REDUCING ESSENTIAL EIGENVALUES IN THE BOUNDARY OF THE NUMERICAL RANGE

NORBERTO SALINAS AND MARIA VICTORIA VELASCO

ABSTRACT. We examine a purely geometric property of a point in the boundary of the numerical range of a (Hilbert space) operator that implies that such a point is a reducing essential eigenvalue of the given operator. Roughly speaking, such a property means that the boundary curve of the numerical range has infinite curvature at that point (we must exclude however linear verteces because they may be reducing eigenvalues without being reducing essential eigenvalues). This result allows us to give an elegant proof of a conjecture of Joel Anderson: *A compact perturbation of a scalar multiple of the identity operator can not have the closure of its numerical range equal to half a disk (neither equal to any acute circular sector).*

In this note, we examine a purely geometric property of a point in the boundary of the numerical range of a (Hilbert space) operator which implies that such a point is a reducing essential eigenvalue of the given operator (see Theorem 1.7 and Corollary 1.8). Roughly speaking, such a property means that the boundary curve of the numerical range has infinite curvature at that point (we must exclude however linear verteces because they may be reducing eigenvalues without being reducing essential eigenvalues). This result allows us to give an elegant proof of a conjecture of Joel Anderson (see [1], [7], [8]):

> *A compact perturbation of a scalar multiple of the identity operator can not have the closure of its numerical range equal to half a disk (neither equal to any acute circular sector).*

Indeed, there are two non-smooth points in the boundary of such a convex set that have infinite curvature and that are not linear verteces. So they are reducing essential eigenvalues (from Corollary 8) which is not possible for a multiple of the identity operator.

The above mentioned theorem also proves in the affirmative a strengthening of a conjecture of Mathias Hübner [6]: such points (of infinite curvature) are in the essential spectrum of the given operator.

1. Reducing Essential Eigenvalues and the Numerical Range

In this section, we recall some basic facts about the numerical range of an operator on Hilbert space, and the notion of reducing essential eigenvalues.

Research at MSRI is supported in part by MSRI grant DMS-9022140.

Definition 1.1. Let T be an operator acting on a (fixed) Hilbert space $\mathcal{H}$. The *numerical range* and the *state numerical range* of T are defined, respectively, as follows

$$W(T) := \{(\langle Tx, x\rangle : \ x \in \mathcal{H}, \ \|x\| = 1\},$$

$$W_s(T) := \{\phi T : \ \phi \in S(C^*(T))\},$$

where $C^*(T)$ denotes the C^*-algebra generated by T and the identity operator I on $\mathcal{H}$. Also, we used the notation $S(\mathcal{A})$ for the state-space of the C^*-algebra $\mathcal{A}$.

Remark 1.2. (a) We recall [10] that $W(T)$ is always convex, and $W_s(T) = \overline{W(T)}$.

(b) The problem of determining the class of bounded convex sets of the form $W(T)$ for some operator T on $\mathcal{H}$ is still open (an easy cardinality argument shows that there are many bounded convex sets which are not in that class). In [1] it is shown that $W(T)$ is always a borel subset of $\mathbb{C}$, and in [8] it is proved that the components of $W(T)$ in the boundary of $W(T)$ must be either singletons or conic arcs. On the other hand, it is not even known whether $W(T)$ can be the union of the open unit disk and a non-trivial open arc in its boundary.

Definition 1.3. Let λ be a point in the boundary of a convex subset C of the complex plane (with non-empty interior). We say that λ is of *infinite curvature* if, after a suitable rotation and translation which identifies λ with zero, we can choose the real axis as supporting line for C (in which case, we shall say that C is in standard position) and then

$$\lim_{\substack{\operatorname{Re}\alpha\to 0\\ \operatorname{Im}\alpha\to 0\\ \alpha\in C}} \frac{|\operatorname{Im}\alpha|}{\operatorname{Re}^2\alpha} = \infty.$$

