ON TOTALLY REAL SPHERES IN COMPLEX SPACE

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Abstract. We shall prove that there are totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ which are not biholomorphically equivalent if $k \geq 5$ and $n = k+2[\frac{k-1}{4}]$. We also show that a smooth manifold M admits a totally real immersion in $\mathbf{C}^n$ with a trivial complex normal bundle if and only if the complexified tangent bundle of M is trivial. The latter is proved by applying Gromov's weak homotopy equivalence principle for totally real immersions to Hirsch's transversal fields theory.

1. Introduction

Let M, N be two totally real and real analytic submanifolds in $\mathbf{C}^n$. We say that M and N are *biholomorphically equivalent* if there is a biholomorphic mapping F defined in a neighborhood of M such that $F(M) = N$. As a standard fact of complexification, one knows that all totally real and real analytic embeddings of M in C^n are biholomorphically equivalent if M is of maximal dimension n. However, the topology of the manifold plays a major role in the existence of totally real immersions or embeddings. For instance, R.O. Wells [19] proved that if an n-dimensional compact and orientable manifold M admits a totally real embedding in $\mathbf{C}^n$, then its Euler number must vanish. It was also observed by Wells that if M is a manifold of dimension n and it admits a totally real immersion in $\mathbf{C}^n$, then its complexified tangent bundle $T^cM = TM \otimes \mathbf{C}$ is trivial. Conversely, the triviality of T^cM also implies the existence of totally real immersions of M in $\mathbf{C}^n$. This was obtained by M.L. Gromov in [8] through the method of convex integration.

The sphere $S^k\colon x_1^2 + \ldots + x_{k+1}^2 = 1$ in $\mathbf{R}^{k+1}$ gives us a trivial totally real embedding of S^k in $\mathbf{C}^{k+1}$. On the other hand, the works of Gromov [8], Ahern-Rudin [1] and Stout-Zame [17] tell us that S^k admits a totally real and real analytic embedding in $\mathbf{C}^k$ if and only if $k = 1, 3$. Our main result is the following.

Theorem 1.1. *If $k \leq 4$, all totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ are biholomorphic equivalent. If $k \geq 5$ and $n_k = k + 2[\frac{k-1}{4}]$, there exist totally real and real analytic embeddings of S^k in $\mathbf{C}^{n_k}$ which are not biholomorphically equivalent, while all totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ are biholomorphically equivalent if $n > n_k$.*

1991 *Mathematics Subject Classification.* Primary 32F25.
Key words and phrases. totally real embedding, complexification, regular homotopy classification.
Research at MSRI is supported in part by NSF grant DMS-9022140.

In fact, we shall prove a slightly stronger result that all totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ are unimodularly equivalent if $k \leq 4$ and $n > k$, i.e. they are biholomorphic equivalent through a mapping F which preserves the holomorphic n-form $dz_1 \wedge \ldots \wedge dz_n$. We should mention that using a transversality argument [5], F. Forstnerič and J.-P. Rosay showed that for a compact manifold M of dimension k, all its totally real and real analytic embeddings in $\mathbf{C}^n$ are biholomorphically equivalent if $n \geq 3k/2$.

The proof of Theorem 1.1 is not constructive. It depends on the weak homotopy equivalence (w.h.e.) principle for totally real immersions established by Gromov in [8]. To show Theorem 1.1, we also need to understand the role which the normal bundle of totally real immersions plays. Recall that a C^1-smooth mapping $f\colon M \to \mathbf{C}^n$ is a *totally real immersion* if f_*T_xM spans a k-dimensional complex linear subspace of $T_{f(x)}\mathbf{C}^n$ for each $x \in M$. We define the *complex* normal bundle of the immersion f, denoted by ν_f, to be the complex vector bundle whose fiber over $x \in M$ is the quotient of $T_{f(x)}\mathbf{C}^n$ by the complex linear span of f_*T_xM. We shall see that two totally real and real analytic embeddings of a manifold are biholomorphically equivalent if and only if their complex normal bundles are topologically equivalent. The normal bundle of immersions plays quite important role in the works of S. Smale [15] and M.W. Hirsch [11]. Analogous to the results of Hirsch [11] about transversal fields of smooth immersions, we obtain the following.

Theorem 1.2. *Let $f\colon M \to \mathbf{C}^n$ be a C^1 totally real immersion. Assume that the complex normal bundle ν_f has a topologically trivial subbundle of rank r. Then there is a regular homotopy $f_t\colon M \to \mathbf{C}^n$ of C^1 totally real immersions such that $f_0 = f$ and $f_1\colon M \to \mathbf{C}^{n-r}$.*

We now draw some conclusions from Theorem 1.2.

Corollary 1.3. *Let M be a smooth manifold of dimension k. Then M admits a totally real immersion in $\mathbf{C}^n$ with a trivial complex normal bundle if and only if there exists a totally real immersion of M in $\mathbf{C}^k$, i.e. the complexified tangent bundle T^cM is trivial.*

A smooth manifold M is said to be *stably parallelizable* if the tangent bundle of $M \times \mathbf{R}$ is trivial. For instance, the boundary of a smooth domain in euclidean space is always stably parallelizable. By a theorem of Hirsch [11], M is stably parallelizable if and only if it is orientable and admits an immersion in $\mathbf{R}^{n+1}$. From Theorem 1.2, we have the following.

Corollary 1.4. *Let M be a manifold of dimension n which is immersible in $\mathbf{R}^{n+1}$. Then T^cM is trivial if M is orientable, or M is non-orientable with $H^2(M, \mathbf{Z}) = 0$.*

In fact, Gromov proved a stronger result that M admits an exact Lagrangian immersion in euclidean space when M is stably parallelizable (see [9], p. 61). One notices that all real surface M can be immersed in $\mathbf{R}^3 \subset \mathbf{C}^3$. On the other hand, Forstnerič [4] proved that a non-orientable compact surface admits a totally real immersion in $\mathbf{C}^2$ if and only if its genus is even. Therefore, the condition that $H^2(M, \mathbf{Z})$ vanishes is essential in Corollary 1.4.

