SURGERY ON POSTCRITICALLY FINITE RATIONAL MAPS BY BLOWING UP AN ARC

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Abstract. Using Thurston's characterization of postcritically finite rational functions as branched coverings of the sphere to itself, we give a new method of constructing new conformal dynamical systems out of old ones. Let $f(z)$ be a rational map and suppose that the postcritical set $P(f)$ is finite. Let α be an embedded closed arc in the sphere and suppose that $f|\alpha$ is a homeomorphism. Define a branched covering g as follows. Cut the sphere open along α. Glue in a closed disc D. Map $S^2 - \mathrm{Int}(D)$ via f and $\mathrm{Int}(D)$ by a homeomorphism to the complement of $f(\alpha)$. We prove theorems which give combinatorial conditions on f and α for g to be equivalent in the sense of Thurston to a rational map. The main idea in our proofs is a general theorem which forces a possible obstruction for g away from the disc D on which the new dynamics is defined.

1. Introduction

A *rational map* $f(z) = p(z)/q(z)$ where p and q are relatively prime complex polynomials determines a holomorphic map of the Riemann sphere $\widehat{\mathbb{C}}$ to itself, and so defines a holomorphic dynamical system. The set of points $C(f)$ where the derivative of f vanishes is the set of *critical points* of f; these are the points where the local degree of the map is larger than one. Counted with multiplicity, there are $2d-2$ critical points where d is the degree of f. The *postcritical set* $P(f)$ of f is defined by $P(f) = \overline{\cup_{n>0} f^n(C(f))}$. The map f is said to be *postcritically finite* if $P(f)$ is finite. The Riemann sphere decomposes into the open *Fatou set* $F(f)$ of points whose asymptotic behavior is stable under small perturbations, and the closed *Julia set* $J(f)$ of points whose behavior under iteration is chaotic.

A *branched covering* $f : S^2 \to S^2$ is an orientation-preserving continuous map such that for every $x \in S^2$, there exist local coordinates near x and $f(x)$ such that, in these coordinates, x and $f(x)$ are the point $0 \in \mathbb{C}$ and f is given by $z \mapsto z^n$, $n \geq 1$. The integer n is called the *local degree* of f near n. The sets $C(f)$ and $P(f)$ are defined analogously as for rational maps.

Convention: We assume throughout this paper that all rational maps and branched coverings are postcritically finite.

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Two branched coverings f and g of the sphere to itself are said to be *equivalent* if there are homeomorphisms ψ_0 and ψ_1 such that $\psi_0 \circ f = g \circ \psi_1$, and such that ψ_0 is isotopic to ψ_1 through homeomorphisms $\psi_t, t \in [0,1]$ with $\psi_t|P(f) = \psi_0|P(f)$ for all t.

Thurston has given necessary and sufficient combinatorial conditions for a postcritically finite branched covering g to be equivalent to a rational map, and has also shown that (with one understood exception) this map is determined up to conformal congugacy by g; see [DH2].

If $f(z)$ is a rational map, suppose by cutting and pasting we obtain a branched covering g. When is g equivalent to a rational map? This question is, by Thurston's theorem, a combinatorial one. In general, given d and N, there are infinitely many distinct equivalence classes of branched coverings f of degree d with $|P(f)| = N$. Only finitely many of these can contain rational maps, so it is useful to have constructions which, when applied to rational maps, yield new rational maps. In this paper we define a certain kind of cutting and pasting process on $f(z)$ which is very robust: with fairly weak hypotheses which do not include restrictions on the degree of f or its global combinatorics, the result g will be equivalent to a rational map.

The rough idea of our construction is the following. Let $f(z)$ be a postcritically finite rational map, let α be a closed, embedded topological arc in $\widehat{\mathbb{C}}$ and suppose that $f|\alpha$ is a homeomorphism. Define a new branched covering g by cutting open the sphere along α, gluing in a disc D, mapping the complement of D by f, and mapping D to the complement of $f(\alpha)$ in the sphere. A local model of g near D is the map $z \mapsto \frac{1}{2}(z + 1/z)$ on a neighborhood of the closed unit disc; this maps the unit circle onto the interval $[-1,1]$. Then $\deg(g) = \deg(f) + 1$, and the local degree of g is one more than the local degree of f at the endpoints of α. More generally, one can map the disc D in a similar fashion n times around the sphere so that $\deg(g) = \deg(f) + n$. We call this construction *blowing up the arc α n times*. Note that by construction $f = g$ away from D. The precise conditions on α for this construction to be well-defined, and the statements of our main theorems, are somewhat technical and are given precisely in Sections 2.3 and 2.4. We prove that under these conditions, the combinatorial class of g depends only on combinatorial data (Proposition 2).

For example, we derive the following as a corollary to our first main theorem, Theorem A. To set up the statement, we recall that in a periodic Fatou component of a postcritically finite rational map f, there is a canonical family of arcs called *internal rays* which are mapped homeomorphically under f to other internal rays (see Section 3.5 for the definition of internal ray).

Theorem: *If α is a periodic or preperiodic internal ray of f, then the branched covering g which is f blown up n times along α is equivalent to a rational map.*

We also apply the blowing up construction to obtain rational maps from Möbius transformations of finite order (see Theorem 4.2).

The central idea in our proofs is Theorem 3.2, *Arcs intersecting obstructions.* This theorem asserts that under suitable hypotheses, an obstruction Γ to the existence of a rational map equivalent to a branched covering g cannot intersect any family of arcs Λ which maps in a certain way. We use this theorem to "push" an obstruction away from the region D in the sphere where we add new dynamics. We then conclude that the obstruction Γ yields an obstruction for the original rational map f, which is impossible.

Our theorems may also be thought of as *combination theorems* between conformal dynamical systems. Other examples of such theorems include *mating* (see [Tan], [ST], [Luo]), where two polynomials of the same degree are glued together along their circles at infinity, and *tuning* (see [Ree], [Ahm]), where the dynamics of a polynomial is glued into the dynamics of a rational map near a superattracting cycle. In both of these examples the degree of the new map is the same as that of the old one. To date, all known combination theorems involving mating and tuning have hypotheses on the degree of the map and on the combinatorics of both the original map $f(z)$ and the new dynamics which is being glued into $f(z)$. Also, a definition of mating or tuning for branched coverings has not yet been given in such a manner as to be defined in terms of purely combinatorial data. Our techniques of proof also allow us to establish that certain blown-up quadratic polynomials may be mated (in a suitably generalized way) with an *arbitrary* quadratic polynomial to yield a rational map; see Section 4.4. Moreover, the technique generalizes easily to similar statements in higher degrees.

Organization of the paper.

- §2: we state needed definitions, give the precise definition of the blowing up construction, state our first two main theorems (Theorems A and B) precisely, and prove that the combinatorial class of g depends only on combinatorial data (Proposition 2).
- §3: we prove Theorems A and B.
- §4: we prove other related results.
- §5: we give examples illustrating our main theorems.
- §6: is an appendix containing a proof of Theorem 3.2, *Arcs intersecting obstructions.*

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Example: blowing up a preperiodic arc in the basilica. Let $f(z) = z^2 - 1$. Let $\alpha = [0, 1]$. The filled-in Julia set of $f(z)$ is the set of points whose orbits are bounded; this set is the black region in Figure 1. Let g be the branched covering which is f blown up once along α. The Julia set of the rational map equivalent to g is shown in Figure 2; see also [Pil1] for a discussion of this example and for a description of g as a branched covering.

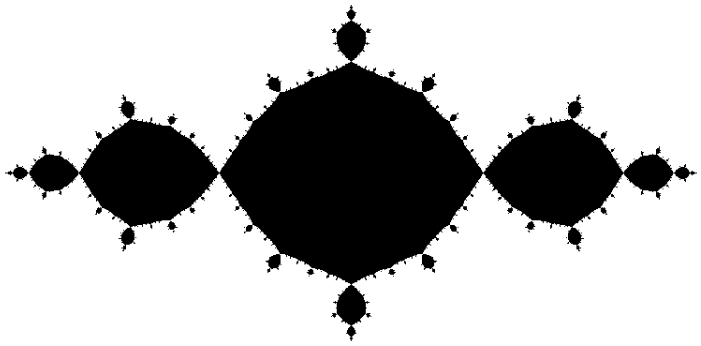

FIGURE 1. The filled-in Julia set of $f(z) = z^2 - 1$.

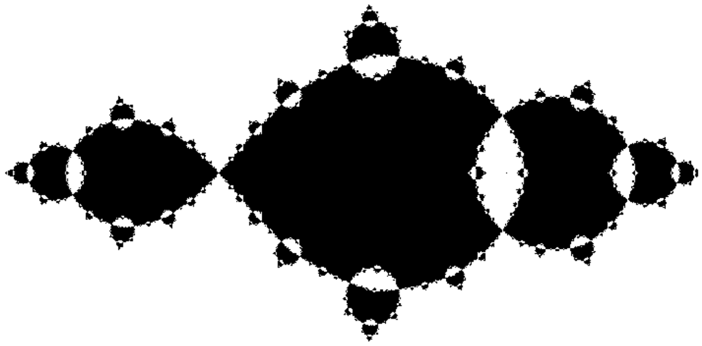

FIGURE 2. Julia set of the rational map corresponding to blowing up $f(z) = z^2 - 1$ along $[0, 1]$. The white regions form the basin of infinity.

2. Blowing up an arc

§§2.1 and 2.2 give definitions needed to formulate our construction and theorems precisely. §2.3 states conditions under which blowing up will be a well-defined operation on equivalence classes. §2.4 gives the precise statements of Theorems A and B. §2.5 gives the construction of blowing up an arc, a proof that the equivalence class of the new map g depends only on combinatorial information, and a proposition (*Orbits off α persist*) which we will use in our proofs.

2.1. Marked branched covers.

Definition 1. *A* **marked branched covering** *is a pair (f, X) where $f : S^2 \to S^2$ is a branched covering and X is a finite set containing $P(f)$ such that $f(X) \subset X$.*

Two marked branched coverings $(f, X), (g, Y)$ are said to be *equivalent* if they satisfy the definition of equivalence of branched coverings with the sets $P(f)$ and $P(g)$ replaced by X and Y respectively. There is a functor from the category of marked branched coverings (f, X) and equivalences to the category of branched coverings and equivalences. The image of a marked brached covering (f, X) under this functor is f; the image of an equivalence $\psi_0 : (S^2, X) \to (S^2, Y)$ is the map of pairs $\psi_0 : (S^2, P(f)) \to (S^2, P(g))$. This functor "forgets" that ψ_0 must send $X - P(f)$ to $Y - P(g)$. We call this the *forgetful functor* sending (f, X) to f.

Examples:

1. Let f be a rational map and X a finite forward-invariant set containing $P(f)$. Then (f, X) is a branched cover marked by the set X. We will call (f, X) a *marked* rational map. Note that if (f, X) is a marked rational map with $\deg(f) = 1$ and $|X| \geq 3$, then f is necessarily a Möbius transformation of finite order.
2. Let $p(z) = z^2 - 1$. Then $p : [-1, 0] \to [-1, 0]$ is an orientation-reversing homeomorphism. Postcompose $p(z)$ with a homeomorphism h isotopic to the identity relative to $P(p)$ such that for $f(z) = h \circ p(z)$, $f([-1/4, 0]) = [-1, -3/4]$ and $f([-1, -3/4]) = [-1/4, 0]$. Let $X = P(f) \cup \{-1/4, -3/4\}$. Then (f, X) is a branched cover marked by the set X.

