EXAMPLES OF DOMAINS WITH NON-COMPACT AUTOMORPHISM GROUPS

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Abstract. We give an example of a bounded, pseudoconvex, circular domain in $\mathbb{C}^n$ for any $n \geq 3$ with smooth real-analytic boundary and non-compact automorphism group, which is not biholomorphically equivalent to any Reinhardt domain. We also give an analogous example in $\mathbb{C}^2$, where the domain is bounded, non-pseudoconvex, is not equivalent to any Reinhardt domain, and the boundary is smooth real-analytic at all points except one.

Let D be a bounded or, more generally, a hyperbolic domain in $\mathbb{C}^n$. Denote by $\mathrm{Aut}(D)$ the group of biholomorphic self-mappings of D. The group $\mathrm{Aut}(D)$, with the topology given by uniform convergence on compact subsets of D, is in fact a Lie group [Kob].

A domain D is called Reinhardt if the standard action of the n-dimensional torus $\mathbb{T}^n$ on $\mathbb{C}^n$,

$$z_j \mapsto e^{i\phi_j} z_j, \qquad \phi_j \in \mathbb{R}, \quad j = 1, \dots, n,$$

leaves D invariant. For certain classes of domains with non-compact automorphism groups, Reinhardt domains serve as standard models up to biholomorphic equivalence (see e.g. [R], [W], [BP], [GK1], [Kod]).

It is an intriguing question whether *any* domain in $\mathbb{C}^n$ with non-compact automorphism group and satisfying some natural geometric conditions is biholomorphically equivalent to a Reinhardt domain. The history of the study of domains with non-compact automorphism groups shows that there were expectations that the answer to this question would be positive (see [Kra]). In this note we give examples that show that the answer is in fact negative.

While the domain that we shall consider in Theorem 1 below has already been noted in the literature [BP], it has never been proved that this domain is not biholomorphically equivalent to a Reinhardt domain. Note that this domain is circular, i.e. it is invariant under the special rotations

$$z_j \mapsto e^{i\phi} z_j, \qquad \phi \in \mathbb{R}, \quad j = 1, \dots, n.$$

Our first result is the following

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Theorem 1. *There exists a bounded, pseudoconvex, circular domain $\Omega \subset \mathbb{C}^3$ with smooth real-analytic boundary and non-compact automorphism group, which is not biholomorphically equivalent to any Reinhardt domain.*

Proof. Consider the domain

$$\Omega = \{|z_1|^2 + |z_2|^4 + |z_3|^4 + (\overline{z_2}z_3 + \overline{z_3}z_2)^2 < 1\}.$$

The domain Ω is invariant under the action of the two-dimensional torus $\mathbb{T}^2$

$$\begin{aligned} z_1 &\mapsto e^{i\phi_1} z_1, && \phi_1 \in \mathbb{R}, \\ z_j &\mapsto e^{i\phi_2} z_j, && \phi_2 \in \mathbb{R}, \quad j = 2, 3, \end{aligned}$$

and therefore is circular. It is also a pseudoconvex, bounded domain with smooth real-analytic boundary. The automorphism group $\mathrm{Aut}(\Omega)$ is non-compact since it contains the following subgroup

$$\begin{aligned} z_1 &\mapsto \frac{z_1 - a}{1 - \overline{a}z_1}, \\ z_2 &\mapsto \frac{(1-|a|^2)^{\frac{1}{4}} z_2}{(1-\overline{a}z_1)^{\frac{1}{2}}}, \\ z_3 &\mapsto \frac{(1-|a|^2)^{\frac{1}{4}} z_3}{(1-\overline{a}z_1)^{\frac{1}{2}}}, \end{aligned} \tag{1}$$

for a complex parameter a with $|a| < 1$.