For convenience sake, we adopt the convention that C is in standard position if it is contained in the upper-half plane, and the point of the boundary under study is the origin. If the real axis is the only tangent line to the boundary of C, then we call the origin a point of *infinite lefthand curvature* whenever

$$\lim_{\substack{\operatorname{Re}\alpha\to 0-\\ \operatorname{Im}\alpha\to 0\\ \alpha\in C}} \frac{|\operatorname{Im}\alpha|}{\operatorname{Re}^2\alpha} = \infty,$$

and we call the origin a point of *infinite righthand curvature* whenever

$$\lim_{\substack{\operatorname{Re}\alpha\to 0+\\ \operatorname{Im}\alpha\to 0\\ \alpha\in C}} \frac{|\operatorname{Im}\alpha|}{\operatorname{Re}^2\alpha} = \infty.$$

Remark 1.4. (a) Obviously, in order to show that a point is of infinite curvature, the suitable rotation will not be unique in general. Of course, this will not be an obstacle for deducing later some analitic properties that such a points may satisfy.

Moreover, it should be noticed that the property of having infinite curvature depends only on the graph of the function defined by the boundary of C near $\lambda = 0$. Indeed, let C be in a standard position. Then it is easily shown that λ is of infinite curvature precisely when

$$\lim_{\substack{\alpha \to 0 \\ \alpha \in \partial C}} \frac{\operatorname{Im} \alpha}{\operatorname{Re}^2 \alpha} = \infty.$$

(b) Of course, a linear vertex in the boundary of a convex set C (i.e., a point λ in ∂C such that C can be written as the convex hull of λ and another convex set $C' \neq C$) is a point of infinite curvature of C. In fact, it readily follows that even a corner in the boundary of a convex set is a point of infinite curvature. (We recall [7] that a point λ in the boundary of a convex set C is called a *corner* if C is a subset of a convex set C' where λ is a linear vertex). But, it is easy to come up with examples of points of infinite curvature where the boundary is continuously differentiable.

(c) To have infinite lefthand curvature means that C, and more precisely the graph given by ∂C, has infinite curvature on the left of zero, while to have infinite righthand curvature means that the curvature on the right of zero is infinite. It is also clear that a point may be of infinite onesided curvature but not of infinite curvature. For instance, this is the case for the origin in the convex set $C := A \cup B$ where $A := \{\alpha \in \mathbb{C} : \ -1 \leq \operatorname{Re} \alpha \leq 0, \ 0 \leq \operatorname{Im} \alpha \leq 1\}$ and $B := \{\alpha \in \mathbb{C} : \ (\operatorname{Re} \alpha)^{3/2} \leq \operatorname{Im} \alpha \leq 1, \ 0 \leq \operatorname{Re} \alpha \leq 1\}$.

Definition 1.5. A point λ is called an *reducing approximate eigenvalue* of an operator T if there exists a sequence u_n of unit vectors in $\mathcal{H}$ such that

$$\lim_{n \to \infty} \|(T - \lambda I)u_n\| + \|(T^* - \overline{\lambda} I)u_n\| = 0.$$

If in addition u_n tends weakly to zero, then λ is called a *reducing essential eigenvalue*. The set of all reducing approximate eigenvalues of the operator T will be denoted by $R(T)$ and the set of reducing essential eigenvalues by $R_e(T)$.

A point $\lambda \in \mathbb{C}$ is called a *normal essential eigenvalue* of T if λ is in $R_e(T)$ and every time that an orthonormal sequence u_n satisfies $\|(T - \lambda I)u_n\| \to 0$, we also have $\|(T^* - \overline{\lambda} I)u_n\| \to 0$.

Remark 1.6. (a) The properties and characterizations of $R(T)$ and $R_e(T)$ were already discussed in [9]. In particular, λ belongs to $R(T)$ if, and only if, there exist a $*$-homomorphism $\phi : C^*(T) \to \mathbb{C}$ such that $\phi T = \lambda$ and, analogously, λ is in $R_e(T)$ if and only if there exist a $*$-homomorphism $\phi : C^*(T) \to \mathbb{C}$ such that $\phi(C^*(T) \cap \mathcal{K}) = 0$ and $\phi T = \lambda$. (Here, we are employing the usual notation $\mathcal{K}$ for the ideal of compat operators) Thus, $R(T)$ (resp. $R_e(T)$) is contained in the intersection of $\sigma(T)$ (resp. $\sigma_e(T)$) and the complex conjugate set of $\sigma(T^*)$ (resp. $\sigma_e(T^*)$).