In both examples, the image of (f, X) under the forgetful functor is combinatorially equivalent to a rational map. In the first case, however, the set $X - P(f)$ of "new" marked points are already preperiodic under the dynamics of a rational map, whereas in the second case we have added new preperiodic points by our choice of representative of the combinatorial class of $p(z)$.

2.2. **Arcs and curves in (S^2, X).** Let X be a finite set in the sphere. An *arc α in (S^2, X)* we define to be the image of the unit interval $[0, 1]$ under a map j into the sphere such that the endpoints $e(\alpha) := j(\{0, 1\}) \subset X$, $j|(0, 1)$ is an embedding, and $j(0, 1) \cap X = \emptyset$. Thus we allow the endpoints $e(\alpha)$ of α to coincide (though we will not cut along such arcs in our construction). A *simple closed curve in (S^2, X)* is a simple closed curve in $S^2 - X$; it is said to be *peripheral* if it is isotopic into every neighborhood of some point in X and *essential* if it is not contractible in $S^2 - X$.

Two arcs (resp. curves) η_0, η_1 in (S^2, X) are said to be *isotopic relative to X*, written $\eta_0 \simeq_X \eta_1$, if there is a continuous, one-parameter family $\eta_t, t \in [0, 1]$ of such arcs (respectively curves) joining η_0 to η_1. The isotopy class relative to X of an arc or simple closed curve η we will denote by $[\eta]_X$, or by $[\eta]$ when the set X is understood.

A set of pairwise nonisotopic arcs (essential curves) in (S^2, X) will be called an *arc (curve) system* in (S^2, X). If Λ and Λ' are arc or curve systems in (S^2, X) we write $\Lambda \simeq_X \Lambda'$ if every element of Λ is isotopic to a unique element of Λ' relative to X, and conversely.

The *intersection number* between a pair α, β consisting of either arcs or curves in (S^2, X) is defined by

$$\alpha \cdot \beta = \min_{\alpha \simeq_X \alpha', \beta \simeq_X \beta'} \#\{(\alpha' - e(\alpha')) \cap (\beta' - e(\beta'))\}.$$

Invariance up to isotopy of arc systems.

Let (f, X) be a marked branched covering.

If λ is an arc in (S^2, X), we will call the closure $\widetilde{\lambda}$ of a connected component of $f^{-1}(\lambda - e(\lambda))$ a *lift* of λ under f. Thus each arc λ in (S^2, X) has $\deg(f)$ distinct lifts $\widetilde{\lambda}$. If the endpoints of λ are distinct, each lift maps homeomorphically under f onto λ. If Λ is an arc system in (S^2, X), a *lift* $\widetilde{\Lambda}$ is defined to be an arc system in (S^2, X) each of whose elements $\widetilde{\lambda}$ is a lift of some element $\lambda \in \Lambda$.

An arc system Λ in (S^2, X) is said to be *forward invariant under f up to isotopy relative to X* if there exists a subset $\Lambda_0 \subset \Lambda$ (possibly all of Λ) and a lift $\widetilde{\Lambda}_0$ of Λ_0 such that $\widetilde{\Lambda}_0 \simeq_X \Lambda$. If $\Lambda \simeq_X \Lambda'$ and Λ is forward invariant up to isotopy relative to X, then so is Λ'. (An isotopy between an element $\lambda \in \Lambda$ and an element $\lambda' \in \Lambda'$ can be lifted to an isotopy between $\widetilde{\lambda} \in \widetilde{\Lambda}$ and an arc $\widetilde{\lambda}'$ which is a lift of $\widetilde{\lambda}'$.)

The next proposition allows us to extract arc systems which are forward-invariant up to isotopy from a collection of arcs which are literally forward-invariant.

Proposition 1. *Suppose $\mathcal{L}$ is a collection of arcs l in (S^2, X) such that for each $l \in \mathcal{L}$, $f|l$ is a homeomorphism and $f(l) \in \mathcal{L}$. Then there is a subset $\Lambda \subset \mathcal{L}$ which is an arc system forward invariant under f up to isotopy, and such that each $l \in \mathcal{L}$ is isotopic relative to X to a unique element of Λ.*

Proof. Let $\Lambda_0 \subset f(\mathcal{L})$ be an arc system chosen arbitrarily so that each arc $f(l), l \in \mathcal{L}$ is isotopic relative to X to a unique element of Λ_0. Then for each $\lambda \in \mathcal{L}$, $f(\lambda) \simeq_X \lambda_0$ for a unique $\lambda_0 \in \Lambda_0$. Hence by lifting isotopies, $\lambda \simeq_X \widetilde{\lambda}_0$, where $\widetilde{\lambda}_0$ is a lift of λ_0. Hence every element of $\mathcal{L}$ is isotopic to a lift of an element of Λ_0. We now enlarge our set Λ_0 so that the last property holds. Let $\mathcal{L}_1$ denote the set of elements of $\mathcal{L}$ which are not isotopic to any element of Λ_0. Let $\Lambda_1 \subset \mathcal{L}_1$ be any arc system for which each element of $\mathcal{L}_1$ is isotopic to a unique element of Λ_1, and for which each element $\lambda_1 \in \Lambda_1$ is isotopic to some element of $\mathcal{L}_1$. Then $\Lambda = \Lambda_0 \cup \Lambda_1$ is an arc system with the desired properties. $\square$

2.3. **Conditions under which blowing up is defined.** Now suppose that (f, X) is a branched covering of degree ≥ 1 marked by the finite set X, and let α be an arc in S^2. Assume that the following conditions are satisfied:

Blowing Up Conditions:

1. $f|\alpha$ is a homeomorphism;
2. α is a union of arcs $\alpha_j, j = 1, 2, ..., L$ in $(S^2, f^{-1}(X))$;
3. $\mathrm{Int}(\alpha) \cap (X \cup C(f)) = \emptyset$.

Note that the arcs $f(\alpha_j)$ are arcs in (S^2, X).

Remarks: Allowing α to consist of more than one arc in $(S^2, f^{-1}(X))$ is indeed useful: for example, consider the example of blowing up the "airplane" in Section 5. More generally, we may apply the blowing up construction to finite collections

$\alpha^1, \alpha^2, \ldots$ of arcs for which $\mathrm{Int}(\alpha^i) \cap \alpha^j = \emptyset$ when $i \neq j$ and for which each arc satisfies the Blowing Up Conditions.

2.4. **Statement of main results.** We assume the following in this subsection. Let (f, X) be a marked rational map with $\deg(f) \geq 2$. Let α be an arc in $\widehat{\mathbb{C}}$ satisfying the Blowing Up Conditions. Let (g, X) be marked branched cover which is the map (f, X) blown up n times along α.

Theorem A *If for all $j = 1, 2, \ldots, L$,*

1. *$f(\alpha_j)$ is isotopic relative to X to an arc which is contained in a finite arc system Λ_j which is forward invariant under f, and*
2. *there is a subset $\Lambda_{j,0} \subset \Lambda_j$ with a lift $\widetilde{\Lambda}_{j,0}$ such that $\widetilde{\Lambda}_{j,0} \simeq_X \Lambda_j$ and $\widetilde{\Lambda}_{j,0} \cap Int(\alpha) = \emptyset$,*

then g is combinatorially equivalent to a rational map.

Corollary 1. *If α is a periodic or preperiodic internal ray of f, then g is equivalent to a rational map.*

Corollary 2. *Suppose α is a finite union of periodic or preperiodic internal rays of f. If for all $n \geq 0$, the intersection number relative to X satisfies $f^n(f(\alpha)) \cdot f(\alpha) \neq 0$ only when $f^n(f(\alpha)) = f(\alpha)$, then g is equivalent to a rational map.*

The definition of internal ray is given in Section 3.5.

Next, assume $f(z)$ is a polynomial. According to work of Douady and Hubbard, there is a canonically defined finite topological tree T contained in the filled-in Julia set of f, depending only on f and $e(\alpha)$, such that $f(T) \subset T$ with vertices mapped to vertices. In Section 3.4 we outline the construction of T.

Theorem B: (Blowing up an arc in a generalized Hubbard tree) *If $\alpha \subset T$, then g is combinatorially equivalent to a rational map.*

Remark: The Blowing Up Conditions strongly restrict the possibilities for α. The novel feature of this theorem, relative to the first, is that we do not require that the α_j are eventually periodic up to isotopy relative to X.

2.5. **Definition of blown-up map.** Let (f, X) and α satisfy the Blowing Up Conditions of Section 2.3 . The construction is outlined in Figure 3 for the case $n = 4$.

We first cut the sphere along the interior of α. Let W be an open disc containing $\mathrm{Int}(\alpha)$ bounded by a Jordan curve such that $\overline{W} \cap f^{-1}(X) = \alpha \cap f^{-1}(X)$ and $\partial W \supset e(\alpha)$. Since $\mathrm{Int}(\alpha) \cap (X \cup C(f)) = \emptyset$, we may choose W so that $f|\overline{W}$ is a homeomorphism. Let $h_t : \widehat{\mathbb{C}} - \mathrm{Int}(\alpha) \hookrightarrow \widehat{\mathbb{C}}$ be a continuous family of embeddings satisfying

1. $h_0 = \mathrm{id}$;
2. $h_t|(\widehat{\mathbb{C}} - W) = \mathrm{id}$ for all t;

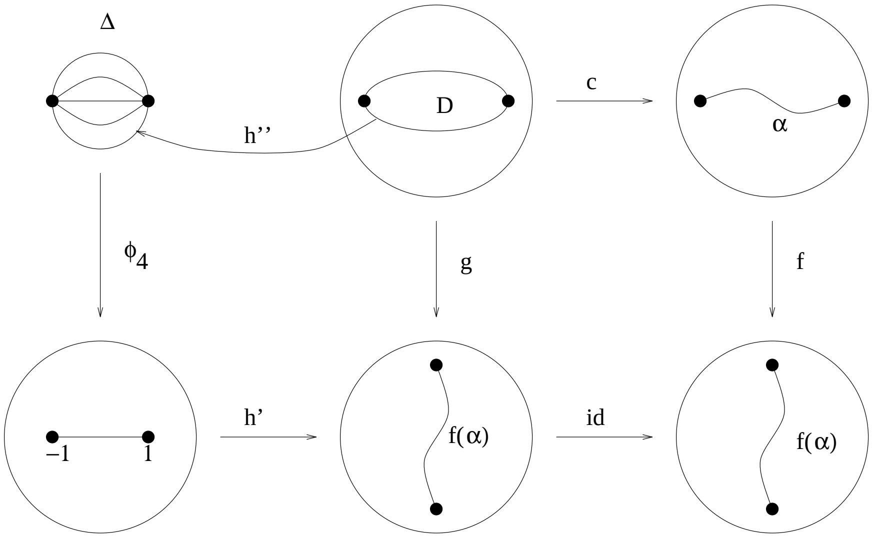

FIGURE 3. Construction of the blown-up map g from f.

3. $h_1(\widehat{\mathbb{C}} - \mathrm{Int}(\alpha)) = \widehat{\mathbb{C}} - (D - e(\alpha))$, where D is a closed disc.

The Jordan curve ∂D is a union of two closed arcs ∂D_+ and ∂D_- with endpoints $e(\alpha)$ which we denote by $\{e_{-1}, e_{+1}\}$. The map h_1^{-1} extends uniquely to a continuous "collapsing" map $c : \widehat{\mathbb{C}} - \mathrm{Int}(D) \to \widehat{\mathbb{C}}$ sending $\widehat{\mathbb{C}} - (D - e(\alpha))$ homeomorphically to $\widehat{\mathbb{C}} - \mathrm{Int}(\alpha)$ and each arc ∂D_+ and ∂D_- homeomorphically to α.

