THE KREIN SPECTRAL SHIFT AND RANK ONE PERTURBATIONS OF SPECTRA

A. Poltoratski

Abstract. We use recent results on the boundary behavior of Cauchy integrals to study the Krein spectral shift of a rank one perturbation problem for self-adjoint operators. As an application, we prove that all self-adjoint rank one perturbations of a self-adjoint operator are pure point if and only if the spectrum of the operator is countable. We also study pairs of pure point operators unitarily equivalent up to a rank one perturbation and give various examples of rank one perturbations of singular spectra.

1. Introduction.

In this paper we use well-known connections between one-dimensional perturbation theory and complex analysis. We apply recent results on the boundary behavior of Cauchy integrals to study the spectral properties of rank one perturbations of self-adjoint operators.

Section 2 discusses the structure of a family of the spectral measures of rank one perturbations of a self-adjoint operator. We establish the relations between such families and the characteristic function of the operator.

Section 3 is devoted to the properties of the Krein spectral shift of a rank one perturbation problem. We study spectral shifts in terms of the boundary behavior of the resolvent function.

In Section 4 we prove that all self-adjoint rank one perturbations of a self-adjoint operator are pure point if and only if the spectrum of the operator is countable.

Section 5 analyzes relative spectral properties of two self-adjoint pure point operators unitarily equivalent up to a rank one perturbation.

In Section 6 we give an example of "the absence of mixed spectra". We construct a family $A + \lambda(\cdot, \varphi)\varphi$ of rank one perturbations of a self-adjoint operator such that the operators corresponding to the coupling constants $\lambda \in [0; 1]$ are singular continuous and the operators corresponding to the coupling constants $\lambda \in \mathbb{R} \setminus [0; 1]$ are pure point.

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2. Families $\mathcal{M}_\varphi$ and rank one perturbations of self-adjoint operators.

We will denote by $\mathcal{M}(\hat{\mathbb{R}})$ the space of Borel complex measures on $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$ with the norm

$$||\mu|| = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d|\mu(t)|}{1+t^2} + |\mu|(\infty).$$

We will also use the notation $\mathcal{M}_+(\hat{\mathbb{R}})$ for the subset consisting of nonnegative measures.

Let ϕ be a nonconstant analytic function in the upper half plane $\mathbb{C}_+$ such that $|\phi| \leq 1$. Then for any α from the unit circle $\mathbb{T}$ function $\frac{\alpha+\phi}{\alpha-\phi}$ has positive real part in $\mathbb{C}_+$. Thus there exists a measure $\mu_\alpha \in \mathcal{M}_+(\hat{\mathbb{R}})$ such that its Poisson integral satisfies

$$(\mathcal{P}\mu_\alpha)(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y d\mu_\alpha(t)}{(x-t)^2+y^2} + y\mu_\alpha(\infty) = \operatorname{Re} \frac{\alpha+\phi}{\alpha-\phi}. \tag{2.1}$$

We will denote by $\mathcal{M}_\phi(\hat{\mathbb{R}})$ the family of such measures $\{\mu_\alpha\}_{\alpha\in\mathbb{T}}$ corresponding to function ϕ.

If ϕ is defined in the unit disk one can replace the Poisson integral in (2.1) by the Poisson integral in the disk and consider the analogous family $\mathcal{M}_\phi(\mathbb{T})$ consisting of positive measures on the unit circle $\mathbb{T}$.

Families $\mathcal{M}_\phi$ possess many interesting properties, see [A1], [A2], [C], [P1], [P2] and [Sa]. As was shown by Clark in [C], if ϕ is an inner function in the unit disk $\mathbb{D}$ then $\mathcal{M}_\phi(\mathbb{T})$ is the system of the spectral measures of all unitary rank one perturbations of the model contraction with the characteristic function ϕ. By adopting the argument of Clark to the case of the upper half-plane, one can make a similar connection between rank one perturbations of self-adjoint operators and families $\mathcal{M}_\phi(\hat{\mathbb{R}})$. Here we present a different way to establish this connection.

If A_0 is a bounded cyclic self-adjoint operator and φ is its cyclic vector, then we can consider the family of rank one perturbations

$$A_\lambda = A_0 + \lambda(\cdot, \varphi)\varphi,$$

$\lambda \in \mathbb{R}$. Let ν_λ be the spectral measure of φ for A_λ. Since A_λ is bounded, $\operatorname{supp} \nu_\lambda \subset \mathbb{R}$ is compact. Thus the Cauchy transform of ν_λ

$$\mathcal{K}\nu_\lambda(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\nu_\lambda(t)}{t-z} = \frac{1}{\pi}((A_\lambda - z)^{-1}\varphi, \varphi) \tag{2.2}$$

converges for any $z \in \mathbb{C}_+$.

The relation for the resolvents

$$(A_0 - z)^{-1} - (A_\lambda - z)^{-1} = \left[\lambda(\cdot, \varphi)\left((A_\lambda - z)^{-1}\varphi\right)\right](A_0 - z)^{-1},$$

for any $z \in \mathbb{C}_+$, gives us

$$\mathcal{K}\nu_\lambda(z) = \frac{\mathcal{K}\nu_0(z)}{1 + \pi\lambda\mathcal{K}\nu_0(z)}, \tag{2.3}$$

see [A].

Since ν_0 is a nonnegative Borel measure on $\mathbb{R}$, $-i\mathcal{K}\nu_0$ is an analytic function with a nonnegative real part in $\mathbb{C}_+$:

$$\operatorname{Re} -i\mathcal{K}\nu_0 = \mathcal{P}\nu_0.$$

Thus

$$-i\mathcal{K}\nu_0 = \frac{1+\phi}{1-\phi} \tag{2.4}$$

for some bounded analytic function ϕ, $|\phi| \le 1$ in $\mathbb{C}_+$.

Hence by (2.3)

$$(\mathcal{P}\nu_\lambda)(x+iy) = \operatorname{Re} -i\mathcal{K}\nu_\lambda(x+iy) = \operatorname{Re} \frac{\frac{1-\phi}{1+\phi}}{1 + i\pi\lambda\frac{1-\phi}{1+\phi}} = c \operatorname{Re} \frac{\beta - \phi}{\beta + \phi} \tag{2.5}$$

where $\beta = \frac{1+i\pi\lambda}{1-i\pi\lambda}$, $c = \frac{1}{1+\pi^2\lambda^2}$. Thus

$$\mathcal{P}\nu_\lambda = c\mathcal{P}\mu_\beta$$

where $\{\mu_\beta\}_{\beta\in\mathbb{T}} = \mathcal{M}_\phi(\hat{\mathbb{R}})$.

Conversely, if ϕ is an analytic function in $\mathbb{C}_+$ with $|\phi| \le 1$ we can consider the operator A_μ of multiplication by z in $L^2(\mu)$ where $\mu = \mu_1 \in \mathcal{M}_\phi(\hat{\mathbb{R}})$. Reversing the above argument we can show that for any $\alpha \in \mathbb{T}$ the measure $\mu_\alpha \in \mathcal{M}_\phi$ is a spectral measure for the rank one perturbation $A + \lambda(\cdot, 1)1$ where $\lambda = \frac{i}{\pi}\frac{1-\alpha}{1+\alpha}$. Therefore, each family $\mathcal{M}_\phi(\hat{\mathbb{R}})$ is a family of spectral measures of rank one perturbations of some self-adjoint operator.

Each measure from $\mathcal{M}_+(\hat{\mathbb{R}})$ belongs to a family $\mathcal{M}_\phi$ for some ϕ (ϕ can be found from (2.1) uniquely up to a Möbius transform). One can also say that two given measures μ and ν from $\mathcal{M}(\hat{\mathbb{R}})$ belong to the same family $\mathcal{M}_\phi$ iff there exist operator A and its rank one perturbation $A_\lambda = A + \lambda(\cdot, \varphi)\varphi$ such that μ and ν are the spectral measures of φ for A and A_λ. Indeed, as we proved above, such A and A_λ exist iff $c_1\mu, c_2\nu \in \mathcal{M}_\psi$ for some function ψ and positive constants c_1 and c_2. Suppose $c_1\mu = \mu_\alpha, c_2\nu = \mu_\beta$ for some $\mu_\alpha, \mu_\beta \in \mathcal{M}_\psi$. Then $\mu, \nu \in \mathcal{M}_\phi$ for $\phi = b \circ \psi$ where b is the Möbius transform of the unit disk $\mathbb{D}$ such that $|b'(\alpha)| = 1/c_1, |b'(\beta)| = 1/c_2$, see [A2].

Further relations between the resolvent functions of the rank one perturbations of a self-adjoint operator are discussed in [A], [D], [M-P], [S-W], [S] and [R-J-L-S].

Remark. The same argument works for the families of unitary operators (in particular for those studied by Clark [C]).

Let $\mathcal{U}_1$ be a unitary cyclic operator. Let probability measure $\mu_1 \in \mathcal{M}_+(\mathbb{T})$ be the spectral measure of some cyclic vector v for $\mathcal{U}_1$. Then we can consider the family of unitary rank one perturbations of $\mathcal{U}_1$:

$$\mathcal{U}_\alpha = \mathcal{U}_1 + (\alpha - 1)(\cdot, \mathcal{U}_1^{-1}v)v,$$

$\alpha \in \mathbb{T}$. For the resolvents we have

$$(\mathcal{U}_1 - z)^{-1} - (\mathcal{U}_\alpha - z)^{-1} = (\mathcal{U}_1 - z)^{-1}[\alpha(\cdot, \mathcal{U}_1^{-1}v)v](\mathcal{U}_\alpha - z)^{-1}.$$

Denote by μ_α the spectral measure of $v/||v||$ for U_α. Let $K\mu_\alpha$ and $P\mu_\alpha$ be the standard Cauchy and Poisson integrals of measure μ_α in the unit disk $\mathbb{D}$. Since $K\mu_\alpha = ((\mathcal{U}_\alpha - z)^{-1}v, v)$ for $z \in \mathbb{D}$, we have that

$$K\mu_\alpha = \frac{\alpha K\mu_\alpha}{1 + (\alpha - 1)K\mu_1}. \tag{2.6}$$

If we consider an analytic function θ in $\mathbb{D}$ such that $|\theta| \leq 1, \theta(0) = 0$ and

$$P\mu_1 = \operatorname{Re} \frac{1+\theta}{1-\theta}$$

in $\mathbb{D}$, then

$$K\mu_1 = \frac{1}{1-\theta}$$

and by (2.6)

$$P\mu_\alpha = 2\operatorname{Re} K\mu_\alpha - 1 = 2\operatorname{Re}\left[\frac{\frac{\alpha}{1-\theta}}{1 + \frac{\alpha-1}{1-\theta}} - \frac{1}{2}\right] = \operatorname{Re}\frac{\alpha+\theta}{\alpha-\theta}.$$

Thus $\{\mu_\alpha\}_{\alpha \in \mathbb{T}} = \mathcal{M}_\theta(\mathbb{T})$.