The paper is organized as follows. In section two we shall discuss Gromov's w.h.e.-principal for totally real immersions. Section three is devoted to the proof of Theorem 1.2 and Corollary 1.4. The proof of Theorem 1.1 will be given in the last section, where we shall also make essential uses of homotopy groups of complex Stiefel manifolds obtained by M.L. Kervaire [13] and F. Sigrist [14].

2. Classification of totally real immersions

In this section, we shall first recall Gromov's w.h.e.-principal for ample differential relations established in [8]. We shall also discuss the group structure on the regular homotopy classes of totally real immersions of S^k in $\mathbf{C}^n$.

Let M be a smooth manifold of dimension k. Assume that $k \leq n$. By a *regular homotopy* f_t of totally real C^1-immersions of M in $\mathbf{C}^n$, one means that for each $t \in [0,1]$, f_t is a totally real C^1-immersion, and $df_t \colon TM \to T\mathbf{C}^n$ depends on $t \in [0,1]$ continuously. Mappings from M to $\mathbf{C}^n$ can be identified with sections of the trivial bundle $X = M \times \mathbf{C}^n \to M$. If $f \colon M \to \mathbf{C}^n$ is a C^1-mapping defined in a neighborhood of $x \in M$, we define the 1-jet of f at x to be $J^1_x f = (f(x), df_x)$. We denote by X^1 the space of 1-jets of C^1 sections of $X \to M$. Then it is easy to see that X^1 is a fibration over M whose fiber over $x \in M$ consists of $\mathbf{R}$-linear mappings from $T_x M$ to $T_z \mathbf{C}^n$ for some $z \in \mathbf{C}^n$. We shall adapt (compact-open) C^0-topology on X^1. Here, we should say a few words about the topologies used in the sequel. For a homotopy of sections or immersions, we shall always use the (compact-open) C^r-topology, i.e. the weak C^r topology. For approximating a mapping or function, we shall always use the fine C^r-topology. The reader is referred to [12] for basic properties of these two kinds of topologies.

By Σ_x, one denotes the set of 1-jets $J^1_x f$ such that $df_x(T_x M)$ spans a complex linear subspace of $T_{f(x)}\mathbf{C}^n$ of rank k. This is equivalent to say that the complexification $df_x \otimes \mathbf{C} \colon T^c_x M \to T_{f(x)}\mathbf{C}^n$ is injective. Let Σ be the union of Σ_x for all $x \in M$. Then $\Omega = X^1 \setminus \Sigma$ is an open subset of X^1, which is called a *totally real differential relation.* Thus, a C^1-mapping $f \colon M \to \mathbf{C}^n$ is totally real if and only if $J^1 f$ maps M into Ω.

For each $x_0 \in M$, we choose local coordinates $u_1, \dots, u_k$ in a neighborhood U of x_0. Fix $x \in U$ and $1 \leq j \leq n$. Let Z be the set of 1-jets $J^1_x f$ satisfying

$$f(x) = z, \quad f_{u_i}(x) = v_i, \quad i \neq j,$$

where z and $v_i (i \neq j)$ are fixed k vectors in $\mathbf{C}^n$. Then either $Z \setminus \Sigma$ is an empty set, or the linear convex hull of each connected component of $Z \setminus \Sigma$ is the whole affine space Z. According to the terminology of Gromov [8], Ω is said to be *ample* in the coordinate directions. To see this, we notice that if $v_1, \dots, \hat{v}_j, \dots, v_k$ are not $\mathbf{C}$-linearly independent, then $Z \subset \Sigma$. Otherwise, $Z \cap \Sigma$ consists of all 1-jets $(z, v_1, \dots, v, \dots, v_k)$ such that v is a $\mathbf{C}$-linear combination of $v_1, \dots, \hat{v}_j, \dots, v_k$. Therefore, $Z \cap \Sigma$ is a subspace of the affine space Z with real codimension $2n - 2(k-1) \geq 2$. This implies that $Z \setminus \Sigma$ is connected and it spans the whole space Z. As a consequence of Theorem 1.3.1 in [8], we can state the following result.

Theorem 2.1 (Gromov, [8]). *Let M be a smooth manifold of dimension $k \leq n$, and let Ω and J^1 be as above. Then J^1 is a one-to-one correspondence between the regular homotopy classes of totally real C^1-immersion of M in $\mathbf{C}^n$ and the homotopy classes of continuous sections of $\Omega \to M$. In particular, M admits a totally real immersion in $\mathbf{C}^n$ if and only if $\Omega \to M$ has a global continuous section.*

We now consider the case that M is the sphere S^k. Here we need the fact that the complexified tangent bundle T^cS^k of S^k is trivial. This follows from the existence of totally real immersion of S^k in $\mathbf{C}^k$. An explicit example of Lagrangian (whence totally real) immersions of S^k was constructed by A. Weinstein [18]. As we mentioned earlier, Gromov showed that all stably parallelizable manifold, such as a sphere, admits an exact Lagrangian immersion (see [9], p. 61). It seems to us that there are no other proof in the literature about the triviality of T^cS^k. Throughout the whole paper, we shall fix a topological trivialization of $T^cS^k = S^k \times \mathbf{C}^k$.