We add the following new dynamics. Let $\Delta = \{z \mid |z| < 1\ \}$, and let $\phi_n : \overline{\Delta} \to \widehat{\mathbb{C}}$ be the map given by

$$\phi_n(z) = A_2(A_1(z))^{2n},$$

where $A_1(z) = -i\frac{z+1}{z-1}$ and $A_2(z) = \frac{z-1}{z+1}$. The Möbius transformation A_1 is a rigid rotation with respect to the spherical metric sending the ordered triple $(-1, 0, 1)$ to $(0, i, \infty)$ and sending $\overline{\Delta}$ to $Im(z) \geq 0$; the map $z \mapsto z^{2n}$ maps the half-plane $Im(z) \geq 0$ n times around the sphere sending the real axis to the nonnegative real axis; the Möbius transformation A_2 is a rigid rotation sending $(0, 1, \infty)$ to $(-1, 0, 1)$. Thus the map ϕ_n wraps the closed unit disc $\overline{\Delta}$ around the sphere n times in such a way that ϕ_n is locally injective on $\overline{\Delta} - \{-1, 1\}$, $\phi_n(-1) = -1, \phi_n(1) = 1$, and $\phi_n(\partial\Delta) = [-1, 1]$. Let $\partial\Delta_+, \partial\Delta_-$ denote respectively the intersection of $\overline{\Delta}$ with $Im(z) \geq 0$ and $Im(z) \leq 0$.

We now describe the gluing of f and ϕ_n. Choose a homeomorphism

$$h' : (\widehat{\mathbb{C}}, [-1, 1]) \to (\widehat{\mathbb{C}}, f(\alpha))$$

which sends -1 to $f(e_{-1})$ and $+1$ to $f(e_{+1})$. The compositions $(\phi_n|_{\partial\Delta_\pm})^{-1} \circ (h')^{-1} \circ f \circ c|_{\partial D_\pm} : \partial D_\pm \to \partial\Delta_\pm$ are homeomorphisms of closed arcs and together define a homeomorphism from ∂D to $\partial\Delta$. Since D and $\overline{\Delta}$ are closed discs we can extend this to a homeomorphism h'' between D and $\overline{\Delta}$.

We now define a new branched covering g by

$$g(z) = \begin{cases} f \circ c(z), & \text{if } z \in \widehat{\mathbb{C}} - \operatorname{Int}(D), \\ h' \circ \phi_n \circ h''(z), & \text{if } z \in \overline{D}. \end{cases}$$

The two definitions agree on the boundary, by construction, hence they give a well-defined branched covering of the sphere to itself such that $\deg(g) = \deg(f) + n$.

We may also apply the above construction simulaneously to finite collections of arcs $\alpha^1, \alpha^2, \ldots$ which satisfy the Blowing Up Conditions and which satisfy $\operatorname{Int}(\alpha^i) \cap \alpha^j = \emptyset, i \neq j$. We omit the details of this more general construction but emphasize that we may assume that for $i \neq j$, $\overline{W^i} \cap \overline{W^j} \subset e(\alpha^i) \cap e(\alpha^j) \subset X$.

Proposition 2 (g depends only on classes). *The combinatorial class of g depends only on the integer n, the combinatorial class of (f, X), and the isotopy classes of the arcs in $(S^2, f^{-1}(X))$ comprising α.*

Proof. We prove in detail only the case of blowing up along a single arc; the case of blowing up along multiple arcs in similar, since we may choose the closed discs $\overline{W^i}$ to intersect only in X.

If (f_1, X_1) is equivalent to (f_2, X_2), then $(f_1, f_1^{-1}(X_1))$ is equivalent to $(f_2, f_2^{-1}(X_2))$: an equivalence $\psi_0 : (S^2, X_1) \to (S^2, X_2)$ may be lifted to an equivalence $\psi_1 : (S^2, f_1^{-1}(X_1)) \to (S^2, f_2^{-1}(X_2))$.

Let $\phi_0, \phi_1 : (S^2, f_1^{-1}(X_1)) \to (S^2, f_2^{-1}(X_2))$ be a pair of homeomorphisms giving an equivalence between $(f_1, f_1^{-1}(X_1))$ and $(f_2, f_2^{-1}(X_2))$. Then $\phi_0 \circ f_1 = f_2 \circ \phi_1$ and $\phi_0 \simeq \phi_1$ relative to $f_1^{-1}(X_1)$. Let $\alpha^2 = \phi_1(\alpha^1)$. Then α^2 is an arc in S^2 which is comprised of arcs in $(S^2, f_2^{-1}(X_2))$. We must prove that if (g_i, X_i) is (f_i, X_i) blown up n times along $\alpha^i, i = 1, 2$, then g_1 and g_2 are combinatorially equivalent. For convenience, if $\beta^i, i = 1, 2$ are arcs which are unions of finitely many arcs β^i_j in (S^2, Y), we will write $\beta^1 \simeq_Y \beta^2$ if $\beta^1_j \simeq_Y \beta^2_j$ for all j.

Step 1: We first show the following. Suppose for $i = 1, 2$, g^i is the map produced from the map f by blowing up an arc α^i, where $\alpha^1 \simeq_{f^{-1}X} \alpha^2$, and W_i, D_i, h'_i, h''_i, and h^i_t are the discs and maps used in the definition of g_i. Then g_1 is equivalent to g_2.

Since $f|\overline{W}_i$ is a homeomorphism, $f(\overline{W}_i)$ is a closed disc. There exists a homeomorphism $\psi_0 : (S^2, X) \to (S^2, X)$ isotopic to the identity relative to X such that $\psi_0(f(\overline{W}_1), f(\alpha^1)) = (f(\overline{W}_2), f(\alpha^2))$. For since $\alpha^1 \simeq_{f^{-1}X} \alpha^2$, $f(\alpha^1) \simeq_X f(\alpha^2)$. By results of Epstein [EP], there is an ambient isotopy of S^2 carrying $\partial f(\overline{W}_1) \cup f(\alpha^1)$ to $\partial(f(\overline{W}_2)) \cup f(\alpha^2)$ fixing X pointwise. Let $\Psi : S^2 \times I \to S^2$ be such an isotopy such that $\Psi(\cdot, 0) = \mathrm{id}$, $\Psi(\cdot, 1) = \psi_0$, and $\Psi(X, t) = \mathrm{id}_X$ for all t.

Since $X \supset P(f)$, $f|S^2 - f^{-1}X \to S^2 - X$ is a covering map. Hence there is a lift $\widetilde{\Psi}$ of Ψ which is an isotopy fixing X pointwise between the identity and a map which we call ψ^1. Note that $\psi^1 : (\overline{W}_1, \alpha^1) \to (\overline{W}_2, \alpha^2)$.

Recall that $g_i|S^2 - \overline{W}_i = f|S^2 - \overline{W}_i, i = 1,2$. We will define a homeomorphism $\psi_1 : (S^2, X) \to (S^2, X)$ such that $g_2\psi_1 = \psi_0 g_1$ and show ψ_0 is isotopic to ψ_1 relative to X. The map ψ_1 will be defined as four maps $\psi^{(1)} = \psi^1|S^2 - W_1$, $\psi^{(2)}, \psi^{(3)}, \psi^{(4)}$ which piece together.

We now define $\psi^{(2)}$. Let $K_i = (g_i|D_i)^{-1}(f(\alpha^i)), i = 1,2$. Then K_i is a graph in S^2 with exactly two vertices which are the endpoints $e(\alpha^i) \subset f^{-1}(X)$ and with $n+2$ edges, each of which joins one vertex to the other. There is a well-ordering on the set of edges of K_i: $E < E'$ if the edge E separates $\mathrm{Int}(E')$ from $\mathrm{Int}(\partial_- D_i)$ in D_i. Write the set of edges in increasing order as $\{E_0^i, ..., E_n^i\}$. Since we glue in the same new dynamics ϕ_n in the construction of g_1 and g_2, there is a homeomorphism $\psi^{(2)} : K_1 \to K_2$ such that $g_2\psi^{(2)} = \psi_0 g_1$ and $\psi^{(2)}(E_l^1) = E_l^2, l = 0,1,...,n$, i.e. $\psi^{(2)}$ preserves the well-ordering of the edges.

We now define $\psi^{(3)}$. For $l = 0,1,...,n-1$, the complementary component U_l^i of $D^i - K_i$ bounded by $E_l^i \cup E_{l+1}^i$ is a Jordan domain mapping homeomorphically under g_i onto $S^2 - f(\alpha^i)$, $i = 1,2$. Hence for each l, there is a homeomorphism $\psi^{(3),l} : U_l^1 \to U_l^2$ such that $g_2\psi^{(3),l}|U_l^1 = \psi_0 g_1|U_l^1$. Since $\psi^{(2)}$ preserves the ordering of edges, the homeomorphisms $\psi^{(3),l}$ and $\psi^{(2)}$ agree on the common boundaries of their domains of definition.

A similar argument shows that there is a homeomorphism $\psi^{(4)} : W_1 - D_1 \to W_2 - D_2$ such that $g_2\psi^{(4)}|W_1 - D_1 = \psi_0 g_1|W_1 - D_1$ and which agrees with $\psi^{(1)}$ on ∂W_1 and with $\psi^{(2)}$ on ∂D_1.

Hence the maps $\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \psi^{(4)}$ define a homeomorphism of the sphere ψ_1 such that $\psi_0 g_1 = g_2\psi_1$.

That ψ_0 and ψ_1 are isotopic relative to X follows from the fact that $\psi_1|S^2 - W_1$ is isotopic (via $\widetilde{\Psi}$) to the identity relative to X and the fact that $X \cap \overline{W}_1 \subset e(\alpha^1) \subset \partial W_1$. This completes the proof of Step 1.

Step 2: We now prove the general statement. Let $(f_1, f_1^{-1}X_1), (f_2, f_2^{-1}X_2)$ be marked covers and ϕ_0, ϕ_1 as above give an equivalence carrying the arc α^1 to α^2. By replacing f_2 with the conjugate $\phi_0 f_2 \phi_0^{-1}$ we may assume that $\phi_0 = \mathrm{id}$ and that $f_1(\alpha^1) = f_2(\alpha^2)$.

By Step 1, the combinatorial class of g_i is independent of the choice of discs W, D. Let W_1, D_1 be any such choice for g_1. Hence we may assume that $(W_2, D_2) = \phi_1(W_1, D_1)$ and that $f_1(\overline{W}_1, \alpha^1) = f_2(\overline{W}_2, \alpha^2)$.

Let $\psi_0 : (S^2, X)) \to (S^2, X)$ be the identity map (after conjugation, we may assume $X = X_1 = X_2$). We will find a homeomorphism ψ_1 such that $\psi_0 g_1 = g_2\psi_1$ and ψ_0 is isotopic to ψ_1 relative to X. As in the proof of the previous step, we define ψ_1 in four pieces $\psi_1|S^2 - W_1 = \psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \psi^{(4)}$. The details are all similar to those in the proof of the previous step. □

Remark: A similar proof shows that when blowing up along any finite collection of arcs $\alpha^1, \alpha^2, ...$ of arcs for which $\mathrm{Int}(\alpha^i) \cap \alpha^j = \emptyset, i \neq j$, the equivalence class of the resulting map is independent of the choices and representatives. Moreover, the resulting class is the same whether we blow up one arc at a time or all of them simultaneously.

The next proposition gives a condition under which arc systems in (S^2, X) forward invariant under f up to isotopy are also forward invariant under g up to isotopy.