As we mentioned before, this result was obtained in [C] for rank one unitary perturbations of the model contraction $T_\theta = \mathcal{S}P_\theta$, where θ is an inner function, $\mathcal{S} : f \mapsto zf$ is a shift operator in the Hardy space H^2 and P_θ is the orthogonal projector from H^2 onto the model space $\theta^*(H^2) = H^2 \ominus \theta H^2$. Any singular cyclic unitary operator can be represented as a rank one perturbation of T_θ for some inner θ. Conversely, if the operator $\mathcal{U}_1$ from the last remark is singular, then $\mathcal{U}_1 - (\cdot, \mathcal{U}_1^{-1}v)v$ is a C_0 completely nonunitary contraction with the characteristic function θ.

3. The Krein spectral shift.

In this section we discuss the notion of the Krein spectral shift in the case of rank one perturbations. For basic results and definitions in this area we refer to [K], [A-D], [M-P] and [S].

Let A be a bounded self-adjoint operator, φ its cyclic vector. Let $\mu \in \mathcal{M}_+(\hat{\mathbb{R}})$ be the spectral measure of φ for A. Since μ is a nonnegative measure, its Cauchy transform has nonnegative imaginary part. Therefore for each $\lambda \in \mathbb{R}$ the function $1 + \pi\lambda\mathcal{K}$ admits the following representation:

$$1 + \pi\lambda\mathcal{K}\mu = \exp[\mathcal{K}u] \tag{3.1}$$

fore some real valued function $u \in L^\infty(\mathbb{R}), ||u|| \leq \pi$ with compact support. If $\lambda \geq 0$ then function u can be chosen to satisfy

$$u = \arg(1 + \pi\lambda\mathcal{K}\mu) \tag{3.2}$$

where arg stands for the principal branch of argument taking values in $(-\pi; \pi]$ (we assume that $\mathcal{K}\mu$ is defined a. e. on $\mathbb{R}$ by its angular boundary values). In this case we will have $0 \leq u \leq \pi$. Similarly if $\lambda < 0$ then u can be chosen to satisfy (3.2) with arg taking values in $[-\pi; \pi)$. In this case we will have $0 \leq u \leq -\pi$.

Definition. Such u is called the Krein spectral shift of the perturbation problem $(A \mapsto A + \lambda(\cdot, \varphi)\varphi)$.

The Krein spectral shift is usually defined in more general settings of compact perturbations $(A \mapsto A + K)$, see [K] and [M-P]. When K is one-dimensional the general definition can be reduced to the one given above, see [M-P].

Let ν be the spectral measure of φ for $A + \lambda(\cdot, \varphi)\varphi$. Even though measure ν does not appear in (3.1), it is uniquely determined by μ and λ and satisfies a similar equation.

Lemma 3.1.
Measures μ, ν and the Krein spectral shift u satisfy

$$1 + \pi\lambda\mathcal{K}\mu = \exp[\mathcal{K}(u)] = [1 - \pi\lambda\mathcal{K}\nu]^{-1}. \tag{3.3}$$

Proof. Apply formulas (3.1) and (2.3) with $\nu_0 = \mu$ and $\nu_\lambda = \nu$. ▲

Since in this paper we will mostly deal with the spectral measures of operators, it will be more convenient for us to call u the phase shift of the pair of measures (μ, ν) for the coupling constant λ. If $\lambda = 1$ we will call u the phase shift of the pair $(\mu; \nu)$. One can use formula (3.3) as an alternative definition of the phase shift.

Note that in this definition u determines μ and ν uniquely. This reflects the fact that among the spectral measures of A and $A + \lambda(\cdot, \varphi)\varphi$ we always choose those corresponding to the vector φ.

Any function $u \in L^\infty(\mathbb{R}), 0 \leq u \leq \pi$ with compact support is the Krein spectral shift of some perturbation problem (the phase shift of some pair of measures). Indeed, for any $\lambda > 0$ one can consider a nonnegative measure $\mu \in \mathcal{M}(\hat{\mathbb{R}})$ with compact support satisfying (3.1). Then u is the Krein spectral shift of the perturbation problem ($A_\mu \mapsto A_\mu + \lambda(\cdot,1)1$). Similarly each $u \in L^\infty(\mathbb{R}), 0 \geq u \geq -\pi$ with compact support is also the Krein spectral shift of some perturbation problem (with $\lambda < 0$).

As was shown in the previous Section, there exist operator A and its cyclic vector φ such that μ and ν are spectral measures for φ of A and $A+(\cdot,\varphi)\varphi$ iff both measures belong to the same family $\mathcal{M}_\phi$ for some function ϕ analytic in $\mathbb{C}_+$. One can show that if we take $\mu_\alpha, \mu_\beta \in \mathcal{M}_\phi$ $\alpha = e^{i\theta_1}, \beta = e^{i\theta_2}, 0 \leq \theta_1 < \theta_2 < 2\pi$ where ϕ is inner then the phase shift u for the pair (μ_α, μ_β) can be defined as

$$u = \pi\chi_{\{x\in\mathbb{R}|\phi(x)=e^{i\theta},\theta_1<\theta<\theta_2\}}$$

where χ_E denotes the indicator function of the set E (we again assume that ϕ is defined a. e. on $\mathbb{T}$ by its angular boundary values).

As before, let u be the phase shift of the pair of measures (μ,ν) and ϕ be a characteristic function: $\mu,\nu \in \mathcal{M}_\phi$. Then u has the following properties.

Lemma 3.2.

Let $K \subset \mathbb{R}$ be a measurable set. Then the following conditions are equivalent

i) the restrictions of μ and ν on K are singular,

ii) $|u|$ is equal to 0 or π a. e. on K,

iii) $|\phi| = 1$ a. e. on K.

Proof. For i)⇔ii) see [M-P] or [S]; for i)⇔iii) see [C].▲

We will denote by Qu the conjugate Poisson integral of u:

$$Qu(x+iy) = -\operatorname{Re}\mathcal{K}u(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}u(t)dt.$$

It is well-known that if $v \in L^\infty(\mathbb{R})$ with a compact support then the difference

$$Qv(x+iy) - \int_{\mathbb{R}\setminus(x-y;x+y)}\frac{v(t)dt}{x-t} \tag{3.4}$$

is $O(1)$ as $x+iy \underset{\not>}{\longrightarrow} x_0$ for any $x_0 \in \mathbb{R}$ and $o(1)$ for any Lebesgue point x_0 of v, see [G].

Let $E \subset \mathbb{R}$ be a measurable set, $\mu \in \mathcal{M}(\mathbb{R})$. We will denote

$$p.\ v. \int_E d\mu(x) = \lim_{\epsilon\to 0}\int_{E\setminus(-\epsilon;\epsilon)} d\mu(x).$$

We will denote by μ^s the singular component of measure μ. The definition of the phase shift and relation (3.4) imply the following

Lemma 3.3.
i) For μ^s-a.e x

$$p.\ v. \int\limits_{\mathbb{R}} \frac{u(x+t)dt}{t} = \infty. \tag{3.5}$$

For ν^s-a.e x

$$p.\ v. \int\limits_{\mathbb{R}} \frac{u(x+t)dx}{t} = -\infty. \tag{3.6}$$

ii) There exist $\alpha, \beta \in \mathbb{T}$ such that for μ^s-a.e x

$$\phi(z) \xrightarrow[z \underset{\not\to}{\longrightarrow} x]{} \alpha;$$

For ν^s-a.e x

$$\phi(z) \xrightarrow[z \underset{\not\to}{\longrightarrow} x]{} \beta.$$

Proof. Follows from formulas (2.1) and (3.3). For i) see [S]; for ii) see [C].▲

Example 3.4. Let $x, y \in \mathbb{R}, x < y$. Let $u = \pi\chi_{(x;y)}$. Then by Lemma 3.2 ii) μ and ν are singular. By Lemma 3.3 i) μ is concentrated on the set $\{x\}$ and ν is concentrated on the set $\{y\}$. Since u is nonconstant, μ and ν are nonzero. Thus $\mu = b\delta_x$ and $\nu = c\delta_y$ for some positive b and c.

To prove the next property of phase shifts we will need the following

Theorem 3.5([P1]). *Let $\sigma \in M(\hat{\mathbb{R}})$ be a singular measure with compact support, $f \in L^1(|\sigma|)$. Define meromorphic function F in $\mathbb{C}_+$ as*

$$F = \frac{1+\mathcal{K}(f\sigma)}{1+\mathcal{K}\sigma}.$$

Then F has nontangential boundary limits equal to f $\quad|\sigma|$-a. e. on $\mathbb{R}$.

This Theorem is proved in [P1] in the case of the unit disk.

We will say that real function f defined in $\mathbb{C}_+$ is less than ∞ (greater than $-\infty$) at point $x \in \mathbb{R}$ if

$$\liminf_{z \underset{\not\to}{\longrightarrow} x} f(z) < \infty \quad (\limsup > -\infty).$$

We will write that

$$p.\ v. \int\limits_E f(x)dx < \infty \quad (> -\infty)$$

if

$$\liminf_{\epsilon \to 0} \int\limits_{E \setminus (-\epsilon;\epsilon)} d\mu(x) < \infty \quad (\limsup > -\infty).$$

Corollary 3.6. *Let $\sigma, \gamma \in \mathcal{M}_+\mathbb{R})$ be singular measures with compact supports. Define holomorphic function F in $\mathbb{C}_+$ as*

$$F = \frac{1+\mathcal{K}\sigma}{1+\mathcal{K}\gamma}.$$

Let K be a measurable subset of $\mathbb{R}$. Then $|F| < \infty$ σ-a. e. on K iff the restriction of σ on K is absolutely continuous with respect to γ.