Recall that the complex Stiefel manifold $V_{n,k}$ consists of k-frames of $\mathbf{C}^n$, i.e. the space of ordered k linear independent vectors in $\mathbf{C}^n$. With the fixed trivialization for T^cS^k, the global sections of $\Omega \to M$ can be identified with mappings from S^k to $\mathbf{C}^n \times V_{n,k}$ as follows. Let $e_1, \dots, e_k$ be the set of $\mathbf{C}$-linearly independent continuous sections which defines the trivialization of T^cS^k. Assume that $\phi \colon M \to \Omega$ is a global section. Then $\phi = (f, \varphi)$ where $f \colon M \to \mathbf{C}^n$ and $\varphi \colon TM \to T\mathbf{C}^n$ satisfy the property that for each $x \in M$, $\varphi(x) \colon T_xM \to T_{f(x)}\mathbf{C}^n$ is $\mathbf{R}$-linear and its complexification is injective. Hence,

$$v(x) = (\varphi(x)(e_1(x)), \dots, \varphi(x)(e_k(x)))$$

is a set of linearly independent k vectors of $T_{f(x)}\mathbf{C}^n$. Denote by $U_{n,k}$ the space of unitary k-frames of $\mathbf{C}^n$, where by a unitary k-frame $(v_1, \dots, v_k)$, one means that $v_1, \dots, v_k$ satisfy the condition $<v_i, v_j> = \delta_{i,j}$ for the standard hermitian inner product $<\cdot, \cdot>$ on $\mathbf{C}^n$. From the well-known normalization, one knows that $V_{n,m}$ is the product of $U_{n,m}$ with the space T of upper-triangle matrices with positive eigenvalues. Thus, a homotopy $\phi_t \colon S^k \to \Omega$ induces a homotopy

$$(f_t, A_t, \varphi_t) \colon S^k \to \mathbf{C}^n \times T \times U_{n,k}.$$

Since $\mathbf{C}^n$ and T are contractible, we see that the set of homotopy classes of sections of $\Omega \to S^k$ is the same as the homotopy classes of mappings from S^k to $U_{n,k}$. It is well-known that the homotopy classes of mappings from S^k to $U_{n,k}$ is the homotopy group $\pi_k(U_{n,k})$ (see p. 88 in [16], p. 211 in [3]). Thus, we obtain a one-to-one mapping

$$j_* \colon I(S^k, \mathbf{C}^n) \to \pi_k(U_{n,k}),$$

where $I(S^k, \mathbf{C}^n)$ stands for the set of regular homotopy classes of totally real immersions of S^k in $\mathbf{C}^n$.

With the fixed trivialization of T^cS^k, j_* is a *canonical* mapping in the sense that there is a commutative diagram

$$\begin{array}{ccc} I(S^k, \mathbf{C}^n) & \xrightarrow{j_*} & \pi_k(U_{n,k}) \\ \downarrow i_* & & \downarrow i'_* \\ I(S^k, \mathbf{C}^N) & \xrightarrow{j'_*} & \pi_k(U_{N,k}), \quad N > n, \end{array} \tag{2.1}$$

where the homomorphism i_* is induced by the inclusion of sending a totally real immersion $f\colon S^k \to \mathbf{C}^n$ to a totally real immersion $(f, 0)\colon S^k \to \mathbf{C}^N$, and i'_* is induced by regarding a unitary k-frame of $\mathbf{C}^n$ as a unitary k-frame of $\mathbf{C}^N$. In other words, the adapted group structure on $I(S^k, \mathbf{C}^m)$ is preserved under the inclusion $\mathbf{C}^n \subset \mathbf{C}^N$. This will be important for us to prove Theorem 1.1.

3. Proof of Theorem 1.2

In this section we shall apply Gromov's w.h.e.-principle to prove Theorem 1.2. With necessary modifications, we shall follow very closely the proof of Theorem 6.4 in [11].

Let M be a manifold of dimension k. By FM, one denotes the (complexified) k-frame bundle of M, which consists of ordered k linearly independent vectors in T^cM with the same base point. For each $e = (v_1, \dots, v_k) \in F_xM$ and $g = (g_{i,j}) \in GL(k, \mathbf{C})$, we define $g \cdot e$ to be the k-frame $(v'_1, \dots, v'_k)$ with $v'_i = \sum_{j=1}^k g_{i,j} v_j$. Then FM is a principal $GL(k, \mathbf{C})$-bundle over M. For $k < m \le n$, we denote by $E_{n,m}$ be the associated bundle of FM with fiber $V_{n,m}$. More precisely, we define a $GL(k, \mathbf{C})$-action on $V_{n,m}$ by

$$(g, (v_1, \dots, v_m)) \mapsto (g \cdot (v_1, \dots, v_k), v_{k+1}, \dots, v_m).$$

Then $E_{n,m}$ is the set of equivalence classes of the relation $\sim$ on $FM \times V_{n,m}$ with $(e, f) \sim (g \cdot e, g \cdot f)$ for all $g \in GL(k, \mathbf{C})$. The bundle projection $p_{n,m}\colon E_{n,m} \to M$ is induced by the composed projection $FM \times V_{n,m} \to FM \to M$. Let $p^{n,m}_{n,k}\colon V_{n,m} \to V_{n,k}$ be the projection of deleting the last $m-k$ vectors from each m-frame. Notice that the $GL(k, \mathbf{C})$-action on $V_{n,m}$ does not affect the last $m-k$ vectors of an m-frame of $\mathbf{C}^n$. Hence, $p^{n,m}_{n,k}$ induces a projection from $E_{n,m}$ to $E_{n,k}$ such that $p_{n,m} = p_{n,k} \circ p^{n,m}_{n,k}$. We further remark that $p^{n,m}_{n,k}\colon E_{n,m} \to E_{n,k}$ is a fiber bundle. In particular, $p^{n,m}_{n,k}\colon E_{n,m} \to E_{n,k}$ has the covering homotopy property, i.e. for any finite polyhedron P, a homotopy $h_t\colon P \to E_{n,k}$ has a lifting $\tilde{h}_t\colon P \to E_{n,m}$, if the initial lifting $\tilde{h}_0$ exists.

To use the covering homotopy property, we consider the set of sections of the fiber bundle $E_{n,m} \to M$. Given a section $s\colon M \to E_{n,m}$, we define a mapping $\varphi\colon FM \to V_{n,m}$ by $\varphi(e) = f$ if $s(x)$ is the equivalence class of $(e, f) \in FM \times V_{n,m}$. Then φ is well-defined. For if (e, f) and (e, f') are in the same equivalence class $s(x)$.