Proposition 3 (Orbits off α persist). *Suppose Λ is an arc system in (S^2, X) which is forward invariant under f up to isotopy relative to X. If, under f, there exists a lift $\widetilde{\Lambda} \simeq_X \Lambda$ such that $\widetilde{\Lambda} \cap Int(\alpha) = \emptyset$, then Λ is forward invariant under g up to isotopy relative to X.*

Proof. Let $\Lambda_0 \subset \Lambda$ be an arc system and $\widetilde{\Lambda}_0$ a lift of Λ_0 under f such that $\widetilde{\Lambda}_0 \simeq_X \Lambda$. Since $g|\widehat{\mathbb{C}} - D = f \circ c$, if $\widetilde{\lambda}_0$ is a lift under f of $\lambda_0 \in \Lambda_0$ such that $\widetilde{\lambda}_0 \cap \mathrm{Int}(\alpha) = \emptyset$, then $h_1(\widetilde{\lambda}_0) = c^{-1}(\widetilde{\lambda}_0)$ is a lift under g of λ_0. Since h_1 is isotopic to the identity relative to X, $\widetilde{\lambda}_0 \simeq_X h_1(\widetilde{\lambda}_0)$. Hence $h_1(\widetilde{\Lambda}_0)$ is a lift of Λ_0 under g such that $h_1(\widetilde{\Lambda}_0) \simeq_X \Lambda_0$. □

3. Proofs of Theorems A and B

In §3.1 we state the characterization of marked rational maps as marked branched coverings as an immediate corollary to the proof of Thurston's characterization of rational maps as branched coverings. In §3.2 we state the main ingredient in our proof, Theorem 3.2, *Arcs intersecting obstructions*; its proof we relegate to an Appendix. In §3.3 and we prove Theorem A; §3.4 gives the proof of Theorem B, which depends on a construction given in §3.5.

3.1. **Characterization of marked rational maps.** Suppose (f, X) is a marked branched covering. To a curve system Γ in (S^2, X) we associate the *Thurston linear transformation* which is the linear map

$$f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$$

given by

$$f_\Gamma(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} \frac{1}{\deg(f : \gamma' \to \gamma)} [\gamma']_\Gamma,$$

where $[\gamma']_\Gamma$ denotes the element of Γ isotopic to γ', if it exists. If there are no such elements, the sum is taken to be zero. Here, we regard elements of Γ as the basis vectors of $\mathbb{R}^\Gamma$. This transformation depends only on the isotopy classes of curves relative to X and is natural with respect to iteration of f. Since the entries of f_Γ are nonnegative, the Perron-Frobenius Theorem ([Gan], Chapter XIII) implies that there is a nonnegative real eigenvalue $\lambda(f_\Gamma)$ equal to the spectral radius of f_Γ.

A nonnegative square matrix A_{ij} is said to be *irreducible* if for each (i, j), there is an $n \geq 0$ such that $A^n_{ij} > 0$. We say that Γ is irreducible if the matrix for f_Γ is irreducible. The Perron-Frobenius theory easily gives that if Γ is any curve system with $\lambda(f_\Gamma) > 0$, then then Γ contains an irreducible curve system Γ' for which $\lambda(f_{\Gamma'}) = \lambda(f_\Gamma)$.

A curve system Γ is said to be a *multicurve* if its elements are nonperipheral and pairwise disjoint. The proof of Thurston's characterization of rational maps as branched coverings of the sphere given in [DH2] extends immediately to the setting of branched covers marked by additional points.

Theorem 3.1 (Characterization of marked rational maps). *Let (f, X) be a marked branched covering. Then (f, X) is combinatorially equivalent to a marked rational map (R, Y) if and only if for every irreducible multicurve Γ in (S^2, X), either*

1. $\lambda(f_\Gamma) = 1$ *and the orbifold associated to $(f, P(f))$ has signature $(2, 2, 2, 2)$, in which case R is covered by an integral endomorphism of the torus, or*
2. $\lambda(f_\Gamma) < 1$*, in which case the rational map R is unique up to conjugation by elements of $Aut(\widehat{\mathbb{C}})$.*

Definition 2 (Thurston obstruction.). *If (f, X) is a marked branched covering for which the orbifold associated to $(f, P(f))$ is not the $(2, 2, 2, 2)$ orbifold, an (irreducible) multicurve Γ for which $\lambda(f_\Gamma) \geq 1$ is called an (irreducible) Thurston obstruction.*

Example: Let $\Gamma = \{\gamma_0, ..., \gamma_{p-1}\}$ be a multicurve such that the following holds: for each $0 \leq j \leq p-1$, γ_j has a (not necessarily unique) lift $\delta \simeq_X \gamma_{j-1(\mathrm{mod}\, p)}$ and $\deg f : \delta \to \gamma_j = 1$. Then Γ is called a *Levy cycle* and is an example of an irreducible Thurston obstruction.

Remarks:

1. See [DH2] for the proof and for the definition of the orbifold associated to a rational map. A rational map R is said to be *covered by a torus endomorphism* if there is a complex torus T, a holomorphic covering map $\widetilde{R} : T \to T$, and a surjective holomorphic map p such that $R \circ p = p \circ \widetilde{R}$. The endomorphism $\widetilde{R}$ is said to be *integral* if it has a lift to an automorphism of $\mathbb{C}$ of the form $z \mapsto nz$, some $n \geq 2$. The first case in the above theorem is rare and does not arise, for example, if there are periodic critical points of f or if $|P(f)| \geq 5$.
2. In [DH2] the curve systems Γ in the theorem are required to be f*-invariant*, i.e. every lift δ of an element $\gamma \in \Gamma$ is either inessential, peripheral, or isotopic to an element of Γ. However, by taking preimages we may always extend an irreducible multicurve Γ to an f-invariant multicurve without decreasing the leading eigenvalue of the corresponding Thurston transformation.

To apply Theorem 3.1 we will need to rule out the first rare case. Let (f, X) be a marked rational map $f(z)$. If f is covered by an integral torus endomorphism, then f is not a polynomial. Moreover, there can be no arcs in (S^2, X) with distinct endpoints which are periodic under f up to isotopy relative to X. For the Euclidean metric on $\mathbb{C}$ pushes down to an (orbifold) metric on $\widehat{\mathbb{C}}$ which is uniformly expanded under f. Hence all inverse branches of f are uniformly contracting with respect to this orbifold metric. This implies that if $\widetilde{\lambda}$ is any lift under f of an arc λ in (S^2, X), then the diameter of $\widetilde{\lambda}$ is strictly less than that of λ. Since the diameter of any arc in (S^2, X) with distinct endpoints is positive, this shows that there are no arcs with distinct endpoints which are periodic under f up to isotopy relative to X. Hence to prove our two theorems, we may assume that the orbifold O_f is not the $(2,2,2,2)$ orbifold.

If g is f blown up along α then the Euler characteristic $\chi(O_g) < \chi(O_f) \leq 0$. The $(2,2,2,2)$ orbifold has Euler characteristic zero, hence O_g cannot be the $(2,2,2,2)$ orbifold. So g is combinatorially equivalent to a rational map if and only if there are no irreducible Thurston obstructions for g.

3.2. **Arcs intersecting obstructions.** Let (f, X) be a marked branched covering, Λ be an arc system in (S^2, X), and Γ a curve system in (S^2, X).

The *unweighted Thurston linear transformation* $f_{\#,\Gamma} : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is defined by

$$f_{\#,\Gamma}(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} [\gamma']_\Gamma \text{ for } \gamma \in \Gamma,$$

where $[\gamma']_\Gamma$ denotes the element in Γ homotopic to γ' and zero otherwise. As before, this map is independent of choice of representatives and natural with respect to iteration. Note that $0 \leq (f_\Gamma)_{i,j} \leq (f_{\#,\Gamma})_{i,j}$ for all i, j and that $(f_\Gamma)_{i,j} = 0$ if and only if $(f_{\#,\Gamma})_{i,j} = 0$. This implies that f_Γ is irreducible if and only if $f_{\#,\Gamma}$ is irreducible.

In analogy with the unweighted Thurston linear transformation, we define

$$f_{\#,\Lambda} : \mathbb{R}^\Lambda \to \mathbb{R}^\Lambda$$

by setting

$$f_{\#,\Lambda}(\lambda) = \sum_{\lambda' \subset f^{-1}(\lambda)} [\lambda']_\Lambda \text{ for } \lambda \in \Lambda,$$

where $[\lambda']_\Lambda$ denotes the element of Λ homotopic to λ' (if it exists). We say that Λ is *irreducible* if $f_{\#,\Lambda}$ is irreducible. It is straightforward to verify that the transformation $f_{\#,\Lambda}$ depends only on $[\Lambda]_X$ and not on the choice of representatives and that $f_{\#,\Lambda}$ is natural with respect to iteration of f, i.e. $(f^n)_{\#,\Lambda} = (f_{\#,\Lambda})^n$.

If α and β are arcs in (S^2, X) we define the *multiplicity* $\operatorname{mult}(f : \alpha \to \beta)$ to be the number of lifts of β which are isotopic to α relative to X. Thus $(f_{\#,\Lambda})_{ij} = \operatorname{mult}(f : \lambda_i \to \lambda_j)$.

The following proposition follows immediately from the decomposition theorem for nonnegative square matrices in [Gan], Chapter XIII.

Proposition 4 (**Decomposition of invariant arc systems**). *Let Λ be a finite arc system forward invariant under f up to isotopy relative to X. Then there are disjoint (as subsets of Λ) irreducible arc subsystems $\Theta_1, \Theta_2, ..., \Theta_s$ of Λ and integers $n_1, n_2, ..., n_s$ such that for each $\lambda \in \Lambda$, there is a (not necessarily unique) Θ_k such that λ is isotopic to a lift of an element of Θ_k under f^{n_k}.*

For Λ a curve system or an arc system in (S^2, X), set $\widetilde{\Lambda}$ (resp. $\widetilde{\Lambda}(f^n)$) to be the union of those components of $f^{-1}(\Lambda)$ (resp. $f^{-n}(\Lambda)$) which are isotopic to elements of Λ. If Λ is irreducible, each component of Λ is isotopic to some (not necessarily unique) component of $\widetilde{\Lambda}(f)$ (resp. $\widetilde{\Lambda}(f^n)$).

The following theorem appears in a different form in [ST].

Theorem 3.2 (Arcs intersecting obstructions). *Let (f, X) be a marked branched covering, Γ an irreducible Thurston obstruction in (S^2, X), and Λ an irreducible arc system in (S^2, X). Suppose furthermore that $\#(\Gamma \cap \Lambda) = \Gamma \cdot \Lambda$. Then either*

1. *$\Gamma \cdot \Lambda = 0$ and $\Gamma \cdot f^{-n}(\Lambda) = 0$, $f^{-n}(\Gamma) \cdot \Lambda = 0$ for all $n \geq 1$; or*
2. *$\Gamma \cdot \Lambda \neq 0$ and*
   (a) *Each component of Γ (resp. Λ) is isotopic to a unique component of $\widetilde{\Gamma}$ (resp. $\widetilde{\Lambda}$), the mapping $f : \widetilde{\Gamma} \to \Gamma$ (resp. $f : \widetilde{\Lambda} \to \Lambda$) is a homeomorphism, $\widetilde{\Gamma} \cap (f^{-1}(\Lambda) - \widetilde{\Lambda}) = \emptyset$, and $\widetilde{\Lambda} \cap (f^{-1}(\Gamma) - \widetilde{\Gamma}) = \emptyset$. More precisely,*
      (i) *for each $\gamma \in \Gamma$, there is exactly one curve $\gamma' \subset f^{-1}(\gamma)$ such that $\gamma' \cap \widetilde{\Lambda} \neq \emptyset$. Moreover, the curve γ' is the unique component of $f^{-1}(\gamma)$ which is isotopic to an element of Γ;*
      (ii) *for each $\lambda \in \Lambda$, there is exactly one connected component λ' of $f^{-1}(\lambda)$ such that $\lambda' \cap \widetilde{\Gamma} \neq \emptyset$. Moreover, the arc λ' is the unique component of $f^{-1}(\lambda)$ which is isotopic to an element of Λ;*
   (b) *the transformations $f_{\#,\Gamma}$ and $f_{\#,\Lambda}$ are transitive permutations of the basis vectors.*
   (c) *The above results remain true if we replace f by f^n, for any $n \geq 1$ (though transitivity may fail).*
   (d) *For any arc λ in (S^2, X) which is isotopic to an arc in $f^{-n}(\Lambda)$ for some $n > 1$ but which is not isotopic to an arc in Λ, $\Gamma \cdot \lambda = 0$. Similarly, for any curve γ in (S^2, X) which is isotopic to a curve in $f^{-n}(\Gamma)$ for some $n > 1$ but which is not isotopic to a curve in Γ, $\Lambda \cdot \gamma = 0$.*

We prove this theorem in the Appendix.