(b) Let $R_{00}(T)$ be the set of finite multiplicity reducing eigenvalues of T which are isolated points of $R(T)$. Here, by a finite-dimensional reducing eigenvalue of T

we mean a complex number λ such that $null(T-\lambda I)\cap null(T^*-\overline{\lambda}I)$ is a non trivial finite dimensional subspace. Then, it is shown in [9], Theorem 6.1, that $R(T)=R_e(T)\cup R_{00}(T)$, where the union (of course) is disjoint.

Theorem 1.7. (a) *Let $0\in\partial W_s(T)$ be a point of infinite curvature and consider $W_s(T)$ in standard position. If u_n is a sequence of unit vectors in $\mathcal{H}$ such that $\langle Tu_n,u_n\rangle\to 0$ then $\|Tu_n\|\to 0$ and $\|T^*u_n\|\to 0$.*

(b) *Let $0\in\partial W_s(T)$ be a point of infinite righthand (resp. lefthand) curvature and consider $W_s(T)$ in standard position. If u_n is a sequence of unit vectors in $\mathcal{H}$ such that $\langle Tu_n,u_n\rangle\to 0$ where, for all n, $\langle Tu_n,u_n\rangle$ is not zero and is contained in a segment of end points given by zero and some α_0 in $W_s(T)$ with* $\operatorname{Re}\alpha_0>0$ *(resp.* $\operatorname{Re}\alpha_0<0$*), then $\|Tu_n\|\to 0$ and $\|T^*u_n\|\to 0$.*

Proof. First of all we will stablish some inequalities which will be useful to our purposes. We begin by observing that from the condition $W_s(T)\subset\{\lambda\in\mathbb{C}:\ \operatorname{Im}\lambda\geq 0\}$ we obtain the following elementary but important inequality:

$$\operatorname{Im}\langle Tu,u\rangle\geq 0,\quad \text{for all } u \text{ in } \mathcal{H}.$$

Also, given arbitrary vectors u,z in $\mathcal{H}$ and an arbitrary $\alpha\in\mathbb{C}$, we have:

$$\langle T(u+\alpha z),u+\alpha z\rangle=\langle Tu,u\rangle+\overline{\alpha}\langle Tu,z\rangle+\alpha\langle Tz,u\rangle+|\alpha|^2\langle Tz,z\rangle.$$

Since $\operatorname{Im}\langle T(u\pm\alpha z),u\pm\alpha z\rangle\geq 0$, we see that

$$\begin{aligned}|\operatorname{Im}[\overline{\alpha}\langle Tu,z\rangle+\alpha\langle Tz,u\rangle]|&\leq\\ \operatorname{Im}[\langle Tu,u\rangle+|\alpha|^2\langle Tz,z\rangle]&\leq|\langle Tu,u\rangle|+|\alpha|^2\|z\|^2\|T\|.\end{aligned}\tag{1.1}$$

Consequently,

$$\begin{aligned}|\operatorname{Im}\langle T(u+\alpha z),u+\alpha z\rangle|\leq 2\operatorname{Im}[\langle Tu,u\rangle+|\alpha|^2\langle Tz,z\rangle]&\leq\\ 2(|\langle Tu,u\rangle|+|\alpha|^2\|T\|\|z\|^2)&.\end{aligned}\tag{1.2}$$

On the other hand,

$$\begin{aligned}|\operatorname{Re}\langle T(u+\alpha z),u+\alpha z\rangle|&\geq\\ |\operatorname{Re}[\overline{\alpha}\langle Tu,z\rangle+\alpha\langle Tz,u\rangle]|-|\operatorname{Re}\langle Tu,u\rangle|-|\alpha|^2|\operatorname{Re}\langle Tz,z\rangle|&\geq\\ |\operatorname{Re}[\overline{\alpha}\langle Tu,z\rangle+\alpha\langle Tz,u\rangle]|-|\langle Tu,u\rangle|-|\alpha|^2|\|T\|\|z\|^2&.\end{aligned}\tag{1.3}$$

