HODGE THEORY IN THE SOBOLEV TOPOLOGY FOR THE DE RHAM COMPLEX

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ABSTRACT. The authors study the Hodge theory of the exterior differential operator d acting on q-forms on a smoothly bounded domain in $\mathbb{R}^{N+1}$, and on the half space $\mathbb{R}^{N+1}_+$. The novelty is that the topology used is not an L^2 topology but a Sobolev topology. This strikingly alters the problem as compared to the classical setup. It gives rise to a boundary value problem belonging to a class of problems first introduced by Višik and Eskin, and by Boutet de Monvel.

CONTENTS

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PRELIMINARIES

0. Introductory Remarks

Fix a smoothly bounded domain $\Omega \subseteq \mathbb{R}^{N+1}$. In classical treatments of the d operator (see [SWE]), one considers the complex

$$\bigwedge^q \xrightarrow{d} \bigwedge^{q+1} \xrightarrow{d} \cdots .$$

Here $\bigwedge^q$ denotes the q-forms of Cartan and de Rham having $L^2(\Omega)$ coefficients and the operator d is understood to be densely defined. One considers the operator $\Box = dd^* + d^*d$. Here the adjoints are calculated in the $L^2(\Omega)$ topology.

Now $\Box$ makes sense on those forms ψ such that $\psi \in \operatorname{dom} d^*$ and $d\psi \in \operatorname{dom} d^*$. One can decompose the space $\bigwedge^q$ into (the closure of) the image of $\Box$ and its orthogonal complement. Then one exploits this decomposition to construct a right inverse for $\Box$. This inverse is easily used to show that $\Box$ and its accompanying boundary conditions form a second order elliptic boundary value problem of the classical (coercive) type.

In 1963, J.J. Kohn [KOH1] determined how to carry out the analog of these last calculations for the $\overline{\partial}$ operator of complex analysis on a strongly pseudoconvex domain Ω in $\mathbb{C}^n$. This is the so-called $\overline{\partial}$-Neumann problem. Of course this analysis, while similar in spirit, is much more complicated. It gave rise to the important "Kohn canonical solution" to the equation $\overline{\partial}u = f$. That is the solution u that is orthogonal to holomorphic functions *in the* $L^2(\Omega)$ *topology.*

Experience in the function theory of several complex variables has shown that it is useful to have many different canonical solutions to the $\overline{\partial}$-problem. For instance, in the strongly pseudoconvex case we profitably study the Kohn solution by comparing it with the Henkin solution (not canonical, but nearly so), see [HEN], and the Phong solution (determined in the $L^2(\partial\Omega)$ topology rather than the $L^2(\Omega)$ topology), see [PHO].

The ultimate goal of the program that we are initiating in this paper is to construct solutions to the $\overline{\partial}$ problem that are orthogonal to holomorphic functions in a Sobolev space W^s inner product. There are *a priori* reasons for knowing that this program is feasible. First, Sobolev space is a Hilbert space, so there must be a minimal solution in the Sobolev topology. Second, Boas [BOA] has studied the space of W^s holomorphic functions as a Hilbert space with reproducing kernel. The associated Bergman projection operator is of course closely related to the Neumann operator for the $\overline{\partial}$ problem.

The present paper carries out the first step of the proposed program. We work out the Hodge theory for the exterior differentiation operator d in the inner product induced by the Sobolev space W^s topology. Of particular interest are the boundary conditions that arise when we calculate the adjoint d^* in this topology, and the elliptic boundary value problem that arises when we consider $\square = dd^* + d^*d$. We calculate a complete existence and regularity theory.

In this paper, we restrict attention to the case $s = 1$. This is done both for convenience and to keep the notation relatively simple—even in this basic case the calculations are often unduly cumbersome. In geometric applications, the case $s = 1$ is already of great interest. We leave the detailed treatment of higher order s to a future paper.

In future work, we will carry out this program of analysis in the Sobolev space topology for the $\overline{\partial}$-Neumann problem on a strongly pseudoconvex domain. Not only will this give rise to new canonical solutions for the $\overline{\partial}$ problem, but it should give a new way to view the Sobolev regularity of the classical $\overline{\partial}$ problem, and of understanding the subelliptic gain of $1/2$ in regularity.

Essentially the paper is divided into two parts. In the first of these we study the boundary value problem that arises from our Hodge theory on the special domain given by the half space, and in the second one we deal with the problem on a smoothly bounded domain. *We would like point out that the second part has been written so that it can be read independently from the first part.* When we use results from the half space case, we give precise reference to them.

