EXTREMAL SOLUTIONS OF THE TWO-DIMENSIONAL L-PROBLEM OF MOMENTS, II

MIHAI PUTINAR

ABSTRACT. All extremal solutions of the truncated L-problem of moments in two real variables , with support contained in a given compact set, are described as characteristic functions of semi-algebraic sets given by a single polynomial inequality. An exponential kernel, arising as the determinantal function of a naturally associated hyponormal operator with rank-one self-commutator, provides a natural defining function for these semi-algebraic sets. We find an intrinsic characterization of this kernel and we describe a series of analytic continuation properties of it which are closely related to the behaviour of the Schwarz reflection function in portions of the boundary of the extremal supporting set.

1. INTRODUCTION

In a previous paper, [P1], a special class of extremal solutions of the L-problem of moments in two variables was related via hyponormal operators to quadrature domains in the plane and to some rational functions related to them. It is the aim of the present paper to apply the same techniques to all extremal solutions of the L-problem and to begin a study of the analytic objects arising from this investigation. Although this programme is more general and and the results below are less precise, a series of facts from operator theory and function theory converge to an interesting picture of all extremal solutions of the L-problem of moments.

To explain the above sentences we have to be more specific. Let K be a compact subset of the complex plane, let L be a fixed positive constant and let N be a fixed positive integer. We are interested in classifying and characterizing in intrinsinc terms the moments

$$a_{kl} = \int_K \phi(x,y) x^n y^m dA, \quad 0 \le m \le m+n \le N,$$

of a measurable function ϕ on K which satisfies $0 \le \phi \le L$, dA-a.e. . Above, and throughout the paper dA stands for the planar Lebesque measure. Note that, because we are working with the two-dimensional Lebesgue measure, the alterations of null measure of the compact set K (such as adding or removing continuous exterior lines

1991 *Mathematics Subject Classification.* 44A60, 47B20, 30E05.

This paper was completed while the author was on leave from University of California at Riverside, visiting the Mathematical Sciences Research Institute in Berkeley; paper supported in part by NSF grant DMS 9500954 and research at MSRI partially supported by NSF grant DMS 9022140.

or internal slits) are not significant for our discussion. As a matter of fact, later in the paper we will restict ourselves to the case when K is the closure of a domain with real algebraic boundary.

Let Σ denote the collection of all vectors $a = (a_{kl})_{k+l \le N} \in \mathbf{R}^d, (d = \frac{(N+1)(N+2)}{2})$ which arise as the moments of a function ϕ as before. It is clear that Σ is a compact convex subset of $\mathbf{R}^d$. Following M.G.Krein and his classical by now convexity theory (cf. [K],[KN]) we infer that a point a is extremal in the set Σ if and only if it corresponds to the moments of a function ψ of the form

$$\psi = L\chi_\Omega, \quad \Omega = \{(x, y) \in K; p(x, y) > 0\},$$

where p is an arbitrary polynomial of degree N (with real coefficients) and χ_D is the characteristic function of the set D. Moreover, we will see below that only in this case the above moment problem has a unique solution. The role of the bound L in some related extremality problem will also appear in the sequel. Thus, following the classical one-variable theory, we may ask how the extremal solution ϕ is encoded in its moments of degree less or equal than N. Although we are far from having a satisfactory answer to this basic question in its full generality, some particular cases are worth being discussed in more detail.

For instance, in the special case when the set Ω above (in the formula of the extremal solution ϕ) is a quadrature domain, the following exponential kernel was the key to the preceding determination problem (and actually to much more):

$$E_\Omega(z, \overline{w}) = exp(\frac{-1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}), \quad (z, w \in \mathbf{C} \setminus \overline{\Omega}).$$

(The reader will notice that we have passed tacitly to complex coordinates; this transformation obviously does not change the moment problem).

The importance of the exponential kernel is two fold: first it is analytic in $z, \overline{w}$ and the moments of the function χ_Ω can be deduced by simple algebraic operations from its Taylor expansion at infinity, and second it admits a canonical factorization as

$$E_\Omega(z, \overline{w}) = 1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi\rangle, \quad (z, w \in \mathbf{C} \setminus \overline{\Omega}).$$

where T is the unique hyponormal operator with one-dimensional commutator

$$([T^*, T] = \xi \otimes \xi)$$

having χ_Ω as principal function. See [P1],[P2] for details and bibliographical references.

The main result of [P1] asserts that Ω is a domain of quadrature (in the sense of [Gu],[Sa],[Sh]) if and only if there is a polynomial $P(z)$ such that the function $P(z)\overline{P(w)}E_\Omega(z, \overline{w})$ is polynomial at infinity,or, if and only if the T^*-invariant subspace generated by the vector ξ is finite dimensional. In this case a simple and rather constructive dictionary relates the above three objects, see [P2]. For what follows it is important to remark that, in the case of a quadrature domain, the associated exponential kernel extends meromorphically in each variable from $\mathbf{C} \setminus \overline{\Omega}$ to the whole plane. As explained in [P2], in the polar parts of the Laurent series of E_Ω at its (finitely many) poles we can read the complete sequence of moments of χ_Ω, the operator T and in particular a defining equation for the domain Ω.

Let us turn now to an arbitrary domain $\Omega \subset K$ which carries an extremal solution of the truncated L-problem of moments supported by K. A first part of the present paper is devoted to the analytic continuation properties of the associated exponential kernel $E_\Omega(z, \overline{w})$. First we will see that E_Ω extends analytically in both variables $z, \overline{w}$ across a point $\lambda \in \partial\Omega$ only if the boundary $\partial\Omega$ is real analytic in a neighbourhood of λ. Then we relate the local analytic continuation of the exponential kernel to the Schwarz function of the boundary of Ω. (See [D],[Sh] for details about the Schwarz function).

Passing from local to global analytic continuation, we prove that the kernel E_Ω extends analytically inside Ω as far as the Schwarz function (of a portion of the boundary) extends along continuous paths.

Suppose now that the supporting compact set K is semi-algebraic, that is it is defined by a simultaneous system of polynomial inequalities. Such examples are the disk, the square, etc. Thus, in view of the aforementioned characterization of extremal solutions ϕ of the L-problem of moments, the supporting domain Ω of ϕ is still semi-algebaric (with one more defining inequality). In that case it is well known that the Schwarz function of the irreducible components of the boundary of Ω is an algebraic function. Thus the kernel E_Ω extends in each variable, along paths, from a connected component C of $\partial\Omega$ to the whole domain Ω, minus the ramification points of the respective Schwarz functions. As a consequence we can produce quadrature formulas for the domain Ω, supported as anaytic functionals on a system of curves (more precisely the cuttings which specify a determination of the multivalued Schwarz function).

We would like to mention that, as in the preceding papers, the operator theory is necessary in several points of the subsequent proofs, although the statements are purely function theoretic.

The paper is organized as follows. Section 2 recalls the essential results of M.G.Krein which characterize the extremal solutions of the truncated L-problem of moments. A great deal of effort was put in this area by statisticians, who have followed A.A.Markov and P.L. Chebyshev in deriving sharp estimates for the distribution of collections of random variables with a set of moments prescribed, see [G],[KS]. Section 3 is devoted to local and global analytic continuation properties of the exponential kernel of a planar domain. In Section 4 we specialize the results to semi-algebraic domains and we obtain a general quadrature formula supported on thin sets. Section 5 is independent of the main body of the paper; we describe there a set of positivity conditions which characterize the exponential kernel. In this section we meet old ideas and techniques (based on the magic of resolvents of linear operators) due to de Branges and his followers. In particular, we interpret in this final section the L-problem of moments as an interpolation problem of the Carathéodory-Fejér type, for a class of analytic functions defined in a polydisk of dimension four.

Acknowledgements. We would like to thank the Mathematical Sciences Research Institue in Berkeley for creating a unique scientific environment which made possible this work. We are indebted to Björn Gustafsson and Harold S.Shapiro for several discussions on related themes and for their interest in this subject. We would like to thank James Rovnyak for his comments on the structure of determinantal functions and the bibliogarphical reference [PR].

2. Convexity results

This first part of the paper contains a survey of a series of known results derived from the work of M.G.Krein and his succesors. They are intended to serve as a motivation for the next sections. For that reason this part is independent of the rest of the paper, and it is presented in a slightly more general setting ($\mathbf{R}^n$ rather than $\mathbf{R}^2$ and an arbitrary support compact K for the moment problem). The basic monographs we refer to for details are [KS] and [KN].

The variable in $\mathbf{R}^n$ is denoted by $x = (x_1, \dots, x_n)$; dx means the volume measure in $\mathbf{R}^n$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ we denote $|\alpha| = \alpha_1 + \dots + \alpha_n$ and put as usual $x^\alpha = {x_1}^{\alpha_1} \dots {x_n}^{\alpha_n}$.

Let K be a compact subset of $\mathbf{R}^n$; in order to avoid some minor complications we assume that $int(K) \neq \emptyset$. Fix a positive integer N and a positive constant L. The *truncated L-problem of moments* supported by the set K consists in finding necesary and sufficient conditions for a sequence $a = (a_\alpha)_{|\alpha| \leq N}$ of real numbers to be represented as:

$$a_\alpha = \int_K x^\alpha \phi(x) dx, \quad (|\alpha| \leq N, \phi \in L^1(K, dx), 0 \leq \phi \leq L). \tag{1}$$

Moreover, it is traditionally of interest to classify all solutions of this problem and to characterize the uniqueness cases, see [KN] Chapter VII.