Now we treat the case where zero is a point of infinite curvature in $\partial W_s(T)$, *i.e.*, assertion (a). Let u_n be a sequence of unit vectors in $\mathcal{H}$ such that $\langle Tu_n,u_n\rangle\to 0$. We may write (uniquely) $Tu_n=\delta_n u_n+\beta_n v_n$ and $T^*u_n=\overline{\delta_n}u_n+\overline{\gamma}_n w_n$, where v_n and w_n are unit vectors orthogonal to u_n, and $\beta_n,\gamma_n,\delta_n$, are given respectively by $\delta_n:=\langle Tu_n,u_n\rangle$, $\beta_n:=\langle Tu_n,v_n\rangle$ and $\gamma_n:=\langle Tw_n,u_n\rangle$, for all $n=1,2,\ldots$. Also, by multiplying by an appropriate complex number of modulus one, if needed, we may assume that $\operatorname{Re}\langle v_n,w_n\rangle\geq 0$ for all n. Let τ_n be a complex number with $|\tau_n|$=1 and such that $|\overline{\tau_n}\beta_n+\tau_n\gamma_n|=|\beta_n|+|\gamma_n|$, $n=1,2,\ldots$. Let $r_n:=|\delta_n|^{\frac{1}{2}}$ if $\delta_n\neq 0$ and $r_n:=\frac{1}{n}$

otherwise. We define $\alpha_n := r_n\tau_n$, for all n. Since $|\overline{\alpha_n}\beta_n + \alpha_n\gamma_n| = r_n(|\beta_n| + |\gamma_n|)$, there exist suitable complex numbers η_n with $|\eta_n| = 1$ such that $\overline{\alpha_n}\beta_n + \alpha_n\gamma_n = r_n(|\beta_n| + |\gamma_n|)\eta_n$, for all n. Let $z_n := v_n + w_n$. Since $\langle Tu_n, z_n\rangle = \beta_n(1 + \langle v_n, w_n\rangle)$ and $\langle Tz_n, u_n\rangle = \gamma_n(1 + \langle v_n, w_n\rangle)$ we obtain:

$$\overline{\alpha_n}\langle Tu_n, z_n\rangle + \alpha_n\langle Tz_n, u_n\rangle = (\overline{\alpha_n}\beta_n + \alpha_n\gamma_n)(1 + \langle v_n, w_n\rangle) = r_n(|\beta_n| + |\gamma_n|)\eta_n(1 + \langle v_n, w_n\rangle)$$

Hence, from (1.1) we deduce that

$$r_n(|\beta_n| + |\gamma_n|)|\operatorname{Im}[\eta_n(1 + \langle v_n, w_n\rangle)]| \leq |\delta_n| + 4{r_n}^2\|T\|$$

so, we obtain:

$$(|\beta_n| + |\gamma_n|)|\operatorname{Im}[\eta_n(1 + \langle v_n, w_n\rangle)]| \to 0. \tag{1.4}$$

We claim that also

$$(|\beta_n| + |\gamma_n|)|\operatorname{Re}[\eta_n(1 + \langle v_n, w_n\rangle)]| \to 0. \tag{1.5}$$

To prove this claim we define $x_n := u_n+\alpha_n z_n$ whenever $\operatorname{Im}[\overline{\alpha_n}\langle Tu_n, z_n\rangle+\alpha_n\langle Tz_n, u_n\rangle] \geq 0$ and $x_n := u_n - \alpha_n z_n$ otherwise. Now we consider two particular cases:

(i) In the case that $\operatorname{Re}\langle Tx_n, x_n\rangle = 0$ for all n it follows from (1.3) that $r_n(|\beta_n| + |\gamma_n|)|\operatorname{Re}[\eta_n(1+\langle v_n, w_n\rangle)]| \leq |\delta_n|+4{r_n}^2\|T\|$. Thus $(|\beta_n|+|\gamma_n|)|\operatorname{Re}[\eta_n(1+\langle v_n, w_n\rangle)]| \to 0$, as desired.