Then there is $g \in GL(k, \mathbf{C})$ such that $g \cdot e = e$ and $g \cdot f = f'$. Obviously, $g \cdot e = e$ implies that $g = \mathrm{id}$. Hence, $f' = f$. Moreover, if (e, f) is in the equivalence class $s(x)$, so is $(g \cdot e, g \cdot f)$. Hence, $\varphi(g \cdot e) = g \cdot \varphi(e)$, i.e. φ is a $GL(k, \mathbf{C})$-*equivariant* mapping. Conversely, given a $GL(k, \mathbf{C})$-equivariant mapping $\varphi\colon FM \to V_{n,m}$, we set $s(x)$ to be the equivalence class of $(e, \varphi(e))$ in $E_{n,m}$ for $e \in F_xM$. Thus, there is a one-to-one correspondence between the set of $GL(k, \mathbf{C})$-equivariant mappings from FM to $V_{n,m}$ and the set of sections of the fiber bundle $E_{n,m} \to M$. Therefore, the covering homotopy property gives us the following.

Theorem 3.1. *Let φ_t be a homotopy of $G(k, \mathbf{C})$-equivariant mappings from FM to $V_{n,k}$. Assume that φ_0 has a lifting $\tilde{\varphi}_0\colon FM \to V_{n,k+r}$ such that $\tilde{\varphi}_0$ is $GL(k, \mathbf{C})$-equivariant. Then there is a homotopy $\tilde{\varphi}_t$ of lifting of φ_t such that each $\tilde{\varphi}_t\colon FM \to V_{n,k+r}$ is $GL(k, \mathbf{C})$-equivariant.*

We now let $p'\colon V_{n,k+r} \to V_{n,r}$ be the mapping of projecting a $(k + r)$-frame to its last r components. Let $\phi\colon FM \to V_{n,k+r}$ be a $GL(k, \mathbf{C})$-equivariant mapping. Then $\varphi = p\phi\colon FM \to V_{n,k}$ is also a $GL(k, \mathbf{C})$-equivariant mapping. Notice that $GL(k, \mathbf{C})$ acts on each fiber of FM transitively and that the last r components of a $(k+r)$-frame of $\mathbf{C}^n$ is fixed under the $GL(k, \mathbf{C})$-action. This implies that $p'\varphi$ is constant along fibers of FM. Therefore, $p'\varphi$ is the lifting of some mapping $\psi\colon M \to V_{n,r}$. Following [11], we shall call $\psi\colon M \to V_{n,r}$ a *transversal* r-field of $\varphi\colon M \to V_{n,k}$. Thus, we identify the set of $GL(k, \mathbf{C})$-equivariant mappings from FM to $V_{n,k+r}$ with the set of $GL(k, \mathbf{C})$-equivalent mappings from FM to $V_{n,k}$ with transversal r-fields. In general, we say that $\psi\colon M \to V_{n,r}$ is *transverse* to a totally real immersion $f\colon M \to \mathbf{C}^n$ if for each $x \in M$ and $\psi(x) = (v_1, \dots, v_r)$, v_j is not contained in $f_*(T^c_xM)$ $(1 \le j \le r)$.

Corollary 3.2. *Let f_t be a homotopy of totally real immersions of M in $\mathbf{C}^n$. Assume that ν_{f_0} has a topological trivial subbundle of rank r. Then ν_{f_1} also contains a topological trivial subbundle of rank r.*

Proof. We identify $f_{t*}\colon T^cM \to T\mathbf{C}^n$ with a homotopy f_t of mappings from M to $\mathbf{C}^n$ and a homotopy φ_t of $GL(k, \mathbf{C})$-equivariant mappings from FM to $V_{n,k}$. By assumptions, φ_0 has a transversal r-field. Hence, Theorem 3.1 implies that φ_1 also admits a transversal r-filed, i.e. ν_{f_1} has a trivial subbundle of rank r. □

We also need the following.

Lemma 3.3. *Let M be a smooth manifold and $f\colon M \to \mathbf{C}^n$ a C^∞-smooth totally real immersion. Assume that ν_f has a topologically trivial subbundle of rank r. Then there is a smooth unitary r-field $\psi\colon M \to U_{n,r}$ which is transverse to f.*

Proof. By assumptions, there is a continuous r-field $\psi_0\colon M \to V_{n,r}$ which is transverse to the immersion f. Using the approximation in fine C^0-topology, we can replace ψ_0 by a smooth r-field ψ_1 which is still transverse to the immersion f. We now project

$\psi_1(x)$ to the orthogonal complement of $f_*T^c_xM$. By the well-known normalization, we readily obtain the desired unitary r-filed. □

Proof of Theorem 1.2. Let $f\colon M \to \mathbf{C}^n$ be a C^1-smooth totally real immersion. Assume that the complex normal bundle ν_f has a topologically trivial subbundle of rank r. We shall seek a homotopy f_t of totally real immersions of M in $\mathbf{C}^n$ such that $f_0 = f$ and $f_1(M) \subset \mathbf{C}^{n-1}$. We further require that the complex normal bundle of $f_1\colon M \to \mathbf{C}^{n-1}$ has a trivial subbundle of rank $r-1$. Thus, Theorem 1.2 follows from the induction.

We first find a smooth mapping $g\colon M \to \mathbf{C}^n$ which is sufficiently close to f in fine C^1-topology such that each $f_t = (1-t)f + tg$ is still a totally real immersion for $0 \leq t \leq 1$. Rename g by f. Then, Corollary 3.2 implies that ν_f still contains a trivial subbundle of rank r. By Lemma 3.3, there exists a smooth unitary r-field $\psi = (\xi_1, \dots, \xi_r)$ which is transverse to the smooth totally real immersion $f\colon M \to \mathbf{C}^n$.