For a first part of this section we can adopt the normalization $L = 1$.

Let us denote, for $L = 1$, the set of all possible moment sequences as follows:

$$\Sigma = \{a(\phi) = (a_\alpha); a_\alpha = \int_K x^\alpha \phi(x) dx, |\alpha| \leq N, \phi \in L^1(K, dx), 0 \leq \phi \leq 1\}. \tag{2}$$

Let $\mathbf{R}[x]$ be the space of polynomials in the variables x, with real coefficients. We put $\mathcal{P}_N = \mathcal{P}_N(\mathbf{R}^n) = \{p \in \mathbf{R}[x], deg(p) \leq N\}$. With these data fixed, we denote $d = dim(\mathcal{P}_N)$ and parametrize the coordinates in the space $\mathcal{P}_N \subset \mathbf{R}^d$ as follows $y = (y_\alpha)_{|\alpha| \leq N}$.

It is clear that Σ is a compact convex subset of $\mathbf{R}^d$. Let a^0 be a boundary point of Σ and let $f(y) = \langle c, y \rangle + d$ be a supporting affine functional to Σ, at the point a^0.

Let us represent the point a^0 as the moment sequence of the function ϕ_0: $a^0 = a(\phi_0)$. Then we have the following relations:

$$\langle c, a\rangle + d \leq 0, \quad (a \in \Sigma),$$

and

$$\langle c, a^0\rangle + d = 0.$$

By substarcting them and representing a as $a = a(\phi)$ we obtain:

$$\int_K p^0(x)(\phi(x) - \phi_0(x))dx \leq 0, \quad (\phi \in L^1(K), 0 \leq \phi \leq 1), \tag{3}$$

where $p^0(x) = \sum_{|\alpha|\leq N} c_\alpha x^\alpha$. But relation (3) is possible for all functions ϕ as above if and only if $p^0(x) > 0$ implies $\phi_0(x) = 1$ and $p^0(x) < 0$ implies $\phi_0(x) = 0$. Since the set of zeroes of a non-trivial polynomial has null volume measure, these latter implications determine an unique element $\phi_0 \in L^1$. In other terms we have proved that $\phi_0 = \chi_{\{p^0>0\}}$ a.e. ,where χ_S is the charactersistic function of the set S.

In fact the above argument can easily be reversed, and we can state the following conclusion.

Theorem 2.1. *A point $(a_\alpha)_{|\alpha|\leq N}$ belongs to the boundary of the set Σ of all moments if and only if there is a non-trivial polynomial $p(x)$ of degree less or equal than N, and with the property:*

$$a_\alpha = \int_{K\cap\{p>0\}} x^\alpha dx, \quad (|\alpha| \leq N).$$

Above we have denoted in short by $\{p > 0\}$ the set of those points x which satisfy $p(x) > 0$. Since we have assumed the compact set K to possess interior points, it is immediate to remark that $a(\phi) \in int(\Sigma)$ for all functions ϕ satisfying $0 < \phi < 1$.

Theorem 2.2.(M.G.Krein). *A necessary and sufficient condition for the truncated 1-problem of moments (1) to be solvable is that, for every polynomial $p(x) = \sum_{|\alpha|\leq N} c_\alpha x^\alpha$, we have:*

$$\sum_{|\alpha|\leq N} a_\alpha c_\alpha \leq \int_K max(p(x), 0)dx.$$

Proof. Let us put $a = (a_\alpha), c = (c_\alpha), p_+ = max(p, 0)$. The necessity follows from the observation:

$$\langle a, c\rangle = \int_K p(x)dx \leq \int_K p_+(x)dx.$$

For proving the sufficiency we will show that the vector a of virtual moments and the point $0 \in \Sigma$ cannot be separated by a supporting hyperplane to the set Σ. Exactly as before, let :

$$\langle c, y\rangle + d \leq 0, \quad (y \in \Sigma),$$
$$\langle c, y^0\rangle + d = 0,$$

be a supporting hyperplane to Σ at the boundary point

$$y^0 = a(\phi_0), \phi_0 = \chi_{p^0>0}, p^0(x) = \sum c_\alpha x^\alpha.$$

In particular

$$d = \langle c, y^0\rangle = -\int_K {p^0}_+(x)dx \leq 0.$$

Therefore, $\langle c, 0\rangle + d \leq 0$ and the proof is complete.

By following the same lines one can obtain generalized Chebysev inequalities. Namely, for a continuous function ψ on K, not belonging to the space of polynomials $\mathcal{P}_N$ and a point $a \in int(\Sigma)$ we want to find the extreme values of $\int_K \psi\phi dx$ over all measurable functions $\phi, 0 \leq \phi \leq 1$, having the finite sequence of moments $a = a(\phi)$ prescribed. Let us denote

$$\Phi(a) = \{\phi \in L^1(K); 0 \leq \phi \leq 1, a(\phi) = a\}.$$

The main result in this area, with important applications to mathematical statistics, can be stated as follows:

There are polynomials $\overline{p}, \underline{p} \in \mathcal{P}_N$ *with the property that* $\chi_{\{\psi>\overline{p}\}}, \chi_{\{\underline{p}>\psi\}} \in \Phi(a)$ *and*

$$sup\{\langle p, a\rangle; p \in \mathcal{P}_N, p \leq \psi\} = \int_{K\cap\{\underline{p}>\psi\}} \psi dx = min_{\phi\in\Phi(a)} \int_K \psi\phi dx \leq$$

$$max_{\phi\in\Phi(a)} \int_K \psi\phi dx = \int_{K\cap\{\psi>\overline{p}\}} \psi dx = inf\{\langle p, a\rangle; p \in \mathcal{P}_N, p \geq \psi\}.$$

Above we identify the polynomial p with the sequence of its coefficients, also denoted by $p \in \mathbf{R}^d$.

For a proof and relevant comments about the above facts, see [KS] Sections VIII.8 and XII.2. For applications of generalized Chebyshev inequalities we also refer to [G].

For the aims of the present paper it is enough to retain from these inequalities the fact that they are attained on some extremal solutions $\phi \in \Phi(a)$ always given by the characteristic function of a sublevel set of a polynomial (plus possibly a multiple of the new function ψ).

Now we slightly change the point of view. Since we assume the supporting compact set given, after a translation and homothety the problem (1) is equivalent to:

$$2a_\alpha - L\int_K x^\alpha dx = \int_K x^\alpha(2\phi(x) - L)dx, \quad (|\alpha| \le N),$$

where the new unknown function $2\phi - L$ satisfies:$-L \le 2\phi - L \le L$. Modulo this transformation we consider henceforth the following moment problem:

$$a_\alpha = \int_K x^\alpha \phi(x)dx, \quad (|\alpha| \le N, \phi \in L^1(K), -L \le \phi \le L). \tag{4}$$

Let us denote $a = (a_\alpha)_{|\alpha|\le N}$ to be the sequence of virtual moments.

Because we have assumed $int(K) \ne \emptyset$, the monomials $(x^\alpha)_{|\alpha|\le N}$ are linearly independent regarded as functions of $x \in K$. Thus, for L large enough the problem (4) always admits a solution ψ.

Let us cosider the embedding $\mathcal{P}_N(\mathbf{R}^n) \subset L^1{}_{\mathbf{R}}(K, dx)$ and consider the linear functional

$$l_a : \mathcal{P}_N \longrightarrow \mathbf{R}, \quad l_a(x^\alpha) = a_\alpha, (|\alpha| \le N).$$

In virtue of Riesz Theorem, any continuous extension of l_a to $L^1{}_{\mathbf{R}}(K)$ is represented by a function $\phi \in L^\infty{}_{\mathbf{R}}(K)$. Hence ϕ is a solution of problem (4) and we have:

$$l_a(p) \le \|p\|_{1,K}\|\phi\|_{\infty,K}, \quad (p \in \mathcal{P}_N). \tag{5}$$

Then it remains to remark that the converse implication is given by Hahn-Banach Theorem. Moreover, the familiar analysis of the equality case in (5) is also relevant for us. Summing up, we can state the next theorem.

Theorem 2.3. *a). Problem (4) admits a solution if and only if*

$$L \ge sup\{\frac{l_a(p)}{\|p\|_{1,K}}; p \in \mathcal{P}_N \setminus \{0\}\}.$$

b). If $L = sup\{\frac{l_a(p)}{\|p\|_{1,K}}; p \in \mathcal{P}_N \setminus \{0\}\}$, then the solution is unique and it coincides with the function $Lsgn(p_0)$, where p_0 is a polynomial of degree less or equal than N.

Below we only sketch the proof of Theorem 2.3. The reader can find more details in [KN] §IX.1-2.