Proof. Let $\sigma = f\gamma + \eta$ where $f \geq 0, f \in L^1(\gamma), \eta \perp \gamma$. Then

$$F = \frac{1+\mathcal{K}\sigma}{1+\mathcal{K}\gamma} = \frac{1+\mathcal{K}\sigma}{1+\mathcal{K}(\sigma+\gamma)} \times \frac{1+\mathcal{K}(\sigma+\gamma)}{1+\mathcal{K}\gamma}. \tag{3.7}$$

By Theorem 3.5 the first fraction has finite nonzero limits $|\sigma|$-a. e. Since

$$\frac{1+\mathcal{K}(\sigma+\gamma)}{1+\mathcal{K}\gamma} = \left[\frac{1+\mathcal{K}\gamma}{1+\mathcal{K}((1+f)\gamma+\eta)}\right]^{-1},$$

by Theorem 3.5 the second fraction in (3.7) has finite limits $|\gamma|$-a. e. and tends to infinity η-a. e. Thus $|F| < \infty$ σ-a. e. on K iff $\eta(K) = 0$. ▲

Lemma 3.7. *Let u_1 and u_2 be the phase shifts of the pairs of measures (μ_1, ν_1) and (μ_2, ν_2) respectively. Let $K \subset \mathbb{R}$ be a measurable set. Then*

i)

$$p.\ v. \int_{\mathbb{R}} (u_1(t+x) - u_2(t+x))\frac{dt}{t} < \infty$$

for μ_1-a. e. $x \in K$ iff the restriction of μ_1^s on K is absolutely continuous with respect to μ_2;

$$p.\ v. \int_{\mathbb{R}} (u_1(t+x) - u_2(t+x))\frac{dt}{t} > -\infty$$

for ν_1-a. e. $x \in K$ iff the restriction of ν_1^s on K is absolutely continuous with respect to ν_2;

ii) if μ_2^s-a. e. $x \in K$ is a Lebesque point of $u_1 - u_2$ and

$$p.\ v. \int_{\mathbb{R}} (u_1(t+x) - u_2(t+x))\frac{dt}{t} = f(x) < \infty$$

for μ_2^s-a. e. $x \in K$ then the restriction of μ_1^s on K is equal to the restriction of $e^f\mu_2^s$ on K; if ν_2^s-a. e. $x \in K$ is a Lebesque point of $u_1 - u_2$ and

$$p.\ v. \int_{\mathbb{R}} (u_1(t+x) - u_2(t+x))\frac{dt}{t} = f(x) > -\infty$$

for ν_2^s-a. e. $x \in K$ then the restriction of ν_1^s on K is equal to the restriction of $e^{-f}\nu_2^s$ on K.

Proof. Notice that by the defining formula (3.3)

$$\exp\left[\mathcal{K}(u_1 - u_2)\right] = \frac{1+\mathcal{K}\mu_1}{1+\mathcal{K}\mu_2} = \frac{1+\mathcal{K}\nu_2}{1+\mathcal{K}\nu_1}.$$

Now Corollary 3.6 and formula (3.4) imply part i). Since the difference (3.4) is $o(1)$ at Lebesgue points of $u_1 - u_2$, part ii) follows from part i) and Theorem 3.5.▲

Lemma 3.8 [M-P]. *Measure μ has a point mass at x iff*

$$\int\limits_{x-1}^{x+1} (\pi\chi_{(x;x+1)} - u(y))\frac{dy}{y-x} < \infty. \tag{3.8}$$

Measure ν has a point mass at x iff

$$\int\limits_{x-1}^{x+1} (\pi\chi_{(x-1;x)} - u(y))\frac{dy}{x-y} < \infty. \tag{3.9}$$

Proof. As was shown in Example 3.5 measure μ_0 corresponding to the phase shift $u_1 = \pi\chi_{(x;x+1)}$ has a point mass at x. It is left to apply statement i) of Lemma 3.7 to $u_1, u_2 = u$, and $K = \{x\}$.

The statement can also be proved directly, see [M-P]. ▲

Remark. In terms of the characteristic function ϕ a point mass of $\mu_\alpha \in \mathcal{M}_\phi$ can be described in the following way, see [C], [A-C] or [Sa]. For any $x \in \mathbb{R}$ $\mu_\alpha(x) > 0$ iff $\phi(z) \to \alpha$ as $z \underset{\not>}{\longrightarrow} x$ and ϕ has a finite nontangential derivative $\phi'(x)$ at x.

In terms of other measures from the family $\mathcal{M}_\phi$, $\mu_1(x) > 0$ iff $\mathcal{K}\mu_{-1}(z) \to 0$ as $z \underset{\not>}{\longrightarrow} x$ and $(x-y)^{-1} \in L^2(\mu_{-1})$, see [C], [S-W] or [S].

4. On the stability of the absence of continuous spectra

Numerous examples given in [A-D], [D], [S] and [R-J-L-S] show that for a general self-adjoint operator the property of not having a continuous part is not stable under rank one perturbations. For instance [D] contains an example of a self-adjoint operator A and its cyclic vector φ such that A is pure point but $A + \lambda(\cdot, \varphi)\varphi$ are singular continuous for all real $\lambda \neq 0$.

Here we prove that in order to have *only* pure point rank one perturbations the operator must have a very "thin" spectrum. We will denote the spectrum of A by $\sigma(A)$.

Theorem 4.1. *Let A be a self-adjoint operator. Then the following two conditions are equivalent:*

1) All self-adjoint rank one perturbations of A are pure point,

2) $\sigma(A)$ is countable.

Remark. We assume that A itself is included in the set of all its rank one perturbations.

In 1) "all" can not be replaced with "almost all" in any reasonable sense: it is not difficult to show that if μ is a standard singular Cantor measure on $[0;1]$ then $A_\mu + \lambda(\cdot, \varphi)\varphi$ is pure point for a.e. $\lambda \in \mathbb{R}$ for any cyclic vector φ.

To prove Theorem 4.1 we will need the following two Lemmas.

Lemma 4.2. *Let $F \subset \mathbb{R}$, $|F| = 0$ be a closed set, $F = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} I_n$ where $I_n = (x_n; y_n)$ are disjoint open intervals. Then one can choose two disjoint sets of positive integers L and M such that for any $y \in F \setminus \{x_1, x_2, x_3, ...\}$*

$$\int_{(y;y+1)\cap \bigcup_{n\in L} I_n} \frac{dx}{x-y} = \int_{(y;y+1)\cap \bigcup_{n\in M} I_n} \frac{dx}{x-y} = \infty. \tag{4.1}$$

and for any $y \in F \setminus \{y_1, y_2, y_3, ...\}$

$$\int_{(y-1;y)\cap \bigcup_{n\in L} I_n} \frac{dx}{y-x} = \int_{(y-1;y)\cap \bigcup_{n\in M} I_n} \frac{dx}{y-x} = \infty. \tag{4.2}$$

Proof. Since $|F| = 0$ we can choose $\{I_{n_l^1}\}_{l=1}^{\infty}$ in such a way that

$$\int_{(y;y+1)\cap \bigcup_{l=1}^{\infty} I_{n_l^1}} \frac{1}{x-y} dx > 1$$

for any $y \in F \setminus \{x_1, x_2, x_3, ...\}$,

$$\int_{(y-1;y)\cap \bigcup_{l=1}^{\infty} I_{n_l^1}} \frac{1}{y-x} dx > 1$$

for any $y \in F \setminus \{y_1, y_2, y_3, ...\}$ and the set of the endpoints of the intervals $I_{n_l^1}$ has no cluster points except $\pm\infty$. Indeed, for each $n \in \mathbb{N}$ we can choose a finite number of intervals covering more than a half (in measure) of the interval $[n; n+1]$. Then we can put $\{I_{n_l^1}\}$ to be the set of all chosen intervals for all $n \in \mathbb{N}$.

Since the set of the endpoints of $I_{n_l^1}$ has no cluster points in $\mathbb{R}$, the set $(0;1) \setminus \bigcup_{l=1}^{\infty} I_{n_l^1}$ is a union of disjoint closed intervals. Therefore in a similar way we can choose intervals $I_{n_l^2} \subset \mathbb{R} \setminus \bigcup_{l=1}^{\infty} I_{n_l^1}$ such that

$$\int_{(y;y+1)\cap \bigcup_{l=1}^{\infty} I_{n_l^2}} \frac{1}{x-y} dx > 1$$

for any $y \in F \setminus \{x_1, x_2, x_3, ...\}$,

$$\int_{(y-1;y)\cap \bigcup_{l=1}^{\infty} I_{n_l^2}} \frac{1}{y-x} dx > 1$$

for any $y \in F \setminus \{y_1, y_2, y_3, ...\}$ and the set of the endpoints of the intervals I_l^2 has no cluster points except possibly the ends of the intervals I_l^1 and $\pm\infty$. Proceeding in this way we will obtain $\{I_{n_l^k}\}_{l=1}^{\infty}$ for $k = 1, 2, 3....$ After that we can put $L = \{n_k^{2l-1}\}_{l,k\in\mathbb{N}}$ and $M = \{n_k^{2l}\}_{l,k\in\mathbb{N}}$. ▲

Lemma 4.3. *Let $F \subset (0;1)$ be a closed set, $|F| = 0$, $F = (0;1) \setminus \bigcup I_n$ where $I_n = (x_n; y_n)$ are disjoint open intervals. Let $\{z_n\}_{n>0}$ be a sequence of points such that $z_n \in I_n$ and $y_n - z_n < |I_n|^2$. Let M be as in the previous Lemma. Then there exists $N \subset M$ such that for all $y \in F$*

i) if $y = y_n$ for some $n \in \mathbb{N}$ then

$$\int\limits_{(y;y+1)\cap \bigcup\limits_{n\in N} I_n} \frac{dx}{x-y} = +\infty. \tag{4.3}$$

ii) if $y \neq x_n, y_n$ for any n and $E = \bigcup\limits_{n\in N} I_n \cup \bigcup\limits_{n\in\mathbb{N}} [z_n; y_n]$, then

$$p.\ v. \int\limits_{(-1;1)} \chi_E(y+x)\frac{dx}{x} < +\infty. \tag{4.4}$$

Proof. WLOG we can assume that the intervals I_n are enumerated in such a way that $|I_1| \geq |I_2| \geq ....$

Let δ_l be a small positive constant (the exact choice of δ_l will be made later). For each $l \in \mathbb{N}$ choose a sequence of intervals $\{I_{n^l_k}\}_{k>0}$ such that for any $k \in \mathbb{N}$ $n^l_k \in M$, $I_{n^l_k} \in (y_l; y_l + \delta_l)$, $\quad y_{n^1_1} > x_{n^1_1} > y_{n^1_2} > x_{n^1_2} > ...$ and

$$\int\limits_{(y_l;y_l+1)\cap \bigcup\limits_{k>0} I^l_k} \frac{dx}{x-y_l} = \infty.$$

Put $N = \{n^l_k |\ l, k \in \mathbb{N}\}$. Then condition $i)$ is satisfied. We will show that if constants δ_l are chosen small enough then N also satisfies $ii)$.

