that the argument for this part does not use Claim 5.2, i.e, the use of Lemma 4.1, neither.

Now, we proceed to the proof of (b):

We first consider the case of $n = 1$. As before, we let M_2 be defined by

$$z_2 + \overline{z_2} + \sum_{j=0}^{\infty} \widetilde{\psi_j}(z_1, \overline{z_1}) z_2^j = 0; \text{or } z_2 + \overline{z_2} + \sum_{j=0}^{\infty} \overline{\widetilde{\psi_j}}(\overline{z_1}, z_1) \overline{z_2^j} = 0.$$

For each $b = (b_1, b_2) \approx 0$, $A'_b = \{(w_1, w_2) : \ Q'_w \cap \mathcal{U}'_2 = Q'_b \cap \mathcal{U}'_2\}$. As did in Lemma 3.3, by letting $z_1 = 0$ in the defining equations of Q'_w and Q'_b, one sees that $w_2 = b_2$. Now, Q_w can also be defined by

$$z_2 = -\overline{b_2} - \sum_{j=0}^{\infty} \overline{\widetilde{\psi}}_j(\overline{w}_1, z_1) \overline{b_2}^j,$$

where the notation $\overline{\widetilde{\psi}}$ is the same as explained before. Write

$$\overline{\widetilde{\psi_j}}(\overline{w_1}, z_1) = \sum_{\alpha > 0} \overline{\Xi_{j\alpha}}(\overline{w_1}) z_1^{\alpha}.$$

Then $z_2 = -\overline{b_2} - \sum_\alpha \sum_j \overline{\Xi_{j\alpha}}(\overline{w}_1) \overline{b_2}^j z_1^\alpha$. So, the equation: $Q'_w = Q'_b$ can be written as

$$\sum_j \overline{\Xi_{j\alpha}}(\overline{w}_1) \overline{b_2}^j = \sum_j \overline{\Xi_{j\alpha}}(\overline{b}_1) \overline{b_2}^j$$

or

$$\sum_j \Xi_{j\alpha}(w_1) {b_2}^j = \sum_j \Xi_{j\alpha}(b_1) {b_2}^j,$$

for any α.

Choose k_0 to be the smallest integer so that for certain α_0, $\Xi_{0\alpha_0}(w_1) = a^* w_1^{k_0} + o(|w_1|^{k_0})$ with $a^* \neq 0$.

Write $P(w_1, b_2) = \Xi_{0,\alpha_0}(w_1) + \sum_{j>0} \Xi_{j,\alpha_0}(w_1) b_2^j$. For (b_1, b_2) close to the origin, we define

$$E = \{(b_1, b_2) : \ \text{for some } w_1^0, \ \ P(w_1^0, b_2) = P(b_1, b_2), \ \frac{\partial P}{\partial w_1}(w_1^0, b_2) = 0\}.$$

Then we first claim that near the origin, $\mathcal{W} \subset E$. For this, let $w = (\xi, b_2) \notin E$, and let $C(w) = \{(z_1, b_2) : \ P(z_1, b_2) = P(\xi, b_2)\}$. Then by the definition of P, one has $A'_w \subset C(w)$. Write $A'_w = \{(\tau_j, b_2)\}_j$ and suppose that $w \in W_0$. Then for some j^*, by the definition of $\mathcal{W}$, there are two sequences $\{(\eta_j, c_j)\}$ and $\{(\xi_j, c_j)\}$ such that (a) $\eta_j \neq \xi_j$; (b) $\eta_j, \xi_j \to \tau_{j^*}$ for some index j^*, $c_j \to b_2$; and (c) $A'_{(\eta_j, c_j)} = A'_{(\xi_j, c_j)}$.

From (c) and the definition of P, it follows that $P(\eta_j, c_j) = P(\xi_j, c_j)$. Moreover, by the Taylor expansion: $P(\eta_j, c_j) = P(\xi_j, c_j) + \frac{\partial P}{\partial w_1}(\xi_j, c_j)(\eta_j - \xi_j) + o(|\eta_j - \xi_j|)$; we obtain $\frac{\partial P}{\partial w_1}(\xi_j, c_j) = o(1)$. Letting $j \to 0$, we therefore conclude that $\frac{\partial P}{\partial w_1}(\tau_{j^*}, b_2) = 0$, which together with the fact $P(\tau_{j^*}, b_2) = P(\xi, b_2)$ implies that $w \in E$. This is a contradiction and thus proves our claim.

We next claim that E is an analytic variety spread over b_2-axis. To see that, we let

$$E^* = \{(b_1, b_2, w_1) \in O(0) : P(w_1, b_2) = P(b_1, b_2), \ \frac{\partial P}{\partial w_1}(w_1, b_2) = 0\}.$$

Then E is the projection of E^* to the first two copies of $\mathbf{C^1}$. This projection map is obviously finite to one near the origin and thus is locally proper. So, E is an analytic variety near 0. Meanwhile, for each b_2, from $\frac{\partial P(w_1, b_2)}{\partial w_1} = 0$, we can solve out only finitely many w_1's with $|w_1|$ small. Moreover, for any solution w_1, we can easily see that $|w_1| \lesssim |b_2|^{\frac{1}{k_0 - 1}}$. Thus it follows that there are also only finitely many b_1's so that $P(w_1, b_2) = P(b_1, b_2)$ with $\frac{\partial P(w_1, b_2)}{\partial w_1} = 0$. Furthermore, one can easily see that $|b_1| \lesssim |b_2|^{1/k_0}$. Therefore, we see that E is spread over b_2-axis. From the Weiestrass preparation theorem, it follows that E can be defined near 0 by an equation with the form as given in Part (a) of Condition S. Meanwhile, it is also easy to see that $E \cap (\mathbf{C^1} \times \{b_2\}) \subset \Delta_{|b_2|^{\alpha^*}} \times \{b_2\}$ for $|b_2|$ sufficiently small, where Δ_r is used to stand for the disk in $\mathbf{C^1}$ with center at the origin and with radius r, α^* is a constant between $\frac{1}{k_0+1}$ and $\frac{1}{k_0}$. By our choice of k_0, it follows easily for sufficiently small $|b_2|$, that $\Delta_{|b_2|^{\alpha^*}} \times \{b_2\} \subset D$ when $b_2 < 0$, and $\Delta_{|b_2|^{\alpha^*}} \times \{b_2\} \subset D^c$ when $b_2 > 0$. This completes the proof of Part (a) in case $n = 1$.

Finally, we show that M_1 satisfies Condition S when M_1 is rigid. For this purpose, we assume that M_1 is defined by an equation of the form: $z_{n+1} + \overline{z_{n+1}} + \sum_{|\alpha|, |\beta| > 0} a_{\alpha\beta} z'^{\alpha} \overline{z'^{\beta}} = 0$. Let $P_\alpha(z') = \sum_\beta \overline{a_{\alpha\beta}} z'^{\beta}$. Then for $a = (a', a_{n+1})$ $b = (b', b_{n+1})$ close to the origin, $Q_a = Q_b$ if and only if $a_{n+1} = b_{n+1}$ and $P_\alpha(a') = P_\alpha(b')$. By the finite type assumption, it follows that the common zero of P'_αs is 0 near the origin. Using the Nullstellantz theorem, we see that for some finitely many indices $\{\alpha_j\}_{j=1}^m$, the locus of $\{P_{\alpha_j}\}_{j=1}^m$ is also zero. Now, for a small neighborhood U of 0, let $U' = U \cap \{(z_1, \cdots, z_n, 0)\}$. Define Λ from a small neighborhood U' of $0'$ to $\mathbf{C^m}$ by $\Lambda(z') = (P_{\alpha_j}(z'))_{j=1}^m$. Then Λ is finite to 0 and proper from U', after suitably shrinking of U. So, by the Remmert theorem, we conclude that the set $E' = \Lambda^{-1}\left(\Lambda(\{z \in U : \mathrm{d}\Lambda \text{ does not have maximal rank at } z\})\right)$ is a proper variety of U' (see [DF1] for a similar argument). It is easy to see that $\mathcal{W} \subset E$, with $E = (E' \times \mathbf{C^1}) \cap \mathcal{U}_1$. Now, after a linear change of coordinates in the $(z_1, \cdots, z_n)$-space, we see that E is contained in an analytic variety defined by an equation of the form: $z_1^{N^*} + \sum_{j<N^*} c_j(z_2, \cdots, z_n) z_1^j = 0$ with $c_j(0) = 0$. Since this equation does not contain z_{n+1}-variable, it follows easily that M_1 satisfies Condition S at the origin.

We now give the following lemma, which indicates how the information on the branching locus can be used to obtain the holomorphic extendibility. We use Δ_a to denote the disc in $\mathbf{C^1}$ centered at 0 with radius a.

Lemma 5.3: Let M be a hypersurface near the origin of $\mathbf{C^{n+1}}$ defined by an equation of the form $\rho(z, \overline{z}) = z_{n+1} + \overline{z_{n+1}} + \phi(z, \overline{z}) = 0$ with $\phi(z, z) = o(|z|)$. Let D be the side defined by $\rho(z, \overline{z}) < 0$. Assume that g is a holomorphic function over D that admits a holomorphic correspondence extension to $\Delta_{\delta_0} \times \Delta_\delta^n$ for some small $\delta_0, \delta > 0$, i.e, there exist holomorphic functions $a_j \in \mathrm{Hol}(\Delta_{\delta_0} \times \Delta_\delta^n)$ such that $P(z, g(z)) = g^N(z) + \sum_{j=0}^{N-1} a_j g^j(z) = 0$ for $z \in D$. Here we assume that $P(z, X)$ is irreducible in X, and $\Delta_{\delta_0} \times \Delta_\delta^n \cap D \subset\subset D$. Denote by Z^* the branching locus of g and write Z for the genuine branching locus of g as defined in Remark 3.4 (d). Assume that Z is defined by $\widetilde{P}(z_2, \cdots, z_{n+1}; z_1) = z_1^{N_1} + \sum_{j=0}^{N_1-1} d_j(z_2, \cdots, z_{n+1}) z_1^j = 0$ for certain $d_j(z_2, \cdots, z_{n+1}) \in \mathrm{Hol}(\Delta_\delta^n)$ with $d_j(0) = 0$. Also, suppose that for each small positive number t, there is a positive number $\epsilon(t)$ such that when $|b_j| < \epsilon(t)$ with $j = 2, \cdots, n$, $Z \cap D^*(b^*, t)$ is contained in some connected open subset of $D \cap D^*(b^*, t)$, where $D^*(b^*, t) = \{z : z = (z_1, b_2, \cdots, b_n, -t) : z_1 \in \mathbf{C^1}\}$. Then g admits a holomorphic extension near 0.