Let $p_0 \in \mathcal{P}_N \setminus \{0\}$ be a polynomial with the property that that

$$L_0 = sup\{\frac{l_a(p)}{\|p\|_{1,K}}; p \in \mathcal{P}_N \setminus \{0\}\} = \frac{l_a(p_0)}{\|p_0\|_{1,K}},$$

and let $\psi_0 \in L^\infty{}_{\mathbf{R}}(K)$ be an extremal element which satisfies $\psi_0(p) = l_a(p), p \in \mathcal{P}_N$, $\|\psi_0\|_{\infty,K} = L_0$ and

$$\int_K p_0\phi_0 dx = \|p_0\|_{1,K}\|\phi_0\|_{\infty,K}.$$

Since $\int_K (p_0\phi_0 - |p_0|L_0)dx = 0$ and $p_0\phi_0 \le |p_0|L_0$ we obtain

$$\phi_0 = L_0 sgn(p_0), a.e..$$

Notice again that the zero set of a polynomial has null measure and therefore the element ϕ_0 is well defined almost everywhere.

Moreover, if q_0 is anoter non-zero element of $\mathcal{P}_N$ satisfying the extremal condition $\frac{l_a(q_0)}{\|q_0\|_{1,K}} = L_0$, then

$$\int_K \frac{q_0}{\|q_0\|_{1,K}}\phi_0 dx = \int_K \frac{p_0}{\|p_0\|_{1,K}}\phi_0 dx = L_0.$$

Whence $\phi_0 = L_0 sgn(q_0)$.

Thus the extremal solution ϕ_0 of the moment problem (4) is indeed unique.

¿From the previous argument it is also clear that the problem (4) has a convex continuum of solutions whenever L is greater than the critical value L_0.

To finish our brief presentation of these convexity methods, let us reverse the problem (4) and present the above results in the following form.

Corollary 2.4. *The function $\phi \in L^{\infty}{}_{\mathbf{R}}(K)$ is uniquely determined in the ball $\{\psi \in L^{\infty}{}_{\mathbf{R}}(K); \|\psi\|_{\infty,K} \le \|\phi\|_{\infty,K}\}$ by its finite sequence of moments $a = a(\phi)$ if and only if*

$$\phi = \|\phi\|_{\infty,K} sgn(p),$$

where $p \in \mathcal{P}_N \setminus \{0\}$.

Conversely, for any non-zero polynomial $p \in \mathcal{P}_N$ the moemnts

$$a_\alpha = \int_{K\cap\{p>0\}} x^\alpha dx - \int_{K\cap\{p<0\}} x^\alpha dx, \quad (|\alpha| \le N),$$

determine the function $sgn(p)$ in the unit ball of $L^{\infty}{}_{\mathbf{R}}$.

For the latter statement, it suffices to remark that, for any function $\phi \in L^{\infty}{}_{\mathbf{R}}(K)$ satisfying $a(\phi) = a(sgn(p))$ we have:

$$\|\phi\|_{\infty,K}\|p\|_{1,K} \ge |\int_K \phi p dx| = |\int_K sgn(p)p dx| = \int_K sgn(p)p dx = \|p\|_{1,K}.$$

If we assume in addition that $\|\phi\|_{\infty,K} = 1$, then we obtain the stated uniqueness result $\phi = sgn(p)$, a.e. .

To draw a conclusion of this section, we have seen from two different perspectives that the extremal solutions of the moment problem (1) are parametrized by the sub-level sets $\{p > 0\}$ of any non-zero polynomial p of degree less or equal than N.

3. Analytic continuation of the exponential kernel

¿From now on we restrict our study to the L-problem of moments in two real variables. As explained in the previous paper [P1] in this dimension the theory of hyponormal operators meets favorably the L-problem. Without entering into all technical details exposed in [P1], [P2] we fix below some conventions and notation. An introduction to the theory of hyponormal operators is given in [MP].

Let ϕ be a measurable function with compact support in the complex plane which satsifies $0 \leq \phi \leq 1$, a.e. .Let T be the unique (up to unitary equivalence) hyponormal operator with rank-one self-commutator ($[T^*, T] = \xi \otimes \xi$) with the principal function equal to ϕ.

The two objects above are related by the following formula:

$$\langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi \rangle = 1 - exp(\frac{-1}{\pi} \int_{\mathbf{C}} \frac{\phi(\zeta) dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}),$$

which is valid for all points z, w in the reslovent set of the operator T. Actually a separately continuous extesion of the above formula to the whole $\mathbf{C}^2$ holds, see [P1]. The importance of this formula lies in the fact that it relates, after taking the Taylor expansions at infinity, the moments of the function ϕ to the moments of the operator T.

We are interested in the sequel in the moments of the extremal solutions discussed in the preceding section. Therefore, assuming the supporting compact set K (introduced in §2) nice, we will study the above relationship only for characteristic functions $\phi = \chi_\Omega$, where Ω is a bounded domain of the complex plane, satisfying $\Omega = int(\overline{\Omega})$. To simplify notation we put:

$$E_\Omega(z, \overline{w}) = exp(\frac{-1}{\pi} \int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}), \quad (z, w \in \mathbf{C} \setminus \overline{\Omega}).$$

This exponential kernel is analytic in z and antianalytic in w. To simplify the terminology an analytic-antianalytic function of several variables denoted as

$$f(z, u, \dots ; \overline{w}, \overline{t}, \dots),$$

will be called analytic in $z, u, \dots ; \overline{w}, \overline{t}, \dots$.

To each domain Ω as above we associate the unique irreducible hyponormal operator T satisfying $[T^*, T] = \xi \otimes \xi$ and having the principal function equal to χ_Ω. We simply call T *the hyponormal operator corresponding to* Ω. Recall that the spectrum of T is the closure of Ω, the essential spectrum is the boundary of Ω, and so on. See [MP] Chapter XI for more details.

The aim of the present section is to investigate the analytic continuation properties of the kernel E_Ω across portions of the boundary of Ω. This study is motivated by the earlier results obtained in the case of quadrature domains, cf. [P1],[P2].

Proposition 3.1. *Let $\lambda \in \partial\Omega$ and assume that there exists an open neighbourhood U of λ in $\mathbf{C}$ with the property that the function $E_\Omega(z,\overline{w})$ extends analytically from $(U \setminus \overline{\Omega}) \times (U \setminus \overline{\Omega})$ to $U \times U$.*

Then there is an open neighbourhood V of λ in U such that $V \cap \partial\Omega$ is a real analytic set, on which the extension of the function $E_\Omega(z,\overline{z})$ vanishes.

Proof. Let $E_\Omega(z,\overline{w})$ be the canonical extension of the exponential kernel to $\mathbf{C} \times \mathbf{C}$. We recall that, denoting by $(T^* - \overline{z})^{-1}\xi$ the unique solution η of minimum norm of the equation:

$$(T^* - \overline{z})\eta = \xi,$$

we obtain a weakly continuous function defined everywhere on $\mathbf{C}$. Then the identity

$$exp(\frac{-1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}) = 1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi \rangle$$

holds everywhere, by a result due to K.Clancey, see [P1].We also know that $\|(T^* - \overline{z})^{-1}\xi\| = 1$ for all points $z \in \Omega$, cf. [MP] Theorem XI.4.1.

Let $F(z,\overline{w})$ be the analytic extension of $E_\Omega(z,\overline{w})$ to $U \times U$, whose existence was assumed in the statement.

Let us consider the closed linear span $K = \bigvee_{z \in U \setminus \overline{\Omega}} (T^* - \overline{z})^{-1}\xi$. Since the statement of Lemma 3.1 is local, we can assume that $U \setminus \overline{\Omega}$ intersects the unbounded component of $\mathbf{C} \setminus \overline{\Omega}$. Thus , according to the resolvent equation, there is a point $a \in U$ belonging to the unbounded component of $\mathbf{C} \setminus \overline{\Omega}$ with the property that the operator $(T^* - \overline{a})^{-1}$ leaves the subspace K invariant. Then the operator

$$(T^* - \overline{b})^{-1} = (T^* - \overline{a}) - 1(1 + (\overline{a} - \overline{b})(T^* - \overline{a})^{-1})^{-1}$$

still leaves K invariant whenever $|b - a| \leq \|(T - a)^{-1}\|^{-1}$. By following a path disjoint of $\overline{\Omega}$ which joins a to infinity we find in finitely many steps that the operator $(T^* - \overline{c})^{-1}$ leaves the subspace K invariant for c in the neighbourhood of infinity. In conclusion, by taking the Taylor series of the resolvent function at infinity we obtain that K is a T^*-closed invariant subspace in the Hilbert space H where T acts.

Let us denote $S = (T^*|_K)^*$, regarded as an operator from K to K. Since F coincides with E_Ω on $(U \setminus \overline{\Omega})^2$, we have:

$$F(z,\overline{w}) = 1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi \rangle, \quad (z, w \in U \setminus \overline{\Omega}),$$

or equivalently

$$F(z,\overline{w}) = 1 - \langle (S^* - \overline{w})^{-1}\xi, (S^* - \overline{z})^{-1}\xi \rangle, \quad (z, w \in U \setminus \overline{\Omega}).$$

The difference between the two equations lies in the fact that the second one makes sense on the whole set $U \times U$.