We are now going to explicitly determine $\mathrm{Aut}(\Omega)$. Let $F = (f_1, f_2, f_3)$ be an automorphism of Ω. Then, since Ω is bounded, pseudoconvex and has real-analytic boundary, F extends smoothly to $\overline{\Omega}$ [BL]. Therefore, F must preserve the rank of the Levi form $\mathcal{L}_{\partial\Omega}(q)$ of $\partial\Omega$ at every $q \in \partial\Omega$. The only points where $\mathcal{L}_{\partial\Omega} \equiv 0$ are those of the form $(e^{i\alpha}, 0, 0)$, $\alpha \in \mathbb{R}$. These points must be preserved by F. This observation implies that $f_j(e^{i\alpha}, 0, 0) = 0$ for all $\alpha \in \mathbb{R}$, $j = 2, 3$. Restricting f_2, f_3 to the unit disc $\Omega \cap \{z_2 = z_3 = 0\}$, we see that $f_j(z_1, 0, 0) = 0$ for all $|z_1| \le 1$, $j = 2, 3$. Therefore, $F(0) = (b, 0, 0)$ for some $|b| < 1$. Taking the composition of F and the automorphism G of the form (1) with $a = b$, we find that the mapping $G \circ F$ preserves the origin. Since Ω is circular, it follows from a theorem of H. Cartan [C] that $G \circ F$ must be linear. Therefore, any automorphism of Ω is the composition of a linear automorphism and an automorphism of the form (1).

The above argument also shows that any linear automorphism of Ω can be written as

$$\begin{aligned} z_1 &\mapsto e^{i\phi_1} z_1, \\ z_2 &\mapsto a z_2 + b z_3, \\ z_3 &\mapsto c z_3 + d z_3, \end{aligned}$$

where $\phi_1 \in \mathbb{R}$, $a, b, c, d \in \mathbb{C}$, and the transformation in the variables (z_2, z_3) is an automorphism of the section $\Omega \cap \{z_1 = 0\}$. Further, since the only points of $\partial\Omega$ where $\operatorname{rank} \mathcal{L}_{\partial\Omega} = 1$ are those of the form $(z_1, w, \pm w)$ with $w \neq 0$ and since automorphisms of Ω preserve such points, it follows that any linear automorphism of Ω is in fact given by

$$\begin{aligned} z_1 &\mapsto e^{i\phi_1} z_1, \\ z_2 &\mapsto e^{i\phi_2} z_{\sigma(2)}, \\ z_3 &\mapsto \pm e^{i\phi_2} z_{\sigma(3)}, \end{aligned}$$

where $\phi_1, \phi_2 \in \mathbb{R}$, and σ is a permutation of the set $\{2, 3\}$.

The preceding description of $\mathrm{Aut}(\Omega)$ implies that $\dim \mathrm{Aut}(\Omega) = 4$. That is to say, each of the four connected components of $\mathrm{Aut}(\Omega)$ is parametrized by the point a from the unit disc and by the rotation parameters ϕ_1, ϕ_2.

Suppose now that Ω is biholomorphically equivalent to a Reinhardt domain $D \subset \mathbb{C}^3$. Since Ω is bounded, it follows that D is hyperbolic. It follows from [Kru] that any hyperbolic Reinhardt domain $G \subset \mathbb{C}^n$ can be biholomorphically mapped onto its normilized form $\tilde{G}$ for which the identity component $\mathrm{Aut}_0(\tilde{G})$ of $\mathrm{Aut}(\tilde{G})$ is described as follows. There exist integers $0 \le s \le t \le p \le n$ and $n_i \ge 1$, $i = 1, \dots, p$, with $\sum_{i=1}^p n_i = n$, and real numbers α_i^k, $i = 1, \dots, s$, $k = t+1, \dots, p$, and β_j^k, $j = s+1, \dots, t$, $k = t+1, \dots, p$ such that if we set $z^i = \left(z_{n_1+\dots+n_{i-1}+1}, \dots, z_{n_1+\dots+n_i}\right)$, $i = 1, \dots, p$, then $\mathrm{Aut}_0(\tilde{G})$ is given by the mappings

$$\begin{aligned} z^i &\mapsto \frac{A^i z^i + b^i}{c^i z^i + d^i}, \quad i = 1, \dots, s, \\ z^j &\mapsto B^j z^j + e^j, \quad j = s+1, \dots, t, \\ z^k &\mapsto C^k \frac{\prod_{j=s+1}^t \exp\left(-\beta_j^k \left(2\overline{e^j}^T B^j z^j + |e^j|^2\right)\right) z^k}{\prod_{i=1}^s (c^i z^i + d^i)^{2\alpha_i^k}}, \quad k = t+1, \dots, p, \end{aligned} \tag{2}$$

where

$$\begin{aligned} &\begin{pmatrix} A^i & b^i \\ c^i & d^i \end{pmatrix} \in SU(n_i, 1), \quad i = 1, \dots, s, \\ &B^j \in U(n_j), \quad e^j \in \mathbb{C}^{n_j}, \quad j = s+1, \dots, t, \\ &C^k \in U(n_k), \quad k = t+1, \dots, p. \end{aligned}$$