Proof of Lemma 5.3: Without loss of generality, we assume that

$$\widetilde{P}(z_2, \cdots, z_{n+1}; X) = 0$$

has no double roots, i.e, $\widetilde{P}(z_2, \cdots, z_{n+1}; X)$ and $\frac{\partial \widetilde{P}}{\partial X}$ are relatively prime. Denote by $\widetilde{Z}$ the branching locus of $\widetilde{P}$, which is a subvariety of Δ_δ^n. Now, we choose a small $t > 0$ and $b^* = (b_2, \cdots, b_n)$ with $|b_j| < \epsilon(t)$ such that $(b^*, -t) \notin \widetilde{Z}$ and $Z^* \cap \mathbf{C^1} \times \{(b^*, -t)\}$ is discrete. Write $B_\eta(b^*, -t)$ for the *closed* ball in $\mathbf{C^n}$ centered at $(b^*, -t)$ with radius η. Then, when η is sufficiently small, $B_\eta(b^*, -t) \cap \widetilde{Z} = \emptyset$ and $0 \notin B_\eta(b^*, -t)$. Construct a pseudoconvex domain $\widetilde{B}(t) \subset\subset \Delta_\delta^n$ such that the following holds: (i) $0 \in \widetilde{B}(t)$, $\widetilde{B}(t) \supset B_\eta(b^*, -t)$; and (ii) there is a contraction map $T(\cdot, \tau) : \widetilde{B}(t) \to \widetilde{B}(t)$ ($\tau \in [0, 1]$) such that $T(\cdot, 0) = \mathrm{id}$, $T(\widetilde{B}(t), 1) = B_\eta(b^*, -t)$, and for any τ, $T(\cdot, \tau)$ is the identical map when restricted to $B_\eta(b^*, -t)$.

Now, using the hypothesis on Z and shrinking δ if necessary, we can find $R > 0$ with $R < \delta_0$ such that $(\Delta_R \setminus \Delta_{\frac{1}{2}R}) \times \Delta_\delta^n \cap Z = \emptyset$. Write $\Omega_0 = \left(\Delta_R \times \widetilde{B}(t)\right) \setminus$

$\overline{\left(\Delta_{\frac{1}{2}R} \times (\widetilde{B}(t) \setminus B_\eta(b^*,-t))\right)}$, which can also be written as

$$(\Delta_R \times \mathrm{int}(B_\eta(b^*,-t))) \cup \left((\Delta_R \setminus \overline{\Delta_{\frac{1}{2}R}}) \times \widetilde{B}(t)\right).$$

Here, for a subset K, we write $\mathrm{int}(K)$ for the collection of the interior points of K. Then, it is easy to see that any (closed) loop in $\Omega_0 \setminus Z$ based at certain $z_0 \in (\Delta_R \times \mathrm{int}(B_\eta(b^*,-t))) \setminus Z^*$ can be deformed, relative to the base point and without cutting Z, to a loop in $(\Delta_R \times B_\eta(b^*,-t)) \setminus Z$. Indeed, let $\gamma = (\gamma_1, \gamma^*)$ be a loop in $\Omega_0 \setminus Z$ based at z_0. Then the above described deformation can be performed by $(\gamma_1(t), T(\gamma^*(t), \tau))$.

We next set up the following notation: For a given connected subset D' (near the origin) with $D' \setminus Z$ pathwise-connected, a point $z_0 \in D' \setminus Z^*$, and a complex number w_0 with $P(z_0, w_0) = 0$, we define $\mathrm{Val}(z_0, w_0; D')$ to be the collection of w's, which is obtained as follows: There are a loop $\gamma \subset D' \setminus Z$, based at z_0, and a continuous section $\mathcal{G}$ from $[0,1]$ to $(\mathcal{O}, \pi, \mathbf{C})$ with $P(z; \mathcal{G}) \equiv 0$ such that each representation of $\mathcal{G}(0)$ takes value w_0 and each representation of $\mathcal{G}(1)$ takes value w at z_0. We write $\mathcal{P}(z) = \{w : \ P(z,w) = 0\}$. When $\mathrm{Val}(z_0, w_0; D')$ is independent of the choice of w_0, we simply use the notation $\mathrm{Val}(z_0; D')$. In particular, this is the case when $\#\mathrm{Val}(z_0, w_0; D') = N$.

Claim 5.4: $\#\mathrm{Val}(z_0; \Delta_R \times B_\eta(b^*,-t)) = N$, i.e,

$$\mathrm{Val}(z_0; \Delta_R \times (B_\eta(b^*,-t)) = \{\text{all solutions of } P(z_0, X) = 0\}.$$

Proof of Claim 5.4: Let Ω_0 be as defined above. We then notice that the holomorphic hull of Ω_0 is $\Delta_R \times \widetilde{B(t)}$. In fact, for any $\phi \in \mathrm{Hol}(\Omega_0)$, the following Cauchy integral gives the holomorphic extension of ϕ to $\Delta_R \times \widetilde{B}(t)$:

$$\frac{1}{2\pi i} \int_{|\xi| = \widetilde{R}} \frac{\phi(\xi, z_2, \cdots, z_{n+1})}{\xi - z_1} \mathrm{d}\xi, \text{ with } \widetilde{R} \to R.$$

Now, we fix a point $z_0 \in \Omega_0 \setminus Z^*$ and a number w_0 with $P(z_0, w_0) = 0$. For $z \in \Omega_0 \setminus Z^*$, let $\gamma \subset \Omega_0 \setminus Z$ be a curve connecting z_0 to z. Denote by w be the value of the branch of $\mathcal{P}$ at z which is continuous along γ and takes the initial value w_0. Write $\mathrm{Val}(z, w; \Omega_0) = \{w_1, \cdots, w_{k(z,w)}\}$. Then, by the monodromy theorem (see Remark 3.4 (d)), one sees that $k(z,w)$ is constant, say k', for all z, w. Also, $\mathrm{Val}(z, w; \Omega_0)$ is independent of the choice of γ, though w does. Now, $P_0(z, X) = \prod_{j=1}^{k'}(X - w_j(z))$ will give a well-defined polynomial in X with coefficients holomorphic and bounded in $\Omega_0 \setminus Z^*$. Applying the Riemann extension theorem, we conclude that $P_0(z, X)$ is a polynomial in X with coefficients holomorphic in Ω_0. The above observation then indicates that the coefficients of P_0 are holomorphic in $\Delta_R \times \widetilde{B}(t)$, too. So, by the irreducibility of P, we conclude that $k' = N$.

Now, from the argument preceding this claim and the monodromy theorem, the proof of Claim 5.4 follows. $\square$

Claim 5.5: Let b^* and t be as chosen above and write $p_0 = (b^*, -t)$. Let $\epsilon(t)$ be sufficiently small. Let $z_0 = (z_1, p_0) \in \Delta_R \times \{p_0\} \setminus Z^*$ for a certain z_1. Then $\#\mathrm{Val}(z_0; \Delta_R \times \{p_0\}) = N$.

We first mention that the proof of Lemma 5.3 follows easily from Claim 5.5. Indeed, by the hypothesis, the finite set $Z \cap \{\Delta_R \times \{p_0\}\}$ stays in some *connected* open subset of $D \cap \{\mathbf{C^1} \times \{p_0\}\}$, say D^*. Since $\#(Z \cap \{\mathbf{C^1} \times \{p_0\}\})$ is finite, we can choose a simply connected smooth domain $D^{**} \subset\subset D^*$ with $Z \cap (\mathbf{C^1} \times \{p_0\}) \subset D^{**} \cap (\mathbf{C^1} \times \{p_0\})$ (see Remark 5.6 (a) below). Therefore, by the Schoenflies theorem, there is a retraction from $\Delta_R \times \{p_0\}$ to $\overline{D^{**}}$ that deforms $\Delta_R \times \{p_0\} \setminus D^{**}$ to the boundary of D^{**}. Thus any loop in Δ_R that is based at $z^* \in D^{**} \setminus Z^*$ can be deformed, relative to the base point, to $\overline{D^{**}}$ without cutting $Z \cap \Delta_R \times \{p_0\}$. Hence, by the monodromy theorem, it follows that $\#\mathrm{Val}(z^*; D^*) = N$ for any given point $z^* \in D^{**}$. But this is impossible; for f is holomorphic in a small neighborhood of D^*. So, to conclude the proof of Lemma 5.3, it suffices for us to prove Claim 5.5.

We now turn to the proof of Claim 5.5. By our choice of $B_\eta(b^*, -t)$, we know that

$$\widetilde{P}(z_2, \cdots, z_{n+1}; z_1) = z_1^{N_1} + \sum_{j<N_1} d_j(z_2, \cdots, z_{n+1}) z_1^j = 0$$

has roots

$$\{\phi_1(z_2, \cdots, z_{n+1}), \cdots, \phi_m(z_2, \cdots, z_{n+1})\},$$

which can be arranged such that $\phi_j(z^*) \in \mathrm{Hol}(B_\eta(b^*, -t))$ and $\phi_j(z^*) \neq \phi_l(z^*)$ for $j \neq l$. So, for a given $z^* \in B_\eta(b^*, -t)$, $(\Delta_R \times \{z^*\}) \cap Z = \{(\phi_j(z^*), z^*)\}_{j=1}^m$. Let $p_j^0 = \phi_j(p_0)$. We will show that there exists a homeomorphism π_{z^*}: $\Delta_R \to \Delta_R$, depending continuously on $z^* \in B_\eta(b^*, -t)$, so that $\pi_{z^*}(\phi_j(z^*)) = p_j^0$. If this can be done, then any loop $\gamma(t) = (\gamma_1(t), \gamma^*(t))$, which is based at z_0 and stays in $(\Delta_R \times B_\eta(b^*, -t)) \setminus Z$ can be deformed to a closed curve in $\Delta_R \times \{p_0\}$ by the following map:

$$\beta(t, \tau) = \left(\pi^{-1}_{\tau p_0 + (1-\tau)\gamma^*(t)} \circ \pi_{\gamma^*(t)}(\gamma_1(t)), \tau p_0 + (1-\tau)\gamma^*(t)\right)$$

with $\beta(t, 0) = \gamma(t)$, $\beta(t, 1) \in \Delta_R \times \{p_0\}$, and $\beta(0, \tau) = \beta(1, \tau) = z_0$. Meanwhile, from the construction, one can verify that $\{\beta(t, \tau)\} \cap Z = \emptyset$. Thus, from the monodromy theorem stated in Remark 3.4 (d), it follows that $\#\mathrm{Val}(z_0; \Delta_R \times \{p_0\}) = N$.