(ii) Assume now that $\operatorname{Re}\langle Tx_n, x_n\rangle \neq 0$ for all n. Also $\operatorname{Im}\langle Tx_n, x_n\rangle \neq 0$ because the curvature at $\lambda = 0$ is infinite so that the only real point of $W(T)$ is zero. Furthermore, it follows that

$$\mu_n := \frac{|\operatorname{Im}\langle Tx_n, x_n\rangle|}{\operatorname{Re}^2\langle Tx_n, x_n\rangle} \to \infty,$$

since $\langle Tx_n, x_n\rangle \to 0$ with $\|x_n\| \to 1$. Therefore by (1.2)

$$|\operatorname{Re}\langle Tx_n, x_n\rangle| = \left(\frac{|\operatorname{Im}\langle Tx_n, x_n\rangle|}{\mu_n}\right)^{\frac{1}{2}} \leq \left(\frac{2(|\delta_n| + 4{r_n}^2\|T\|)}{\mu_n}\right)^{\frac{1}{2}}.$$

However by (1.3)

$$|\operatorname{Re}\langle Tx_n, x_n\rangle| \geq r_n(|\beta_n| + |\gamma_n|)|\operatorname{Re}\eta_n(1 + \langle v_n, w_n\rangle)]| - |\delta_n| - 4{r_n}^2\|T\|.$$

Hence,

$$r_n(|\beta_n| + |\gamma_n|)|\operatorname{Re}[\eta_n(1 + \langle v_n, w_n\rangle)]| \leq \left(\frac{2(|\delta_n| + 4{r_n}^2\|T\|)}{\mu_n}\right)^{\frac{1}{2}} + |\delta_n| + 4{r_n}^2\|T\|$$

and so $(|\beta_n|+|\gamma_n|)|\operatorname{Re}[\eta_n(1+\langle v_n,w_n\rangle)]|\to 0$, as desired.
Thus, from (i) and (ii), the above claim follows. Therefore, by (1.4), we see that

$$(|\beta_n|+|\gamma_n|)|(1+\langle v_n,w_n\rangle)| = (|\beta_n|+|\gamma_n|)|\eta_n(1+\langle v_n,w_n\rangle)| =$$
$$(|\beta_n|+|\gamma_n|)\left(\operatorname{Re}^2[\eta_n(1+\langle v_n,w_n\rangle)]+\operatorname{Im}^2[\eta_n(1+\langle v_n,w_n\rangle)]\right)^{\frac{1}{2}}\to 0.$$

But $|1+\langle v_n,w_n\rangle|\geq 1$ because $\operatorname{Re}\langle v_n,w_n\rangle\geq 0$, for every n. Thus $|\beta_n|+|\gamma_n|\to 0$ which means precisely that $\|Tu_n\|\to 0$ as well as $\|T^*u_n\|\to 0$, so that (a) is proved.

In order to prove (b), assume for example that zero is a point of infinite righthand curvature (the case of infinite lefthand curvature may be handled similarly). Let α_0 be in $W_s(T)$ such that $\operatorname{Re}\alpha_0>0$. Also $\operatorname{Im}\alpha_0>0$ since otherwise zero would not be a point of infinite righthand curvature. Let ε_n be a sequence of positive real numbers such that $\varepsilon_n\to 0$ and let u_n be a sequence of unit vectors in $\mathcal{H}$ such that $\langle Tu_n,u_n\rangle=\varepsilon_n\alpha_0$. As in the proof of (a), we write $Tu_n=\delta_n u_n+\beta_n v_n$ and $T^*u_n=\overline{\delta_n}u_n+\overline{\gamma}_n w_n$, where v_n and w_n are as before. However, in this case we have the following additional information $\delta_n=\varepsilon_n\alpha_0$. Let $z_n:=v_n+w_n$ and M be a constant greater than $\frac{|\alpha_0||\operatorname{Re}\langle Tz_n,z_n\rangle|}{\operatorname{Re}\alpha_0}$ (take for instance $M=\frac{5|\alpha_0|\|T\|}{\operatorname{Re}\alpha_0}$). We define again $\alpha_n:=r_n\tau_n$, for every n, where τ_n is as before but now $r_n:=\left(\frac{|\delta_n|}{M}\right)^{\frac{1}{2}}$. Then,