From the standard embedding $S^{2n-1} \subset \mathbf{C}^n$, one gets a complex vector bundle $T^{(1,0)}S^{2n-1}$ whose fiber over $v \in S^{2n-1}$ consists of vectors in $\mathbf{C}^n$ which are orthogonal to v with respect to the standard hermitian metric on $\mathbf{C}^n$. Let E be the frame bundle over S^{2n-1} whose fiber consists of linear independent vectors $v_1, \dots, v_{k+r-1}$ of $T^{(1,0)}S^{2n-1}$ with the same base point. Consider the mapping

$$\tilde{\xi}_r(x)\colon e \mapsto (df_x(e), \xi_1(x), \dots, \xi_{r-1}(x)), \quad e \in F_xM. \tag{3.1}$$

Since ξ_r is orthogonal to f_*T^cM and ξ_j $(j < r)$, then $\tilde{\xi}_r(x)$ maps F_xM into $E_{\xi_r(x)}$. Hence, $\tilde{\xi}_r\colon FM \to E$ is a $GL(k,\mathbf{C})$-equivariant mapping which covers the mapping $\xi_r\colon M \to S^{2n-1}$. Since the dimension of M is less than $2n-1$, the smoothness of the mapping ξ_r implies that there is $y_0 \notin \xi_r(M)$. Set $Y = S^{2n-1} - \{y_0\}$. Since Y is contractible, then there is a homotopy $h_t\colon M \to Y$ such that $h_0 = \xi_r$ and $h_1 \equiv y_1 \in Y$. Also, there is a trivialization

$$T^{(1,0)}S^{2n-1}|_Y = Y \times \tilde{\mathbf{C}}^{n-1}, \quad \tilde{\mathbf{C}}^{n-1} = T^{(1,0)}_{y_1}S^{2n-1}. \tag{3.2}$$

Therefore, $\tilde{\xi}_r$ can be written as (h_0, ϕ_0) with ϕ_0 a $GL(k,\mathbf{C})$-equivariant mapping from FM to E_{y_1}. Put $\tilde{h}_t = (h_t, \phi_0)$. Returning to $T^{(1,0)}S^{2n-1}|_Y$ from the trivialization (3.2), we obtain a homotopy $\tilde{h}_t$ of $GL(k,\mathbf{C})$-equivariant mappings from FM to $E|_Y$. Returning to the ambient space $\mathbf{C}^n$, we then have $T^{(1,0)}S^{2n-1} \subset S^{2n-1} \times \mathbf{C}^n$. Thus, we obtain a homotopy $\tilde{h}_t$ of $GL(k,\mathbf{C})$-equivariant mappings from FM to $\mathbf{C}^n \times V_{n,k+r-1}$ such that $\tilde{h}_1\colon FM \to E_{y_1}$.

Let us identify y_1 with a point in $\mathbf{C}^{n-1}$. Put $T_{y_1}\mathbf{C}^{n-1} = \tilde{\mathbf{C}}^{n-1}$. Then we have $E_{y_1} = y_1 \times V_{n-1,k+r-1}$. We now write $\tilde{h}_t$ as (h_t, φ_t, ψ_t), where each $\varphi_t\colon FM \to V_{n,k}$ is $GL(k,\mathbf{C})$-equivariant, and ψ_t is a transversal $(r-1)$-field of φ_t. From (3.1), it is clear that $\varphi_0\colon FM \to V_{n,k}$ is just the $GL(k,\mathbf{C})$-equivariant mapping induced by $f_*\colon T^cM \to T\mathbf{C}^n$. This implies that f_* and (h_1, φ_1) are homotopic as fiberwisely injective $\mathbf{C}$-linear mappings from T^cM to $T\mathbf{C}^n$. Notice that $h_1 \equiv y_1 \in \mathbf{C}^{n-1}$ and

$\varphi_1\colon FM \to V_{n-1,k}$. Now Theorem 2.1 implies that there is a totally real immersion $g\colon M \to \mathbf{C}^{n-1}$ such that g_* and (h_1, φ_1) are joined by a homotopy k_t of fiberwisely injective $\mathbf{C}$-linear mappings from T^cM to $T\mathbf{C}^{n-1}$. Thus, we have proved that $f_*\colon T^cM \to T\mathbf{C}^n$ is homotopic to $g_*\colon T^cM \to T\mathbf{C}^{n-1}$ as fiberwisely injective $\mathbf{C}$-linear mappings from T^cM to $T\mathbf{C}^n$. Using Theorem 2.1 again, we know that there is a homotopy of totally real immersions joining f and g. To complete the proof of Theorem 1.2, we need to show that the complex normal bundle of $g\colon M \to \mathbf{C}^{n-1}$ has a topologically trivial subbundle of rank $r-1$. To this end, we notice that φ_1 has a transversal $(r-1)$-field ψ_1 in $\mathbf{C}^{n-1}$ since $\tilde{h}_1\colon FM \to E_{y_1} \equiv V_{n-1,k+r-1}$. By applying Corollary 3.2 to the homotopy k_t, we see that the complex normal bundle of g has a trivial subbundle of rank $r-1$. The proof of Theorem 1.2 is complete.

Proof of Corollary 1.4. Assume that M is immersed in $\mathbf{R}^{n+1} \subset \mathbf{C}^{n+1}$. When M is orientable, the real normal line bundle of the immersion of M in $\mathbf{R}^{n+1}$ is trivial. Let ν be a normal vector field of M. It is obvious that ν is still transverse to T^cM. Therefore, Theorem 1.2 implies that M admits a totally real immersion in $\mathbf{C}^n$, i.e. T^cM is trivial. We now consider the case that M is non-orientable and $H^2(M, \mathbf{Z}) = 0$. By a theorem of Whitney, a smooth manifold has a compatible analytic structure. Using the analytic approximations of smooth mappings in fine C^r-topology [7], we can replace the original immersion of M in $\mathbf{R}^{n+1}$ by an analytic immersion $f\colon M \to \mathbf{R}^{n+1}$. Now the complexification of f gives us a holomorphic mapping from M^c to $\mathbf{C}^{n+1}$, where M^c is a complexification of M depending on f. By Grauert's Tube theorem [7], there is a Stein neighborhood X of M in M^c such that M is a strong deformation retraction of X. Since X is Stein, then all holomorphic line bundle over X is determined by its first Chern class. On the other hand, $H^2(X, \mathbf{Z}) = H^2(M, \mathbf{Z}) = 0$, since M is a deformation retraction of X. Hence, the normal bundle of X in $\mathbf{C}^{n+1}$ is trivial. Now, Theorem 1.1 implies that M admits a totally real immersion in $\mathbf{C}^n$. The proof of Corollary 1.4 is complete.