To construct the above mentioned $\pi_{z^*} : \Delta_R \to \Delta_R$. We notice that

$$\min_{j \neq l, z^* \in B_\eta(b^*, -t)} \{|p_j(z^*) - p_l(z^*)|\} > 0.$$

By using a Möbius transform, we can assume that $p_1(p_0) = p_1^0 = 0$. Suppose that for an index $j < m$, we can find a homeomorphic self map $\sigma_j(\cdot, z^*)$ of Δ_R, which depends continuously on z^* and maps $p_l(z^*)$ to p_l^0 for $l \leq j$. For the brevity of the notations, let us still write $p_j(z^*)$ for the transformed points: $\sigma_j(p_j(z^*), z^*)$. We then wish to find a homeomorphic self map $\widetilde{\sigma_{j+1}}(\cdot, z^*)$ of Δ_R, depending continuously on $z^* \in B_\eta(b^*, -t)$, such that $\widetilde{\sigma_{j+1}}(p_l^0, z^*) = p_l^0$ for $l \leq j$ and $\widetilde{\sigma_{j+1}}(p_{j+1}(z^*), z^*) = p_{j+1}^0$. If this can be done, letting $\sigma_{j+1}(\cdot, z^*) = \widetilde{\sigma_{j+1}}(\cdot, z^*) \circ \sigma_j(\cdot, z^*)$, and applying the induction argument, we see the existence of the aforementioned map π_{z^*}. Indeed, the existence of $\widetilde{\sigma_{j+1}}(\cdot, z^*)$ clearly follows from the following Remark 5.6 (b) and by making $\epsilon(t)$ sufficiently small:

Remark 5.6: In this remark, we would like to say a few words about two simple facts in the set topology that were used in the proof of Lemma 5.2:

(a) Let D be a connected domain in $\mathbf{C^1}$. Then, for any given finitely many points $\{p_j\}_{j=1}^k \subset D$, there is a smoothly bounded simply connected domain $D^* \subset\subset D$ such that $\{p_j\}_{j=1}^k \subset D^*$.

Indeed, we can find a curve γ in D connecting all of the p_j's, which, by the piecewise linear approximation, can be assumed to be piecewise linear. Making further division to the image of γ if necessary, we can assume that $C = \gamma([0,1])$ is a linear graph (or, a finite CW-complex of dimension 1), which contains $\{p_j\}$ in the set of its vertices. Now, a well-known result in Topology indicates that we can find a maximal tree $\widetilde{C}$ inside C, which, by definition, is simply connected and contains all the vertices of C. Furthermore, the tubular neighborhood basis theorem shows the existence of a small neighborhood $\widetilde{D}(\subset D^*)$ of $\widetilde{C}$, that can be retracted to $\widetilde{C}$ and thus is simply connected, too. (See for example, J. F. Hudson: *Piecewise Linear Topology*, W. A. Benjamin, Inc, 1969; in particular, Theorem 2.11, pp57). Now, we have a Riemann mapping σ from $\widetilde{D}$ to Δ. Then obviously, $\sigma^{-1}(\Delta_r)$ does our job when $r < 1$ is sufficiently close to 1. $\square$.

(b) Let $\{a, a_1, \cdots, a_k\}$ be $k+1$ distinct points in Δ. Suppose that $p(z)$ is a continuous map from the ball the closed unit ball $B_n \subset \mathbf{C^n}$ to Δ such that $\epsilon = \min_{j=1,\cdots,k; z\in B_n} |p(z) - a_j|$ is a sufficiently small positive number, and $|p(z) - p(z_0)| < \frac{1}{4}\epsilon$. Then there exists a homeomorphic self map $\sigma(\cdot, z)$ of Δ_R, which depends continuously on z, such that $\sigma(p(z), z) = a$ and $\sigma(a_j, z) = a_j$ for $j = 1, \cdots, k$.

The proof of this fact can be seen as follows: Pick a continuous retraction π from B_n to z_0. We will then construct a homeomorphic self map $\sigma(\cdot, z)$ of Δ, that depends continuously on $z \in B_n$, such that the following holds: (a) $\sigma(a_j, z) = a_j$ $(j = 1, \cdots, k)$; (b) $\sigma(\cdot, z_0) = \mathrm{id}$; (c) $\sigma(p(z), z) = p(z_0)$ for $z \in B_n$. To this aim, we assume that $p(z_0) = 0$ to simplify the notation. Notice that $|p(z) - p(z_0)| < \epsilon/4$ and thus $p(z)$ is contained in the disk $\Delta_{\epsilon/4}$. We also observe, by the hypothesis, that $a_j \notin \Delta_\epsilon$. Now, we let $\tau(\cdot, z)$ be the Möbius transform from $\Delta_{\epsilon/4}$ to itself that maps $p(z)$ for z to $p(z_0)$. Meanwhile, we impose the condition that $\tau'(p(z), z) > 0$ to make it unique. From the explicit formula of τ, it is clear that $\tau(\cdot, z)$ depends continuously on $z \in B_n$.

For $w = \epsilon/4e^{\sqrt{-1}\theta}$ with $0 \le \theta < 2\pi$, write $\tau(w, z) = \frac{\epsilon}{4}e^{\sqrt{-1}r(\theta,z)}$, where $r(0, z) \in [0, 2\pi)$, $r(2\pi, z) - r(0, z) = 2\pi$, and $r(\theta, z)$ is a strictly increasing function of θ for fixed z. Moreover, $r(\theta, z)$ depend continuously on (θ, z). The homeomorphism $\sigma(\cdot, z)$ from Δ to itself can be defined as follows:

(i) $\sigma(\cdot, z_0) = \mathrm{id}$. For $z \in B_n$, we let $\sigma(w, z) = w$, for $w \notin \Delta_\epsilon$; $\sigma(w, z) = \tau(w, z)$, for $w \in \Delta_{\epsilon/4}$; and otherwise

$$\sigma(w, z) = |w| \exp\left(\sqrt{-1}\frac{4}{3\epsilon}\left((|w| - \epsilon/4)\theta + (\epsilon - |w|)h(\theta, z)\right)\right),$$

where $w = |w|e^{\sqrt{-1}\theta}$ with $|w| \in [\epsilon/4, \epsilon]$.

It can be easily verified that σ possesses all the properties we imposed.

Completion of the proof of Theorem 1.1. We now are ready to apply the previously established results to present a proof of Theorem 1.1. Indeed Theorem 1.1 follows from the following Theorem 1.1$'$ and Lemma 5.1 (b).

Theorem 1.1$'$: Let $M_1 \subset \mathbf{C^{n+1}}$ and $M_2 \subset \mathbf{C^{n+1}}$ be real analytic hypersurfaces of finite D-type. Suppose that f is a CR mapping from M_1 to M_2. Let $p \in M_2$. Assume that f extends as a holomorphic correspondence near p and assume furthermore that M_1, M_2 satisfy Condition S at p, and $f(p)$, with respect to D and Ω as introduced before, respectively. Then f admits a holomorphic extension near p.

Proof of Theorem 1.1$'$: For each component f_j of f, we see that the genuine branching locus of the polynomial equation defining f_j, denoted by $\mathcal{E}_0^{(j)}$, is contained in $\mathcal{E}_0$. By Lemma 5.1, it then follows that for sufficiently small $t > 0$, one can always find $\epsilon(t) > 0$ such that when $|b_j| < \epsilon(t)$ $(j = 2, \cdots, n)$, $\mathcal{E}_0^{(j)} \cap \{(z_1, b_2, \cdots, b_n, -t)\}$ is contained in some connected component of $D \cap \{(z_1, b_2, \cdots, b_n, -t)\}$. Now, applying Lemma 5.3, we conclude that f_j extends holomorphically across 0 for each j. $\square$

Remark 5.7: By examing the proof, it can be seen that Theorem 1.1$'$ also holds when M_1 and M_2 are merely assumed to be essentially finite at the origin in the sense that $A_0 = A_0' = \{0\}$ (see [BJT] or [DW]).

6. Proofs of Theorem 1.2 and Theorem 1.4

Now, we pass to the proofs of Theorem 1.2 and Theorem 1.4. We first let M_1, M_2, and f be as in Theorem 1.2. We also assume that f is not constant. Then we claim that f must be an algebraic map. To see this, we first stratify M_2 into the disjoint union of the sets $M_2^\pm$, S_2, S_1, and S_0. Here $M_2^\pm$ is the set of points in M_2 where the Levi form is non-zero; S_2 is a locally finite union of $2-$dimensional real analytic totally real submanifolds in $M_2 \setminus M_2^\pm$; S_1 is a locally finite union of 1-dimensional real analytic submanifolds in $M_2 \setminus (M^\pm \cap S_2)$; and S_0 is a locally finite subset. For the existence of such a semi-analytic stratification, we refer the reader to a related explanation appeared in [DFY]. Assume that $0 \in M_1$ and $f(0) = 0$. Since the algebraicity is a global property, we need only to show the map is algebraic at some small piece of M_1. Using the finite type assumption of M_1, we can assume the existence of some non empty open piece U_0 in $M_1^{\pm}$, where $M_1^{\pm}$ denotes the set of points in M_1 where the Levi form do not vanish. When $f(U_0) \cap M_2^{\pm} \neq \emptyset$, by results of Pinchuk-Tsyganov [PS], Lwey [Le], and Pinchuk [Pi], we see that f is actually real analytic at some point p in U_0. Thus applying a result of Baouendi-Rothschild [BR3], we see $J_f \not\equiv 0$ near p. Now, the algebraicity follows from a theorem of Webster [We].