Denote $J_l = \bigcup\limits_{k>0} I_{n^l_k}$. Choose constants δ_l to satisfy both

$$\delta_l < |I_l|^2 \tag{4.5}$$

and

$$\delta_l < \frac{1}{4}(y_l - z_l)\,\mathrm{dist}(y_l; (y_l;\infty) \cap \bigcup_{k<l} J_k) \tag{4.6}$$

for every $l \in \mathbb{N}$ (notice that the distance in (4.6) is always nonzero by the construction). Let now $y \in F, y \neq x_n, y_n$ for any $n > 0$. Consider the sets of integers $M' = \{l|x_l > y\}$, $M'' = \{l|\ y_l + \delta_l < y\}$ and $M''' = \{l|\ y_l < y,\ y_l + \delta_l > y\}$. Then $\mathbb{N} = M' \cup M'' \cup M'''$. Put

$$J' = \bigcup_{l\in M'} J_l, \quad J'' = \bigcup_{l\in M''} J_l \quad \text{and} \quad J''' = \bigcup_{l\in M'''} J_l.$$

Then, since $J_l \subset (y_l; y_l + \delta_l)$,

$$\left| \int\limits_{J'} \frac{dx}{x-y} \right| < \infty$$

by (4.5). Also obviously

$$\int\limits_{J''} \frac{dx}{x-y} < 0 < \infty$$

since $J'' \subset (y-1; y)$.

Let $\{n_k\}_{k\in\mathbb{N}}$ be some enumeraton of M''' such that $n_1 < n_2 < n_3 < \dots$. Put $\epsilon_k = y - y_{n_k} + \delta_{n_k}$ and consider

$$C_k = \int\limits_{(y-1;y+1)\setminus(y-\epsilon_k;y+\epsilon_k)} \chi_E(x)\frac{dx}{x-y}$$

(notice that $0 < \epsilon_k < 2\delta_{n_k} \to 0$ as $k \to \infty$). Then by (4.6) $\delta_{n_k} < \frac{1}{4}(y_{n_k} - z_{n_k})$ and $\delta_{n_k} < \frac{1}{4}\operatorname{dist}(y_{n_k}; J_l)$ for all $k \in \mathbb{N}$. Hence for any $k \in \mathbb{N}$ we have that

$$\operatorname{dist}(y; (y;\infty) \cap \bigcup_{l<k} J_{n_l}) > \operatorname{dist}(y_{n_k}; (y_{n_k};\infty) \cap \bigcup_{l<k} J_{n_l}) - \delta_{n_k} >$$

$$> \frac{3}{4}\operatorname{dist}(y_{n_k}; (y_{n_k};\infty) \cap \bigcup_{l<k} J_{n_l}) > 3\frac{\delta_{n_k}}{(y_{n_k} - z_{n_k})}.$$

Since $|\bigcup\limits_{l<k} J_{n_l}| < 1$ and $\delta_{n_k} < \frac{1}{4}(y_{n_k} - z_{n_k})$,

$$\frac{|(y;\infty) \cap \bigcup\limits_{l<k} J_{n_l}|}{\operatorname{dist}(y; (y;\infty) \cap \bigcup\limits_{l<k} J_{n_l})} < \frac{(y_{n_k} - \delta_{n_k}) - z_{n_k}}{2\delta_{n_k}} < \frac{|(z_{n_k}; y_{n_k} - \delta_{n_k})|}{\operatorname{dist}(y; (z_{n_k}; y_{n_k} - \delta_{n_k}))}.$$

Therefore

$$\left| \int\limits_{z_{n_k}}^{y_{n_k}-\delta_{n_k}} \frac{dx}{x-y} \right| > \left| \int\limits_{(y;\infty)\cap\bigcup\limits_{l<k} J_{n_l}} \frac{dx}{x-y} \right|. \tag{4.7}$$

Since $y_n - z_n < |I_n|^2$,

$$\int\limits_{(y;\infty)\cap\bigcup\limits_{n\in\mathbb{N}}(z_n;y_n)} \frac{dx}{x-y} < C < +\infty$$

and by (4.7)

$$\left| \int_{z_{n_k}}^{y_{n_k}-\delta_{n_k}} \frac{dx}{x-y} \right| > \left| \int_{(y;\infty)\cap\left[\bigcup_{n\in\mathbb{N}}(z_n;y_n)\cup\bigcup_{l<k}J_{n_l}\right]} \frac{dx}{x-y} \right| - C. \tag{4.8}$$

Since $\delta_{n_l} < \frac{1}{4}(y_{n_l} - z_{n_l})$ for all l and $|I_{n_1}| \geq |I_{n_2}| \geq ...$, we have that $J_{n_l} \subset (y-\epsilon_k; y+\epsilon_k)$ for all $l \geq k$. Thus

$$\left\{\frac{\chi_E(x)}{x-y} > 0\right\} \setminus (y-\epsilon_k; y+\epsilon_k) = (y;\infty) \cap \left[\bigcup_{n\in\mathbb{N}}(z_n;y_n) \cup \bigcup_{l<k} J_{n_l}\right].$$

Since

$$\frac{\chi_E(x)}{x-y} < 0$$

on $(z_{n_k}; y_{n_k} - \delta_{n_k})$, (4.8) implies $C_k < C$ for any $k \in \mathbb{N}$. Hence

$$\liminf_{\epsilon\to 0} \int_{(-1;1)\setminus(-\epsilon;\epsilon)} \chi_E(x)\frac{dx}{x} < +\infty$$

and condition (ii) is satisfied. ▲

We will need the following

Definition. We will say that two disjoint sets of real numbers $\mathcal{A}$ and $\mathcal{B}$ are well-mixed if they satisfy the following conditions:

1) for any two points $x, y \in \mathcal{A}$ each of the sets $(x;y)$ and $\mathbb{R} \setminus [x;y]$ contains at least one point from $\mathcal{B}$.

2) for any two points $x, y \in \mathcal{B}$ each of the sets $(x;y)$ and $\mathbb{R} \setminus [x;y]$ contains at least one point from $\mathcal{A}$.

Proof of Theorem 4.1. 2) $\Rightarrow$ 1). As follows from the Weyl-von Neumann theorem on the stability of spectra, the essential spectrum is stable under rank one perturbations.

1) $\Rightarrow$ 2). Suppose $\sigma(A)$ is uncountable. We will show that there exists a self-adjoint rank one perturbation with nontrivial continuous part.

If A is not cyclic then we can always choose a cyclic subspace such that the restriction of A on this subspace has an uncountable spectrum. Indeed, A is unitarily equivalent to the multiplication by z in a direct integral of Hilbert spaces H_ξ

$$H = \int \oplus H_\xi d\mu(\xi)$$

where μ is a scalar measure with uncountable support. If we choose a one-dimensional subspace l_ξ in each of these Hilbert spaces and consider the direct integral

$$L = \int \oplus l_\xi d\mu(\xi)$$

then the restriction of multiplication by z on L will be cyclic with the spectral measure μ. Since $\operatorname{supp}\mu$ is uncountable, the spectrum of such a restriction will be uncountable.

Notice that if some rank one perturbation of this restriction has a nontrivial continuous part then the corresponding rank one perturbation of the whole operator also has a nontrivial continuous part. Therefore, WLOG we can assume that A is cyclic.

Denote by $\mathcal{A}$ the set of all eigenvalues of A. WLOG $\mathcal{A} \subset [0;1]$. Then there exists a closed uncountable set $F \subset (0;1), m(F) = 0$ such that for any $x \in F$ and any $\epsilon > 0$ both sets $(x-\epsilon; x) \cap \mathcal{A}$ and $(x; x+\epsilon) \cap \mathcal{A}$ are nonempty.

Let $I_1 = (x_1; y_1), I_2 = (x_2; y_2), \ldots$ be disjoint open intervals such that $F = (0;1) \setminus \bigcup I_n$. We always can assume that F contains no isolated points i.e. $x_i \neq y_j$ for any i, j.

First, using Lemmas 4.2 and 4.3 we will construct a phase shift u_0 of a pair $(\mu_0; \nu_0)$ such that $\mu_0 = \sum c_n \delta_{a_n}$ for some $a_n \in \mathcal{A}, c_n > 0$ and ν_0 is continuous with $\operatorname{supp}\nu_0 \subset F$.

By Lemma 4.2 we can choose two sets of integers L and M satisfying (4.1) and (4.2). For each $n \in \mathbb{N}$ choose $z_n \in I_n \cap \mathcal{A}$ such that $y_n - z_n < |I_n|^2$. By Lemma 4.3 we can choose $N \subset M$ satisfying (4.3) and (4.4). Denote $O = \mathbb{N} \setminus N$. For each $k \in O$ put $a_k = z_k$. For each $k \in N$ choose $a_k \in I_k \cap \mathcal{A}$ such that $a_k - x_k < |I_k|^2$. After that put $u_0 = \pi$ on $\bigcup (a_k; y_k)$ and $u_0 = 0$ elsewhere. Then u_0 has compact support.

Consider the pair of measures $(\mu_0; \nu_0)$ such that u_0 is its phase shift. By Lemma 3.2 μ_0 and ν_0 are singular. To prove that $\mu_0 = \sum a_n \delta_{a_n}$ we need to show that $\mu_0(F) = 0$.

To show that $\mu_0(F) = 0$ let us notice that if $y \in F$ then

$$0 \geq \int_{(y-1;y)\cap \bigcup_{k\in N}(x_k;a_k)} \frac{dx}{x-y} \geq -C_1 > -\infty$$

because $a_k - x_k < |I_k|^2$ for any $k \in N$. Thus

$$\int_{\{u_0=\pi\}} \frac{dx}{x-y} < \int_{\bigcup_{k\in\mathbb{N}}(z_k;y_k)\cup\bigcup_{k\in N} I_k} \frac{dx}{x-y} + C_1. \tag{4.9}$$

If $y \neq y_i, x_i$ for any $i \in \mathbb{N}$ then (4.9) and Lemma 4.3 imply that there exist positive $\epsilon_1, \epsilon_2, \ldots$ such that $\epsilon_n \to 0$ as $n \to \infty$ and

$$\int_{\{u_0=\pi\}\setminus(y-\epsilon_n;y+\epsilon_n)} \frac{dx}{x-y} < C_2 < \infty.$$

for any n. Hence,

$$p.\ v. \int_{\mathbb{R}} \frac{u_0(y+x)dx}{x} < \infty.$$

Therefore by Lemma 3.3 $\mu_0(F\setminus\{x_1, y_1, x_2, y_2, ...\}) = 0$. Also, $\mu_0(x_k) = \mu_0(y_k) = 0$ for all k because $u_0(x) \to 0$ as $x \to y_k+$ and $u_0(x) \to \pi$ as $x \to x_k-$ which means that condition (3.8) is not satisfied. Hence, $\mu_0 = \sum c_n\delta_{a_n}$ for some positive c_n.