The normalized form $\tilde{G}$ is written as

$$\begin{aligned} G = \Bigg\{ &\left|z^1\right| < 1, \dots, |z^s| < 1, \\ &\left(\frac{z^{t+1}}{\prod_{i=1}^s \left(1 - |z^i|^2\right)^{\alpha_i^{t+1}} \prod_{j=s+1}^t \exp\left(-\beta_j^{t+1} |z^j|^2\right)}, \dots, \right. \\ &\left. \frac{z^p}{\prod_{i=1}^s \left(1 - |z^i|^2\right)^{\alpha_i^p} \prod_{j=s+1}^t \exp\left(-\beta_j^p |z^j|^2\right)} \right) \in \tilde{G}_1 \Bigg\}, \end{aligned} \tag{3}$$

where $\tilde{G}_1 := \tilde{G} \bigcap \left\{z^i = 0,\, i = 1, \dots, t\right\}$ is a hyperbolic Reinhardt domain in $\mathbb{C}^{n_{t+1}} \times \dots \times \mathbb{C}^{n_p}$.

It is now easy to see that, for any hyperbolic Reinhardt domain $D \subset \mathbb{C}^3$ written in a normilized form $\tilde{D}$, $\mathrm{Aut}_0(\tilde{D})$ given by formulas (2) cannot have dimension equal to 4.

This completes the proof. ∎

Remark. The theorem can be easily extended to $\mathbb{C}^n$ for any $n \geq 3$ (just replace $|z_1|^2$ in the defining function of Ω by $\sum_{j=1}^{n-2} |z_j|^2$, z_2 by z_{n-1}, z_3 by z_n).

There is considerable evidence that, in complex dimension two, an example such as that constructed in Theorem 1 does not exist. Certainly the example provided above depends on the decoupling, in the domain Ω, of the variables z_2, z_3 from the variable z_1. Such decoupling is not possible when the dimension is only two.

The work of Bedford and Pinchuk (see [BP] and references therein) suggests that the only smoothly bounded domains in $\mathbb{C}^2$ with non-compact automorphism groups are (up to biholomorphic equivalence) the complex ellipsoids

$$\Omega_\alpha = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2\alpha} < 1\},$$

where α is a positive integer. It is also a plausible conjecture that any bounded domain in $\mathbb{C}^2$ with non-compact automorphism group and a boundary of finite smoothness C^k for $k \geq 1$, is biholomorphically equivalent to some Ω_α, where $\alpha \geq 1$ and is not necessarily an integer. Of course all the domains Ω_α are pseudoconvex and Reinhard.

However, as the following theorem shows, if we allow the boundary to be non-smooth at just one point, then the domain may be non-pseudoconvex and be non-equivalent to any Reinhardt domain.

Theorem 2. *There exists a bounded, non-pseudoconvex domain $\Omega \subset \mathbb{C}^2$ with non-compact automorphism group such that $\partial\Omega$ is smooth real-analytic everywhere except one point (this exceptional point is an orbit accumulation point for the automorphism group action), and such that Ω is not biholomorphically equivalent to any Reinhardt domain.*

For the proof of Theorem 2, we first need the following lemma.

Lemma A. *If $\Omega \subset \mathbb{C}^2$ is a bounded, non-pseudoconvex, simply-connected domain such that the identity component $Aut_0(\Omega)$ of the automorphism group $Aut(\Omega)$ is non-compact, then Ω is not biholomorphically equivalent to any Reinhardt domain.*

Proof of Lemma A. Suppose that Ω is biholomorphically equivalent to a Reinhardt domain D. Since Ω is bounded, it follows that D is hyperbolic. Also, since $\mathrm{Aut}_0(\Omega)$ is non-compact, then so is $\mathrm{Aut}_0(D)$. We are now going to show that any such domain D is either pseudoconvex, or not simply-connected, or cannot be biholomorphically equivalent to a bounded domain. This result clearly implies the lemma.