When $f(U_0) \subset S_2$, by shrinking U_0 if necessary, we assume that f extends to a certain side, say D, of M_1. As in the proof of Lemma 2.1, we can find a totally real submanifold $E \subset U_0$ and a wedge $W^+ \subset D$ with edge E. Moreover, we can assume that $f(E)$ is contained in a connected piece of S_2. Now, by the reflection principle, we conclude that f admits a holomorphic extension near E. Since f maps a neighborhood of E in U_0 to a two dimensional manifold, we easily see that $J_f \equiv 0$ near E. This contradicts the non constant assumption for f, by [BR3]. Similarly, we can exclude the case: $f(U_0) \subset S_1 \cup S_0$. Summarizing the above arguments, we can conclude the algebraicity of f.

We also recall a result in [DF2], which states that any proper holomorphic map between two bounded algebraic domains in $\mathbf{C^n}$ $(n > 1)$ is continuous up to the boundary. Meanwhile, it is also algebraic by [We] (see also [DF2]). Thus, the proofs of Theorem 1.2, Corollary 1.2$'$, and Theorem 1.4 now clearly follow from Theorem 1.1 and the following general result:

Lemma 6.1: Let M_1 and M_2 be two connected algebraic hypersurfaces of finite D-type in $\mathbf{C^n}$ $(n > 1)$ and assume that D is bounded domain with M_1 as part of its boundary. Suppose that f is an algebraic holomorphic map from D to $\mathbf{C^n}$, that is continuous up to $D \cup M_1$, and maps M_1 into M_2. Then for each point $p \in M_1$, f extends as a holomorphic correspondence to a neighborhood of p.

Proof of Lemma 6.1: Let M_1 and M_2 be defined by two real polynomials $\rho_1(z, \overline{z})$ and $\rho_2(z, \overline{z})$, respectively. Without loss of generality, we assume that $p = 0$ and $f(0) = 0$. By the hypothesis, we suppose that the j^{th} component f_j of f satisfies the irreducible polynomial equation $P_j(z, X) = \sum_{l=0}^{N_j} a_{jl}(z) X^l = 0$. As mentioned before, by a result of Baouendi-Treves [BT] (see also [Tr]), we can assume, without loss of generality, that any CR function defined over M_1 can be extended to D.

Write V for the irreducible variety in $\mathbf{C^n} \times \mathbf{C^n}$, defined by $P_j(z, w) = 0$ for $j = 1, \cdots, n$, which contains the graph Γ_f of f over D. Write π, π' for the nature projections from V to the first and the second copies of $\mathbf{C^n}$, respectively. Define E to be the collection of points which are either the zeros of $a_{jN_j}(z)$ for some j, or are the branching points of of $P_j(z, X)$ for certain j. Notice that E is an analytic variety of codimension ≥ 1.

Claim 6.2: Fix a point $z_0 \in (M_1 \setminus E) \cap \mathcal{U}_1$ and choose a sufficiently small ball B centered at z_0. Consider the multiple-valued extension $\mathcal{F}_j$ of f_j from $B \cap D$ to

$B \cup (D^c \cap \mathcal{U}_1) \setminus E$ with $\mathcal{U}_1$ as chosen before (see §3 for a precise definition of $\mathcal{F}_j$) . If there is a constant C such that $|\widetilde{f}(z)| \leq C$ for any $z \in B \cup (D^c \cap \mathcal{U}_1)$ and $\widetilde{f}(z) \in \mathcal{F}_j(z)$, then f_j, near 0, satisfies a polynomial equation with leading coefficient 1.

Proof of Claim 6.2: For each $z \in B \cup (D^c \cap \mathcal{U}_1) \setminus E$, let

$$\mathcal{F}_j(z) = \{f_j^{(1)}(z), \cdots, f_j^{(k(z))}(z)\}$$

. First, by the monodromy theorem, $k(z)$ is constant. Now, suppose that all elements in $\mathcal{F}_j(z)$ is bounded by a constant independent of z. As we did before, using the Riemann extension theorem, we conclude that $\widetilde{P_j}(z, X) = \prod_l (X - f_j^{(l)}(z))$ is a polynomial in X with coefficients holomorphic and bounded in $D^c \cap \mathcal{U}_1$. Now, by the aforementioned result of Baouendi-Treves and Trepreau, one sees that the coefficients of $\widetilde{P_j}$ can be extended holomorphically to a neighborhood of 0. $\widetilde{P_j}$ has leading coefficient 1 and annihilates $f_j(z)$ when $z \in \mathcal{U}_1$. □

Now, seeking a contradiction, suppose that Lemma 6.1 is false.

Then Claim 6.2 indicates that no matter how we shrink $\mathcal{U}_1$ and make z_0 close to 0, there exists a fixed index j_0 such that one can always find some point $z^1 \in B \cup (D^c \cap \mathcal{U}_1) \setminus E$ with $|\hat{f}_{j_0}(z^1)| > 1$ for certain $\hat{f}_{j_0}(z) \in \mathcal{F}_{j_0}(z)$. Observe that for a given curve $\gamma \subset B \cup (D^c \cap \mathcal{U}_1) \setminus E$ with $\gamma(0) = z_0$ and $\gamma(1) = z^1$, there is a certain holomorphic branch $\widetilde{f_{j_0}}$ of $\mathcal{F}_{j_0}$ along γ, which takes value $f_{j_0}(z_0)$ at z_0 and takes value $\hat{f}_{j_0}(z^1)$ at z^1.

On the other hand, by the Whitney approximation theorem (see J. Milnor: *Differential Topology*, Lecture Notes by Munkress; in particular, Theorem 1.28 and Lemma 1.29) and by noting that $B \cup (D^c \cap \mathcal{U}_1) \setminus E$ can be retracted to $D^c \cap \mathcal{U}_1 \setminus E$, we notice the existence of a smooth simple curve γ^* connecting z_0 to z with $\gamma((0, 1]) \subset (D^c \cap \mathcal{U}_1) \setminus E$ and γ transversal to M_1 at z_0. Moreover γ^* is homorpotic to γ (relative to the base point) in $B \cup (D^c \cap \mathcal{U}_1) \setminus E$.

So, using the Monodromy theorem we can simply assume that $\gamma = \gamma^*$. As what we did in Lemma 3.1, we can thicken $\widetilde{g} = \gamma \cup \mathcal{R}(\gamma)$ and then apply the uniqueness theorem of holomorphic functions and the invariant property of Segre varieties to conclude that $f(Q_z \cap O(z^*)) \subset Q'_{\widetilde{f}(z)}$, for any $z \in \gamma$, where the branch $\widetilde{f}$ of $\mathcal{F}$ is determined by the initial value condition $\widetilde{f}(z_0) = f(z_0)$.

Since $f(0) = 0$, for any small $\epsilon > 0$, by changing the size of $\mathcal{U}_1$, we see the existence of a sequence $\widetilde{p}_j \to 0$ such that $|\widetilde{f}(\widetilde{p}_j)| = \epsilon$. In particular, we see that the cluster set $Cl_{\widetilde{f}}(0)$ of $\widetilde{f}$ at 0 contains points with norm ϵ. Here, the branch $\widetilde{f}$ is as explained above. We now are ready to use the following argument to show that this is impossible, which was first used in [DF2] for a different purpose.

Let $\hat{V}$ be the algebraic closure of V in $\mathbf{P^n} \times \mathbf{P^n}$ (see [Wh]). Without loss of generality, we can further assume that $\hat{V}$ is irreducible and contains Γ_f. Let $\hat{\pi}$ and $\hat{\pi}'$

be the nature projection from $\hat{V}$ to the first and the second copies of $\mathbf{P^n}$, respectively. Since $\hat{V}$ is of dimension n, away from a proper subvariety, we see that $\hat{\pi}$ and $\hat{\pi}'$ are local biholomorphisms. For each z, let $Q_z = \{w \in \mathbf{C^n} \cap \mathcal{U}_1 : \ \rho_1(w, \overline{z}) = 0\}$ as before, and let $\hat{Q}_z$ be the compactification of Q_z in $\mathbf{P^n}$. Now, for each $w \in Cl_{\widetilde{f}}(0)$, the cluster set of $\widetilde{f}$ at 0, let $z_j \to 0$ with $\widetilde{f}(z_j) = w_j \to w_0$. Then $z_j \in D^c$ and $f(Q_{z_j} \cap O(z_j^*)) \subset Q'_{w_j}$. This gives that $\hat{\pi}' \circ \hat{\pi}^{-1}(\hat{Q_{z_j}}) \supset Q'_{w_j} \cap \mathcal{U}'_2$, where $\mathcal{U}'_2$ is as defined in §2. Now, letting $j \to \infty$, we see that $\hat{\pi}' \circ \hat{\pi}^{-1}(\hat{Q_0}) \supset Q_{w'} \cap \mathcal{U}'_1$. Notice that $\pi' \circ \hat{\pi}^{-1}(\hat{Q_0})$ is an analytic variety of $\mathbf{P}^n$ of codimension 1, by the Remmert mapping theorem. Hence, it follows that $\hat{Q}_{w'} \cap \mathcal{U}'_2$ has to be contained in one of the finite irreducible components of $\pi' \circ \pi^{-1}(\hat{Q_0})$. Since $\#A'_w < \infty$ ($w \in \mathcal{U}'_2$), for each connected component of $\pi' \circ \pi^{-1}(\hat{Q_0})$, there are only finitely many w's with $|w|$ small such that the above property holds. This implies that $\#(Cl_{\widetilde{f}}(0) \cap O(0)) < \infty$ and contradicts our assumption. The proof of Lemma 6.2 is complete. $\square$

7. Extending proper holomorphic mappings in $\mathbf{C^2}$- Proof of Theorem 1.3

With Theorem 1.1 at our disposal, we now use the ideas appeared in [DFY] and, in particular, [DP] to present a proof of the following Theorem 1.3 $'$, which together with Theorem 1.1 gives Theorem 1.3. We would like to mention again the whole strategy is similar to that in [DP] (see also closely related work in [DFY]) for the study of biholomorphic maps.

Theorem 1.3$'$: Let D and D' be two smoothly bounded domains with real analytic boundaries in $\mathbf{C^2}$. Let f be a proper holomorphic map from D to D'. Suppose that f extends continuously up to ∂D. Then f extends as a holomorphic correspondence across ∂D. By Theorem 1.1, it thus follows that f admits a holomorphic extension to $\overline{D}$.