Corollary 3. *If Λ contains more than one periodic cycle, or if the multiplicity of f from some element $\alpha \in \Lambda$ to $\beta \in \Lambda$ is larger than one, then there are no irreducible Thurston obstructions which intersect Λ.*

Proof. If so then $f_{\#,\Lambda}$ is not a transitive permutation of the basis vectors. □

3.3. **Proof of Theorem A.** By Theorem 3.1, *Characterization of marked rational maps*, and our discussion at the end of Section 3.1, (g, X) is combinatorially equivalent to a marked rational map if there exist no irreducible Thurston obstructions Γ for g. Let Γ be such an obstruction and let $\widetilde{\Gamma}$ be the union of those components of $g^{-1}(\Gamma)$ isotopic to elements of Γ, relative to X.

By the second hypothesis of the theorem and Proposition 3, *Orbits off α persist*, for each j, Λ_j is forward invariant under g up to isotopy relative to X.

By Proposition 4, *Decomposition of invariant arc systems*, for each j, there is an irreducible arc system $\Theta_j \subset \Lambda_j$ and an integer n_j such that $f(\alpha_j)$ is isotopic relative to X to a lift of an element of Θ_j under g^{n_j}.

We now show that we may choose representatives of Γ so that $\widetilde{\Gamma} \cap D = \emptyset$. This will complete the proof of the theorem: for since $g|\widehat{\mathbb{C}} - (D - e(\alpha)) = f \circ c$ and $c : \widehat{\mathbb{C}} - (D - e(\alpha)) \to \widehat{\mathbb{C}} - Int(\alpha)$ is isotopic to the identity relative to X, Γ is then an irreducible Thurston obstruction for the marked rational map (f, X). By Theorem 3.1, *Characterization of marked rational maps*, and the fact that f is not covered by an integral torus endomorphism, this gives a contradiction.

First, we may assume Γ minimizes the number of intersections with $f(\alpha_j)$ and Θ_j for all j. Write $\partial D = \cup_j \alpha^{\pm}_{0,j}$, where $\alpha^{\pm}_{0,j}$ are the two unique components of $(g|D)^{-1}(f(\alpha_j))$ which are contained in $\partial_{\pm} D$ respectively. The proof now breaks down into several cases:

Case 1: $\Gamma \cdot f(\alpha_j) = 0$. Then $\Gamma \cap f(\alpha_j) = \emptyset$, hence $\widetilde{\Gamma} \cap g^{-1}(f(\alpha_j)) = \emptyset$, and so $\widetilde{\Gamma} \cap \alpha^{\pm}_{0,j} = \emptyset$.

Case 2: $\Gamma \cdot f(\alpha_j) \neq 0$. We will again show that $\widetilde{\Gamma} \cap \alpha^{\pm}_j = \emptyset$.

We first show that in this case $f(\alpha_j) \in \Theta_j$ up to isotopy. By our choice of Θ_j, $f(\alpha_j)$ is isotopic to a lift of an element $\theta \in \Theta_j$ under f^{n_j}. First, if $\Gamma \cdot \Theta_j = 0$, then by an application of Theorem 3.2, *Arcs intersecting obstructions*, Part 1, we would have $\Gamma \cdot f(\alpha_j) = 0$, contradicting our assumption. Hence $\Gamma \cdot \Theta_j \neq 0$. Next, if $f(\alpha_j) \notin \Theta_j$ up to isotopy, then by Theorem 3.2, *Arcs intersecting obstructions*, Part 2(d) (with $\Lambda = \Theta_j$ and $\lambda = f(\alpha_j)$), $\Gamma \cdot f(\alpha_j) = 0$, contradicting our assumption. Hence $f(\alpha_j) \in \Theta_j$ up to isotopy.

Neither of the arcs $\alpha^{+}_{0,j}, \alpha^{-}_{0,j}$ can be isotopic to elements of Θ_j. For if so, then α must consist of a single arc in (S^2, X), i.e. $L = 1$ in the definition of the blown up map g. Then α itself is isotopic to an element of Θ_j since Θ_j is irreducible. It then follows that $\partial D \subset g^{-1}(f(\alpha))$ consists of two arcs α^{+}_0 and α^{-}_0 in (S^2, X) which are isotopic relative to X, by construction of the blown-up map g. Hence the multiplicity of $g : \alpha^{\pm}_0 \to f(\alpha)$ is at least $n \geq 2$, where n is the number of times we blow up α. Hence the transformation $g_{\#,\Theta}$ is not a transitive permutation of the basis vectors. By Theorem 3.2, *Arcs intersecting obstructions*, $\Gamma \cdot \Theta = 0$, hence $\Gamma \cdot f(\alpha) = 0$, contradicting our assumption.

By the previous paragraph $\alpha^{\pm}_{0,j} \notin \Theta_j$ up to isotopy. By Theorem 3.2, Part 2a(ii), there is exactly one lift λ' of $f(\alpha_j)$ which intersects (as a subset of S^2) the set $\widetilde{\Gamma}$, and this lift λ' is isotopic to an element of Θ_j. Hence $\alpha^{\pm}_{0,j} \cap \widetilde{\Gamma} = \emptyset$.

In both cases we have shown that for all j, $\widetilde{\Gamma} \cap \alpha_{0,j}^{\pm} = \emptyset$. Thus $\widetilde{\Gamma} \cap \partial D = \emptyset$, which implies that $\widetilde{\Gamma} \cap D = \emptyset$: there can be no components of Γ contained entirely in D since $\mathrm{Int}(D) \cap X = \emptyset$. $\square$

Proof of Corollary 1 Suppose α is a periodic or preperiodic internal ray of f. Points in the interior of an internal ray lie in Fatou components and have infinite forward orbits under f. Hence no points in the interior of an internal ray can be preperiodic under f. We may therefore assume that α consists of a single arc in $(S^2, f^{-1}X)$, and hence that $f(\alpha)$ is a periodic or preperiodic internal ray with endpoints in X.

Two internal rays either coincide, or intersect only in their endpoints. Hence if $l_i = f^i(f(\alpha)), i > 0$, then $l_i \cap \mathrm{Int}(f(\alpha)) \neq \emptyset$ if and only if $\alpha = l_i$ for some l_i, i.e. $f(\alpha)$ is a periodic internal ray.

Let $\mathcal{L} = \{l_i\}_{i>0}$. Then $\mathcal{L}$ is a finite collection of arcs in (S^2, X) which, as a subset of the sphere, is forward invariant under f. By Proposition 1 there exists a subset $\Lambda \subset \mathcal{L}$ which is an arc system forward invariant under f such that each element of $\mathcal{L}$ is isotopic relative to X to a unique element of $\mathcal{L}$. By the definition of forward invariance there is a subset $\Lambda_0 \subset \Lambda$ and a lift $\widetilde{\Lambda}_0 \simeq_X \Lambda$.

If $f(\alpha) \notin \Lambda$, then $\mathrm{Int}(f(\alpha)) \cap \Lambda = \emptyset$, hence $\mathrm{Int}(\alpha) \cap f^{-1}(\Lambda) = \emptyset$. Since $\widetilde{\Lambda}_0 \subset f^{-1}(\Lambda)$, $\widetilde{\Lambda}_0 \cap \mathrm{Int}(\alpha) = \emptyset$. The Corollary then follows by Theorem A.

If $f(\alpha) \in \Lambda$, let λ' be any arc which is isotopic to $f(\alpha)$ relative to X and such that $\mathrm{Int}(f(\alpha)) \cap \lambda' = \emptyset$. Let Λ' be the arc system $(\Lambda - \{f(\alpha)\}) \cup \{\lambda'\}$. Then $\Lambda \simeq_X \Lambda'$ and so Λ' is forward invariant under f up to isotopy relative to X. By construction $\Lambda' \cap \mathrm{Int}(f(\alpha)) = \emptyset$, hence $f^{-1}(\Lambda') \cap \mathrm{Int}(\alpha) = \emptyset$. If Λ'_0 is the unique subset of Λ' for which $\Lambda'_0 \simeq_X \Lambda_0$, and if $\widetilde{\Lambda}'_0$ is the lift of Λ'_0 obtained by lifting isotopies between elements of Λ'_0 and Λ_0, then $\widetilde{\Lambda}'_0 \simeq_X \widetilde{\Lambda}_0 \simeq_X \Lambda$. Since $\widetilde{\Lambda}'_0 \subset f^{-1}(\Lambda')$, $\widetilde{\Lambda}'_0 \cap \mathrm{Int}(\alpha) = \emptyset$. The Corollary then follows by Theorem A.

$\square$

Proof of Corollary 2 The argument is similar to the one given above. The hypothesis guarantees the existence of the arc λ' used in the last paragraph. $\square$

3.4. **Proof of Theorem B.** Let $f(z)$ be a postcritically finite monic polynomial and X a finite set such that $f(X) \subset X$ and $X \supset P(f)$. Let $K(f)$ and $F(f)$ denote the filled-in Julia set and Fatou set of f, respectively.

In [Poi], §I.1 (originally in [DH1]), it is shown that $X - \{\infty\}$ determines uniquely a finite topological tree T with vertices $V(T)$ satisfying the following properties.

1. $T \subset K(f)$ and $T \cap F(f)$ only in internal rays;
2. $f(T) \subset T$ and $f(V(T)) \subset V(T)$;
3. $V(T) \supset X - \{\infty\}$.

The definition of internal and external rays is given in the following section.

Our first step in the proof is to construct a "spider" S for T. The construction of S is given in the following section. S is a finite topological graph with vertices $V(S)$ and edges $E(S)$ satisfying the following properties:

1. $V(S) = X \supset P(f)$;
2. Every edge of S is a topological arc joining a finite vertex to infinity;
3. $f(S) \subset S$;
4. f maps edges of S homeomorphically to edges of S;
5. $S \cap T = X - \{\infty\}$;

Note that the edges of S are arcs in (S^2, X).

Given the existence of the spider S with the properties above, we now prove Theorem B.

The edges $E(S)$ form a finite collection of arcs l in (S^2, X) such that $f(l) \in E(S)$ for each $l \in E(S)$. By Proposition 1, there is an arc system $\Lambda \subset E(S)$ which is forward invariant under f up to isotopy relative to X, i.e. there is a subset $\Lambda_0 \subset \Lambda$ and a lift $\widetilde{\Lambda}_0 \simeq_X \Lambda$. Note that since $\widetilde{\Lambda}_0 \simeq_X \Lambda \subset X$, the endpoints of $\widetilde{\Lambda}$ are points in X.