For ν_0 we obviously have $\operatorname{supp}\nu_0 \subset F$. Let $y \in F, y = y_i$ for some $i > 0$. Then

$$\int_{(y;y+1)\cap \bigcup_{k\in N} I_k} \frac{dx}{x-y} = \infty$$

because N satisfies (4.3). Since for $k \in N$ we have $a_k - x_k < (y_k - x_k)^2$, this implies that

$$\int_{(y;y+1)\cap \bigcup_{k\in N} (a_k;y_k)} \frac{dx}{x-y} = \infty.$$

Since on $(a_k; y_k)$ $u_0 = \pi$, condition (3.9) is not satisfied. Thus $\nu_0(y) = 0$. If $y \in F$ and $y \neq y_i$ for any i, then

$$\int_{(y-1;y)\cap \bigcup_{k\in O} I_k} \frac{dx}{y-x} = \infty$$

by (4.2) because $M \subset O$. Since for $k \in O$ we have $y_k - a_k < (y_k - x_k)^2$, this implies that

$$\int_{(y-1;y)\cap \bigcup_{k\in O} (x_k;a_k)} \frac{dx}{y-x} = \infty.$$

Again, since $u_0 = 0$ on $(x_k; a_k)$, condition (3.9) does not hold and $\nu_0(y) = 0$. Hence, ν_0 is continuous.

Our next goal is to transform u_0 into the phase shift u of a pair $(\mu; \nu)$ such that μ is a spectral measure of A and ν has a nontrivial continuous part (note that μ_0 does not have point masses at some points of $\mathcal{A}$, therefore it is not a spectral measure of A).

Let $\{b_n\}_{n=1}^\infty$ be some enumeration of the set $\mathcal{A} \setminus \{a_n\}_{n=1}^\infty$. For each b_n let us choose c_n such that if $b_n \in I_k$ then $c_n \in I_k$,

$$|b_n - c_n| < \frac{\operatorname{dist}(b_n; (\mathbb{R} \setminus I_k) \cup \{b_1, b_2, ..., b_{n-1}\})}{2^{n+1}} \tag{4.10}$$

and for each $n \in \mathbb{N}$ the sets $\mathcal{B}_n = \{a_i\}_{i=1}^\infty \cup \{b_1, b_2, ..., b_n\}$ and $\mathcal{C}_n = \{c_1, c_2, ..., c_n\} \cup F$ are well-mixed.

For each $k \in \mathbb{N}$ define the function u_k in the following way:

1) $|u_k| = 0$ or π everywhere on $\mathbb{R}$;

2) u_k is continuous everywhere except $\mathcal{B}_k \cup \mathcal{C}_k$;

3) u_k jumps from 0 to π at each point of $\mathcal{B}_k$ and from π to 0 at each point of $\{c_1, c_2, ..., c_k\}$.

Let u_k be the phase shift of a pair $(\mu_k;\nu_k)$. Then

$$\mu_k=\sum_{n=1}^{\infty}\alpha_n^k\delta_{a_n}+\sum_{n=1}^{k}\beta_n^k\delta_{b_n}$$

for some positive constants α_i^k and β_i^k and

$$\nu_k=\sum_{n=1}^{k}\gamma_n^k\delta_{c_n}+f_k\nu_0$$

for some positive constants γ_1^k and some positive function $f_k\in L^1(\nu_0)$. Condition (4.10) implies that the sequence $\{u_k\}$ converges in measure to some function u. Since $\operatorname{supp} u_k\subset[-1;2]$ for any $k\in\mathbb{N}$, u has compact support. Let u be the phase shift of a pair $(\mu;\nu)$. Then $\mu_k\to\mu$ and $\nu_k\to\nu$ in the $*$-weak topology.

Since by (4.10)

$$\int_{\mathbb{R}}\left|\frac{u_k(x)-u_{k+1}(x)}{x-y}dx\right|<\frac{1}{2^{k+1}} \tag{4.11}$$

for any $y\in\mathcal{B}_k$, by part ii) of Lemma 3.7

$$\mu_{k+1}=g_k\mu_k+\beta_{k+1}^{k+1}\delta_{b_{k+1}}$$

where $g_k\in L^\infty(\mu_k)$,

$$1-\frac{1}{2^k}<g_k<1+\frac{1}{2^k} \tag{4.12}$$

μ_k-a. e. Also by (4.11) and part ii) of Lemma 3.7

$$1-\frac{1}{2^k}<\frac{f_k}{f_{k+1}}<1+\frac{1}{2^k} \tag{4.13}$$

ν_0-a. e.

Since

$$||\mu_k||=\operatorname{Im}\exp\left[\mathcal{K}u_k(i)\right]\to\operatorname{Im}\exp\left[\mathcal{K}u(i)\right],$$

$||\mu_k||\to||\mu||$ and $\beta_{k+1}^{k+1}\to 0$ as $k\to\infty$. Since each μ_k is pure point, that implies, together with (4.12), that μ is pure point and that

$$\mu=\sum_{n=1}^{\infty}\alpha_n\delta_{a_n}+\sum_{n=1}^{\infty}\beta_n\delta_{b_n}$$

where

$$0<\alpha_i^1\prod_{k\in\mathbb{N}}g_k(a_i)=\alpha_i<\infty$$

and

$$0<\beta_i^i\prod_{k>i}g_k(b_i)=\beta_i<\infty.$$

Thus A is unitarily equivalent to A_μ. Also, since $\nu=\eta+\sigma$ where η is some positive measure and σ is a $*$-weak limit of the sequence $\{f_k\nu_0\}$, (4.13) implies that $\sigma=f\nu_0$ for some $f\in L^1(\nu_0)$,

$$0<f_1(x)\prod_{k>1}f_k(x)=f(x)<3f_1(x)$$

ν_0-a. e. Thus $A_\mu+(\cdot,1)1$ has a nontrivial continuous part. ▲

Corollary 4.4. *Let A be a self-adjoint operator. Then all trace class perturbations of A are pure point iff $\sigma(A)$ is countable.*

Proof.
The "if" part follows from the stability of the essential spectrum under trace class perturbations.
The "only if" part follows from Theorem 4.1.
The statement also follows from the Theorem of Carey and Pincus [C-P] on the equivalence modulo the trace class. ▲

5. The problem of two spectra

Definition. We will say that two operators A and B are equivalent up to a rank one perturbation if there exist operators A' and B' acting in the same space such that A is unitarily equivalent to A', B is unitarily equivalent to B' and $\operatorname{rank}(A' - B') = 1$.

In this section we will give a partial answer to the following question.

The problem of two spectra. Let μ and ν be two finite Borel measures on $\mathbb{R}$. When do there exist two cyclic self-adjoint operators A and B equivalent up to a rank one perturbation such that μ and ν are spectral measures of A and B respectively?

Definition. If such A and B exist, we will say that μ and ν solve the problem of two spectra (PTS).

We will say that μ is equivalent to ν and write $\mu \sim \nu$ if there exists $f \in L^1(\mu)$ such that $f > 0 \quad \mu$-a. e. and $f\mu = \nu$. We will say that pairs of measures (μ, ν) and (μ', ν') are equivalent, $(\mu, \nu) \sim (\mu', \nu')$, if $\mu \sim \mu'$ and $\nu \sim \nu'$.
In terms of the families M_ϕ, we can say that μ and ν solve PTS if and only if there exists ϕ such that $(\mu, \nu) \sim (\mu', \nu')$ for some $\mu', \nu' \in \mathcal{M}_\phi$. In terms of the phase shift, μ and ν solve PTS if and only if there is an equivalent pair possessing a phase shift i. e. satisfying (3.3) for some real function $u \in L^\infty(\mathbb{R})$, $||u||_\infty \leq \pi$ and $\lambda \in \mathbb{R}$.
In this section we will discuss the case of pure point measures μ and ν.
The first result in this direction is the following theorem, proved by Gelfand and Levitan.

Theorem 5.1. *Let $\mathcal{A} = \{a_n\}_{n=1}^\infty$ and $\mathcal{B} = \{b_n\}_{n=1}^\infty$ be two disjoint sequences of real numbers, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c \in \mathbb{R}$. There exist unitarily equivalent up to a rank one perturbation cyclic self-adjoint operators A and B such that $\sigma(A) = \operatorname{Clos} \mathcal{A}$ and $\sigma(B) = \operatorname{Clos} \mathcal{B}$ if and only if the sets $\{a_1, a_2, ...\}$ and $\{b_1, b_2, ...\}$ are well-mixed.*
If such A and B exist then they are unique up to a unitary equivalence.

Proof. Define function u on $\mathbb{R}$ to satisfy the following conditions:
1) u is continuous and $|u|$ is equal to π or 0 everywhere on $\mathbb{R} \setminus (\{a_n\} \cup \{b_n\})$,
2) at the points $\{a_n\}$ u jumps from 0 to π,
3) at the points $\{b_n\}$ u jumps from π to 0.
WLOG $\inf_{n\in\mathbb{N}} a_n \leq \inf_{n\in\mathbb{N}}$ and $\sup_{n\in\mathbb{N}} a_n \leq \sup_{n\in\mathbb{N}} b_n$. Then u has compact support. As we discussed before, there are unique measures $\mu, \nu \in \mathcal{M}_+(\mathbb{R})$ such that u, μ and ν satisfy formula (3.3) with $\lambda = 1$. Put $A = A_\mu$ and $B = A_\nu$.