$$|\alpha_n|^2|\langle Tz_n,z_n\rangle| = \varepsilon_n\frac{|\alpha_0||\operatorname{Re}\langle Tz_n,z_n\rangle|}{M} < \varepsilon_n\operatorname{Re}\alpha_0 = \operatorname{Re}\langle Tu_n,u_n\rangle$$

so that

$$\operatorname{Re}\langle Tu_n,u_n\rangle + |\alpha_n|^2\operatorname{Re}\langle Tz_n,z_n\rangle > 0.$$

On the other hand, for every n, there exists a complex number η_n with $|\eta_n|=1$ such that

$$\overline{\alpha_n}\langle Tu_n,z_n\rangle+\alpha_n\langle Tz_n,u_n\rangle = \left(\frac{|\delta_n|}{M}\right)^{\frac{1}{2}}(|\beta_n|+|\gamma_n|)\eta_n(1+\langle v_n,w_n\rangle).$$

Now we define $x_n:=u_n+\alpha_n z_n$ if $\operatorname{Re}\eta_n(1+\langle v_n,w_n\rangle)>0$ and $x_n:=u_n-\alpha_n z_n$ otherwise. Then, $\operatorname{Re}\langle Tx_n,x_n\rangle>0$. Consequently, since there are not positive real numbers in $W_s(T)$ we deduce that $\operatorname{Im}\langle Tx_n,x_n\rangle\neq 0$ for all n. Also, again because zero is a point of infinite righthand curvature and $\langle Tx_n,x_n\rangle\to 0$ with $\|x_n\|\to 1$, it follows that $\frac{|\operatorname{Im}\langle Tx_n,x_n\rangle|}{\operatorname{Re}^2\langle Tx_n,x_n\rangle}\to\infty$. Then, by proceding as above in the corresponding part of the proof of (a), we deduce that $(|\beta_n|+|\gamma_n|)|\operatorname{Re}[\eta_n(1+\langle v_n,w_n\rangle)]|\to 0$. On the other hand, from (1.2) we obtain $(|\beta_n|+|\gamma_n|)|\operatorname{Im}[\eta_n(1+\langle v_n,w_n\rangle)]|\to 0$. Therefore we conclude that $(|\beta_n|+|\gamma_n|)|\eta_n(1+\langle v_n,w_n\rangle)|\to 0$ and so $|\beta_n|+|\gamma_n|\to 0$. This proves that $\|Tu_n\|\to 0$ and $\|T^*u_n\|\to 0$. $\square$

Assertion (a) in the next corollary constitutes an improvement of the main result in [6]. Also assertion (b) shows in particular that *points of infinite one-sided curvature of*

$\partial W_s(T)$ which are not of infinite curvature are reducing essential eigenvalues. Finally the assertion (c) improves [7] , Corollary 4.

Corollary 1.8. (a) *If λ is either a point of infinite curvature or a point of infinite one-sided curvature of $\partial W_s(T)$, then $\lambda \in R(T)$.*

(b) *If λ is a point of infinite one-sided curvature in $\partial W_s(T)$ but it is not a linear vertex, then $\lambda \in R_e(T)$.*

(c) *If λ is a point of infinite curvature in $\partial W_s(T)$, but it is not a linear vertex of $W(T)$, then λ is a normal essential eigenvalue of T.*