We now show that Λ is forward invariant under g up to isotopy relative to X. By Proposition 3, *Orbits off α persist*, it is enough to show that $\widetilde{\Lambda}_0 \cap \mathrm{Int}(\alpha) = \emptyset$. Since $\alpha \subset T$ and $f(T) \subset T$, $f(\alpha) \subset T$. Since $T \cap S \subset X$, $f(\alpha) \cap S \subset X$. Hence $\alpha \cap f^{-1}(S) \subset f^{-1}(X)$. Now, $\widetilde{\Lambda}_0 \subset f^{-1}(S)$, and as remarked above, the endpoints of $\widetilde{\Lambda}_0$ are points in X, hence $\alpha \cap \widetilde{\Lambda}_0 \subset X$. But $\mathrm{Int}(\alpha) \cap X = \emptyset$, hence $\mathrm{Int}(\alpha) \cap \widetilde{\Lambda}_0 = \emptyset$.

Hence Λ is forward invariant under g up to isotopy relative to X. By Proposition 4, there are disjoint (as subsets of Λ) irreducible arc systems $\Theta_k \subset \Lambda$ and integers n_k such that each element of Λ is a lift of an element of Θ_k under n_k for some k.

Let Γ be an irreducible Thurston obstruction for g and let $\widetilde{\Gamma}$ be the union of components of $g^{-1}(\Gamma)$ which are isotopic relative to X to elements of Γ. We may assume that Γ minimizes the number of intersections with each $f(\alpha_j)$ and with each $\lambda \in \Lambda$. As in the previous proof it suffices to show that $\widetilde{\Gamma} \cap \partial D = \emptyset$, where D is the disc used in the construction of g.

Let $\mathcal{K} = (g|D)^{-1}(\Lambda)$; this is a finite, connected graph in D with vertices equal to $g^{-1}(X) \cap D \subset \mathcal{K}$.

We first reduce to the case when $\widetilde{\Gamma} \cap \mathcal{K} = \emptyset$. As in the previous proof, there are two cases. Let $\lambda \in \Lambda$.

Case 1: $\Gamma \cdot \lambda = 0$. Then $\Gamma \cap \lambda = \emptyset$ by our choice of representatives of Γ, so $\widetilde{\Gamma} \cap (g|D)^{-1}(\lambda) = \emptyset$.

Case 2: $\Gamma \cdot \lambda \neq 0$.

Let $\widetilde{\lambda}$ be any lift of λ under g which is contained in D (and hence in $\mathcal{K}$). By Proposition 4, λ is isotopic to a lift of an element of some irreducible arc system $\Theta_j \subset \Lambda$ under f^{n_k}, some n_k.

Since $\lambda \in \Lambda \subset E(S)$, λ is an edge of the spider S and hence joins a point of $X - \{\infty\}$ to the point ∞. By construction $D \cap \infty = \emptyset$. Since $\widetilde{\lambda} \subset D$ is a lift of λ under g, one of the endpoints of $\widetilde{\lambda}$ is contained in D and maps onto ∞. Hence this endpoint is strictly preperiodic under g. Hence $\widetilde{\lambda}$ cannot be contained in the irreducible arc system Θ_k. By Theorem 3.2, *Arcs intersecting obstructions*, $\widetilde{\Gamma} \cap \widetilde{\lambda} = \emptyset$.

Together, the two cases show that $\widetilde{\Gamma} \cap \mathcal{K} = \emptyset$.

So we may assume $\widetilde{\Gamma} \cap \mathcal{K} = \emptyset$. We conclude the proof by showing that if $\widetilde{\Gamma} \cap \partial D \neq \emptyset$, then Γ does not have minimal intersection number with $f(\alpha)$, contrary to our assumption.

Suppose $\widetilde{\gamma} \subset \widetilde{\Gamma}$ intersects ∂D. Since $\widetilde{\Gamma} \cap \mathcal{K} = \emptyset$, $\widetilde{\gamma} \cap \mathrm{Int}(D)$ is contained in a component V of $D - \mathcal{K}$. Since $V(S) = X \supset P(f)$, and since each edge $E(S)$ is isotopic relative to X to an element of some Λ_k, each component of $S^2 - \Lambda$ is an open disc containing no critical values of g. Hence each component of $g^{-1}(S^2 - \Lambda)$ maps homeomorphically under g to its image. Since V is contained in such a component, $g|V$ is a homeomorphism. Let $W \subset V$ be any component of $V - \widetilde{\gamma}$ which "outermost", i.e. not separated from ∂D by any component of $\widetilde{\gamma} \cap D$. Then W is a disc in $S^2 - g^{-1}X$ bounded by exactly one subarc of $\widetilde{\gamma}$ and one subarc of $\partial_+ D$ or $\partial_- D$. Since $g|V$ is a homeomorphism and $W \subset V$, $g(W)$ is a disc in $S^2 - X$ bounded by one subarc of $f(\alpha)$ and one subarc of $\gamma \subset \Gamma$, contradicting the assumption that $\Gamma \cap f(\alpha)$ is minimized. □

3.5. **Construction of S.** We first recall some facts about the dynamics of postcritically finite polynomials from [Mil].

Let $f(z)$ be a postcritically finite polynomial. The *filled-in Julia set* $K(f) = \{z | f^n(z) \not\to \infty\}$ is a connected, compact subset of $\mathbb{C}$ whose complement is connected. The Julia set $J(f) = \partial K(f)$ is locally connected. For each Fatou component Ω, there is exactly one point $x \in \Omega$ such that $f^n(x) \in P(f)$ for some n. Let Ω_x denote the Fatou component containing such a point x. A classical theorem of Böttcher implies the following: there are holomorphic isomorphisms $\phi_x : (\Delta, 0) \to (\Omega_x, x)$ such that for all $w \in \Delta$, $\phi_{f(x)}(w^{d_x}) = f(\phi_x(w))$, where d_x is the local degree of f near x. Since $J(f)$ is locally connected, each Fatou component has locally connected boundary, and a theorem of Carathéodory implies that the maps ϕ_x extend continuously to $S^1 = \partial\Delta$. Let R_t denote the radial line $\{r \exp(2\pi i t) | 0 \leq r \leq 1\}$. The image $R_{x,t} = \phi_x(R_t)$ we call the *ray of angle t in Ω_x*; if $x = \infty$ $R_{x,t}$ is called an *external ray*; otherwise we call it an *internal ray*. Each periodic point $x \in J(f)$ is the landing point of at least one and at most finitely many external rays. The landing point of an external ray is periodic if and only if x is periodic, though the periods may differ. Internal and external rays map homeomorphically onto their images.

Construction of S. Our spider will be a union

$$S = \bigcup_{x \in X - \{\infty\}} E_x,$$

where E_x denotes the set of edges of the graph S incident to x.

- If $x \in J(f) \cap (X - \{\infty\})$, each element $e \in E_x$ will be an external ray landing at x, and E_x will be the union of all external rays landing at x.
- If $x \in F(f) \cap (X - \{\infty\})$, each element $e \in E_x$ will be the union of exactly one internal ray in Ω_x and one external ray landing at a common point $q \in \partial\Omega_x$. The set E_x will be a finite union of such pairs.

Since external and internal rays map homeomorphically onto their images, and the image of an internal (external) ray is again an internal (external) ray, edges will map homeomorphically to edges.

Each $x \in J(f) \cap (X - \{\infty\})$ is eventually periodic, so the set E_x of external rays landing at x is finite (see [Mil], §18), and $f(E_x) = E_{f(x)}$. Since $T \subset K(f)$, $E_x \cap T \subset X - \{\infty\}$.

We now define E_x for $x \in F(f) \cap (X - \{\infty\})$. First, choose one element x from each periodic cycle in $(X - \{\infty\}) \cap F(f)$. Let p be the period of x. Then $T \cap \Omega_x$ consists of a finite collection of eventually periodic internal rays. Thus there exists a periodic internal ray R_x, of some period $kp > 0$, such that $R_x \cap T = \{x\}$. Let q be the landing point of the internal ray R_x; then q is periodic of period k. There exists a periodic external ray R_∞ landing at q. Since q is not a critical point, f^k is a local homeomorphism near q sending R_x to itself, hence the ray R_∞ also has period k. Let $e_x = R_x \cup R_\infty$; this will be one edge in the collection E_x. Define

$$E_x = \bigcup_{n=0}^{k-1} f^{np} e(x).$$

E_x is finite since q is periodic. For $y \in f^n(x), 1 \leq n < p$, define

$$E_y = f^n(E_x).$$

Thus E_y is finite. Since R_x is periodic, $R_x \cap T = \{x\}$, $f(X - \{\infty\}) \subset X - \{\infty\}$, and $f(T) \subset T$, we have that $E_y \cap T \subset X - \{\infty\}$ for all y.

We now inductively define E_x for strictly preperiodic $x \in X - \{\infty\} \cap F(f)$. Suppose the collection E_y has been defined, where $f(x) = y$. Choose any element $e_y \in E_y$. The edge e_y is a union of an internal ray R_y and external ray R_∞; let q be their common landing point. Choose any preimage R_x of R_y joining x to a point q' in $\partial\Omega_x$. Since a small disc near q pulled back to q' is again a disc, there is a preimage R'_∞ of R_∞ joining ∞ to q'. Let $e_x = R_x \cup R'_\infty$ and $E_x = \{e_x\}$. Then $f : e_x \to e_y$ and $E_x \cap T = \{x\}$.

This completes the construction of S.

4. Complements

4.1. **Blowing up multiple arcs.** Theorems A and B generalize in the obvious way to blowing up along finite collections $\alpha^1, \alpha^2, ...$ of arcs for which $\mathrm{Int}(\alpha^i) \cap \alpha^j = \emptyset, i \neq j$.

4.2. **Blowing up Möbius transformations.** Let G be a finite connected graph in S^2 with vertex set X and edge set E. Suppose in addition that no edge joins a vertex to itself and that no two vertices are joined by more than one edge. Assign to each edge $e \in E$ an integer $m(e) \geq 0$. Let (M, X) be a marked Möbius transformation of finite order and suppose $M(G) = G$.

Theorem: (Blowing up Möbius transformations) *Let (g, X) be the marked branched covering (M, X) blown up $m(e)$ times along the edge $e \in E$. Let G' denote the subgraph of G which is the union of edges e' (and their vertices) for which $m(M^i e') > 0$ for some i. Then (g, X) is equivalent to a marked rational map if and only if*

1. *G' is connected, and*
2. *if G' is nonempty, no component of $S^2 - G'$ contains more than one point of X.*

Proof. If G' is empty there is nothing to prove. Otherwise, there is at least one edge which is blown up. The vertices of this blown up edge become periodic critical points for g, hence the orbifold of the (unmarked) branched covering g is not the $(2, 2, 2, 2)$ orbifold. Hence by Theorem 3.1, *Characterization of marked rational maps*, (g, X) is equivalent to a marked rational map if and only if (g, X) has no Thurston obstructions.

Note that G' is invariant under M. Let X' denote the vertices of G'. Then for each connected component G'' of G', G'' has at least two vertices.

We now prove the necessity of (1) and (2). First, assume that (g, X) is equivalent to a marked rational map. Then $(g^{\circ n}, X)$ is also equivalent to a marked rational map for every $n \geq 1$.

If (2) fails, then there is a component U of $S^2 - G'$ such that $|U \cap (X - X')| \geq 2$. Let γ be any simple closed curve in U separating G' from $U \cap (X - X')$. Then γ is nonperipheral since $|X'| \geq 2$ and $|U \cap (X - X')| \geq 2$ by assumption. Choose $n \geq 1$ so that $M^{\circ n}(\gamma) = \gamma$. We may further assume that γ is disjoint from the discs W within which we alter the map (M, X) to obtain the map (g, X). It then follows that $g^{\circ n}(\gamma) = \gamma$ and that $(g^{\circ n})|\gamma$ is a homeomorphism. Then γ is a Levy cycle for $(g^{\circ n}, X)$, so $(g^{\circ n}, X)$ is not equivalent to a rational map.