Then B is unitarily equivalent to $A + (\cdot, 1)1$. By Lemma 3.3 i) $\operatorname{supp}\mu \subset \operatorname{Clos}\mathcal{A}$ and $\operatorname{supp}\nu \subset \operatorname{Clos}\mathcal{B}$. Also by Lemma 3.8 the measures μ and ν have point masses at points a_i and b_i respectively, therefore $\operatorname{supp}\mu = \operatorname{Clos}\mathcal{A}$ and $\operatorname{supp}\nu = \operatorname{Clos}\mathcal{B}$. Thus the operators A and B satisfy the condition of the Theorem. Since the sequences are well-mixed and have only one cluster point there exists a unique function u satisfying properties 1)-3). Therefore A and B are unique up to a unitary equivalence. ▲

Remark. If $\mu = \sum \alpha_n \delta_{a_n}$ and $\nu = \sum \beta_n \delta_{b_n}$, where $\alpha_i, \beta_i > 0$, solve PTS then the sets $\mathcal{A} = \{a_1, a_2, ...\}$ and $\mathcal{B} = \{b_1, b_2, ...\}$ are well-mixed. To prove it one can, for instance, notice that if $\mathcal{A}$ and $\mathcal{B}$ are not well-mixed then the function u satisfying properties 1)-3) from the above proof does not exist.

Note that Theorem 5.1 provides us with operators such that μ and ν are only *absolutely continuous* with respect to spectral measures of A and $A + (\cdot, \phi)\phi$ but not necessarily *equivalent* to them. Therefore it does not imply that μ and ν solve PTS. As shown in Example 5.2 below, the condition that $\mathcal{A}$ and $\mathcal{B}$ are well-mixed is not sufficient for μ and ν to solve PTS.

Example 5.2. *Put $a_n = (-1)^n/2^n$ for $n = 1, 2, 3, ...$, $b_1 = -1$, $b_n = a_{n-1} + (-1)^n/4^n$ for $n = 2, 3, ...$ Then the sets $\{a_1, a_2, ...\}$ and $\{b_1, b_2, ...\}$ are well-mixed.*

Let μ and ν be defined as in the last remark. Suppose that μ and ν solve PTS. Then there exist a pair $(\mu, \nu) \sim (\mu', \nu')$ possessing a phase shift u. Such u must be continuous on $\mathbb{R} \setminus (\{a_n\} \cup \{b_n\})$, jump from 0 to π at any a_n and jump from π to 0 at any b_n. Thus $u = 0$ on $(1/2^{2n} - 1/4^{2n+1}; 1/2^{2n})$ and on $(-1/2^{2n-1} + 1/4^{2n}; -1/2^{2n+1})$ for $n = 1, 2, ...$ and $u = \pi$ on the rest of $\mathbb{R}$.

But then Lemma 3.8 implies that $\mu(0) > 0$. Since $0 \neq a_n$ for any $n \in \mathbb{N}$ we have a contradiction.

Hence in Theorem 5.1 one can not prescribe arbitrarily whether or not spectral measures of A and B have a point mass at c.

If the sequences $\{a_1, a_2, ...\}$ and $\{b_1, b_2, ...\}$ are finite, disjoint and well-mixed then μ and ν obviously solve PTS. More interesting example is provided by the following theorem.

Theorem 5.3 ([A3]). *Let $\mathcal{A}$ and $\mathcal{B}$ be two disjoint countable sets on the unit circle $\mathbb{T}$. Suppose that $\operatorname{Clos}\mathcal{A} = \operatorname{Clos}\mathcal{B} = \mathbb{T}$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be some enumerations of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then there exist sequences of positive real numbers $\{\alpha_n\}$ and $\{\beta_n\}$ such that the measures $\sum \alpha_n \delta_{a_n}$ and $\sum \beta_n \delta_{b_n}$ belong to the same family M_φ*

In terms of operator theory Theorem 5.3 means that any two cyclic pure point self-adjoint operators whose spectrum is equal to $\mathbb{R}$ are unitarily equivalent up a rank one perturbation. Note that the condition $\operatorname{Clos}\mathcal{A} = \operatorname{Clos}\mathcal{B} = \mathbb{T}$ automatically implies that $\mathcal{A}$ and $\mathcal{B}$ are well-mixed. However the following example shows that in general the condition $\operatorname{Clos}\mathcal{A} = \operatorname{Clos}\mathcal{B}$ does not imply that that the corresponding measures solve PTS even if $\mathcal{A}$ and $\mathcal{B}$ are well-mixed.

Example 5.4. *Let C be the standard Cantor null set on the unit interval $[0; 1]$, $C = [0; 1] \setminus \bigcup I_n$ where $I_n = (x_n; y_n)$ are disjoint open intervals. Let $\mathcal{A}$ and $\mathcal{B}$ be* *two disjoint well-mixed countable sets of points of C such that*

$$0, x_1, x_2, .. \in \mathcal{A}, 1 \in \mathcal{B}. \tag{5.1}$$

Then $\operatorname{Clos}\mathcal{A} = \operatorname{Clos}\mathcal{B} = C$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be some enumerations of $\mathcal{A}$ and $\mathcal{B}$ respectively. Define $\mu = \sum \frac{1}{2^n}\delta_{a_n}$ and $\nu = \sum \frac{1}{2^n}\delta_{b_n}$. Suppose that μ and ν solve PTS. Then there exists the phase shift u of the pair $(\mu';\nu')$ where $(\mu,\nu) \sim (\mu',\nu')$. Function u must be constant on each I_n. Condition (5.1) implies that $u = 0$ a. e. on $\mathbb{R}$. This means that μ' and ν' are trivial and we obtain a contradiction.

It turns out that this example is, in a certain sense, typical: the following Theorem shows that such μ and ν solve PTS for any two well-mixed dense in K sequences if and only if the set K is not too "porous".

Theorem 5.5. *Let $K \subset \mathbb{R}$ be a closed set. Denote by $I_1, I_2, ...$ the disjoint open intervals $I_k = (x_k; y_k)$ such that $K = \mathbb{R} \setminus \bigcup I_n$.*

Then the following two conditions are equivalent:

i)Any two cyclic self-adjoint pure point operators A and B such that the sets of their eigenvalues are well-mixed and $\sigma(A) = \sigma(B) = K$ are unitarily equivalent up to a rank one perturbation.

ii) If $y \in K \setminus \{x_1, y_1, x_2, y_2, ...\}$ then

$$\int\limits_{\bigcup_{n\in\mathbb{N}} I_n} \frac{dx}{|y-x|} < \infty; \tag{5.2}$$

if $y = x_k$ or $y = y_k$ for some $k \in \mathbb{N}$ then

$$\int\limits_{\bigcup_{n\in\mathbb{N}, n\neq k} I_n} \frac{dx}{|y-x|} < \infty. \tag{5.3}$$

Example 5.6. Condition ii) fails for $K = C$ for any standard Cantor set C (even if $|C| > 0$). Condition ii) obviously holds if ∂K is a finite set. We will show that there exist more complicated examples of such K, namely that for any nowhere dense set $L \subset \mathbb{R}$ there exists a set K such that K satisfies ii) and $\partial K \supset L$.

Denote by $J_i = (x_i; y_i)$ the disjoint open intervals such that $\operatorname{Clos} L = \mathbb{R} \setminus \bigcup J_n$. On each J_i choose a sequence of intervals $\{\Delta_i^k\}_{-\infty}^{\infty}$, $\Delta_i^k = (x_i^k; y_i^k)$ such that $x_i^k \to x_i$ as $k \to -\infty$, $x_i^k \to y_i$ as $k \to \infty$ and

$$\int\limits_{\cup_k \Delta_i^k} \frac{dx}{|x-y|} < \frac{1}{2^i}$$

for all $y \in \mathbb{R} \setminus J_i$. Then we can put $K = \mathbb{R} \setminus \bigcup_{i,k} \Delta_i^k$.

Similarly, one can show that for any closed $L \subset \mathbb{R}$ there exists a closed $K \supset L$ satisfying ii) and such that $\partial K \supset \partial L$.

Proof of Theorem 5.5. ii)$\Rightarrow$i) WLOG $K \subset (0;1)$. Let $\mathcal{A}$ and $\mathcal{B}$ be the sets of eigenvalues of A and B respectively. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be some enumerations of $\mathcal{A}$ and $\mathcal{B}$ respectively.

a) Let us first consider the case when the endpoints of the intervals I_k belong to neither $\mathcal{A}$ nor $\mathcal{B}$.

Let us find reenumerations $\{a_{n_k}\}$ and $\{b_{m_k}\}$ of the sequences $\mathcal{A} = \{a_n\}$ and $\mathcal{B} = \{b_m\}$ such that for any $k \in \mathbb{N}$

1) the sets $\{a_{n_1}, ..., a_{n_k}\}$ and $\{b_{m_1}, ..., b_{m_k}\}$ are well-mixed and

2) the distance $|a_{n_k} - b_{m_k}|$ is at least 2^{k+1} times less than the distance between the a_{n_k} and any of the points $a_{n_1}, ..., a_{n_{k-1}}, b_{m_1}, ..., b_{m_{k-1}}$.

To find such reenumerations we can act in the following way.

WLOG $a_1 > b_1$. Choose $n_1 = 1$ and $m_1 = 1$. After that for $k = 2, 3, ...$ consequently do the following. Choose a point x in $\mathcal{A} \cup \mathcal{B} \setminus \{a_{n_1}, ..., a_{n_{k-1}}, b_{m_1}, ..., b_{m_{k-1}}\}$ with minimal index in the initial enumeration. Suppose $x \in \mathcal{A}$. Then put $a_{n_k} = x$ and choose $b_{m_k} \in \mathcal{B}$ to satisfy conditions 1) and 2) above. Similarly, if $x \in \mathcal{B}$ then put $b_{m_k} = x$ and choose $a_{n_k} \in \mathcal{A}$ to satisfy conditions 1) and 2) above.

Note that it is always possible to choose b_{m_k} or a_{n_k} to satisfy 2) because $a_{n_k}, b_{m_k} \in K \setminus \{x_1, y_1, x_2, y_2, ...\}$.

Now for each k define a function u_k on $\mathbb{R}$ in such a way that $|u_k| \in \{0, \pi\}$ on $\mathbb{R}$, u_k is continuous everywhere except $a_{n_1}, a_{n_2}, ...a_{n_k}, b_{n_1}, b_{n_2}, ..., b_{n_k}$, u_k jumps from 0 to π at each $a_{n_i}, 0 < i \le k$ and u_k jumps from π to 0 at each $b_{n_i}, 0 < i \le k$ (this construction is possible because the sequences are well-mixed). Since $a_1 > b_1$, u_k has a compact support. Therefore u_k is the phase shift of a pair of measures $(\mu_k; \nu_k)$. It is easy to see that $\mu_k = \sum_{i=1}^{k} \alpha_i^k \delta_{a_{n_i}}$ and $\nu_k = \sum_{i=1}^{k} \beta_i^k \delta_{b_{m_i}}$ for some positive constants $\alpha_1^k, ..., \alpha_k^k, \beta_1^k, ..., \beta_k^k$.