We can now assume that the domain D is written in its normalized form $\tilde{D}$ as in (3), and $\mathrm{Aut}_0(\tilde{D})$ is given by formulas (2). Then, since $\mathrm{Aut}_0(\tilde{D})$ is non-compact, it must be that $t > 0$. Next, if $p = t$, then $\tilde{D}$ is either non-hyperbolic (for $s < t$), or (for $s = t$) is the unit ball or the unit polydisc and therefore is pseudoconvex. Thus we can assume that $t = 1$, $p = 2$, $n_1 = n_2 = 1$.

Let $\tilde{D}_1 \subset \mathbb{C}$ be the hyperbolic Reinhardt domain analogous to $\tilde{G}_1$ that was defined above (see (3)). Clearly, there are the following possibilities for $\tilde{D}_1$:

(i) $\tilde{D}_1 = \{0 < |z_2| < R\}$, $0 < R < \infty$;
(ii) $\tilde{D}_1 = \{r < |z_2| < R\}$, $0 < r < R \leq \infty$;
(iii) $\tilde{D}_1 = \{|z_2| < R\}$, $0 < R < \infty$.

For the cases **(i)**, **(ii)**, $\tilde{D}$ is always not simply-connected, and therefore we will concentrate on the case **(iii)**. If $s = 0$, then $\tilde{D}$ is not hyperbolic since it contains the complex line $\{z_2 = 0\}$. Thus we can assume that $s = 1$. Next observe that, for $\alpha_1^2 \geq 0$, the domain $\tilde{D}$ is always pseudoconvex. Thus we may take $\alpha_1^2 < 0$. Then the domain $\tilde{D}$ has the form

$$\tilde{D} = \left\{ |z_1| < 1,\, |z_2| < \frac{R}{(1 - |z_1|^2)^\gamma} \right\}, \qquad \gamma > 0.$$

We will now show that the above domain $\tilde{D}$ cannot be biholomorphically equivalent to a bounded domain. More precisely, we will show that any bounded holomorphic function on $\tilde{D}$ is independent of z_2.

Let $f(z_1, z_2)$ be holomorphic on $\tilde{D}$ and $|f| < M$ for some $M > 0$. For every ρ such that $|\rho| \leq \frac{R}{2}$, the disc $\Delta_\rho = \{|z_1| < 1,\, z_2 = \rho\}$ is contained in $\tilde{D}$. We will show that $\partial f / \partial z_2 \equiv 0$ on every such Δ_ρ, which implies that $\partial f / \partial z_2 \equiv 0$ everywhere in $\tilde{D}$.

Fix a point $(\mu, \rho) \in \Delta_\rho$ and restrict f to the disc $\Delta'_\mu = \{z_1 = \mu,\, |z_2| < R_\mu\}$, where $R_\mu = R/2(1 - |\mu|^2)^\gamma$. Clearly, $(\mu, \rho) \in \Delta'_\mu$ and $\overline{\Delta'_\mu} \subset \tilde{D}$. By the Cauchy Integral Formula

$$f(\mu, z_2) = \frac{1}{2\pi i} \int_{\partial \Delta'_\mu} \frac{f(\mu, \zeta)}{\zeta - z_2}\, d\zeta,$$

for $|z_2| < R_\mu$, and therefore

$$\frac{\partial f}{\partial z_2}(\mu, \rho) = \frac{1}{2\pi i} \int_{\partial \Delta'_\mu} \frac{f(\mu, \zeta)}{(\zeta - \rho)^2}\, d\zeta.$$

Hence

$$\left| \frac{\partial f}{\partial z_2}(\mu, \rho) \right| \leq \frac{M R_\mu}{(R_\mu - |\rho|)^2}.$$

Letting $|\mu| \to 1$ and taking into account that $R_\mu \to \infty$, we see that $|\partial f / \partial z_2(\mu, \rho)| \to 0$ as $|\mu| \to 1$. Therefore, $\partial f / \partial z_2 \equiv 0$ on Δ_ρ.

The lemma is proved. ■

Proof of Theorem 2. We will now present a domain that satisfies the conditions of the lemma. Set

$$\Omega = \left\{ |z_1|^2 + |z_2|^4 + 8|z_1 - 1|^2 \left(\frac{z_2^2}{z_1 - 1} - \frac{3}{2} \frac{|z_2|^2}{|z_1 - 1|} + \frac{\overline{z_2}^2}{\overline{z_1} - 1} \right)^2 < 1 \right\}.$$

The domain Ω is plainly bounded since the third term on the left is non-negative. Next, the identity component $\mathrm{Aut}_0(\Omega)$ of its automorphism group is non-compact since it contains the subgroup

$$\begin{aligned} z_1 &\mapsto \frac{z_1 - a}{1 - az_1}, \\ z_2 &\mapsto \frac{(1-a^2)^{\frac{1}{4}} z_2}{(1-az_1)^{\frac{1}{2}}}, \end{aligned}$$

where $a \in (-1, 1)$.