If (1) fails, then there is a simple closed curve γ in $S^2 - (G' \cup X)$ separating G' into two subsets, each of which has at least two elements of X. Hence γ is nonperipheral. The same argument as in the previous paragraph then shows that $(g^{\circ n}, X)$ is not equivalent to a rational map.

We now prove the sufficiency of (1) and (2). We may assume $|X| \geq 4$, since otherwise there are no nonperipheral simple closed curves in (S^2, X) and hence no

Thurston obstructions. Next, we may assume that for every $e \in E$ and for every $i > 0$, $M^i e \simeq_X e$ if and only if $M^i e = e$, by our requirement that there is at most one edge between any two vertices and the assumption $M(G) = G$.

For each orbit of an edge in E' under M, choose one edge e'_j. Let $\epsilon_j \simeq_X e'_j$ be an arc in (S^2, X) chosen so that $\epsilon_j \cap G = e(\epsilon_j)$. Then in particular, $\mathrm{Int}(\epsilon_j) \cap e' = \emptyset$ for each $e' \in E'$. Let Λ_j denote the arc system which is the orbit of ϵ_j under M. Then Λ_j is forward-invariant under M and the interior of every element of Λ_j is disjoint from the edges along which we blow up. Hence by Proposition 3, *Orbits off α persist*, each Λ_j is forward-invariant up to isotopy under (g, X).

The transformations $g_{\#,\Lambda_j}$ are irreducible but are not transitive permutations of the basis vectors. The transformation $M_{\#,\Lambda_j}$ is a transitive permutation of the basis vectors since Λ_j is the orbit of a single arc under M and is therefore irreducible. Hence as matrices $g_{\#,\Lambda_j} \geq M_{\#,\Lambda_j} \geq 0$, so $g_{\#,\Lambda_j}$ is irreducible since $M_{\#,\Lambda_j}$ is irreducible. Moreover, if $\lambda \in \Lambda_j$ and $\lambda \simeq_X e'$ where $m(e') \geq 1$, then $\mathrm{mult}(g : \lambda \to g(\lambda)) = m(e') + 1 \geq 2$, and so the entry of $g_{\#,\Lambda_j}$ corresponding to the ordered pair $(\lambda, g(\lambda))$ is at least two.

Let Γ be an irreducible Thurston obstruction for (g, X). We may assume that the elements of Γ are chosen so that $\Gamma \cdot \Lambda_j = |\Gamma \cap \Lambda_j|$ for each j. By Theorem 3.2, *Arcs intersecting obstructions*, $\Gamma \cap \Lambda_j = \emptyset$ for each j. Since $\cup_j \Lambda_j = G'$, $\Gamma \subset S^2 - G'$. Conditions (1) and (2) imply that every component of $S^2 - (G' \cup X)$ is either simply-connected or is a once-punctured disc. Hence there are no nonperipheral simple closed curves in $S^2 - (G' \cup X)$, and so there are no Thurston obstructions for (g, X). $\square$

Example: Let $f_{p/q}(z) = z^2 + c_{p/q}$, $q \geq 2$, where $p < q$ and p and q are relatively prime, where $c_{p/q}$ is the center of the hyperbolic component tangent to the main cardioid in the p/q limb of the Mandelbrot set. We call such a polynomial "starlike". (There is a topological tree T which is a union of q internal rays, each joining a point in the finite superattracting cycle containing 0 to a common repelling fixed point. Thus T is a "star". The tree T is mapped homeomorphically under f into itself with rotation number p/q.) Let $M_{p/q} = e^{2\pi i p/q} z$ and $X_{p/q} = \{\infty\} \cup_{n \in \mathbb{Z}} e^{2\pi i n p/q}$. Let G be the orbit of the arc $[1, \infty]$ under $M_{p/q}$. Then $f_{p/q}$ is equivalent to $(M_{p/q}, X_{p/q})$ blown up once along $\alpha = [1, \infty] \subset G$.

Remark: Let $M_{p/q}$ be as above. Let $Y_{p/q} = X_{p/q} - \{\infty\}$ and let G be the unit circle. Let α be the circular segment joining 1 and $e^{2\pi i \frac{1}{q}}$. Then blowing up $(M, Y_{p/q})$ once along α yields a quadratic non-polynomial rational map. Mary Rees (personal communication) has informed us that all quadratic non-polynomial maps with obstructed tunings arise in this fashion.

4.3. **Killing obstructions by blowing up.** The blowing up construction can be applied to marked branched covers which are not equivalent to rational maps in such a way that the result is equivalent to a rational map.

Let $p(z) = z^2 - 1$. Extend p to a degree two branched cover of the closed disc D by adjoining the map $z \to z^2$ along the circle at infinity S^1_∞. Now let f be the branched cover obtained by gluing the action of p on two copies D_+, D_- of D along S^1_∞. The map f is called the *mating* of f with itself and is not equivalent to a rational map. For let R_t denote the external ray of angle t for p. Then after adjoining on S^1_∞, R_t becomes a topological arc R_t^∞ in D joining $\infty \cdot e^{2\pi i t}$ to the landing point of R_t in $J(p)$. For the map f, the unions of arcs $R^\infty_{1/3}$, $R^\infty_{2/3}$ in D_+ and D_- are joined together to form a simple closed curve $\gamma \subset S^2 - P(f)$ which maps to itself homeomorphically by degree one, and hence is an obstruction. Note that the interval $[-1, 0] \subset D_-$ is mapped homeomorphically to itself under f. Consider the map $g = (f, P(f))$ blown up once along $\alpha = [-1, 0] \subset D_-$. Then by Theorem 3.2, *Arcs Intersecting obstructions*, no obstruction can intersect α. Hence any obstruction for g is isotopic relative to $P(g)$ into D_+, where there are no such obstructions. So g is equivalent to a rational map while f is not.

4.4. **Generalized matings.** One may think of the map g in the previous example as the polynomial $z^2 - 1$ mated with the rational map which is $z^2 - 1$ blown up along α. (The resulting branched covering is well- defined since there are no critical points in the interior of the disc D in which we glue new dynamics.) Using techniques similar to the ones employed in the proofs of our theorems one may prove the following. To set up the statement, suppose $f_1(z)$ is a starlike quadratic polynomial with tree T. Let α be the union of two consecutive edges in T. Then α satisfies the hypothesis of Corollary 2, so blowing up n times along α yields a branched covering equivalent to a rational map $r_n(z)$. The map $r_n(z)$ has a fixed critical point of local degree two which we may take to be at infinity.

Theorem 4.1 (Matings with blown-up starlike polynomials exist)**.** *Let $f_2(z)$ be an arbitrary postcritically finite quadratic polynomial. Then the mating of $f_2(z)$ with $r_n(z)$ is equivalent to a rational map $g(z)$.*

Similar results can be formulated for higher-degree maps as well as for generalizations of tunings. The proofs are all similar to those given here: an obstruction is forced to lie outside the region on the sphere in which the new dynamics is glued by Theorem 3.2, *Arcs intersecting obstructions.*

4.5. **Special maps.** The following theorem is useful since there are interesting examples of maps produced by blowing up arcs to which Theorems A' and B' do not apply.

Theorem: (Special maps) *If g is (f, X) blown up along α, and if $|P(g)| = 3$, then g is equivalent to a rational map.*

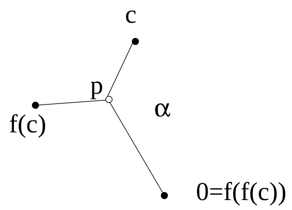

FIGURE 4. The tree for the Douady Rabbit.

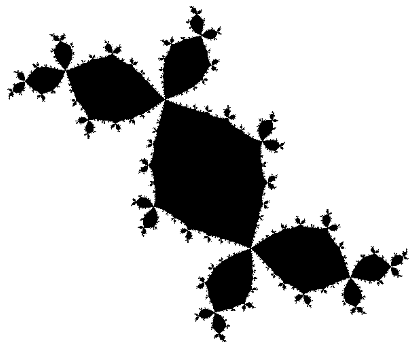

FIGURE 5. Douady's Rabbit.

Proof. The orbifold of g is hyperbolic, so g is equivalent to a rational map if and only if there are no Thurston obstructions, which do not exist since there are no nonperipheral curves in $(S^2, P(G))$. □

5. EXAMPLES

In this section we give some examples and applications.

5.1. Examples illustrating Theorems.

Blowing up a periodic arc in the "rabbit". Let $f(z) = z^2 + c$, where $c \approx -0.12256117 + 0.74486177i$ is chosen so that 0 is periodic of period three and $\mathrm{Im}(z) > 0$. Let $X = \{0, c, f(c)\}$. Then f is "starlike", so there is a tree T (shown in Figure 4) with vertices consisting of the set X and the alpha fixed point p of f and which is mapped homeomorphically to itself under f with rotation number $1/3$. T is a union of the three internal rays for f which are fixed under $f^{\circ 3}$.

We let α be the union of the two internal rays joining 0 and c to p. By Corollary 2, blowing up once along α yields a branched covering equivalent to a rational map. The Julia set of f is shown in Figure 5, and that of f blown up along α is shown in Figure 6.

Blowing up the "airplane". Let $f(z) = z^2 + c$, where $c \approx -1.75488$ is chosen so that the critical point at zero is periodic of period three with orbit lying in the

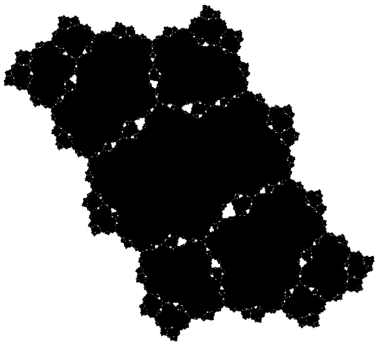

FIGURE 6. The Douady Rabbit blown up along α.

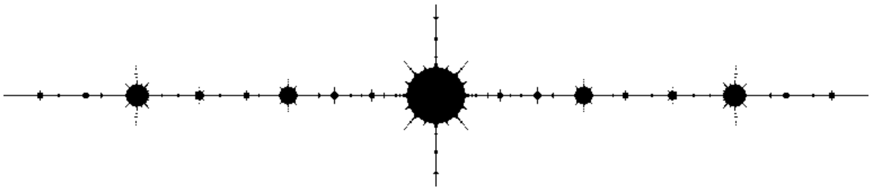

FIGURE 7. The airplane.

real line. Then $c < 0 < f(c)$. Let $X = \{c, 0, f(c)\}$. The tree T_X is the interval $[c, f(c)]$ with vertices X. The polynomial f sends the invervals $[c, 0]$ and $[0, f(c)]$ homeomorphically to the interval $[c, f(c)]$. We let $\alpha = [c, 0] \subset T_X$. Then the Blowing Up conditions are satisfied, and by Theorem B, (f, X) blown up once along α is equivalent to a rational map.

The Julia set of f is shown in Figure 7. The Julia set of the result of blowing up f along α is shown in Figure 8; the map is given by $g(z) = a\frac{(z-1)^3}{1-3z}$, where $a \approx -2.37123$, normalized so that $1 \mapsto 0$ by local degree three, $0 \mapsto -a$ by local degree two, and $-a \mapsto 1$ by local degree one.