Indeed, by the resolvent equation we obtain, for all $w, z, u \in U \setminus \overline{\Omega}, z \neq u$:

$$\langle (S^* - \overline{w})^{-1}\xi, (S^* - \overline{z})^{-1}(S^* - \overline{u})^{-1}\xi \rangle =$$

$$(z-u)^{-1}[\langle (S^*-\overline{w})^{-1}\xi, (S^*-\overline{z})^{-1}\xi\rangle - \langle (S^*-\overline{w})^{-1}\xi, (S^*-\overline{u})^{-1}\xi\rangle] = \frac{F(z,\overline{w})-F(u,\overline{w})}{z-u}.$$

This shows that for any vector $x \in K$ of the form $x = (S^*-\overline{u})^{-1}\xi$, the K-valued function $(S^*-\overline{z})^{-1}x$ extends analytically from $U \setminus \overline{\Omega}$ to U. However, the operator $S-z$ may not be invertible on K, for all values $z \in U$.

Since

$$(T^*-\overline{z})(S^*-\overline{z})^{-1}\xi = \xi, \quad (z \in U),$$

we find

$$\|(S^*-\overline{z})^{-1}\xi\| \geq \|(T^*-\overline{z})^{-1}\xi\|, \quad (z \in U).$$

Consequently

$$F(z,\overline{z}) = 1 - \|(S^*-\overline{z})^{-1}\xi\|^2 \leq 1 - \|(T^*-\overline{z})^{-1}\xi\|^2 = 0, \quad (z \in U \cap \Omega).$$

On the other hand,

$$F(z,\overline{z}) = E_\Omega(z,\overline{z}) > 0, \quad (z \in U \setminus \overline{\Omega}).$$

In conclusion $F(z,\overline{z}) = 0$ for every point $(\partial\Omega) \cap U$, which shows that the set $(\partial\Omega)\cap U$ is included in a real analytic subset of U.

It remains to invoke the local structure of real analytic sets (cf. for instance [M] §III.5.C) and to note that $\partial\Omega$ is the boundary of an open set satisfying $\Omega = int\overline{\Omega}$. Indeed, for a small ball V centered at λ, the real analytic set $\{z \in V; F(z,\overline{z}) = 0\}$ consists of finitely many analytic arcs passing through λ , some possibly being singular at Λ. These arcs divide V into finitely many chambers, each being included either in Ω or in $\mathbf{C}\setminus\Omega$. Then $(\partial\Omega)\cap V$ consists of exactly those analytic semi-arcs starting from λ and dividing interior from exterior chambers. Thus the set $(\partial\Omega)\cap V$ consists of finitely many full analytic semi-arcs starting from the point λ and ending on the boundary of the ball V.

With the notation used in the proof of Proposition 3.1 we isolate the following result. For details concerning local spectral theory and the significance of such a result in the context of abstract spectral decomposition theories, we refer to [DS].

Corollary 3.2. *Assume that the kernel E_Ω extends analytically to a connected neighbourhood U of a point $\lambda \in \partial\Omega$ and U intersects the unbounded connected component of $\mathbf{C}\setminus\overline{\Omega}$. Then the localized resolvent $(S^*-\overline{z})^{-1}x$ exists as an analytic function in $z \in U$, for all vectors $x = (S^*-\overline{u})^{-1}\xi, |u| \gg 0$.*

Later in this section we will see that in fact the unbounded connected component of the complement of $\overline{\Omega}$ plays no special role. Any other component can substitute it.

The following result is a partial converse to Proposition 3.1.

Theorem 3.3. *Let Ω be a bounded planar domain and let $\lambda \in \partial\Omega$ a point in whose neighbourhood $\partial\Omega$ is real analytic and smooth.*

Then there is an open neighbourhood U of λ in $\mathbf{C}$, such that the kernel $E_\Omega(z, \overline{w})$ extends analytically from $z, w \in U \setminus \overline{\Omega}$ to $z, w \in U$.

Moreover, there is an invertible analytic function $f(z, \overline{w})$ defined for $z, w \in U$, with the property that the analytic extension E'_Ω of the kernel E_Ω satisfies:

$$E'_\Omega(z, \overline{w}) = (z - w^*) f(z, \overline{w}), \quad (z, w \in U), \tag{6}$$

where $w \mapsto w^*$ is the Schwarz reflection function in $U \cap \partial\Omega$.

In the above statement it is implicit that we shrink U to a neighbourhood of λ where the anti-analytic Schwarz reflection map is defined, see [D],[Sh].

Proof. We begin by recalling a simple computation from [P1], Scetion 5. Namely for the open unit disk $D \in \mathbf{C}$,we have:

$$E_D(z, \overline{w}) = 1 - \frac{1}{z\overline{w}}, \quad (z, w \in \mathbf{C} \setminus \overline{D}).$$

The idea of the proof is to transfer, via a conformal map, this identity to a neighbourhood in Ω of the smooth point $\lambda \in \partial\Omega$. The necessary computations are rather long, but elementary. We also mention that in the following proof we do not use the hyponormal operator attached to the domain Ω.

Let U denote an open ball centered at the point λ, such that $U \cap \Omega$ is connected and simply connected and $(\partial\Omega) \cap U$ is a smooth real analytic curve. Let $g : D \longrightarrow U \cap \Omega$ be a conformal map. By Carathéodory Theorem and Schwarz Reflection Principle we can assume, after possibly shrinking the radius of the ball U, that $g(1) = \lambda$, g can be extended conformally to a neighbourhood of the point $1 \in \partial D$ and hence $g^{-1}((\partial\Omega) \cap U) \subset \partial D$. Let B be an open ball centered at the point 1, such that the conformal map g is defined on B and $U \subset g(B)$.

Since

$$E_\Omega(z, \overline{w}) = E_{\Omega \cap U}(z, \overline{w}) E_{\Omega \setminus \overline{U}}(z, \overline{w}) \quad (z, w \in \mathbf{C} \setminus \overline{\Omega}),$$

and $E_{\Omega \cap \overline{U}}(z, \overline{w})$ is an analytic invertible function on $U \times U$, it suffices to prove the statement for $\Omega \cap U$.

We change the variables as follows:$\zeta = g(u), z = g(s), w = g(t)$. For $u \in B$ and $s, t \in B \setminus \overline{D}$ we have:

$$E_{\Omega \cap U}(z, \overline{w}) = exp(\frac{-1}{2\pi i} \int_{\Omega \cap U} \frac{d\overline{\zeta} \wedge d\zeta}{(\zeta - z)(\overline{\zeta} - \overline{w})}) =$$

$$exp(\frac{-1}{2\pi i} \int_{B \cap D} \frac{|g'(u)|^2 d\overline{u} \wedge du}{(g(u) - g(s))(\overline{g(u)} - \overline{g(t)})}) =$$

$$exp(\frac{-1}{2\pi i} \int_{B \cap U} h(u, \overline{u}; s, \overline{t}) \frac{d\overline{u} \wedge du}{(u - s)(\overline{u} - \overline{t})}),$$

where

$$h(u,\overline{u};s,\overline{t}) = \frac{|g'(u)|^2(u-s)(\overline{u}-\overline{t})}{(g(u)-g(s))(\overline{g(u)}-\overline{g(t)})}.$$

The function h is analytic in all four variables, and by possibly shrinking the radius of the ball B we can write:

$$h(u,\overline{u};s,\overline{t}) = 1 + (u-s)h_1(u,s) + (\overline{u}-\overline{t})h_2(\overline{u},\overline{t}) + (u-s)(\overline{u}-\overline{t})h_3(u,\overline{u};s,\overline{t}),$$

where the new functions h_1, h_2, h_3 depend analytically on the respective variables (running through B).

In what follows we replace h by this additive decompositon in the above expression of $E_{\Omega\cap U}$. The first term 1 in h produces the factorization:

$$E_{B\cap D}(s,\overline{t}) = (1-\frac{1}{s\overline{t}})E_{D\setminus\overline{B}}(s,\overline{t}) \quad (s,t\in B\setminus\overline{D}).$$

For the second term, consider an analytic function $H_1(u,s)$ with the property that $\partial/\partial u H_1(u,s) = h_1(u,s)$ for $u,s\in B$. Then the corresponding integral (under the exponential function in the expression of $E_{\Omega\cap U}$) is:

$$I_1(s,\overline{t}) = \frac{-1}{2\pi i}\int_{B\cap D}\frac{h_1(u,s)}{\overline{u}-\overline{t}}d\overline{u}\wedge du = \frac{1}{2\pi i}\int_{\partial(B\cap D)}\frac{H_1(u,s)}{\overline{u}-\overline{t}}d\overline{u} =$$

$$\frac{1}{2\pi i}[\int_{D\cap\partial B}\frac{H_1(u,s)d\overline{u}}{\overline{u}-\overline{t}} - \int_{B\cap\partial D}\frac{H_1(u,s)du}{u-u^2\overline{t}}] =$$

$$\frac{1}{2\pi i}\int_{D\cap\partial B}[\frac{H_1(u,s)d\overline{u}}{\overline{u}-\overline{t}} - \frac{H_1(u,s)du}{u\overline{t}(u-\overline{t}^{-1})}] + \frac{1}{2\pi i}\int_{\partial(D\cap B)}\frac{H_1(u,s)du}{u\overline{t}(u-\overline{t}^{-1})}.$$

In the last sum, the first intergral is analytic in s,t belonging to a neighbourhood of 1 (such that $\overline{t}^{-1}\in D\setminus\partial B$). The second integral can be evaluated by Cauchy's Formula, for $t\in B\setminus\overline{D}$ and it is equal to $H_1(\overline{t}^{-1},s)$. Thus, after shrinking the radius of the ball B, if necessary, the function $I_1(s,\overline{t})$ extends analytically from $B\setminus\overline{D}$ to B.