To start with, we let M be a real analytic hypersurface of finite type in $\mathbf{C^2}$, and let Ω be a domain with M as part of its smooth boundary. Then one has the following semi-analytic stratification as introduced in [DFY]:

$$M = M^+ \cup M^- \cup T_2^+ \cup T_2^- \cup T_2^{\pm} \cup T_1 \cup T_0.$$

Here
(a) M^+ (M^-) is the set of strongly pseudoconvex (strongly pseudoconcave, respectively) boundary points of $\Omega \cup M$;
(b) T_2^+ (T_2^-) is a locally finite union of 2-dimensional totally real analytic submanifolds, where the vanishing order of the Levi form of M, as the boundary of $\Omega \cup M$, is even and positive (negative, respectively). Therefore, Ω is pseudoconvex near T_2^+ and pseudoconcave near T_2^-;

(c) $T_2^{\pm}$ is a locally finite union of 2-dimensional totally real submanifolds, where the vanishing order of the Levi form is odd. It is known (see [DF3] or [BCT]) that any CR functions defined near $T_2^{\pm}$ can be holomorphically extended to both sides of M near $T_2^{\pm}$;
(d) T_1 is a locally finite union of one dimensional real analytic curves; and T_0 is a locally finite subset of M.

Now, let D and D' be as in Theorem 1.3$'$. After making the above type of the stratifications, we have the following disjoint unions:

$$D = \partial D^{+} \cup \partial D^{-} \cup T_2^{+} \cup T_2^{-} \cup T_2^{\pm} \cup T_1 \cup T_0,$$

$$D' = \partial D'^{+} \cup \partial D'^{-} \cup T'^{+}_2 \cup T'^{-}_2 \cup T'^{\pm}_2 \cup T'_1 \cup T'_0.$$

In what follows, we write $\Sigma' = \partial D'^{+} \cup \partial D'^{-} \cup T'^{+}_2 \cup T'^{-}_2 \cup T'^{\pm}_2$, and write Σ for the boundary points in ∂D where f extends holomorphically.

Lemma 7.1: (a) For any boundary point $p \in \partial D$, if $f(p) \in \Sigma'$, then f extends holomorphically across p. (b) f extends almost everywhere in ∂D and $\Sigma \cap T_2^{+}$ is dense in T_2^{+}.

Proof of Lemma 7.1: We first note that in case $q = f(p) \in \partial D'^{-} \cup T'^{\pm}_2 \cup T'^{-}_2$, then p stays inside the holomorphic hull of D (see [BHR2]). Thus f extends automatically at p. We now assume that $q = f(p) \in \partial D'^{+} \cup T'^{+}_2$. Pick a small pseudoconvex piece $M' \subset \partial D$ of q, and let M be the connected component of $f^{-1}(M')$ which contains p. Then since $f(\partial D) \subset \partial D'$, it is clear that $f^{-1}(q) \subset\subset M$. We notice that M has to be pseudoconvex, by the properness and boundary continuity of f. Hence, using a result of Bell-Catlin [BC], it follows that f is smooth at p. By [BBR], we therefore conclude that f extends holomorphically across p.

Next, if for some point p and a small open subset $U_p \subset \partial D$ near p, $f(T_2^{+} \cap U_p) \subset T'_1 \cup T'_0$, then it is easy to see that there is a wedge W^{+} with edge $U_p \cap T_2^{+}$ such that $W^{+} \subset D$. Indeed, after a local change of coordinates, one can assume that $p = 0$, $T_2^{+} = i\mathbf{R}^2$, and $T_0^{(1,0)}$ is given by z_1-axis. Then, one can simply take $W^{+} = \{(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2) \approx (0,0) :\ x_1 < 0, |x_j| > |y_j|\}$.

Notice that $T'_1 \cup T'_0$ is contained in the union of finitely many analytic arcs and finitely many points. Using the continuity of f and applying the reflection principle, one sees that f extends holomorphically in a small neighborhood of some point $z \in U_p \cap T_2^{+}$. Now, notice that $J_f \not\equiv 0$ for $w \in U_p \cap T_2^{+}$ $(\approx z)$. This gives a contradiction; for f maps the two dimensional manifold $U_p \cap T_2^{+}$ into a one dimensional semi-analytic subset. Now, from the above argument, it follows that $\Sigma \cap T_2^{+}$ is dense in T_2^{+}.

Similarly, one can see that f extends holomorphically across almost every point in ∂D, as did in the beginning of §6. $\square$

Now, we fix $p_0 \in \partial D$ and $q_0 = f(p_0)$. We will show that f extends as a holomorphic correspondence near p_0. We can also assume, without loss of generality [BT] [Tr], that every CR function defined on ∂D near p_0 can be extended holomorphically to the side in D.

For each $p \approx p_0$ and $q \approx q_0$ with $q_0 = f(p_0)$, we introduce Segre neighborhood systems near p and q, as did in §2: $\mathcal{U}_3(p) \supset\supset \mathcal{U}_2(p) \supset\supset \mathcal{U}_1(p) \ni p$; $\mathcal{U}_3'(q) \supset\supset \mathcal{U}_2'(q) \supset\supset \mathcal{U}_1'(q) \ni q$. As in [DFY] and [DP] (in particular, see [DP]), we define the "potential" correspondence of f near p_0 as follows:

$$\mathcal{V}(p_0, q_0) = \{(w, w') \in \mathcal{U}_1(p_0) \times \mathcal{U}_1'(q_0) : \ f(Q_w(p_0) \cap \mathcal{U}_2(p_0) \cap D) \supset \ {}_{w'^*}Q'_{w'}(q_0)\}.$$

Here, as in §2, we write w'^* for the reflection point $\mathcal{R}'(w')$ of w'; and we use ${}_{w'*}Q'_{w'}$ to denote the germ of the variety $Q'_{w'}$ at ${w'}^*$. Also, the Segre varieties are chosen in the indicated Segre systems.

In what follows, we will drop the reference points for the Segre systems and Segre varieties to simplify the notation, when there is no confusion arising.

By the invariant property of Segre surfaces, it can be seen that if f extends holomorphically in a small neighborhood G of some point $p \in \mathcal{U}_1(p_0) \cap \partial D$, then $\mathcal{V}(p_0, q_0) \supset \{(w, w') \approx (p, f(p)) : \ w' = f(w), \ w \in G \setminus \overline{D}\}$.

The key point in proving Theorem 1.3$'$ is to show that $\mathcal{V}(p_0, q_0)$ extends as an analytic variety in $\mathcal{U}_1(p_0) \times \mathcal{U}_1'(q_0)$ with the first projection map $\pi : \mathcal{V}(p_0, q_0) \to \mathcal{U}_1(p_0)$ locally proper. For this purpose, we will use the strategy of Diederich-Pinchuk [DP] of applying the extension lemma of Bishop ([Chi]).

Before arguing further, we need recall some useful facts:

(7.a)

$$\partial D \setminus (\mathcal{U}_2(p_0) \cap \partial D \cap Q_T)$$

is dense in $\partial D \cap \mathcal{U}_1(p_0)$, where $Q_T = \cup_{z \in (T_1 \cup T_0) \cap \mathcal{U}_1(p_0)} Q_z(p_0)$.

(7.b) ([DP]): Making $\mathcal{U}_1'$ small, then $(T_1' \cup T_0') \cap \mathcal{U}_1'(q_0)$ is contained in a pluripolar set $\widetilde{T}$ in $\mathcal{U}_1'$. That is, there is a plurisubharmonic function h ($\not\equiv -\infty$) such that $T_0' \cup T_0' \subset \widetilde{T} = \{z \in \mathcal{U}_1' : \ h(z) = -\infty\}$. (This follows from the fact that the semi-analytic subset $(T_1' \cup T_0') \cap \mathcal{U}_1'$ is contained in a one dimensional real analytic subset [BM], which can be complexified.)

(7.c) Fix $\mathcal{U}_1'$. Using the properness and the boundary continuity of f, one can always choose $\mathcal{U}_2$ small enough so that $\mathcal{V}$ has no limit points in $(\mathcal{U}_1 \setminus \overline{D}) \times \partial \mathcal{U}_1'$.

(7.d) ([DP]) For each $p \in T_2^+$, after making $\mathcal{U}_2(p)$ small, it holds that $Q_p \cap \mathcal{U}_2(p) \cap \partial D \subset \partial D^+ \cup \{p\}$.

(7.e) For each $p \in \partial D^+$, after making $\mathcal{U}_2(p)$ small and then making $\mathcal{U}_1(p)$ sufficiently small, it then holds that $Q_{\widetilde{p}} \cap \mathcal{U}_2(p) \cap \partial D = \{\widetilde{p}\}$ for any $\widetilde{p} \in \mathcal{U}_1(p)$.

(7.f) ([BR2]) Let M_1 and M_2 be two connected oriented real analytic hypersurfaces in $\mathbf{C^n}$ of finite D-type. Let f be a non constant holomorphic map from M_1 into M_2 with $f(M_1) \subset M_2$. Then f is locally finite and the normal component of f has non

vanishing normal derivative at each point of M_1. So, by the standard result, it follows that f is locally proper at each point in M_1 and preserves the sides.
(7.h) (Bishop's Lemma [Chi]): Let E be a (complete) pluripolar set in the domain $U\times U'\subset \mathbf{C}^4$, where U and U' are domains in $\mathbf{C^2}$. Let $A\subset (U\times U')\setminus E$ be an analytic set of pure dimension 2 with no limit points in $U\times \partial U'$. Suppose that there exists an open subset $V\subset U$ such that $\overline{A}\cap((V\times U'))$ is an analytic variety in $V\times U'$ of pure dimension 2. Then $\overline{A}$ is an analytic set in $U\times U'$.

Remark $(7.1)'$(a): We observe that (7.d) can be strengthened as follows: For any point $\widetilde{p}(\in T_2^+)\approx p$, it also holds that $Q_{\widetilde{p}}\cap\mathcal{U}_2(p)\cap\partial D\subset\partial D^+\cup\widetilde{p}$. The proof of this fact is the same as that for (7.d) (see [DP]). However, since this fact will be important in our later discussion, for the convenience of a reader, we give the following details (see a similar argument in [DP]):

After an appropriate holomorphic change of coordinates, we can assume that $p=0$ and T_2^+ near p is $i\mathbf{R}^2=\{(iy_1,iy_2)\}$. We also choose z_1-axis for the complex tangent space of ∂D at 0. By the way we chose T_2^+, we can assume that D is defined near 0 by $\rho=2x_2+(2x_1)^k a(z_1,\overline{z_1},y_2)$, where $m>1$ and $a|_{i\mathbf{R}^2}\not\equiv 0$. By the way we chose T_2^+, we can see that k is even and $a(0)>0$ (see [DP]). Now, for $w=(ib_1,ib_2)\in i\mathbf{R}^2$, Q_w is defined by the equation: $z_2-ib_2+(z_1-ib_1)^{2m}a(z,\overline{w})=0$. So, it can be seen that near the origin, $Q_w\cap i\mathbf{R}^2=\{w\}$. This completes the proof of the above fact.