A degree four map with S_3 symmetry. Let G be a topological graph in S^2 which is a triangle with vertices X and edges E. Let M be the identity map. By Theorem 4.2, *Blowing up Möbius transformations*, the marked covering (g, X) blown up once along each edge in E is equivalent to a degree four rational map g with three fixed critical points each of mulitiplicity two. The full mapping class group of orientation-preserving homeomorphisms h of $S^2 - X$ up to isotopy is isomorphic to S_3 and commutes with g up to equivalence since the blowing up construction in this case is performed symmetrically. That is, $h \circ g$ is equivalent to $g \circ h$. It follows that the conformal automorphism group of g is S_3. A formula for this map is given in [McM] and is used there for giving a generally convergent iterative algorithm for solving the cubic.

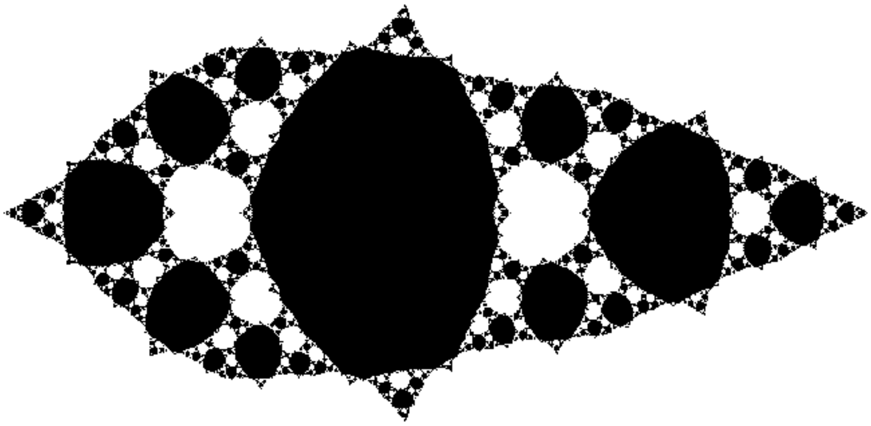

FIGURE 8. The airplane blown up along α.

Remark: More generally one can apply Theorem 4.2, *Blowing up Möbius transformations* to other symmetric graphs and the identity map to produce symmetric postcritically finite rational maps; cf. [DM] for a detailed discussion of rational maps with icosahedral symmetry.

6. Appendix: Proof of Theorem *Arcs intersecting obstructions*

We will use the following easy fact:

Lemma 1. *Let f be a branched covering from S^2 to S^2. Let B be a subset of S^2 such that $f|_B : B \to f(B)$ is a bijection (resp. a k-to-1 mapping). Let A be any subset of S^2. Then $\#f^{-1}(A) \cap B = \#A \cap f(B)$ (resp. $\#f^{-1}(A) \cap B = k \cdot \#A \cap f(B)$).*

For Λ a curve system or an arc system in $(S^2, X))$, set $\widetilde{\Lambda}$ (resp. $\widetilde{\Lambda}(f^n)$) to be the union of those components of $f^{-1}(\Lambda)$ (resp. $f^{-n}(\Lambda)$) which are isotopic to elements of Λ. If Λ is irreducible, each component of Λ is isotopic to some (possibly non-unique) component of $\widetilde{\Lambda}(f)$ (resp. $\widetilde{\Lambda}(f^n)$).

By the definition of irreducible arc system, there exists at least one subset $\widetilde{\Lambda}'$ of $\widetilde{\Lambda}$ such that $\widetilde{\Lambda}'$ is isotopic to Λ and $f : \widetilde{\Lambda}' - X \to \Lambda - X$ is a homeomorphism. Then

$$f_{\#,\Gamma}(\gamma_j) \cdot \Lambda \leq \sum_{\gamma' \subset f^{-1}(\gamma_j)} \#(\gamma' \cap \widetilde{\Lambda}') = \#(f^{-1}(\gamma_j) \cap \widetilde{\Lambda}') = \#(\gamma_j \cap \Lambda) = \gamma_j \cdot \Lambda \tag{1}$$

where the first inequality follows since $f : \widetilde{\Lambda}' - X \to \Lambda - X$ is a homeomorphism and the second to last equality follows by the lemma. Assume $f_{\#,\Gamma}(\gamma_j) = \sum_i b_{ij}\gamma_i$. Then by (1), $(\sum_j \sum_i b_{ij}\gamma_i) \cdot \Lambda \leq \Gamma \cdot \Lambda$. But

$$(\sum_j \sum_i b_{ij}\gamma_i) \cdot \Lambda = (\sum_i (\sum_j b_{ij})\gamma_i) \cdot \Lambda \geq \sum_i \gamma_i \cdot \Lambda = \Gamma \cdot \Lambda \ , \tag{2}$$

where the last inequality is obtained as follows. By irreducibility, f_Γ has no zero row no zero column, so the same is true for $f_{\#,\Gamma}$. But each entry of $f_{\#,\Gamma}$ is a non negative integer, so $b_{ij} \geq 1$ if $b_{ij} \neq 0$. We have then either A) $\Gamma \cdot \Lambda = 0$ or B) $\Gamma \cdot \Lambda \neq 0$ and all inequalities above become equalities.

In case that $\Gamma \cdot \Lambda = 0$, we claim that $\Gamma \cdot f^{-1}(\Lambda) = 0$. Let $\gamma_i \in \Gamma$. Then $\gamma_i \sim \gamma'$ with γ' a curve in $f^{-1}(\gamma_j)$ for some $\gamma_j \in \Gamma$. Then $\gamma_i \cdot f^{-1}(\Lambda) \leq \#(\gamma' \cap f^{-1}(\Lambda)) = k\#(\gamma_j \cap \Lambda) = 0$. The case for $n > 1$ is proved similarly. This prove part one of the theorem.

In case B), equalities in Equation 2 imply that for each i, there is a unique j such that $b_{ij} \neq 0$ and that $b_{ij} = 1$. Thus for each γ_i, there is a unique γ_j such that some component of $f^{-1}(\gamma_j)$ is isotopic to γ_i. Also, there is a unique such component γ' of $f^{-1}(\gamma_j)$ isotopic to γ_i. Since $f_{\#,\Gamma}$ is irreducible, this implies that $f_{\#,\Gamma}$ is a transitive permutation of the basis vectors. As a consequence, γ' is also the unique curve in $f^{-1}(\gamma_j)$ isotopic to an element of Γ.

Equalites in Equation 1 tells us that $f_{\#,\Gamma}(\gamma_j) \cdot \Lambda = \#(f^{-1}(\gamma_j) \cap \widetilde{\Lambda}') = \gamma_j \cdot \Lambda$. Combining with the facts that $f_{\#,\Gamma}(\gamma_j) = [\gamma']_\Gamma = \gamma_i$, $\gamma' \cdot \Lambda \leq \#(\gamma' \cap \widetilde{\Lambda}')$ and $\gamma' \cap \widetilde{\Lambda}' \subset f^{-1}(\gamma_j) \cap \widetilde{\Lambda}'$, we get $\gamma_i \cdot \Lambda = \gamma_j \cdot \Lambda$ and $\gamma' \cap \widetilde{\Lambda}' = f^{-1}(\gamma_j) \cap \widetilde{\Lambda}'$. As a consequence, γ' is also the unique curve in $f^{-1}(\gamma_j)$ such that $\gamma' \cap \widetilde{\Lambda}' \neq \emptyset$. Moreover $f^{-1}(\Gamma) \cap \widetilde{\Lambda}' = \widetilde{\Gamma} \cap \widetilde{\Lambda}'$.

The matrix f_Γ is a permutation with entry $a_{ij} = 1/k_j$, where k_j is the degree of $f : \gamma' \to \gamma_j$. Since $\lambda(f_\Gamma) \geq 1$, we have $k_j = 1$ for all j and $f_\Gamma = f_{\#,\Gamma}$. Thus $f : \widetilde{\Gamma} \to \Gamma$ is a homeomorphism.

Note that in the above argument we only used the following properties of the triple $(\Gamma, \Lambda, \widetilde{\Lambda}')$: the matrix f_Γ has neither zero row nor zero column, with leading eigenvalue at least 1, the set $\widetilde{\Lambda}'$ is isotopic to Λ and $f : \widetilde{\Lambda}' - X \to \Lambda - X$ is a homeomorphism. By what we just proved, the triple $(\Lambda, \Gamma, \widetilde{\Gamma})$ verifies also these properties. So we can redo the same argument for this new triple. We conclude then for each $\lambda \in \Lambda$, there is exactly one curve λ' in $f^{-1}(\lambda)$ with $\lambda' \cap \widetilde{\Gamma} \neq \emptyset$. Moreover, λ' is the unique curve in $f^{-1}(\lambda)$ which is isotopic to an element of Λ. The matrix $f_{\#,\Lambda}$ is a transitive permutation. As a consequence, $\widetilde{\Lambda}' = \widetilde{\Lambda}$.

Replace $\widetilde{\Lambda}'$ by $\widetilde{\Lambda}$ in the beginning of the proof, we conclude finally that $\widetilde{\Gamma} \cap f^{-1}(\Lambda) = \widetilde{\Gamma} \cap \widetilde{\Lambda} = f^{-1}(\Gamma) \cap \widetilde{\Lambda}$ and $\#\widetilde{\Gamma} \cap \widetilde{\Lambda} = \#\Gamma \cap \Lambda = \Gamma \cdot \Lambda$. As a consequence, $\widetilde{\Gamma} \cap (f^{-1}(\Lambda) - \widetilde{\Lambda}) = \emptyset$ and $(f^{-1}(\Gamma) - \widetilde{\Gamma}) \cap \widetilde{\Lambda} = \emptyset$.

Now take any arc λ isotopic to some arc in $f^{-1}(\Lambda)$ but to none of the arcs in Λ, it is isotopic to an arc $\lambda' \in f^{-1}(\Lambda) - \widetilde{\Lambda}$. So $\Gamma \cdot \lambda \leq \#(\widetilde{\Gamma} \cap \lambda') = 0$.

This proves the part $2(a), 2(b)$ and $(2(d), n = 1)$ of the theorem. To get $2(c)$ and $(2(d), n > 1)$, we just need to note that if Γ, Λ are irreducible for f, they are also irreducible for f^n, for all $n \geq 1$. This gives the theorem. $\square$

The following theorem can be proved in exactly the same manner as Theorem 3.2.

Theorem 6.1 (Arcs intersecting arcs). *Let Λ_1, Λ_2 be two irreducible systems of arcs. Then either*

1. $\Lambda_1 \cdot \Lambda_2 = 0$ and $f^{-n}(\Lambda_1) \cdot f^{-n}(\Lambda_2) = 0$ for all $n \geq 1$, or

2. $\Lambda_1 \cdot \Lambda_2 \neq 0$ and $(f^{-n}(\Lambda_1) - \widetilde{\Lambda}_1(f^n)) \cdot \Lambda_2 = 0 = \Lambda_1 \cdot (f^{-n}(\Lambda_2) - \widetilde{\Lambda}_2(f^n))$ for all $n \geq 1$. In this case we also have (a) The transformations $f_{\sharp,\Lambda_i}$ are transitive permutations of the basis vectors; (b) for each $\lambda \in \Lambda_i$, there is a unique component of $f^{-1}(\lambda)$ intersecting $\Lambda_{i'}$, $i \neq i' \in \{1, 2\}$; (c) (a) and (b) remain true if we replace f by $f^n, n \geq 1$.

Corollary 4. *If Λ_1 contains more than one periodic cycle, or if some element of Λ maps with multiplicity greater than one, then $\Lambda_1 \cdot \Lambda_2 = 0$ for any irreducible arc system Λ_2.*

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