Proof. Part (a) is an immediate consequence of the the definition of $R(T)$ and Theorem 1.7. In order to prove (b) and (c) we first claim that a point λ in $W(T)$ which is not a linear vertex belongs to $R_e(T)$. To show the claim we consider T decomposed in the (direct) sum $T = A \oplus B$, acting on $H = H_0^{\perp} \oplus H_0$ where $H_0 := \mathrm{Ker}(T - \lambda I) \cap \mathrm{Ker}(T^* - \overline{\lambda} I)$. Since A and B are the restriction of T to $H_0^{\perp}$ and H_0 respectively, it is clear that λ is not a reducing eigenvalue of A so, in particular $\lambda \notin R_{00}(A)$. Notice that it is easily proved that $W(T) = \mathrm{Convhull}(\{\lambda\}, W(A))$ (see also [8]). Hence λ must be either in $\partial W(A)$ or in the complement of $W_s(A)$. In the first case, $W(T) = W(A)$. The second alternative is not possible since, in that case, λ would be a linear vertex of $W_s(T)$ which contradicts our working assumption. Since λ is a point of infinite curvature in $\partial W_s(A)$ we deduce, from (a), that $\lambda \in R(A)$. Because $\lambda \notin R_{00}(A)$, we conclude that $\lambda \in R_e(A)$. Consequently $\lambda \in R_e(T)$ and the claim is shown. To complete the proof of the theorem we merely observe that if u_n is a orthonormal sequence such that $\|(T - \lambda I)u_n\| \to 0$ then $\langle Tu_n, u_n\rangle \to \lambda$. Hence, as a consequence of Theorem 1.7, we see that also $\|(T - \lambda)^* u_n\| \to 0$. □

The following result proves a strengthening of a conjecture of Mathias Hübner [6].

Corollary 1.9. *Assume that λ is a point of infinite one-sided curvature of $\partial W_s(T)$, for some operator T, and suppose that there is only one tangent line to $\partial W_s(T)$ at the point λ. Then $\lambda \in R_e(T)$. If, in addition, λ is a point of infinite curvature of $W_s(T)$ then λ is a normal essential eigenvalue of T.*

Proof. It follows immediately from part (c) of the previous corollary because λ can not be a linear vertex of $\partial W_s(T)$. □

2. Joint Reducing Essential Eigenvalues and the Joint Numerical Range

Now, we shall find a mild extension of the last corollary to n-tuples.

Definition 2.10. Let $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ be an n-tuple of operators acting on $\mathcal{H}$. The *joint numerical range* and the *joint state numerical range* of $\mathbf{T}$ are defined, respectively, as follows:

$$W(\mathbf{T}) := \{(\langle T_1 x, x\rangle, \langle T_2 x, x\rangle, \dots, \langle T_n x, x\rangle) : \ x \in \mathcal{H}, \ \|x\| = 1\},$$

$$W_s(\mathbf{T}) := \{(\phi T_1, \phi T_2, \dots, \phi T_n) : \ \phi \in S(C^*(\mathbf{T}))\},$$

where $C^*(\mathbf{T})$ denotes the C^*-algebra generated by $\mathbf{T}$ and the identity operator $\mathbf{I}$ on $\mathcal{H}$.

Remark 2.11. (a) $W_s(\mathbf{T})$ is always convex but, in general, $W_s(\mathbf{T}) = \overline{W(\mathbf{T})}$ only for $n = 1$. Indeed, for $n > 1$ the above equality does not always hold and $W(\mathbf{T})$ is not convex in general [3].

(b) If λ is an extreme point of $\partial W_s(\mathbf{T})$, for a given n-tuple of operators $\mathbf{T}$, then $\lambda \in \overline{W(\mathbf{T})}$. Indeed, the proof of this fact uses a standard argument: let Σ_λ be the subset of $S(C^*(\mathbf{T}))$ consisting of those states ϕ on $C^*(\mathbf{T})$ such that $\phi(\mathbf{T}) = \lambda$. Since Σ_λ is compact and convex in the w^*-topology, there exists an extreme point ψ in Σ_λ. Using the fact that λ is an extreme point of $W(\mathbf{T})$, it readily follows that ψ is a pure state of $C^*(\mathbf{T})$. By Glim's Lemma (see [4] and [5]), we obtain that ψ is in the w^*-closure of the set of vector states of $C^*(\mathbf{T})$. Thus, evaluating at $\mathbf{T}$, we conclude that $\lambda \in \overline{W(\mathbf{T})}$, as desired.