(b) To see $(7.a)$, we notice the facts that $Q_z\cap\partial D$ is at most a real analytic subset of dimension 1 and $T_1\cup T_0$ is contained in a real analytic set of at most dimension 1 [BM]. Therefore, $\mathcal{U}_2(p_0)\cap\partial D\cap Q_T$ is contained in a real analytic space which can be fibered along a one dimensional real analytic curve and each fiber has at most dimension 1. Thus, $\mathcal{U}_2(p_0)\cap\partial D\cap Q_T$ has at most dimension 2.

(c) (7.e) can be seen as follows: For each $\widetilde{p}$ close to p, after a normal holomorphic change of coordinates (see related discussions in §2) and using the strong pseudoconvexity of ∂D at $\widetilde{p}$, one sees clearly that $Q_{\widetilde{p}}$ cuts ∂D only at $\widetilde{p}$. Since this process can be done uniformly, we can achieve (7.e) after shrinking the size of $\mathcal{U}_2$.

Lemma 7.2: Shrinking $\mathcal{U}_1(p)$ if necessary, one can make $\mathcal{V}(p,q)$ having no limit points in $(\mathcal{U}_1(p)\setminus\overline{D})\times\Sigma'$.

Proof of Lemma 7.2: Suppose that $(w_v,w_v')\in\mathcal{V}$ with $(w_v,w_v')\to(w_0,w_0')\in(\mathcal{U}_1\setminus\overline{D})\times\Sigma'$. Then $f(Q_{w_v}\cap\mathcal{U}_2\cap D)\supset{}_{w'^*_v}Q'_{w'_v}$. In particular, from the properness of f, it can be seen that there exists a point $\xi_v^*\in Q_{w_v}\cap\overline{\mathcal{U}}_2\cap D$ such that $f(\xi_v^*)=w_v'^*$ and $f(Q_{w_v}\cap O(\xi_v^*))\supset_{w'^*_v}Q'_{w'_v}$ Assume, without loss of generality, that $\xi_v^*\to\xi_0\in\overline{\mathcal{U}}_2\setminus D$ and $w_v^*\to w_0'$. Then $f(\xi_0)=w_0'$ and $\xi_0\in\partial D$. By Lemma 7.1, it follows that f extends holomorphically to a neighborhood of ξ_0. Therefore, there is a small open subset Ω near ξ_0 such that f is a proper holomorphic map from Ω to $f(\Omega)$ and $f|_{\overline{\Omega}}^{-1}(f(\xi_0))=\xi_0$ (see (7.f)). Meanwhile, $f(\Omega\cap D)\subset D'$, $f(\Omega\cap D^c)\subset D'^c\cap\mathcal{U}_2'$. Now, let $\eta_v\in\Omega$ be such that $f(\eta_v) = w'_v$. Then, for $v \gg 1$, it holds that $f(Q_{\eta_v} \cap O(\eta_v^*)) \subset Q'_{w'_v}$. Obviously, $\eta_v \to \xi_0$ and $f(Q_{\eta_v} \cap \Omega) \supset f(Q_{w_v} \cap O(\xi_0))$. Write $f|_\Omega^{-1} = \{\sigma_1, \cdots, \sigma_k\}$. Hence, we see that $Q_{w_v} \cap O(\xi_0) \subset \cup_{j=1}^k \sigma_j(f(Q_{\eta_v} \cap \Omega))$.

Similar to Claim 5.2, we have the following

Claim 7.2′: For v sufficiently large, $Q_{\eta_v} \cap \mathcal{U}_2 = Q_{w_v} \cap \mathcal{U}_2$. Thus, by passing to the limit, it holds that $w_0 \in A_{\xi_0}$.

Proof of Claim 7.2′: Let ρ and ρ' be the defining functions of D and D', respectively. Then we notice that $\rho'(f(z), \overline{f}(w)) = \rho(z, \overline{w})h(z, \overline{w})$ for $z, w \in \Omega$ (see (7.f)) and $h \neq 0$ (we might need to shrink Ω here).

For $z \in \cup_{j=1}^k \sigma_j(f(Q_{\eta_v} \cap \Omega))$, there is a point $\tilde{z} \in Q_{\eta_v} \cap \Omega$ such that $f(z) = f(\tilde{z})$. Now, from

$$\rho'(f(z), \overline{f(\eta_v)})) = \rho(z, \overline{\eta_v})h(z, \overline{\eta_v})$$

and $\rho'(f(\tilde{z}), \overline{f(\eta_v)}) = \rho(\tilde{z}, \overline{\eta_v})h(\tilde{z}, \overline{\eta_v}) = 0$, it follows that $\rho(z, \overline{\eta_v}) = 0$. Thus, we conclude that $z \in Q_{\eta_v}$. Since both Q_{w_v} and Q_{η_v} are connected complex curves in $\mathcal{U}_2$, we see the $Q_{\eta_v} \cap O(\xi_0) = Q_{w_v} \cap O(\xi_0)$

Passing to a limit, we now conclude that $Q_{\xi_0} \cap \mathcal{U}_2 = Q_{w_0} \cap \mathcal{U}_2$.

Next, $w_0 \in A_{\xi_0}$. Since $\xi_0 \in \partial D$, $w_0 \in \partial D$. This is a contradiction and thus completes the proof of Lemma 7.2. $\square$

In what follows, we always make $\mathcal{U}_1$ small such that Lemma 7.2 holds. Following the strategy of Diederich-Pinchuk, we now prove the following lemma.

Lemma 7.3: Suppose that for certain $\mathcal{U}_2(p)$ small enough, it holds that $(Q_p \cap \partial D \cap \mathcal{U}_2(p)) \setminus \{p\} \subset \Sigma$. Then after shrinking $\mathcal{U}_1$ and $\mathcal{U}'_1$ if necessary, $\mathcal{V}(p, q)$ is an analytic sub-variety of $(\mathcal{U}_1(p) \setminus \overline{D}) \times (\mathcal{U}'_1(p) \setminus \overline{D'})$ which has no limit points in $\Big((\mathcal{U}_1 \setminus \overline{D}) \times \partial \mathcal{U}'_1\Big) \cup \Big((\mathcal{U}_1 \setminus \overline{D}) \times \Sigma'\Big)$. We mention that in this lemma, $(Q_p \cap \partial D \cap \mathcal{U}_2(p)) \setminus \{p\}$ can be empty, and all Segre surfaces are defined in the Segre system as chosen here.

Proof of Lemma 7.3: Part of the proof is greatly motivated by an argument in [DP] for the study of biholomorphic mappings.

Let $\mathcal{U}_2$ be sufficiently small so that $(Q_p \cap \overline{D} \cap \overline{\mathcal{U}}_2) \setminus \{p\} \subset \Sigma$. We also assume the property in (7.c). When $(Q_p \cap \overline{D} \cap \partial \mathcal{U}_2) \setminus \{p\} \neq \emptyset$, then by the hypothesis, there is a small $\epsilon > 0$ such that f extends holomorphically to $\overline{B_{3\epsilon}(E_p)}$, where $E_p = Q_p \cap \overline{D} \cap \partial \mathcal{U}_2$ and $B_\delta(E_p) = \{z : \operatorname{dist}(z, E_p) < \delta\}$ for $\delta > 0$. Moreover, by (7.f), we can assume that f is finite to one from $B_{3\epsilon}(E_p)$ and $p \notin B_{4\epsilon}(E_p)$. Now, we let $R'(p) = \{w' \in \mathcal{U}'_1 \setminus D' : f(Q_p \cap B_{2\epsilon}(E_p)) \supset {}_{w'^*}Q'_{w'}\}$. Then we claim that $R'(p)$ is a finite set. Indeed, this can be seen as follows: Let $\{w'_v (\in R'(p))\}$ with $w'^*_v \to w'^*$. Choose $w_v \in Q_p \cap B_{2\epsilon}(E_p)$ such that $f(w_v) = w'^*_v$. Assume also that $w_v \to w \in Q_p \cap \overline{B_{2\epsilon}(E_p)}$. Then as above,

f is a proper holomorphic map from a small neighborhood of w. This implies that $f(Q_p \cap O(w))$ is an analytic curve near w'^*. Thus, it follows that any $Q'_{w'_v}$ coincides with $Q'_{w'}$ when $v \gg 1$. So, by the finiteness of $A'_{w'}$, it follows that $\{w'_v\}$ is a finite sequence.

Let $R(p) = \{w \in Q_p \cap B_{2\epsilon}(E_p) : f(w) = w'^* \text{ for some } w' \in R'(p)\}$. Then $R(p)$ is also finite. Slightly shrinking $\mathcal{U}_2$ and ϵ if necessary, we can assume that $R(p) \cap \partial\mathcal{U}_2 \cap \overline{D} = \emptyset$.

To complete the proof of Lemma 7.3, we need to show that after shrinking $\mathcal{U}_1$, $\mathcal{V}$ is an analytic sub-variety in $(\mathcal{U}_1 \setminus \overline{D}) \times (\mathcal{U}'_1 \setminus \overline{D'})$, which has no limit points in $\Big((\mathcal{U}_1 \setminus \overline{D}) \times \partial\mathcal{U}'_1\Big) \cup \Big((\mathcal{U}_1 \setminus \overline{D}) \times \Sigma'\Big)$.

We first claim that $\mathcal{V}$ is a local variety. To this aim, we will prove that after shrinking the size of $\mathcal{U}_1$, for any $(w, w') \in \mathcal{V}$, then $E^*(w, w') \cap \partial\mathcal{U}_2 \cap Q_w \cap \overline{D} = \emptyset$, where $E^*(w, w') = \{\xi \in (\mathcal{U}_2 \cap D) \cup B_\epsilon(E_w) \cap Q_w : f({}_\xi Q_w) \supset_{w'^*} Q'_{w'}\}$. Here $E_w = Q_w \cap \overline{D} \cap \partial\mathcal{U}_2$ and $B_\epsilon(E_w)$ is defined in a similar way. Also, by $f({}_\xi Q_w) \supset_{w'^*} Q'_{w'}$, we mean that for each small neighborhood $O(\xi)$ of ξ, it holds that $f(Q_w \cap O(\xi)) \supset_{w'^*} Q'_{w'}$.