Further, Ω is simply-connected, since the family of mappings $F_\tau(z_1, z_2) = (z_1, \tau z_2)$, $0 \le \tau \le 1$, retracts Ω inside itself, as $\tau \to 0$, to the unit disc $\{|z_1| < 1,\ z_2 = 0\}$ (which is simply-connected).

To show that Ω is not pseudoconvex, consider its unbounded realization. Namely, under the mapping

$$\begin{aligned} z_1 &\mapsto \frac{z_1 + 1}{z_1 - 1}, \\ z_2 &\mapsto \frac{\sqrt{2} z_2}{\sqrt{z_1 - 1}}, \end{aligned} \tag{4}$$

the domain Ω is transformed into the domain

$$\Omega' = \left\{ \mathrm{Re}\, z_1 + \frac{1}{4}|z_2|^4 + 2\left(z_2^2 - \frac{3}{2}|z_2|^2 + \overline{z_2}^2 \right)^2 < 0 \right\}.$$

It is easy to see that at the boundary point $(-\frac{3}{4}, 1) \in \partial\Omega'$ the Levi form of $\partial\Omega'$ is equal to $-|z_2|^2$, and thus is negative-definite. Therefore, Ω is non-pseudoconvex.

Hence, by Lemma A, Ω is not biholomorphically equivalent to any Reinhardt domain.

Next, if ϕ denotes the defining function of Ω, the following holds at every boundary point of Ω except $(1, 0)$:

$$\frac{\partial \phi}{\partial z_1} = \frac{1}{z_1 - 1} \left(-\frac{z_2}{2} \frac{\partial \phi}{\partial z_2} + 1 - \overline{z_1} \right),$$

and therefore $\mathrm{grad}\, \phi$ does not vanish at every such point. Hence, $\partial\Omega$ is smooth real-analytic everywhere except at $(1, 0)$.

The theorem is proved. ∎

Remarks.

1. The hypothesis of simple connectivity in Lemma A is automatically satisfied if, for example, the boundary of the domain is locally variety-free and smooth near some orbit accumulation point for the automorphism group of the domain (see e.g. [GK2]). For a smoothly bounded domain it would follow from a conjecture of Greene/Krantz [GK3].

2. Tedious calculations show that the boundary of the domain Ω in Theorem 2 is quite pathological near the exceptional point $(1,0)$. It is not Lipschitz-smooth of any positive degree. It would be interesting to know whether there is an example with Lipschitz-1 boundary at the bad point.

In fact, many more examples similar to that in Theorem 2 can be constructed in the following way. Let

$$\Omega' = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re} z_1 + P(z_2) < 0\}, \tag{5}$$

where $P = |z_2|^{2m} + Q(z_2)$ is a homogeneous non-plurisubharmonic polynomial, m is a positive integer, and $Q(z_2)$ is positive away from the origin. Then, by a mapping analogous to (4), Ω' can be transformed into a bounded domain Ω. The domain Ω is simply-connected, non-pseudoconvex, $\operatorname{Aut}_0(\Omega)$ is non-compact, and $\partial\Omega$ is smooth real-analytic everywhere except at the point $(1,0)$. For all such examples, $\partial\Omega$ is not Lipschitz-smooth of any positive degree at $(1,0)$.

It is also worth noting that, in the example contained in Theorem 2, the point $(-1,0)$ is also an orbit accumulation point, but $\partial\Omega$ is smooth real-analytic at this point.

3. It is conceivable that the domain Ω as in Theorem 2 has an alternative, smoothly bounded realization, but it looks plausible that if in formula (5) we allow $P(z_2)$ to be an arbitrary homogeneous polynomial positive away from the origin with no harmonic term, then domain (5) does not have a bounded realization with C^1-smooth boundary, unless $P(z_2) = c|z_2|^{2m}$, where $c > 0$ and m is a positive integer.

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