We notice that a real surface can be immersed in $\mathbf{R}^3$. Thus, Corollary 1.4 gives another proof that all orientable surfaces admit totally real immersions in $\mathbf{C}^2$, which is due to Forstnerič [4] when the surfaces are compact. Corollary 1.4 is inconclusive when M is a non-orientable compact surface, since $H^2(M, \mathbf{Z}) = \mathbf{Z}_2$. It was proved by Forstnerič [4] that a non-orientable compact surface admits a totally real immersion in $\mathbf{C}^2$ if and only if its genus is even. In view of Corollary 1.3, we see that a totally real immersion of a non-orientable compact surface in $\mathbf{C}^3$ has a trivial complex normal bundle if and only if the genus of the surface is even. This also indicates that the hypothesis that $H^2(M, \mathbf{Z}) = 0$ is needed in Corollary 1.4, although it is not a necessary condition for the triviality of T^cM.

4. Proof of Theorem 1.1

In this section we shall prove Theorem 1.1 by using Theorem 1.2. We shall see that the proof of Theorem 1.1 is eventually related to homotopy groups of complex Stiefel manifolds.

Let us first consider the case $k \leq 4$. We need the following lemma.

Lemma 4.1. *Let M and N are two totally real and real analytic submanifolds of $\mathbf{C}^n$. Let ν_M and ν_N be the complex normal bundles of M and N respectively. Then M and N are biholomorphically equivalent if and only if there is an analytic diffeomorphism $f\colon M \to N$ such that the pull-back $f^*\nu_N$ is topologically isomorphic to ν_M.*

Proof. If M, N are biholomorphically equivalent by F, then it is clear that dF maps T^cM into T^cN and df also induces an isomorphism from ν_M to ν_N. We now assume that there is an analytic diffeomorphism $f\colon M \to N$ such that ν_M is topologically isomorphic to $f^*\nu_N$. Let f^c be an biholomorphic extension of f which sends a Stein neighborhood M^c of M onto $N^c = f^c(M^c)$. We may assume that M is a strong deformation retraction of M^c, i.e. there is a continuous mapping $r\colon M^c \to M^c$ with $r|_M = \mathrm{id}$ and a homotopy $r_t\colon M^c \to M$ with $r_0 = r$ and $r_1 = \mathrm{id}$. This implies that any complex vector bundle V over M^c is isomorphic to $r^*(V|_M)$. Therefore, ν_{M^c} is topologically isomorphic to $f^*\nu_{N^c}$. By a theorem of Grauert [6], ν_{M^c} and ν_{N^c} are also holomorphically isomorphic. Let s be the zero section of ν_{M^c}. By a theorem of Docquier and Grauert (see [10], p. 257), the zero-section mapping can be extends to a biholomorphic mapping from a neighborhood of M^c in $\mathbf{C}^n$ to a neighborhood of zero section of ν_{M^c}. Therefore, f^c extends to biholomorphic mapping defined in a neighborhood of M in $\mathbf{C}^n$. □

To start the proof of Theorem 1.1, we first notice that Forstnerič and Rosay [5] proved that any two totally real immersions of a smooth manifolds M in $\mathbf{C}^n$ are regularly homotopic through totally real immersions, if $\dim M \leq 2n/3$. Consequently, their argument also showed that such two totally real and real analytic embeddings are also biholomorphically equivalent if M is compact. In the special case of spheres, we notice that the standard embedding of S^k in $\mathbf{C}^{k+1}$ has a trivial complex normal bundle, then Lemma 4.1 and Corollary 3.2 also imply that any two totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ are biholomorphically equivalent when $n \geq 3k/2$. In particular, all totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ are biholomorphically equivalent when $k = 1, 2$. We also notice that complex line bundles on S^k is classified by the group $\pi_{k-1}(U_1)$ (see [3]). Hence, complex line bundles on S^k are trivial if $k \neq 2$. Now it is easy to see that all totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ are biholomorphically equivalent if $k \leq 4$.

We now consider the case of $k > 4$. Assume first that for some $n > k$, all totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ are biholomorphically equivalent. This implies that all totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ have trivial complex normal bundle. Notice that when $n > k$, a C^1-smooth totally real immersions of S^k in C^n can be connected to a totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ by C^1 totally real immersions. Thus, Corollary 3.2 implies that for any totally real immersion of S^k in $\mathbf{C}^n$, its complex normal bundle is trivial. Therefore, the commutative diagram (2.1) and Theorem 1.2 imply that

$$i_*\colon \pi_k(U_k) \to \pi_k(U_{n,k}) \tag{4.1}$$

is an epimorphism, where i_* is induced by the inclusion $U_k \subset U_{n,k}$. Therefore, the proof of Theorem 1.1 will be complete if we can show that (4.2) is not an epimorphism for $n = n_k$, and also that

$$i_*\colon \pi_k(U_{n_k,k}) \to \pi_k(U_{n,k}) \tag{4.2}$$

is an epimorphism for all $n > n_k$. Notice that $\pi_k(U_{n,k}) = 0$ for $n \geq 3k/2$. This also follows from the fact that all totally real immersions of S^k in $\mathbf{C}^n$ are regularly homotopic. Hence, it suffices to show that (4.2) holds for $n_k < n < 3k/2$.

It is well-known from the Bott periodicity theorem that $\pi_{2l}(U_n) = 0$ and $\pi_{2l+1}(U_n) = \pi_{2l+1}(U_{l+1}) = \mathbf{Z}$ for $n > l$. We shall discuss in two cases.