We also make $\overline{B_\epsilon(E_w)} \subset B_{2\epsilon}(E_p)$ by shrinking $\mathcal{U}_1$.

Indeed, when $E_p = (Q_p \cap \overline{D} \cap \partial\mathcal{U}_2) = \emptyset$, a continuity argument then shows that $E_w = (Q_w \cap \overline{D} \cap \partial\mathcal{U}_2) = \emptyset$ for $w \approx p$.

Now, if the above assertion does not hold in case $E_p \neq \emptyset$, after shrinking $\mathcal{U}_1$ again, we can then find several sequences $\{z_j\} \subset \partial\mathcal{U}_2 \cap \overline{D} \cap E^*(w_j, w'_j)$ with $z_j \to z_0$, $\{w_j\} \subset \mathcal{U}_1 \setminus \overline{D}$ with $w_j \to p$, and $\{w'_j\} \subset \mathcal{U}'_1 \setminus \overline{D}'$ with $w'_j \to w'_0 \in \mathcal{U}'_1$ such that the following holds:

(a) $z_j \in Q_{w_j}$, (b) $f(z_j) = w'_j$, and (c) $f({}_{z_j}Q_{w_j} \cap B_\epsilon(E_{w_j})) \supset_{w'^*_j} Q_{w'_j}$ when $j >> 1$. We observe that f extends to a proper holomorphic map from $O(z_0) \cap B_\epsilon(E_p)$, Hence, by passing to the limit, we can see that $f({}_{z_0}Q_p \cap B_\epsilon(E_p)) \supset_{w'^*_0} Q'_{w'_0}$, $f(z_0) = w'_0$, and $z_0 \in Q_p \cap \partial\mathcal{U}_2 \cap \overline{D}$. Hence, we have a contradiction; for it implies that $z_0 \in R(p) \cap \partial\mathcal{U}_2$.

Now, by making the size of $\mathcal{U}_1$ sufficiently small, we can assume that for each $(w, w') \in \mathcal{V}$, $E^*(w, w') \cap D \cap \overline{\mathcal{U}}_2 \subset \mathcal{U}_2 \cap D$. Write $E^*(w, w') \cap D \cap \overline{\mathcal{U}}_2 = \{a_1, \cdots, a_m\} \subset D \cap \mathcal{U}_2$. Let $(\widetilde{w}, \widetilde{w}')(\in \mathcal{V}) \approx (w, w')$. Then, since $E(\widetilde{w}, \widetilde{w}') \cap D \cap \partial\mathcal{U}_2 = \emptyset$, it can be seen that there exists a point $\widetilde{\xi}$ so that $\widetilde{\xi} \in Q_{\widetilde{w}} \cap \mathcal{U}_2 \cap D$, $f({}_{\widetilde{\xi}}Q_{\widetilde{w}}) \supset_{\widetilde{w}'^*} Q'_{w'}$, and $f(\widetilde{\xi}) = \widetilde{w'}^*$. Indeed, let $\{\widetilde{\xi}_j\} = f^{-1}(\widetilde{w'^*}) \cap \overline{\mathcal{U}}_2 \cap D$. Then since f is locally proper near each $\widetilde{\xi}_j$, the choice of our w' indicates that $f(O(\widetilde{\xi}_l) \cap Q_{\widetilde{w}}) \supset_{\widetilde{w}'^*} Q'_{w'}$ for a certain l. So, we can take $\widetilde{\xi}$ to be this $\widetilde{\xi}_l$, which has to stay in $\mathcal{U}_2 \cap D$.

By the properness of f and by the continuous dependence of Segre surfaces on the base points, it follows that $\widetilde{\xi}$ has to be close to one of the $a'_j s$ when $\widetilde{w}$ is close to w; for otherwise we would have $E^*(w, w') \cap \overline{D} \cap \partial\mathcal{U}_2 \neq \emptyset$. We let ρ_2 be a real analytic defining function of $\partial D'$ near $q = f(p)$, and we choose a coordinates system near p so that the tangent space $T_p^{(1,0)}\partial D$ is the z_1-axis. Then $Q_{\widetilde{w}}$ is given by $z_2 = h(z_1, \overline{\widetilde{w}})$ for some holomorphic function h. We claim that the condition that $f(Q_{\widetilde{w}}\cap D\cap\mathcal{U}_2)\supset {}_{\widetilde{w'}^*}Q'_{\widetilde{w'}}$ for $(\widetilde{w},\widetilde{w'})\approx(w,w')$, can be expressed by the equation $\rho_2(f(z),\overline{\widetilde{w}}')=0$ with $z\in Q_{\widetilde{w}}\cap O(a_j)$ for some j. Indeed, $\rho_2(f(z),\overline{\widetilde{w}}')=0$ implies that $f(Q_{\widetilde{w}}\cap O(a_j))\subset Q'_{\widetilde{w'}}$. Since we can make $O(a_j)$ sufficiently small so that f is proper from $O(a_j)$ to a neighborhood, say O'_j, of w'^*, $f(Q_z\cap O(a_j))$ is also an analytic variety of O'_j of codimension 1. Since we can choose O_j and O'_j suitably so that $O'_j\cap Q'_{\widetilde{w'}}$ is connected for $\widetilde{w'}\approx w'$, thus $f(Q_{\widetilde{w}}\cap O(a_j))=Q_{\widetilde{w'}}\cap O'_j$. Now, if we make $\widetilde{w'}$ close enough to w', then by the continuity of the $\mathcal{R}$-operator, $\widetilde{w}'^*\in f(Q_{\widetilde{w}}\cap O(a_j))$. This verifies our claim.

On the other hand, we have the following Taylor expansion near $a_l^{(1)}$ for each l, where $a_l^{(1)}$ is the first coordinate of a_l:

$$\rho_2(f(z),\overline{\widetilde{w}}')=\sum_j b_j^l(\overline{\widetilde{w}},\overline{\widetilde{w}'})(z_1-a_l^{(1)})^j$$

Let V_l be the common zeros of $b_j^l(\overline{\widetilde{w}},\overline{\widetilde{w}'})$ for $j=0,1,\cdots$. Then clearly near (w,w'), $\mathcal{V}$ is the finite union of V_l's. This shows that $\mathcal{V}$ is a local variety.

We next let $(w_v,w'_v)(\in\mathcal{V})\to(w_0,w'_0)\in(\mathcal{U}_1\setminus\overline{D})\times(\mathcal{U}'_1\setminus\overline{D}')$. We then show that $(w_0,w'_0)\in\mathcal{V}$, too. Notice that $f(Q_{w_v}\cap\mathcal{U}_2)\supset {}_{w'^*_v}Q'_{w'_v}$. So, as argued before, there is a point $\xi_v\in Q_{w_v}\cap\mathcal{U}_2\cap D$ such that $f({}_{\xi_v}Q_{w_v})\supset_{w'^*_v}Q'_{w'_v}$. Assume, without loss of generality, that $\xi_0=\lim\xi_v$. Letting $v\to\infty$, since $\xi_0\in Q_{w_0}\cap\overline{\mathcal{U}}_2\cap D$ and since f is locally proper near ξ_0, it follows that $f(O(\xi_0)\cap Q_{w_0}\cap(B_\epsilon(E_{w_0})\cup(\mathcal{U}_2\cap D))\supset_{w'^*_0}Q'_{w'_0}$. Thus, by the above choice of $\mathcal{U}_2$, it follows that $\xi_0\in\mathcal{U}_2$. We therefore conclude that $f(Q_{w_0}\cap\mathcal{U}_2\cap D)\supset {}_{w'^*_0}Q'_{w'_0}$. This shows that $(w_0,w'_0)\in(\mathcal{U}_1\setminus\overline{D})\times(\mathcal{U}'_1\setminus\overline{D'})$.

Next, applying Lemma 7.2 and (7.b), we conclude that $\mathcal{V}$ has no limit points in $(\mathcal{U}_1\setminus\overline{D})\times\partial\mathcal{U}'_1\cup(\mathcal{U}_1\setminus\overline{D})\times\Sigma'$. □

We still keep the above notation and assume the hypothesis in Lemma 7.3. Notice that $\mathcal{V}$ contains points of dimension 2 (see the observation made after Lemma 7.1). We see that $\mathcal{V}$ is an analytic variety of dimension 2 in $\Big((\mathcal{U}_1\setminus\overline{D})\times\mathcal{U}'_1\Big)\setminus A)$. Here $A=(\mathcal{U}_1\setminus\overline{D})\times(T'_1\cup T'_0)$.

Decompose $\mathcal{V}$ into a locally finite union of its irreducible components. In what follows, we will be only interested in the locally finite union of those two dimensional components, which we will still denote by $\mathcal{V}$, for the brevity of notations. (The new $\mathcal{V}$ can be formed by considering all smooth points of dimension 2 in the old $\mathcal{V}$ and then adding new points by taking the closure). Now, we can say that $\mathcal{V}$ is an analytic variety in $((\mathcal{U}_1\setminus\overline{D})\times\mathcal{U}'_1))\setminus A$, which has pure dimension 2.

Next, we let $\widetilde{A}=(\mathcal{U}_1\setminus\overline{D})\times\widetilde{T}(\supset A)$, which is a pluripolar set in $(\mathcal{U}_1\setminus\overline{D})\times\mathcal{U}'_1$ by (7.b). We also write $\widetilde{\mathcal{V}}=\mathcal{V}\setminus\Big((\mathcal{U}_1\setminus\overline{D})\times\widetilde{T})\Big)$. We now think of $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ as subsets in $(\mathcal{U}_1 \setminus \overline{D}) \times \mathcal{U}_1'$. Then, we see that $\widetilde{\mathcal{V}}$ is an analytic variety of pure dimension 2 in $\Big((\mathcal{U}_1 \setminus \overline{D}) \times \mathcal{U}_1'\Big) \setminus \widetilde{A}$.

Lemma 7.4: As before, let $\{\mathcal{U}_j\}_{j=1}^3$ be a Segre neighborhood system at p and assume the hypothesis in Lemma 7.3. If there exists a sequence $\{p_j\} \subset \Sigma$ with $p_j \to p$, such that $Q_{p_j} \cap \partial D \cap \mathcal{U}_2 \subset \Sigma$, then $\mathcal{V}$ extends as an analytic variety of pure dimension 2 in $\mathcal{U}_1 \times \mathcal{U}_1'$, after shrinking $\mathcal{U}_1$ and $\mathcal{U}_1'$. Thus, by Theorem 1.1, f admits a holomorphic extension at p.