Similarly one proves that the term h_2 produces an extendable analytic function, from $s,t\in B\setminus\overline{D}$ to $s,t\in B$. The integral corresponding to the term h_3 is obviously analytic in B.

Summing up these computations, we have obtained an analytic function $A(s,\overline{t})$, defined for $s,t\in B$, with the property that:

$$E_{\Omega\cap U}(g(s),\overline{g(t)}) = (1-\frac{1}{s\overline{t}})expA(s,\overline{t}), \quad (s,t\in B\setminus\overline{D}).$$

But the function $log(s)$ is well defined for $s\in B$, whence we can write:

$$E_{\Omega\cap U}(g(s),\overline{g(t)}) = (s-\overline{t}^{-1})exp(A(s,\overline{t})-log(s)),$$

or changing back the variables:

$$E_{\Omega\cap U}(z,\overline{w}) = (g^{-1}(z)-g^{-1}(w)^*)expA_1(z,\overline{w}), \quad (z,w\in U\setminus\overline{\Omega}),$$

where $A_1(z,\overline{w})$ is analytic in $z, w \in U$.

It remains to remark that:

$$g^{-1}(z) - g^{-1}(w)^* = g^{-1}(z) - g^{-1}(w^*) = (z - w^*)k(z,\overline{w}),$$

where the function $k(z,\overline{w})$ is invertible (and actually admits an analytic logarithm) in the domain $z, w \in U$.

This completes the proof of Theorem 3.3.

An analysis of the preceding proof shows that all computations make sense before taking the exponential. In particular, let us remark that the function $log(z - w^*)$ is well defined for $z, w \in U \setminus \overline{\Omega}$. Thus we can state the following result.

Corollary 3.4. *Let $\lambda \in \partial\Omega$ be a smooth, real analytic point of the boundary of the domain Ω. There exists an open neighbourhood U of λ in $\mathbf{C}$ and an analytic function $a(z,\overline{w})$ defined for $z, w \in U$, such that:*

$$\frac{-1}{2\pi i}\int_{\Omega} \frac{d\overline{\zeta} \wedge d\zeta}{(\zeta - z)(\overline{\zeta} - \overline{w})} = log(z - w^*) + a(z,\overline{w}), \quad (z, w \in U \setminus \overline{\Omega}). \tag{7}$$

In the opposite direction to Theorem 3.3 we give below an example of real analytic singular boundary point across which the exponential kernel does not extend analytically.

Proposition 3.5. *Let Ω be a bounded domain of the complex plane and let $\lambda \in \partial\Omega$ be a point in whose neighbourhood the boundary $\partial\Omega$ is real analytic.*

If an irreducible component of $\partial\Omega$ at λ intersects $\mathbf{C} \setminus \overline{\Omega}$, then the kernel $E_\Omega(z,\overline{w})$ does not extend analytically to a full neighbourhood of the point $(z,w) = (\lambda,\lambda)$.

Proof. Let X be the germ at λ of an analytic set, whose part of half-branches define the germ of $\partial\Omega$ at λ. Let X_0 denote an irreducible component of X which intersects the set $\mathbf{C} \setminus \overline{\Omega}$.

Suppose that $F(z,\overline{w})$ is an analytic extension of $E_\Omega(z,\overline{w})$, from a component of $\mathbf{C}\setminus\overline{\Omega}$ adjiacent to X_0 to a full neighbourhood of (λ,λ) in $\mathbf{C}^2$. According to Proposition 3.1, $F(z,\overline{z}) = 0$ for $z \in X_0 \cap \partial\Omega$, hence for z belonging to a half-branch of X_0.

Let Y denote the germ of analytic set at λ defined by the equation $Y = \{z; F(z,\overline{z}) = 0\}$. By elementary dimension theory (see for instance [M] §III.5), $X_0 \cap Y$ is either $\{\lambda\}$ or X_0. Since the first alternative is excluded, we find that $X_0 \subset Y$. But for a point $\mu \in X_0 \setminus \overline{\Omega}$ we have:

$$0 < E_\Omega(z,\overline{z}) = F(z,\overline{z}) = 0,$$

a contradiction!

Therefore the analytic set X_0 cannot intersect $\mathbf{C} \setminus \overline{\Omega}$.

We turn now to global analytic extensions of the exponential kernel of a domain. We prove below the extendability of the exponential kernel along any continuous path on which the Schwarz function of a portion of the boundary extends.

Let A be a smooth real analytic arc in the exterior boundary of the bounded domain Ω. Let C denote the unbounded connected component of $\mathbf{C} \setminus \overline{\Omega}$. Let $u(z)$ denote the Schwarz function of the arc A; u is an anaytic function in a neighbourhood of A which coincides with $\overline{z}$ on A, see [D],[Sh].

Fix a point $\lambda \in A$ and consider a differentiable path without self-intersection points, which joins λ , within the domain Ω, to a point $a \in \Omega$. Let U be a tubular neighbourhood of P in $\Omega \cup A$, such that U is a connected, simply connected domain with piecewise boundary, and $A' = \partial U \cap \partial\Omega$ is an open analytic arc in A , containing of course the point λ. We assume that the Schwarz function u extends analytically to $\overline{U}$. Thus the domain $C \cup A' \cup U$ is connected and simply connected. (Visually, we attach the appendix U to the simply connected component C).

Let $z, w \in C$ be fixed points. Then:

$$\int_\Omega \frac{d\overline{\zeta} \wedge d\zeta}{(\zeta - z)(\overline{\zeta} - \overline{w})} = \int_{\Omega \setminus U} \frac{d\overline{\zeta} \wedge d\zeta}{(\zeta - z)(\overline{\zeta} - \overline{w})} + \int_U \frac{d\overline{\zeta} \wedge d\zeta}{(\zeta - z)(\overline{\zeta} - \overline{w})}.$$

Moreover, for a fixed determination of the logarithm ,defined ,say for $\Re(w) > 0, |w| > sup_{\zeta \in U}|u(\zeta)|$, we obtain:

$$\int_U \frac{d\overline{\zeta} \wedge d\zeta}{(\zeta - z)(\overline{\zeta} - \overline{w})} = \int_{\partial U} \frac{log(\overline{\zeta} - \overline{w})d\zeta}{\zeta - z} =$$

$$\int_{A'} \frac{log(\overline{\zeta} - \overline{w})d\zeta}{\zeta - z} + \int_{(\partial U)\setminus A'} \frac{log(\overline{\zeta} - \overline{w})d\zeta}{\zeta - z} =$$

$$\int_{A'} \frac{log(u(\zeta) - \overline{w})d\zeta}{\zeta - z} + \int_{(\partial U)\setminus A'} \frac{log(\overline{\zeta} - \overline{w})d\zeta}{\zeta - z} =$$

$$\int_{(\partial U)\setminus A'} [\frac{log(\overline{\zeta} - \overline{w})d\zeta}{\zeta - z} - \frac{log(u(\zeta) - \overline{w})d\zeta}{\zeta - w}].$$

These computations show that, for every $|w| \gg 0, \Re(w) > 0$, the function $z \mapsto E_\Omega(z, \overline{w})$ extends analytically from $z \in C$ to $z \in C \cup A' \cup U$. Let us denote this extension by $F(z, \overline{w})$.

At this moment the operator theory interpreation of the kernel E_Ω is again invoked to help. Let T be the hyponormal operator attached to the domain Ω. In particular we know that:

$$F(z, \overline{w}) = 1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi \rangle, \quad (z, w \in C).$$

Now we repeat the trick contained in the proof of Proposition 3.1. Let

$$K = \bigvee_{|w|\gg 0, \Re(w)>0} (T^* - \overline{w})^{-1}\xi$$

be the T^*-invariant subspace generated by the vector ξ. Let S^* be the restriction of the operator T^* to K. Then formula

$$F(z,w) = 1 - \langle (S^* - \overline{w})^{-1}\xi, (S^* - \overline{z})^{-1}\xi\rangle$$

holds for $|w| \gg 0, \Re(w) > 0$ and $z \in C$.

Because the function $F(.,\overline{w})$ extends analytically to $C \cup A' \cup U$, we infer, exactly as in the proof of Proposition 3.1, that the function $F(z,\overline{w})$ extends analytically from $z, w \in C$ to $z, w \in C \cup A' \cup U$.

Soon we will consider only domains with real algebraic boundary, so that the above suppositions are not excessive.

In conclusion we have proved the following result.

Theorem 3.7. *Let Ω be a bounded domain and let A be an analytic arc in the exterior boundary of Ω. If the Schwarz function of the arc A extends analytically in the interior of Ω to an open set $U \subset \Omega$, then the kernel $E_\Omega(z,\overline{w})$ extends analytically from the unbounded component C of $\mathbf{C} \setminus \overline{\Omega}$ to $z, w \in C \cup A \cup U$.*

There are examples which show that, to different analytic continuations of the Schwarz function, correspond in general different analytic continuations of the exponential kernel.

In order to extend the results of this section to other connected components of the comlement of Ω, it is sufficient to see how the kernel E_Ω changes under linear fractional transformations.