Case 1. $k = 2l$ $(l \geq 3)$. In this case, $\pi_{2l}(U_{2l}) = 0$ implies that n_k is the largest integer n such that $\pi_{2l}(U_{n,2l})$ is non-trivial. From [14, p. 127], one sees that

$$\pi_{2l}(U_{3l-1,2l}) = \begin{cases} 0, & \text{if } l \text{ is even and } l \geq 2, \\ \mathbf{Z}_2, & \text{if } l \text{ is odd and } l \geq 3. \end{cases}$$

Hence, $n_{2l} = 3l - 1$ when l is odd and $l \geq 3$. We now assume that l is even. Then, one has

$$\pi_{2l}(U_{3l-2,2l}) = \begin{cases} \mathbf{Z}_2, & l = 4, \\ \mathbf{Z}_{48/U(l+1,3)}, & l \geq 6, \end{cases}$$

where $U(\cdot, 3)$ is a James number which divides 24 [14, p. 128]. In particular, $\pi_{2l}(U_{3l-2,2l}) \neq 0$, if l even and $l \geq 4$. This showed that if l is even and $l \geq 4$, $n_{2l} = 3l - 2$ satisfies the property stated in Theorem 1.1.

Case 2. $k = 2l + 1$ $(l \geq 2)$. In this case, one has $\pi_{2l+1}(U_{2l+1}) = \mathbf{Z}$. From [14], we find

$$\pi_{2l+1}(U_{3l+1,2l+1}) = \mathbf{Z}, \tag{4.3}$$

$$\pi_{2l+1}(U_{3l,2l+1}) = \begin{cases} \mathbf{Z}, & \text{if } l \text{ is even and } l \geq 2, \\ \mathbf{Z} + \mathbf{Z}_2, & \text{if } l \text{ is odd and } l \geq 3. \end{cases} \tag{4.4}$$

We now consider the homomorphism

$$i_*\colon \pi_{2l+1}(U_{3l,2l+1}) \to \pi_{2l+1}(U_{3l+1,2l+1}), \tag{4.5}$$

where i_* is induced by the inclusion $U_{3l,2l+1} \subset U_{3l+1,2l+1}$. We need the following.

Lemma 4.2. *Let i_* be defined by* (4.5)*. Then i_* is an epimorphism if and only if l is odd.*

Let us postpone the proof of Lemma 4.2 for a while and finish our proof of Theorem 1.1. Assume first that l is odd. It is clear that there is no epimorphism from $\mathbf{Z}$ to $\mathbf{Z}+\mathbf{Z}_2$. Hence, (4.1) is not onto for $k=2l+1$ and $n=3l$. On the other hand, Lemma 4.2 and the vanishing of $\pi_k(U_{n,k})$ for $n>3l+1$ imply that (4.2) is onto for all $n>3l$. Therefore, $n_{2l}=3l$ satisfies the property stated in Theorem 1.1. Next, we assume that l is even. Then Lemma 4.2 implies that (4.1) is not onto for $n=3l+1$. On the other hand, (4.2) is onto for $n=3l+2$ and $k=2l+1$ because of the vanishing of $\pi_{2l+1}(U_{3l+2,2l+1})$. Therefore, we conclude that for even l, $n_{2l+1}=3l+1$ satisfies the property stated in Theorem 1.1.

We now turn to the proof of Lemma 4.2. Let $U_{3l}\to U_{3l,2l+1}$ be the standard fibration with fiber U_{l-1}, and $U_{3l+1}\to U_{3l+1,2l+1}$ the fibration with fiber U_l. Then the inclusion $U_{3l,2l+1}\subset U_{3l+1,2l+1}$ induces the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
\pi_{2l+1}(U_{3l}) & \xrightarrow{j_*} & \pi_{2l+1}(U_{3l,2l+1}) & \longrightarrow & \pi_{2l}(U_{l-1}) & \to & \pi_{2l}(U_{3l}) \\
\| & & \downarrow i_* & & \downarrow i'_* & & \| \\
\pi_{2l+1}(U_{3l+1}) & \xrightarrow{j'_*} & \pi_{2l+1}(U_{3l+1,2l+1}) & \xrightarrow{\delta} & \pi_{2l}(U_l) & \to & \pi_{2l}(U_{3l+1}). \\
\| & & & & \downarrow p_* & & \| \\
\mathbf{Z} & & & & \pi_{2l}(S^{2l-1}) & & 0
\end{array}$$

It is clear that if i_* is epimorphic, so is i'_*. However, one knows that $\pi_{2l}(U_l)=\mathbf{Z}_{l!}$ (see [2]), and $\pi_{2l}(U_{l-1})=\mathbb{Z}_{l!/2}$ for l even (see [13]). Thus i'_* is not epimorphic when l is even. We now assume that l is odd. In this case, Kervaire showed that $p_*=0$ (see [13], Lemma I.1). Hence, i'_* is an epimorphism, i.e. δi_* is an epimorphism. Using (4.3) and (4.4), we can write more explicitly the following diagram

$$\begin{array}{ccccc}
\mathbf{Z} & \xrightarrow{j_*} & \mathbf{Z}+\mathbf{Z}_2 & & \\
\| & & \downarrow i_* & & \\
\mathbf{Z} & \xrightarrow{j'_*} & \mathbf{Z} & \xrightarrow{\delta} & \mathbf{Z}_{l!}\to 0.
\end{array}$$

Write $j_*(1)=g+\epsilon$, where $g\in\mathbf{Z}$ and $\epsilon\in\mathbf{Z}_2$. Clearly, $i_*(\epsilon)=0$. Hence, we get

$$g\cdot i_*(e)=i_*j_*(1)=j'_*(1)=\pm l!$$

for $e=1\in\mathbf{Z}\subset\mathbf{Z}+\mathbf{Z}_2$. Thus, $i^*(e)$ divides $l!$. On the other hand, $\delta\circ i_*$ is an epimorphism. Hence, $i_*(e)\delta(1)$ must be a generator of $\mathbf{Z}_{l!}$. Therefore $i_*(e)=\pm1$, i.e. i_* is an epimorphism. The proof of Lemma 4.2 is complete.