Proof of Lemma 7.4: The main idea of the proof is to apply the Bishop lemma, as first used in the work of Diederich-Pinchuk [DP]. Without loss of generality, we also assume that any CR function defined near $p \in \partial D$ can be holomorphically extended to D (near p) (see [BT] and [Tr]).

We let z be a certain p_j with $j \gg 1$. Slightly shrinking $\mathcal{U}_2$, we observe that $Q_z \cap \partial D \cap \overline{\mathcal{U}}_2 \subset \Sigma$. It thus follows that for a small neighborhood G of z, one also has $Q_{\tilde{z}} \cap \partial D \cap \overline{\mathcal{U}}_2 \subset \Sigma$ for any $\tilde{z} \in \overline{G}$. Now, we consider $\mathcal{V}^* = \Big(\mathcal{V} \cap ((G \setminus \overline{D}) \times \mathcal{U}_1')\Big)$. Then as did in Lemma 7.2, one sees that $\mathcal{V}^*$ has no limit points in $(G \setminus \overline{D}) \times \Sigma'$. We claim that $\mathcal{V}^*$ has no limit points in $(G \setminus \overline{D}) \times (T_1' \cup T_0')$, neither. To see this, we let $(w_v, w_v')(\in \mathcal{V}^*) \to (w_0, w_0') \in (G \setminus \overline{D}) \times (T_1' \cup T_0')$. Then, as in Claim 7.2 $'$, one sees that for certain $\xi_0 \in Q_{w_0} \cap \partial D \cap \overline{\mathcal{U}}_2$, it holds that $f(\xi_0) = w_0'$. However, by our choice of G, we know that $\xi_0 \in \Sigma$. Thus, a similar argument as in Claim 7.2$'$ indicates that $w_0 \in A_{\xi_0} \in \partial D$. This is a contradiction.

Hence, from the above argument, it follows that $\overline{\mathcal{V}^*} \cap ((G \setminus \overline{D}) \times \mathcal{U}_1') = \mathcal{V}^*$ is an analytic variety of pure dimension 2 in $(G \setminus \overline{D}) \times \mathcal{U}_1'$ without limit points in $\mathcal{U}_1 \times \partial \mathcal{U}_1'$. Now, we let $\widetilde{\mathcal{V}}^* = \mathcal{V}^* \setminus \widetilde{A}$. Then $\overline{\widetilde{\mathcal{V}}^*} \cap ((G \setminus \overline{D}) \times \mathcal{U}_1') = \overline{\widetilde{\mathcal{V}}} \cap ((G \setminus \overline{D}) \times \mathcal{U}_1')$.

Claim 7.5: $\overline{\widetilde{\mathcal{V}}^*} \cap ((G \setminus \overline{D}) \times \mathcal{U}_1') = \mathcal{V}^*$. Thus, $\overline{\widetilde{\mathcal{V}}} \cap ((G \setminus \overline{D}) \times \mathcal{U}_1')$ is an analytic variety of pure dimension 2 in $(G \setminus \overline{D}) \times \mathcal{U}_1'$.

Proof of Claim 7.5: Let $\pi_G : \mathcal{V}^* \to G$ and $\pi_G' : \mathcal{V}^* \to \mathcal{U}_1'$. We first show that π_G' is finite to one, after shrinking G if necessary. To this aim, assume that $w' \in \mathcal{U}_1'$ and let $\{z_j\}$ be a sequence in $\pi_G({\pi_G'}^{-1}(w'))$ with $z_j \to z_0(\in \overline{G})$. As in Lemma 7.3, we can then assume that for a certain sequence $\{\xi_j(\in Q_{z_j})\}$, it holds that $\xi_j(\in Q_{z_j} \cap \overline{\mathcal{U}}_2) \to \xi_0 \in Q_{z_0} \cap \overline{\mathcal{U}}_2 \cap \overline{D}$, $f(\xi_j) \to w'^*$, and $f({}_{\xi_j}Q_{z_j}) \supset {}_{w'^*}Q_{w'}'$. As before, by our choice of G, if follows that there is a small neighborhood of ξ_0 which is mapped properly by f to a neighborhood of w'^*. Thus, $f^{-1}({}_{w'^*}Q_{w'})$ has only finitely many components near ξ_0. Therefore, we conclude that $\{Q_{z_j}\}$ is a finite sequence. Making use of the fact that A_z is finite for each $z \in G$, we see that $\pi_G^{-1}(w')$ is a finite set and thus π_G is a local analytic covering map. In particular, π_G' is an open mapping.

Next, we consider $\mathcal{V}^* \cap ((G \setminus \overline{D}) \times \widetilde{T})$. Then the above fact indicates that it does not contain any non-empty open subset of $\mathcal{V}^*$. In fact, if this is not the case, $\pi'_G(\mathcal{V}^* \cap ((G \setminus \overline{D}) \times \widetilde{T}) \subset \widetilde{T}$ would contain open subsets of $\mathcal{U}'$, which contradicts the property of $\widetilde{T}$ (see (7.b)). Hence, we see that $\overline{\widetilde{\mathcal{V}^*}} \cap ((G \setminus \overline{D}) \times \mathcal{U}'_1) = \mathcal{V}^*$. $\square$

Now, we can apply the Bishop extension lemma (7.h) to conclude that $\widetilde{\mathcal{V}}$ extends as an analytic variety of dimension 2 in $(\mathcal{U}_1 \setminus \overline{D}) \times \mathcal{U}'_1$, which has no limit points in $\mathcal{U}_1 \times \partial \mathcal{U}'_1$. Let us still write the extended variety as $\mathcal{V}$. Since any compact analytic variety in $\mathbf{C^2}$ has to be a finite set, we can see, by using the above mentioned property of $\mathcal{V}$, that $\pi^{-1}(z)$ is finite for each $z \in \mathcal{U}_1 \setminus \overline{D}$, and π is proper. Hence, $\mathcal{V}$ can have only finitely many irreducible components, $\{\mathcal{V}\}_{j=1}^m$, (of dimension 2); and each of them has to be a sheeted analytic covering space over $\mathcal{U}_1 \setminus \overline{D}$ ([Wh]).

So, each $\mathcal{V}_i$ can be defined by equations of the form ([Wh]):

$$P_{ik}(w, w'_k) = {w'_k}^{N_{ik}} + \sum_{0 \le j < N_{ik}} a_{ikj}(w) {w'}_k^j$$

with $a_{ikj}(w)$ holomorphic and bounded in $\mathcal{U}_1 \setminus \overline{D}$. By the result of Baouendi-Treves [BT] and Trepreau [Tr], we see that a_{ikj} can be holomorphically extended to a neighborhood of p. So each $\mathcal{V}_i$ extends to a neighborhood of p. Notice that there are points arbitrarily close to p where f extends, we conclude that $\Gamma_f \subset \cup \mathcal{V}_i$. This tells that f extends as a holomorphic correspondence near p. Applying Theorem 1.1, we see that f extends holomorphically across p. $\square$

We are now ready to complete the proof of Theorem 1.3$'$ by the argument appeared in [DFY]: First, $\Sigma \supset \partial D^- \cup T_2^- \cup T_2^{\pm}$. Since f extends almost everywhere, from (7.e), Lemma 7.3, and Lemma 7.4, it follows that $\partial D^+ \subset \Sigma$. Applying Lemma 7.1, Lemma 7.3 and Lemma 7.4 with Remark 7.1$'$(a), we conclude, from the fact $\partial D^+ \subset \Sigma$, that $T_2^+ \subset \Sigma$. Now, we can use (7.a) to show that $T_1 \cup T_0 \subset \Sigma$:

Let $z (\approx p_0) \in T_1$. Since A_{p_0} is finite, it follows that only finitely many z's can make $Q_z \cap \partial D$ to be a real analytic set of dimension 1 in T_1. Thus, by Lemma 7.3, Lemma 7.4, and (7.a); it follows that f extends at most except at finitely many points near p_0. Applying again Lemma 7.3 and Lemma 7.4, we see that f extends also as a holomorphic correspondence at those isolated points, too. The proof of Theorem 1.3$'$ is now complete. $\square$

Remark 7.6: Except the first part of Lemma 7.1, the entire proof of Theorem 1.3$'$ is purely a local argument. So, using the following Lemma 7.1$''$ to replace part of Lemma 7.1, we also see the proof of the following local version of Theorem 1.3

Theorem 1.3$''$: Let D_1 and D_2 be two bounded domains in $\mathbf{C^2}$. Assume that M_1 and M_2 are contained in the real analytic boundaries of D_1 and D_2, respectively. Suppose that f is a proper holomorphic mapping from D_1 to D_2, that is continuous on $D_1 \cup M_1$ and sends M_1 into M_2. Also assume that M_1 and M_2 are of finite type. Then f admits a holomorphic extension across M_1.

Lemma 7.1$''$: Assume the notation and hypothesis in Theorem 1.3$''$. Let $p_0 \in M_1$ be such that D_2 is pseudoconvex near $q_0 = f(p_0)$. Then f admits a holomorphic extension at p_0.

Proof of Lemma 7.1$''$: Construct a pseudoconvex domain of finite type $\Omega_2 \subset D_2$ with $\partial\Omega_2$ containing a small piece of M_2 near q_0. Let Ω_1 be a connected component of $f^{-1}(\Omega_2)$ that contains p_0 in part of its smooth boundary. Then f is proper from Ω_1 to Ω_2 and maps M_1 near p_0 to M_2 near q_0. Meanwhile, it also follows that Ω_1 is pseudoconvex. Now, by examining the proof of Theorem 2 of [BC] and by [BBR], to see the extension of f at p_0, it suffices for us to show the for $z(\approx z_0) \in \Omega_1$, it holds that $\operatorname{dist}(f(z), M_2)^\gamma \lesssim \operatorname{dist}(z, M_1)$, for some positive integer γ. But this can be seen by applying the Hopf lemma, a classical argument of Henkin, and using the existence of family of plurisubharmonic peaking functions near p_0; as did in [Ber] (pp 629, line -15, pp 629, line 11) and [BC] (pp359 line -6, pp 360 line 7).

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