Condition 2) implies that for any $l > k$ $|\{u_k \ne u_l\}| \le \sum_{i=k}^{l} 1/2^i$. Thus the sequence $\{u_k\}$ converges in measure to some function u. That implies that the sequences of measures $\{\mu_k\}$ and $\{\nu_k\}$ converge in the $*$-week topology to some measures μ and ν (respectively) such that u is the phase shift of the pair $(\mu; \nu)$. To complete the proof we need to show that $\mu = \sum \alpha_n \delta_{a_n}$ and $\nu = \sum \beta_n \delta_{b_n}$ for some positive constants α_i and β_i.

Since

$$||\mu_k|| = \operatorname{Im} \exp\left[\mathcal{K} u_k(i)\right] \to \operatorname{Im} \exp\left[\mathcal{K} u(i)\right],$$

we have that

$$||\mu_k|| - ||\mu_l|| \to 0$$

as $k, l \to \infty$. Thus

$$\sum_{i=k}^{l} \alpha_l^i \le |\ ||\mu_k|| - ||\mu_l||\ | + \sum_{i=1}^{k} |\alpha_k^i - \alpha_l^i| \le |||\mu_k|| - ||\mu_l||| + k/2^{k-1} \to 0 \tag{5.4}$$

as $k, l \to \infty$. But condition 2) implies that

$$\int_{\{u_k \ne u_{k+1}\}} \frac{dx}{|x - a_i|} < \frac{1}{2^k}$$

for any k and any $i \leq k$. Hence, by part ii) of Lemma 3.7

$$|\alpha_i^k/\alpha_i^{k+1} - 1| < 1/2^{k-1} \tag{5.5}$$

for any k and any $i \leq k$. Since $\operatorname{supp} u_k \subset (0;1)$, $\alpha_i^{k+1} \leq ||\mu_k|| \leq 1$. Hence by (5.5)

$$|\alpha_i^k - \alpha_i^{k+1}| < 1/2^{k-1}. \tag{5.6}$$

Together (5.4) and (5.6) imply that

$$||\mu_k - \mu_l|| \to 0$$

as $k, l \to \infty$. Put $\alpha_i = \lim_{k\to\infty} \alpha_i^k$, and $\mu' = \sum_{i=1}^\infty \alpha_i \delta_{a_{n_i}}$. Since the sequence $\{\mu_k\}$ converges in norm, it must converge to μ'. But since μ is the $*$-weak limit of $\{\mu_k\}$, $\mu = \mu' = \sum_{i=1}^\infty \alpha_i \delta_{a_{n_i}}$. It is left to notice that (5.5) implies that

$$\alpha_i = \alpha_i^i \prod_{l>i} \frac{\alpha_i^l}{\alpha_i^{l+1}} > 0$$

for each i. In the same way we can prove that $\nu = \sum \beta_i \delta_{b_i}$ for some positive β_i.

b) In the general case, denote $N = \{n | \partial I_n \cap K \neq \emptyset\}$. Consider disjoint countable sets $\mathcal{A}' = \mathcal{A} \cup \mathcal{C}$ and $\mathcal{B}' = \mathcal{B} \cup \mathcal{D}$ where $\mathcal{C} = \{c_n\} \subset [\cup_{n\in N} I_n] \cap (0;1)$ and $\mathcal{D} = \{d_n\} \subset [\cup_{n\in N} I_n] \cap (0;1)$ satisfy:

1) $\mathcal{C}, \mathcal{D}$ are dense in $[\cup_{n\in N} I_n] \cap (0;1)$ and

2) if for some $n \in N$ one of the endpoints x of the interval I_n belongs to $\mathcal{A}$ ($\mathcal{B}$) but the other endpoint y of I_n is not in $\mathcal{A} \cap \mathcal{B}$ then $y \in \mathcal{A}'$ ($\mathcal{B}'$) (i. e. both x and y are in the same set $\mathcal{A}'$ or $\mathcal{B}'$).

(Notice that since $\operatorname{Clos} \mathcal{A} = \operatorname{Clos} \mathcal{B} = K$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$, K does not have isolated points. Hence $x_k \neq y_l$ for any $k, l \in \mathbb{N}$.)

Then the set $\mathcal{A}' \cup \mathcal{B}'$ contains all the endpoints of the intervals I_n, $n \in \mathbb{N}$.

Then $K' = K \cup \bigcup_{n\in N} I_n$, $\mathcal{A}'$ and $\mathcal{B}'$ satisfy the conditions of part a) and we can construct a phase shift v corresponding to measures $\mu' = \sum \alpha_i \delta_{a_i} + \sum \sigma_i \delta_{c_i}$ and $\nu' = \sum \beta_i \delta_{b_i} + \sum \eta_i \delta_{d_i}$ for some positive $\alpha_i, \sigma_i, \beta_i, \eta_i$. Define $N_\pi = \{n | x_n \in \mathcal{A} \text{ or } y_n \in \mathcal{B}\}$ and $N_0 = \{n | x_n \in \mathcal{B} \text{ or } y_n \in \mathcal{A}\}$. Then $N_\pi \cap N_0 = \emptyset$ because the sets $\mathcal{A}$ and $\mathcal{B}$ are well-mixed and $N = N_\pi \cup N_0$. Let function u be such that $u = v$ outside $\cup_{n\in N} I_n$, $u = \pi$ on each $I_n, n \in N_\pi$ and $u = 0$ on each $I_n, n \in N_0$. Let μ and ν be the measures such that u is the phase shift of the pair $(\mu;\nu)$. Then ii) together with Lemma 3.7 imply that the restriction of μ on $K \setminus \{x_1, y_1, x_2, y_2, ...\}$ is absolutely continuous with respect to μ'. Also, obviously $\mu(\cup_{n\in N} I_n) = 0$ because u is constant on each I_n, $n \in N$. It is left to check the endpoints of the intervals $I_n, n \in N$.

If for some $n \in N_\pi$ $x_n \in \mathcal{A}$, then μ' has a point mass at x_n. Thus v satisfies the condition (3.8) at the point $y = x_n$. Since $u = \pi$ on I_n, by (3.8) and ii) we have

$$\int\limits_{\{u\neq v\}} \frac{dx}{|x - x_n|} < \infty.$$

Hence by (3.8) μ also has a point mass at x_n.

If for some $n \in N_\pi$ x_n belongs to neither $\mathcal{A}$ nor $\mathcal{B}$ then ν' has a point mass at x_n. Thus v satisfies (3.9) at $y = x_n$. Since $u = \pi$ on I_n, by (3.8) and ii) we have that

$$\int\limits_{\{u\neq\pi\}} \frac{dx}{|x - x_n|} < \infty.$$

Thus by (3.8) and (3.9) $\mu(x_n) = 0$ and $\nu(x_n) = 0$.

Other endpoints of I_n, $n \in N$ can be checked in the same way.

Thus $\mu = \sum \alpha_n \delta_{a_n}$ for some positive α_n. Similarly we can show that $\nu = \sum \beta_n \delta_{b_n}$ for some positive β_n.

i)⇒ii)

Suppose condition ii) is not satisfied at some $y \in K$. Let $\mathcal{A}$ and $\mathcal{B}$ be disjoint, dense in K and well-mixed sequences such that $y \in \mathcal{B}$ and $y_i \in \mathcal{A}$ for all i such that $y_i \neq \infty$. Then the phase shift u of the pair $(\mu; \nu)$ is equal to 0 on $\cup I_n$. Thus condition (3.9) is not satisfied at y and $\nu(y) = 0$. ▲

Here is one more way to avoid the situation of Example 5.4.

Theorem 5.6. *Let $K \subset \mathbb{R}$ be a closed set. Denote by $I_1 = (x_1; y_1), I_2 = (x_2; y_2), ...$ disjoint open intervals such that $K = \mathbb{R} \setminus \bigcup I_n$. Let A and B be two self-adjoint cyclic pure point operators, $\mathcal{A}$ and $\mathcal{B}$ be the sets of all eigenvalues of A and B respectively. Suppose that $\sigma(A) = \sigma(B) = K$ and $\mathcal{A} \cap \{x_1, y_1, x_2, y_2, ...\} = \mathcal{B} \cap \{x_1, y_1, x_2, y_2, ...\} = \emptyset$. Then A and B are equivalent up to a rank one perturbation.*

Proof. Notice that in part a) of the implication ii)⇒i) in the previous proof we did not use condition ii). ▲

In the rest of this Section we are going to discuss the following question.

Remark. Let A be a singular self-adjoint cyclic operator, φ and ψ its noncollinear cyclic vectors ($\varphi \neq c\psi$). Is it possible that operators $A^\varphi = A + (\cdot, \varphi)\varphi$ and $A^\psi = A + (\cdot, \psi)\psi$ are unitarily equivalent?

If A is a finite rank operator, then the answer is negative. Indeed, denote by μ and ν the spectral measures of φ for A and A^φ respectively; denote by μ' and ν' the spectral measures of ψ for A and A^ψ respectively. If μ and ν are linear combinations of point masses at points $a_1, a_2, .., a_n$ and $b_1, b_2, .., b_n$ respectively, $a_1 < b_1 < a_2 < b_2 < ... < a_n < b_n$, then the phase shift u of the pair $(\mu; \nu)$ depends only on the sequences $\{a_n\}$ and $\{b_n\}$: $u = 0$ a. e. on $(-\infty; a_1) \cup (b_1; a_2) \cup ... \cup (b_{n-1}; a_n) \cup (b_n; \infty)$ and $u = \pi$ elsewhere. If A^φ is unitarily equivalent to A^ψ then $\nu \sim \nu'$. Also we have that $\mu \sim \mu'$. Thus measures μ' and ν' have point masses at the same points $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$. Hence the phase shift u' of the pair (μ', ν') must be equal to u a. e. on $\mathbb{R}$. Since the phase shift determines the pair of measures uniquely, we have $\mu = \mu'$ and $\nu = \nu'$. Therefore, $\varphi = c\psi$ and we have a contradiction. In the same way one can prove that if all eigenvalues of A are isolated, then any two different (corresponding to noncollinear vectors) rank one perturbations of A can not be unitarily equivalent.

However if $\sigma(A) = \sigma_s(A)$ is more complicated there may exist two different rank one perturbations of A which are unitarily equivalent.

To show that, it is enough to give an example of two equivalent pairs of singular measures (μ,ν) and (μ',ν') with different phase shifts. Indeed, if such pairs exist then $A_\mu+(\cdot,1)1$ and $A_\mu+(\cdot,\sqrt{f})\sqrt{f}$, where $f\in L^1(\mu)$, $f\geq 0$, $\mu'=f\mu$, are unitarily equivalent and $f\neq const$.