Next, we want to show that all totally real and real analytic embeddings of S^k in $\mathbf{C}^n$ are unimodularly equivalent if $n>k$ and $k\le 4$, i.e. they are equivalent through a biholomorphic mapping F satisfying $F^*\Omega=\Omega$. To this end we shall use the Cauchy-Kowalewski theorem to prove a slightly general result.

Proposition 4.3. *Let M, N be two totally real and real analytic submanifolds of $\mathbf{C}^n$ which are biholomorphically equivalent. Assume that for some complexification M^c, the holomorphic normal bundle of M^c contains a holomorphic subbundle of rank one. Then M and N are unimodularly equivalent.*

Proof. Assume that M and N are equivalent by a biholomorphic mapping φ defined near M. It suffices to show that there is a biholomorphic mapping ψ defined near M such that $\psi^*\Omega = \varphi^*\Omega$ and $\psi(M) = M$. By shrinking M^c if necessary, we may assume that $M^c \subset \mathbf{C}^n$ is a Stein manifold. Now we have the decomposition $\nu_{M^c} = \nu' \oplus L$, where L is a line bundle. Moreover, we may assume that ν is a subbundle of $M^c \times \mathbf{C}^n$. Thus, a neighborhood U of the zero section of ν_{M^c} is identified with a neighborhood of M^c. We now want to show that there is a holomorphic mapping

$$\psi\colon (u,v) \mapsto (u, \lambda(u,v)v), \quad u \in \nu',\ v \in L$$

such that $\psi^*\Omega = \varphi^*\Omega$. Let $w = (w_1, \cdots, w_k)$ be local coordinates on M^c, and let $\xi = (\xi_1, \dots, \xi_{n-k-1})$ and t be local trivializations of ν' and L respectively. Then in local coordinates, ψ must be in the form $(w, \xi, t) \to (w, \xi, t'(w, \xi, t))$ with $t'|_{t=0} = 0$. Put

$$\varphi^*\Omega = f(w,\xi,t)dw \wedge d\xi \wedge dt, \quad \Omega = a(w,\xi,t)dw \wedge d\xi \wedge dt,$$

where $dw = dw_1 \wedge \ldots \wedge w_k$, $d\xi = d\xi_1 \wedge \ldots \wedge d\xi_{n-k-1}$. Then $\psi^*\Omega = \varphi^*\Omega$ is equivalent to the equation $a(w,\xi,t')\partial t'/\partial t = f(w,\xi,t)$. By the Cauchy-Kowalewski theorem, we know that for small $|t|$ the solution t' exists uniquely. This means that the required mapping ψ is uniquely determined in local coordinates. Therefore, there is a holomorphic mapping defined in a neighborhood of M^c such that $\psi^*\Omega = \varphi^*\Omega$. Since the restriction of ψ to M^c is the identity mapping, it is easy to see that ψ is one-to-one in some neighborhood of M^c. □

Notice that all totally real and real analytic embeddings of a compact surface M in $\mathbf{C}^n$ $(n \geq 3)$ are biholomorphically equivalent. Also, M admits a totally real and real analytic embedding in $\mathbf{C}^3$. Hence, the complex normal bundle of a totally real embeddings of M in $\mathbf{C}^n$ $(n \geq 3)$ is the direct sum of a line bundle and a trivial bundle. From Proposition 4.3, we have the following.

Corollary 4.4. *All totally real and real analytic embeddings of a compact surface in $\mathbf{C}^n$ $(n \geq 3)$ are unimodularly equivalent.*

References

[1] P. Ahern and W. Rudin, *Totally real embeddings of S^3 in $\mathbf{C}^3$*, Proc. Amer. Math. Soc., **94**(1983), no.3, 360-462.

[2] R. Bott, *The stable homotopy of the classical groups*, Proc. Nat. Acad. Sci. U.S.A., **43**(1957), 933-935.

[3] R. Bott, *Differential Forms in Algebraic Topology*, Grad. Text in Math. 82, Springer-Verlag, New York, 1982.

[4] F. Forstnerič, *Complex tangents of real surfaces in complex surfaces*, Duke Math. J., no.2, **67**(1992), 353-376.
[5] F. Forstnerič and J.-P. Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbf{C}^n$, Invent. Math., **112**(1993), 323-349.
[6] H. Grauert, *Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen*, Math. Ann., **133**(1957), 450-472.
[7] H. Grauert, *On Levi's Problem and the embedding of real-analytic manifolds*, Ann. Math., **68**(1958), 460-472.
[8] M.L. Gromov, *Convex integration of differential relations* I, Math. USSSR, Izv., **7**(1973), 329-343.
[9] M.L. Gromov, *Partial Differential Relations*, Ergeb. der Math. und ihrer Grenz., 3 Folge, Bd. 9, Springer-Verlag, Berlin Heidelberg, 1986.
[10] R.C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965.
[11] M.W. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc., **93**(1959), 242-276
[12] M.W. Hirsch, *Differential Topology*, Graduate Text in Math. **33**, Springer-Verlag, New York, 1976.
[13] M.L. Kervaire, *Some nonstable homotopy groups of Lie groups*, Illinois J. Math., no. 2, **4**(1960), 161-169.
[14] F. Sigrist, *Groupes d'homotopie des variétés de Stiefel complexes*, Comment. Math. Helv., Fasc. 2, **43**(1968), 121-131.
[15] S. Smale, *Classifications of immersions of spheres in Eulidean Space*, Ann. of Math., **69**(1959), 327-344.
[16] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, New Jersey, 1951.
[17] E.L. Stout and W.R. Zame, *A Stein manifold topologically but not holomorphically equivalent to a domain in* $\mathbf{C}^N$, Adv. in Math., **60**(1986), 154-160.
[18] A. Weinstein, *Lectures on Symplectic Geometry*, Reg. Conf. Ser. Math. **29**, Amer. Math. Soc., Providence, 1977.
[19] R.O. Wells, *Compact real submanifolds of a complex manifold with non-degenerate holomorphic tangent bundles*, Math. Ann., **179**(1969), 123-129.

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