Proposition 3.8. *Let Ω be a bounded planar domain and let a, b be complex numbers, $b \neq 0$. Suppose that $0 \in \mathbf{C} \setminus \overline{\Omega}$. Then for every pair $z, w \in \mathbf{C} \setminus \overline{\Omega}$:*

a). $E_{\Omega^{-1}}(z^{-1}, \overline{w}^{-1}) = \frac{E_\Omega(0,0)E_\Omega(z,\overline{w})}{E_\Omega(0,\overline{w})E_\Omega(z,0)}$;

b). $E_{\Omega+a}(z+a, \overline{w+a}) = E_\Omega(z,\overline{w})$;

c). $E_{b\Omega}(bz, \overline{bw}) = E_\Omega(z,\overline{w})$.

We have denoted $\Omega^{-1} = \{z^{-1}; z \in \Omega\}$ and so on. The proof of Proposition 3.8 consists in simple changes of variables in the integral defining the exponential kernel. The details are left to the reader.

It is important to remark that the denominator in formula a) above does not vanish. Indeed,

$$E_\Omega(z,0) = 1 - \langle T^{*-1}\xi, (T^* - \overline{z})^{-1}\xi\rangle,$$

and $max\{\|T^{*-1}\xi\|, \|(T^* - \overline{z})^{-1}\xi\|\} < 1$ for $z \in \mathbf{C} \setminus \overline{\Omega}$.

4. Domains with real algebraic boundary

In view of the general convexity results of Section 2, the extremal solutions of the L-problem of moments are among the characteristic functions of semialgebraic domains. In this section we begin a study of some specific properties of these domains.

We start with a couple of examples. Let $0 < r < R$ be fixed real numbers and let $A(r, R)$ be the annulus $A(r, R) = \{z \in \mathbf{C}; r < |z| < R\}$. Proposition 3.8 shows that:

$$E_{A(r,R)}(z, \overline{w}) = \begin{cases} \frac{1-R^2 z\overline{w}}{1-r^2 z\overline{w}} & |z|, |w| > R \\ \frac{1-r^2 z\overline{w}}{1-R^2 z\overline{w}} & |z|, |w| < r. \end{cases} \tag{8}$$

In the second example we do not compute explicitly the exponential kernel, but instead we locate the singularities of the analytic extensions of it. Take for instance a non-degenerated triangle T in $\mathbf{C}$. Then, according to Theorem 3.7, the kernel E_T extends analytically from the complement of T to the complement of the union of two arbitraily chosen sides of T. Indeed, in this case the Schwarz function of each side is entire, see [D] Chapter 10. As a consequence, formula

$$\int_T f dA = u(f), \quad (f \in \mathcal{O}(\overline{T})),$$

holds, where u is an analytic functional (that is a linear continuous functional on $\mathcal{O}(\overline{T})$) carried by the respective two sides of T. (The reader can actually find explicitly the functional u in the above quadrature formula).For the distinction between the carrier of an analytic functional and the support of a distribution we refer to [H] Section IX.9.1 or [Ma].

Next we prove that similar phenomena occur for arbitrary semi-algebraic domains. Let Ω be a bounded planar domain with real algebraic boundary. Let C denote the unbounded component of $\mathbf{C} \setminus \overline{\Omega}$ and let $C_1, \dots, C_n$ be the bounded connected components of the same set.

The Schwarz function of each analytic arc contained in the boundary of Ω is an algebraic function. Let A denote the set of the ramification points in $\overline{\Omega}$ of all these local Schwarz functions, union with the singular or multiple points of the boundary of Ω. Thus, A is a finite subset of $\overline{\Omega}$. We claim that there is a finite set E of continuous curves contained in $\overline{\Omega}$ and passing through all points of the set A, such that $\Omega' = (\overline{C} \cup \Omega) \setminus E$ is a connected open set and the function $\overline{z}$ extends analytically from $\partial\Omega \setminus E$ to $\Omega \setminus E$. The reader can either consult [AG] Chapter V for the existence of such continuous cuttings or prove directly the claim.

With this preparation and Theorem 3.7 we can state the following result.

Theorem 4.1. *Let Ω be a bounded planar domain with real-algebraic boundary and let Ω' be the domain consisting of the unbounded component of the complement of $\overline{\Omega}$ union with Ω, minus the cuttings explaind above.*

Then the kernel $E_\Omega(z,\overline{w})$ extends analytically from z, w belonging to the unbounded component of the complement of $\overline{\Omega}$ to $z, w \in \Omega'$.

As the preceding examples show, the kernel E_Ω may extend analytically across some of the curves in the set E, but in general these extensions cannot be glued.

Corollary 4.2. *With the notation in Theorem 4.1 we obtain the following quadrature identity:*

$$\int_\Omega f\,dA = u(f), \qquad (f \in \mathcal{O}(\overline{\Omega})),$$

where u is an analytic functional carried by $\partial\Omega'$.

Proof. Indeed, let $u(z)$ be the analytic function which extends $\overline{z}$ from $\partial C \cap \Omega'$ to Ω'. Then for an analytic function in a neighbourhood of $\overline{\Omega}$ we obtain :

$$\frac{1}{2i}\int_\Omega f\,dA = \int_{\partial\Omega} f(z)\overline{z}dz =$$

$$\sum_{j=1}^n \int_{C_j} f(z)\overline{z}dz + \int_{\partial C\cap E} f(z)\overline{z}dz + \int_{\partial C\setminus E} f(z)u(z)dz.$$

By Cauchy's Theorem we can replace the integration curve in the last integral by a curve which is arbitrarily close to E union with the interior boundaries of Ω and Corollary 4.2 follows.

According to Proposition 3.8 similar results hold for any connected component of the complement of the domain Ω. However, the first example above shows that the analytic extensions of the exponential kernel across different connecetd components of the boundary of Ω do not coincide.

As a final remark we return to the original unsolved question which has motivated the whole paper . Namely, from Section 2 we know that the exponential kernels of the extremal domains exhibited there are finitely determined by their Taylor coefficients at infinity, up to a certain prescribed degree; on the other hand Theorem 4.1 above shows that they have an analytic continuation configuration consisting essentially of finitely many ramification points. The question is whether these data suffice for finding some closed expression for the exponential kernels of these extremal domains (involving most likely some special functions).

5. A characterization of the exponential kernel

To reveal another face of the same exponential kernel, we derive below a characterization of it in terms of positive definite functions. The principle, of obtaining such a characterization in terms of postive definite kernels is not new. It goes back to the work of de Branges on Hilbert spaces of analytic functions [dB]; the same technique was exploited later by Pincus and Rovnyak [PR], Carey and Pincus [CP] and several other authors whose interests came in contact with determining or charactersitic functions of various classes of operators. For another notable example see also Livšic [L]. In spite of several close similarities with the mentioned works, we believe that the details contained in this section are new.

To simplify notation we put $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}, \mathbf{D} = \{z \in \mathbf{C}; |z| < 1\}$ and $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. For a measurable function $g : \mathbf{D} \longrightarrow [0,1]$ we denote:

$$E_g(z, \overline{w}) = exp(\frac{-1}{\pi} \int_{\mathbf{D}} \frac{g(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})} dA(\zeta)), \quad (|z|, |w| > 1).$$

Let us also remark that the above function extends analytically to a function

$$E_g : (\hat{\mathbf{C}} \setminus \overline{\mathbf{D}})^2 \longrightarrow \mathbf{C}^*.$$

The question we address in this section is to find a set of minimal conditions which characterizes an analytic function
$E : (\mathbf{C} \setminus \overline{\mathbf{D}})^2 \longrightarrow \mathbf{C}^*$ to be of the form $E = E_g$ for a measurable function g as above.

One obvious condition is:

$$E(\infty, \overline{w}) = E(z, \infty) = 1, \quad (z, w \in \hat{\mathbf{C}} \setminus \overline{\mathbf{D}}). \tag{9}$$

In order to state the next condition we define a new kernel
$F : (\hat{\mathbf{C}} \setminus \overline{\mathbf{D}})^4 \longrightarrow \mathbf{C}$ by the formula:

$$F(z_1, \overline{z_2}; w_1, \overline{w_2}) = \frac{E(z_1, \overline{w_2})E(w_1, \overline{z_2}) - E(z_1, \overline{z_2})E(w_1, \overline{w_2})}{(w_1 - z_1)(\overline{w_2} - \overline{z_2})E(z_1, \overline{w_2})}. \tag{10}$$

Whenever we encounter an analytic function $h(z)$, the quotient

$$\frac{h(z) - h(w)}{z - w}, \quad (z \neq w),$$

is extended analytically across the diagonal $(z = w)$ by the value $h'(z)$.As a matter of terminology the inequality $K(z_1, \overline{z_2}; w_1, \overline{w_2}) \succ 0$ means that the kernel K is non-negatively definite, that is

$$\sum_{k,l=1}^{N} K(s_k, \overline{t_k}; t_l, \overline{s_l}) \lambda_k \overline{\lambda_l} \geq 0,$$

for every finite set of points $\{(s_k, t_k), 1 \leq k \leq N\}$ in the domain of K and every complex numbers $\lambda_k, 1 \leq k \leq N$.