(c) In the next theorem we use the notions of joint reducing approximate point spectrum and of joint reducing essential spectrum of an n-tuple of operators. These are the natural extension to n-tuples of the corresponding notions for single operators.

Theorem 2.12. *Given an n-tuple $\mathbf{T} = (T_1, \cdots T_n)$ of operators acting on $\mathcal{H}$, assume that $\lambda = (\lambda_1, \cdots, \lambda_n) \in \partial W_a(\mathbf{T})$ satisfies λ_j is a point of one-sided infinite curvature of $\partial W_s(T_j)$, for $1 \leq j \leq n$. Then $\lambda \in R(\mathbf{T})$. If, furthermore, λ is not a linear vertex of $\partial W(\mathbf{T})$, then $\lambda \in R_e(\mathbf{T})$. If, in addition, λ_j is a point of infinite curvature of $\partial W_s(T_j)$, for $1 \leq j \leq n$, then λ is a joint normal essential eigenvalue for $\mathbf{T}$.*

Proof. The proof consits of a repeated application of Corollary 1.8, by projecting in each coordinate. We should point out that under the present hypotheses, λ is an extreme point of $\overline{W(\mathbf{T})}$. □

Remark 2.13. (a) It would be interesting to weaken the hypotheses of Corollary 1.8. For example, is it possible to simply assume that λ is a point of upper-infinite curvature in $\partial W_s(T)$ (in the sense that we replace lim by lim sup in the definition of infinite curvature)?

(b) We recall that the m-th matricial range of an operator $\mathbf{T}$ on $\mathcal{H}$ is the set of $m \times m$ matrices of the form $\phi(\mathbf{T})$, where $\phi : \ C^*(\mathbf{T}) \to \mathcal{M}_m$ is a unital completely positive linear map (see, for instance, [2]). Would it be possible to define a notion of points of infinite matricial curvature in the boundary of the matricial range of $\mathbf{T}$, so that to be able to conclude that such points are actually in the approximate reducing matricial spectrum of $\mathbf{T}$ (those $\phi(\mathbf{T})$ for which ϕ is actually a *-homomorphism)?

References

1. J. Agler, *Geometric and topological properties of the numerical range*, Indiana Univ. Math. J. 31 (1982), 767-777.
2. J. Bunce & N. Salinas, *Completely positive maps on C^*-algebras and the left matricial spectra of an operator*, Duke Math. J. 74 (1976), 747-773.
3. A. T. Dash, *Joint numerical ranges*, Glasnik Matematicki (1972) 75-81.
4. J. Dixmier, *Les C^*-algèbres at leurs represéntations*, Gauthier-Villars, Paris (1964).
5. J. Glimm, *A Stone-Weierstrass theorem for C^*-algebras*, Ann. Math. 72 (1960), 216-244.
6. M. Hübner, *Spectrum where the boundary of the numerical range is not round*, To appear in Rocky Mountain Journal of Math..
7. J. Lancaster, *The boundary of the numerical range*, Proc. Amer. Math. Soc. 49 (1975), 393-398.
8. M. Radjabalipour & H. Radjavi, *On the geometry of numerical ranges*, Pacific J. Math. 61 (1975), 507-511.
9. N. Salinas, *Reducing essential eigenvalues*, Duke. Math. J. 40 (1973), 561-580.
10. J. G. Stampfli & J.P. Willians, *Growth conditions and the numerical range in a Banach algebra*, Tôhoku Math. Jour. 20 (1968), 417-424.

Norberto Salinas, Department of Mathematics, University of Kansas, Lawrence KS 66045

E-mail address: `norberto@@kuhub.cc.ukans.edu`

Maria Victoria Velasco, Department of Analysis, University of Granada, Granada, Spain

E-mail address: `vvelasco@@goliat.ugr.es`