To construct such an example one can notice, for instance, that function u in the proof of Theorem 5.5 is not unique. Indeed, the reenumeration of the sequences $\mathcal{A}$ and $\mathcal{B}$ in part a) of the proof of the implication i)$\Rightarrow$ ii) can be done in many different ways. It can be shown that some of the enumerations will give us different phase shifts u at the end of our construction. For instance, let $n_k, m_k\in\mathbb{N}$ be such as in part a) of the proof of the implication i)$\Rightarrow$ ii). Then we can choose $n'_k, m'_k\in\mathbb{N}$ to satisfy the conditions analogous to 1) and 2) and such that $n'_1=1, m'_k=1$ and $|a_{n'_k}-b_{m'_k}|<|a_{n_k}-b_{m_k}|/2$ for $k=2,3,\ldots$. Proceeding in the same way as in part a) of the proof of the implication i)$\Rightarrow$ ii), we can define functions u'_k and consider the limit u' of the sequence $\{u'_k\}$. Then

$$\left|\{\chi_{(a_1;b_1)}\neq u'\}\right| = |\{u'_1\neq u'\}| < |\{u_1\neq u\}|/2 = \left|\{\chi_{(a_1;b_1)}\neq u\}\right|.$$

Therefore $|\{u\neq u'\}|>0$. At the same time we again can prove that u' is the phase shift of the pair $(\sum\alpha'_i\delta_{a_i},\sum\beta'_i\delta_{b_i})$ for some positive α'_i,β'_i. Therefore u and u' are different phase shifts of equivalent pairs of measures.

In terms of the families M_φ, we can say that if $(\mu,\nu)\in\mathcal{M}_\varphi$ and $(\mu',\nu')\in\mathcal{M}_\phi$ are equivalent pairs of measures and φ is a finite Blacshke product, then φ is equal to $b\circ\phi$ for some Möbius transform b. However, if φ is more complicated then φ and ϕ can be completely different functions.

In fact, modifying the construction from the proof of Theorem 5.5 one can show that there exist infinitely many different phase shifts such that the corresponding pairs of measures are equivalent to $(\sum\alpha_i\delta_{a_i},\sum\beta_i\delta_{b_i})$. It means that if A and B are two pure point operators whose eigenvalues are dense in the same set $K\subset\mathbb{R}$ satisfying condition ii) of Theorem 5.5, then there exist infinitely many different (corresponding to noncollinear vectors) rank one perturbations of A unitarily equivalent to B. However, it is absolutely unclear if such situation is possible in the case when A or B has a nontrivial singular continuous part.

6. One example of the absence of the mixed spectrum

Let as usual A be a cyclic self-adjoint operator, ϕ its cyclic vector, $A_\lambda = A+\lambda(\cdot,\phi)\phi\quad\lambda\in\mathbb{R}$. Let μ_λ be the spectral measure of ϕ for A_λ. Let us denote by Π and Σ the sets of λ for which μ_λ has nontrivial pure point and nontrivial singular continuous part on $[0;1]$ respectively. Then the set $\Pi\cap\Sigma$ will consist of those λ for which the corresponding measures are "mixed" on the interval $[0;1]$.

The paper [D] of Donoghue gives examples in which $\Pi=\{0\}, \Sigma=\mathbb{R}\setminus\{0\}$ and $\Sigma=\{0\}, \Pi=\mathbb{R}\setminus\{0\}$. Therefore the set $\Pi\cap\Sigma$ can be empty when both Π and Σ are nonempty.

One of the natural questions which arise from the recent results on rank one perturbations (see [R-J-L-S]) is *whether the set* $\Pi\cap\Sigma$ *can be empty (or almost empty) when the sets* Π *and* Σ *are sufficiently big (topologically or in measure).*

The following example gives a partial answer to this question.

Example 6.1.

We will show that there exist a self-adjoint cyclic operator A such that for some cyclic vector ϕ the operators $A+\lambda(\cdot,\phi)\phi$ are singular continuous on $[0;1]$ for all $0\geq\lambda\geq 1$ and pure point for all other $\lambda\in\mathbb{R}$.

We will first construct the Krein spectral shift u.

Let $\{a_n\}$ be a sequence of real numbers $0<a_n<1$ monotonically decreasing to 0 and such that

$$\prod_{n=1}^{\infty}(1-a_n)=c>0 \tag{6.1}$$

and

$$1-\prod_{k=n}^{\infty}(1-a_k)\geq\frac{1}{n}. \tag{6.2}$$

Consider the Cantor set C corresponding to the sequence $\{a_n\}$: let

$$C_0=I_0^0=[0;1],\quad C_1=I_1^1\cup I_2^1,...,\quad C_n=I_1^n\cup...\cup I_{2^n}^n,...$$

where

$$I_{2k}^n\cup I_{2k-1}^n=I_k^{n-1}\setminus\Delta_k^{n-1}$$

and Δ_k^n is the open interval placed in the center of the interval I_k^n and such that $|\Delta_k^n|=a_n|I_k^n|$ and let $C=\bigcap_{n=0}^{\infty}C_n$.

The Cantor set C has the following properties:

$$|C|=c \tag{6.3}$$

and

$$\frac{|I_n^k\cap C|}{|I_n^k|}=\prod_{n}^{\infty}(1-a_n)\geq\frac{1}{n}\geq\frac{\ln 2}{-\ln|I_n^k|} \tag{6.4}$$

for any $n,k\in\mathbb{N}$.

Define $u=\pi$ on C and $u=0$ elsewhere on $\mathbb{R}$. Denote $U(z)=\mathcal{K}u(z)$ for $z\in\mathbb{C}_+$.

Claim. *U has a finite nontangential derivative $U'(x)=\lim_{z\underset{\not\to}{\longrightarrow}x}\frac{U(x)-U(z)}{x-z}$ at a point $x\in\mathbb{R}$ if and only if $x\notin C$.*

Proof. Since u is locally constant on $\mathbb{R}\setminus C$, U' obviously exists there.

Let intervals I_k^n be the same as in the construction of C above. Let $x\in C$. Let $\{I_{n_k}^k\}_{k=1}^{\infty}$ be the sequence of intervals containing x. Denote by x_k the middle of the interval $I_{n_k}^k$ and put $y_k=|x-x_k|$. Then (6.4) imply that

$$|\pi-\mathcal{P}u(x_n+iy_n)|\to 0$$

(since $\prod_n^\infty(1-a_n)\to 0$) but

$$|\pi - \mathcal{P}u(x_n+iy_n)| \geq \frac{d}{|\ln y_n|}$$

for some $d>0$. Thus $U(x)-U(z)\neq O(x-z)$ as $z \underset{\not\to}{\to} x$ and $U'(x)$ does not exist.▲

Consider a real measure ν_0 such that u is the phase shift of a pair (ν_0,ν) for some measure ν. Put $A=A_\nu, A_\lambda = A+\lambda(\cdot,1)1$. Let ν_λ be the spectral measure of 1 for A_λ (then $\nu=\nu_1$).

By Lemma 3.2 ν is singular. Therefore all ν_λ are singular. Hence

$$|\mathcal{K}\nu_\lambda| \xrightarrow[\substack{z\longrightarrow x\\ \not\to}]{} \infty$$

for ν_λ-a. e. x. By formula (2.3) this means that

$$1+\pi\mathcal{K}\nu_0 = \exp U(z) \xrightarrow[\substack{z\longrightarrow x\\ \not\to}]{} 1-\frac{1}{\lambda}$$

for ν_λ-a. e. x. Thus by the definition of the phase shift for all $\lambda\in(0;1)$

$$\arg(1+\pi\mathcal{K}\nu_0(z)) = \mathcal{P}u(z) \xrightarrow[\substack{z\longrightarrow x\\ \not\to}]{} \pi$$

for ν_λ-a. e. x. Thus all $\nu_\lambda, \lambda\in(0;1)$ are concentrated on the set C. Since U' does not exist on C and

$$\mathcal{K}\nu_0 = \exp U - 1 \xrightarrow[\substack{z\longrightarrow x\\ \not\to}]{} 1-1/\lambda \neq 0$$

ν_λ-a. e., $(\mathcal{K}\nu_0)'$ does not exist on C for all $\lambda\in(0;1)$. Hence the characteristic function ϕ defined in $\mathbb{C}_+$ by formula (2.4) does not have a nontangential derivative on C. Thus by the Remark after Lemma 3.8 all $\nu_\lambda, \lambda\in(0;\infty)$ are singular continuous. Similarly all ν_λ for $\lambda\in\mathbb{R}\setminus[0;1]$ are concentrated on $\mathbb{R}\setminus C$. Since U' exists everywhere on $\mathbb{R}\setminus C$, μ_λ for $\lambda\in\mathbb{R}\setminus[0;1]$ are pure point.

To prove that ν_0 is continuous, let us notice that since by Lemma 3.3

$$p.\ v. \int_{\mathbb{R}} \frac{u(t+x)dt}{t} = \infty$$

for ν_0-a. e. x, ν_0 is concentrated on C.

Let $y\in C$ and let $I^k_{n_k}$ be the sequence of intervals from the construction of C containing y. Then (6.4) implies that condition (3.8) is not satisfied at y. Thus ν_0 can not have a point mass at x. Hence ν_0 is continuous. Similarly, ν_1 is continuous.

Remark. It is still unclear if there exist examples such that the sets Π and Σ are big (topologically or in measure) but $\Pi \cap \Sigma = \emptyset$ when $\sigma(A)$ contains the interval $[0;1]$. Modifying the above example, we can obtain a dense in $[0;1]$ set C' by inserting smaller Cantor sets into each complimentary interval I_n^k, then inserting smaller Cantor sets into each new complimentary interval and so on. If the size of these Cantor sets decreases to 0 fast enough then replacing C in the above example with C' we will obtain an example of A and ϕ such that $\sigma(A) = [0;1]$, $A + \lambda(\cdot, \phi)\phi$ is continuous on $[0;1]$ *for all* $\lambda \in [0;1]$ and pure point *for almost all* $\lambda \in \mathbb{R} \setminus [0;1]$.

The case $\sigma(A) \supset [0;1]$ became especially interesting after it was shown in [Go] and [R-M-S] that $\sigma(A) \supset [0;1]$ implies $\Sigma \setminus \Pi$ is a dense G_δ.

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MSRI, 1000 Centennial Drive, Berkeley, California 94720
E-mail address: agp@msri.org