Theorem 5.1. *Let $E : (\hat{\mathbf{C}} \setminus \overline{\mathbf{D}})^2 \longrightarrow \mathbf{C}^*$ be an analytic function with the normalization property (9) and let F be the associated kernel (10).*

There is a measurable function $g : \mathbf{D} \longrightarrow [0,1]$ with the property that $E = E_g$ if and only if

$$F(z_1, \overline{z_2}; w_1, \overline{w_2}) \succ z_1\overline{w_2}F(z_1, \overline{z_2}; w_1, \overline{w_2}) - (z_1\overline{w_2}F(z_1, \overline{z_2}; w_1, \overline{w_2}))_{z_1=\infty} -$$
$$(z_1\overline{w_2}F(z_1, \overline{z_2}; w_1, \overline{w_2}))_{w_2=\infty} + (z_1\overline{w_2}F(z_1, \overline{z_2}; w_1, \overline{w_2}))_{z_1=w_2=\infty} \succ 0. \tag{11}$$

Note that the second term in the above positivity condition is a second order difference at infinity of the function F.

Proof. The necessity. For this part of the proof we use the known factorization of the kernel $1 - E_g$ in terms of the associated hyponormal operator with rank one self-commutator, see [P1],[P2].

Let g be a measurable function as in the statement and let T denote the irreducible hyponormal operator with rank one self-commutator which has the principal function equal to g, almost everywhere. Let us denote (as before) $[T^*, T] = \xi \otimes \xi$ and let us recall the basic formula:

$$E_g(z, \overline{w}) = 1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi \rangle. \tag{12}$$

Note that we have tacitly made the normalization $supp(g) \subset \overline{\mathbf{D}}$, hence $\sigma(T) \subset \overline{\mathbf{D}}$ and $\|T\| \leq 1$, (because T is a hyponormal operator, see [MP], Corollary 3.1.4).

Next we need a few elementary identities with resolvents,all stated for the current variables $u, v, z_1, \ldots$ outside the closed unit disk:

$$(T-u)^{-1}(T-v)^{-1} = \frac{(T-u)^{-1} - (T-v)^{-1}}{u-v} \tag{13}$$

and

$$(T^* - \overline{u})^{-1}(T-v)^{-1} = (T-v)^{-1}(T^* - \overline{u})^{-1} + \tag{14}$$
$$(T^* - \overline{u})^{-1}(T-v)^{-1}(\xi \otimes \xi)(T-v)^{-1}(T^* - \overline{u})^{-1}.$$

In particular, from formula (14) we obtain:

$$\langle (T^* - \overline{u})^{-1}(T-v)^{-1}\xi, \xi \rangle = \langle (T-v)^{-1}(T^* - \overline{u})^{-1}\xi, \xi \rangle +$$
$$\langle (T^* - \overline{u})^{-1}(T-v)^{-1}\xi, \xi \rangle \angle (T-v)^{-1}(T^* - \overline{u})^{-1}\xi, \xi \rangle,$$

or equivalently:

$$(1 - \langle (T-v)^{-1}(T^* - \overline{u})^{-1}\xi, \xi \rangle)(1 + \langle (T^* - \overline{u})^{-1}(T-v)^{-1}\xi, \xi \rangle) = 1,$$

that is:

$$1 + \langle (T^* - \overline{u})^{-1}(T-v)^{-1}\xi, \xi \rangle = \frac{1}{E_g(v, \overline{u})}. \tag{15}$$

We claim that:

$$F(z_1,\overline{z_2};w_1,\overline{w_2}) = \langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi, (T-w_2)^{-1}(T^*-\overline{w_1})^{-1}\xi\rangle. \tag{16}$$

Indeed, according to these identities we obtain (denoting $E=E_g$):

$$\langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi, (T-w_2)^{-1}(T^*-\overline{w_1})^{-1}\xi\rangle =$$
$$\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}(T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle =$$
$$\langle (T-w_1)^{-1}(T-z_1)^{-1}(T^*-\overline{w_2})^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle +$$
$$\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}(T-z_1)^{-1}\xi,\xi\rangle\langle (T-z_1)^{-1}(T^*-\overline{w_2})^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle =$$
$$\frac{\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle - \langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle}{\overline{w_2}-\overline{z_2}} =$$
$$\frac{\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle - \langle (T-w_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle}{(w_1-z_1)(\overline{w_2}-\overline{z_2})} +$$
$$\frac{\langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle - \langle (T-z_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle}{(w_1-z_1)(\overline{w_2}-\overline{z_2})} +$$
$$(\langle (T-w_1)^{-1}(T-z_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle +$$
$$\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}(T-z_1)^{-1}\xi,\xi\rangle\langle (T-z_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle)\times$$
$$\frac{\langle (T-z_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle - \langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle}{\overline{w_2}-\overline{z_2}} =$$
$$\frac{E(w_1,\overline{z_2})+E(z_1,\overline{w_2})-E(z_1,\overline{z_2})-E(w_1,\overline{w_2})}{(w_1-z_1)(\overline{w_2}-\overline{z_2})} +$$
$$\frac{\langle (T-w_1)^{-1}(T-z_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle(E(z_1,\overline{z_2})-E(z_1,\overline{w_2}))}{E(z_1,\overline{w_2})(\overline{w_2}-\overline{z_2})} =$$
$$\frac{E(w_1,\overline{z_2})+E(z_1,\overline{w_2})-E(z_1,\overline{z_2})-E(w_1,\overline{w_2})}{(w_1-z_1)(\overline{w_2}-\overline{z_2})} +$$
$$\frac{(E(z_1,\overline{w_2})-E(w_1,\overline{w_2}))(E(z_1,\overline{z_2})-E(z_1,\overline{w_2}))}{E(z_1,\overline{w_2})(w_1-z_1)(\overline{w_2}-\overline{z_2})} =$$
$$\frac{E(z_1,\overline{w_2})E(w_1,\overline{z_2})-E(w_1,\overline{w_2})E(z_1,\overline{z_2})}{E(z_1,\overline{w_2})(w_1-z_1)(\overline{w_2}-\overline{z_2})}.$$

Thus relation (16) is verified. It remains to remark that :

$$T(T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi =$$
$$(T^*-\overline{z_2})^{-1}\xi + z_1(T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi =$$
$$z_1(T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi - (z_1(T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi)_{z_1=\infty}.$$

In conclusion the positivity conditions (11) in the statement become:

$$0 \prec \langle T(T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi, T(T-w_2)^{-1}(T^*-\overline{w_1})^{-1}\xi\rangle \prec$$

$$\prec \langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi, (T-w_2)^{-1}(T^*-\overline{w_1})^{-1}\xi\rangle.$$

Since T is a contraction these two positivity conditions are evidently true.

The sufficiency. Let E be an analytic function which satisfies the normalization and positivity conditions in the statement. We want to prove that $E = E_g$, where $g : \mathbf{D} \longrightarrow [0,1]$ is a measurable function. This in turn is equivalent in finding a hyponormal operator T with rank-one self-commutator $[T^*, T] = \xi \otimes \xi$ which has g as pricipal function and hence E as associated determinantal function.

Since the kernel F was supposed to be non-negatively definite, Kolmogorov's factorization theorem implies the existence of a separable, complex Hilbert space H and an H-valued analytic function:

$$\rho : (\hat{\mathbf{C}} \setminus \overline{\mathbf{D}})^2 \longrightarrow H,$$

such that:

$$F(z_1, \overline{z_2}; w_1, \overline{w_2}) = \langle \rho(z_1, \overline{z_2}), \rho(w_2, \overline{w_1})\rangle, \quad (z_j, w_j \in \hat{\mathbf{C}} \setminus \overline{\mathbf{D}}; j = 1, 2). \tag{17}$$

In addition, we can assume without loss of generality that the image of the function ρ spans the whole Hilbert space H.

By assumption, $F(\infty, \overline{z_2}; w_1, \overline{w_2}) = F(z_1, \infty; w_1, \overline{w_2}) = 0$, therefore:

$$\rho(\infty, \overline{z}) = \rho(z, \infty) = 0, \quad (z \in \hat{\mathbf{C}} \setminus \overline{\mathbf{D}}).$$

In particular, both limits $lim_{z_1\to\infty} z_1\rho(z_1, \overline{z_2})$ and $lim_{z_2\to\infty} \overline{z_2}\rho(z_1, \overline{z_2})$ exist.

We define a linear transformation on the range of ρ by the formula:

$$T\rho(z_1, \overline{z_2}) = z_1\rho(z_1, \overline{z_2}) - (z_1\rho(z_1, \overline{z_2}))_{z_1=\infty}. \tag{18}$$

Let n be a positive integer and let us choose arbitrary elements $z_1(k), z_2(k) \in \hat{\mathbf{C}} \setminus \overline{\mathbf{D}}, \lambda_k \in \mathbf{C}, 1 \le k \le n$. In view of condition (11) in the statement of Theorem A.1 we have:

$$\|T \sum_{k=1}^n \lambda_k \rho(z_1(k), \overline{z_2(k)})\| \le \| \sum_{k=1}^n \lambda_k \rho(z_1(k), \overline{z_2(k)})\|.$$

Therefore, the map T extends linearly to a contraction defined on the whole space H. We denote its extension by the same symbol T.

Our next aim is to prove the formula:

$$T^*\rho(z_1, \overline{z_2}) = \overline{z_2}\rho(z_1, \overline{z_2}) - E(z_1, \overline{z_2})(\overline{z_2}\rho(z_1, \overline{z_2}))_{z_2=\infty}. \tag{19}$$

To this end we choose arbitrary points $z_1, z_2, w_1, w_2 \in \hat{\mathbf{C}} \setminus \overline{\mathbf{D}}$ and compute:

$$\langle T^*\rho(z_1, \overline{z_2}), \rho(w_2, \overline{w_1})\rangle - \langle \rho(z_1, \overline{z_2}), T\rho(w_2, \overline{w_1})\rangle =$$

$$\langle \overline{z_2}\rho(z_1, \overline{z_2}), \rho(w_2, \overline{w_1})\rangle - E(z_1, \overline{z_2})(\langle \overline{z_2}\rho(z_1, \overline{z_2}), \rho(w_2, \overline{w_1})\rangle)_{z_2=\infty} -$$

$$\overline{w_2}\langle \rho(z_1, \overline{z_2}), \rho(w_2, \overline{w_1})\rangle + (\langle \rho(z_1, \overline{z_2}), w_2\rho(w_2, \overline{w_1})\rangle)_{w_2=\infty} =$$

$$(\overline{z_2}-\overline{w_2})F(z_1,\overline{z_2};w_1,\overline{w_2})-E(z_1,\overline{z_2})(\overline{z_2}F(z_1,\overline{z_2};w_1,\overline{w_2}))_{z_2=\infty}+$$

$$(\overline{w_2}F(z_1,\overline{z_2};w_1,\overline{w_2}))_{w_2=\infty}=$$

$$\frac{-E(z_1,\overline{w_2})E(w_1,\overline{z_2})+E(z_1,\overline{z_2})E(w_1,\overline{w_2})+E(z_1,\overline{z_2})(E(z_1,\overline{w_2})-E(w_1,\overline{w_2}))}{(w_1-z_1)E(z_1,\overline{w_2})}+$$

$$\frac{E(w_1,\overline{z_2})-E(z_1,\overline{z_2})}{w_1-z_1}=0.$$

Thus formula (19) is verified.

Let us remark that, denoting:

$$\xi=(z_1\overline{z_2}\rho(z_1,\overline{z_2}))_{z_1=z_2=\infty},$$

we have:

$$(T^*-\overline{z_2})(T-z_1)\rho(z_1,\overline{z_2})=(T^*-\overline{z_2})(-z_1\rho(z_1,\overline{z_2}))_{z_1=\infty}=$$

$$(E(z_1,\overline{z_2})(z_1\overline{z_2}\rho(z_1,\overline{z_2}))_{z_2=\infty})_{z_1=\infty}=\xi,$$

(because $E(\infty,\overline{z_2})=1$). Whence we find the formula:

$$\rho(z_1,\overline{z_2})=(T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\quad (z_1,z_2\in\hat{\mathbf{C}}\setminus\overline{\mathbf{D}}). \tag{20}$$

Consequently we obtain:

$$[T^*,T]\rho(z_1,\overline{z_2})=[T^*-\overline{z_2},T-z_1](T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi=$$

$$\xi-E(z_1,\overline{z_2})\xi=(1-E(z_1,\overline{z_2}))\xi.$$

Therefore the operator T has rank-one self-commutator, and the vector ξ spans the range of $[T^*,T]$.

Finally we return to formula (10) and remark that:

$$1-E(z_1,\overline{z_2})=(w_1\overline{w_2}F(z_1,\overline{z_2};w_1,\overline{w_2}))_{w_1=w_2=\infty}=\langle\rho(z_1,\overline{z_2}),\xi\rangle,$$

so that

$$[T^*,T]\rho(z_1,\overline{z_2})=\langle\rho(z_1,\overline{z_2})\xi,\xi\rangle\xi.$$

This proves that $[T^*,T]=\xi\otimes\xi$.

In conclusion $E=E_g$, where g is the principal function of the operator T.

This finishes the proof of Theorem 5.1.

Remark 5.2. By changing the variables $u_j = \frac{1}{z_j}, v_j = \frac{1}{w_j}, j = 1, 2$, we can define the function:

$$G(u_1, \overline{u_2}; v_1, \overline{v_2}) = \frac{F(z_1, \overline{z_2}; w_1, \overline{w_2})}{z_1 \overline{w_2}},$$

so that G is analytic in the polydisk $\mathbf{D}^4$.

For an analytic function $h(z), z \in \mathbf{D}$, we define the difference of h at zero by:

$$\Delta_z h(z) = \frac{h(z) - h(0)}{z}.$$

Then condition (11) becomes:

$$0 \prec \Delta_{u_1} \Delta_{\overline{v_2}} G(u_1, \overline{u_2}; v_1, \overline{v_2}) \prec G(u_1, \overline{u_2}; v_1, \overline{v_2}). \tag{21}$$

.

Let $\mathcal{G}$ denote the class of all analytic functions $G : \mathbf{D}^4 \longrightarrow \mathbf{C}$ which satisfy the positivity conditions (21) and have the structure derived from fomula (10), where E is subject to the normalization (9).

Then the truncated, or full, L-problem of moments treated in the previous sections (and in the papers [P1],[P2]) is equivalent to the following interpolation problem for the class $\mathcal{G}$:

$$G \in \mathcal{G}$$

and

$$((\partial/\partial u_1)^m (\partial/\partial \overline{u_2})^n G)(0, 0; 0, 0) = b_{mn}, \quad (0 \leq m \leq m + n \leq N).$$

This is a two-dimensional variant of the classical Carathéodory-Fejér problem (see for instance [FF]). On the basis of our previous results obtained for the L-problem of moments, we know for the the above interpolation problem how to describe its solvability in positivity terms (in the case $N = \infty$), while for the corresponding truncated interpolation problem we know a description of all its extremal solutions (for N finite). In view of the bijection between the class of functions $\mathcal{G}$ and the measurable functions $g : \mathbf{D} \longrightarrow [0, 1]$, the class $\mathcal{G}$ has a natural convex structure hidden in the free parameter g.

References

[AG] Appel,P.; Goursat,E., *Théorie des fonctions algébriques et de Leurs Intégrales*, Tome I, Paris 1929, Reprint New York, 1976.

[BCR] Bochnak,J., Coste,M., Roy,M.-F., *Géométrie algebrique réele*, Springer, Berlin, 1987.

[CP] Carey,R.W., Pincus,J.D., An exponential formula for determining functions, Indiana Univ. Math. J. 23(1974), 1031-1042.

[D] Davis, P., *The Schwarz function and its applications*, Carus Math. Mono. vol. 17, Math. Assoc. of America, 1974.

[dB] de Branges,L., *Hilbert spaces of entire functions*, Prentice-Hall, N.J., 1968.

[DS] Dunford,N., Schwarz,J.T., *Linear Operators*, Part III, Interscience, New York, 1971.

[FF] Foiaş,C., Frazho,A.E., *The commutant lifting approach to interpolation problems*, Birkhäuser Verlag, Basel, 1990.

[G] Godwin, H.J., *Inequalities on Distribution Functions*, Griffin's Statistical Monographs and Courses, Hafner, New York, 1964.

[Gu] Gustafsson,B., Quadrature identities and the Schottky double, Acta Appl. Math. 1(1983), 209-240.

[H] Hörmander,L., *The Analysis of Linear Partial Differential Operators.I*, 2-nd Edition, Springer, Berlin, 1990.

[KS] Karlin,S., Studden,W.J., *Tchebycheff systems: with applications in Analysis and Statistics*, Interscience Publ., New York, 1966.

[K] Krein,M.G., The ideas of P.L. Chebyshev and A.A.Markov in the theory of limiting values of integrals and their further developments (in Russian), Uspehi Mat. Nauk. 6(1951), 3-120.

[KN] Krein,M.G., Nudelman,A.A., *Markov moment problem and extremal problems*, Transl. Amer. Math. Soc. vol. 50, Amer. Math. Soc. ,Providence, R.I., 1977.

[L] Livsic,M.S., Commuting nonselfadjoint operators and collective motions of systems, in Lect. Notes in Math., vol. 1272, Springer, Berlin et al., pp. 4-38.

[M] Malgrange,B., *Ideals of Differentiable Functions*, Oxford Univ. Press, Oxford, 1966.

[Ma] Martineau,A., Sur les fonctionnelles analytiques et la transformation de Fourier-Borel, J. d'Analyse Math. 11(1963), 1-164.

[MP] Martin,M., Putinar,M., *Lectures on hyponormal operators*, Birkhäuser, Basel, 1989.

[PR] Pincus,J.D., Rovnyak,J., A representation for determining functions, Proc. Amer. Math. Soc. 22(1969), 498-502.

[P1] Putinar,M., Extremal solutions of the two-dimensional L-problem of moments, J. Funct. Analysis, to appear.

[P2] Putinar,M., Linear analysis of quadrature domains, Ark. för Mat., to appear.

[Sa] Sakai.M., *Quadrature domains*, Lect. Notes Math. vol. 934, Springer, Berlin, 1982.

[Sh] Shapiro,H.S., *The Schwarz function and its generalization to higher dimensions*, Univ. Arkansas Lect. Notes Math. vol. 9, John Wiley and Sons, New York, 1992.

Department of Mathematics, University of California, Riverside, CA 92521
E-mail address: mputinar@ucrmath.ucr.edu