In detail, the plan of the paper is as follows: **Section 1** introduces basic notation and definitions, while in **Section 2** we formulate the problem and states the main results. **Sections 3** through **6** are devoted the problem on the half space.

In **Section 3** we calculate the operator d^* on 1-forms and also calculate its domain when the region under study is the upper half space $\mathbb{R}^{N+1}_+$. **Section 4** completes the detailed calculation of d^*. **Section 5** applies a pseudodifferential formalism developed by Boutet de Monvel to the study of our elliptic boundary value problem. This section is included essentially to show how the boundary value problem under investigation can be view as an example of a very general kind of problem first introduced in [BDM1]-[BDM3] and in [ESK]. In **Section 5** we explicitly construct the solution of the boundary value problem on the half space in the case of functions. **Section**

6 studies the problem for q-forms. **Sections 7** through **12** deals with the problem on a bounded smooth domain. In **Section 7** we set up the problem and calculate the domain of d^* and the semi-explicit expression for d^*. In **Section 8** we introduce notation and some technical facts needed in the sequel. **Section 9** contains the proof of the coercive estimate, and the proof of our result about existence of solutions for the boundary value problem. **Section 10** is devoted to the proof of the *a priori* estimate in the case of functions. **Section 11** gives the proof of the regularity result in the case of q-forms. Finally, in **Section 12**, we conclude the proof of our main result in the case of a smoothly bounded domain.

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1. Basic Notation and Definitions

We use the symbol d to denote the usual operator of exterior differentiation acting on q-forms. We let Ω denote a smoothly bounded domain in $\mathbb{R}^{N+1}$. Usually, for simplicity only, a domain is assumed to be connected. The symbol $\bigwedge^q(\Omega)$ denotes the q-forms on Ω with smooth coefficients. The symbol $\bigwedge_0^q(\Omega)$ denotes the forms with coefficients that are C^∞ and compactly supported in Ω. We let $\bigwedge^q(\overline{\Omega})$ denote the q-forms with coefficients that are smooth on $\overline{\Omega}$, and $\bigwedge_0^q(\overline{\Omega})$ denote the q-forms with coefficients in $C^\infty(\overline{\Omega})$ and having compact support in $\overline{\Omega}$ (that is, the support may not be disjoint from the boundary of Ω).

Some of our explicit calculations will be performed on the special domain $\mathbb{R}^{N+1}_+ = \{x = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^{N+1} : x_0 > 0\}$. The half space is of course unbounded, so the function spaces we deal with must take into account the integrability at infinity. On the other hand the half space has the advantage of allowing explicit calculations.

If $\mathcal{L}$ is an operator on forms, then $\mathcal{L}'$ denotes its *formal adjoint*, that is, its adjoint calculated when acting on elements of $\bigwedge_0^p(\Omega)$.

In the discussion that follows we let Ω denote either a smoothly bounded domain Ω, or the half space $\mathbb{R}^{N+1}_+$. If s is a non-negative integer then the s order Sobolev space norm on functions on Ω is given by

$$\|f\|_s^2 = \sum_{|\alpha|\le s} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|^2_{L^2(\Omega)} .$$

The associated inner product is

$$\langle f, g\rangle_s = \sum_{|\alpha|\le s} \int_\Omega \frac{\partial^\alpha f}{\partial x^\alpha} \, \overline{\frac{\partial^\alpha g}{\partial x^\alpha}} \, dV(x).$$

Here dV stands for ordinary Lebesgue volume measure.

For s a non-negative integer we define the Sobolev space $W^s(\Omega)$ as the closure of $C_0^\infty(\overline{\Omega})$. When $s \in \mathbb{R}_+$ we define $W^s(\Omega)$ by interpolation (see [LIM] for instance). Moreover, for $s \in \mathbb{R}_+$, we denote by $\overset{\circ}{W}{}^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^s(\Omega)$. When

$s < 0$ we define the negative Sobolev space $W^s(\Omega)$ to be the dual of $\overset{\circ}{W^s}(\Omega)$ with respect to the standard L^2-pairing.

On the Euclidean space $\mathbb{R}^{N+1}$ we consider the Fourier transform defined initially for a testing function $f \in C_0^\infty$ as

$$\widehat{f}(\xi) \equiv \mathcal{F}f(\xi) \equiv \int_{\mathbb{R}^{N+1}} f(x)e^{-2\pi i x\cdot\xi}\,dx.$$

We will also consider the tangential Fourier transform of functions defined on the half space $\mathbb{R}^{N+1}_+$: If $f \in C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$ we set

$$\hat{f}(x_0, \xi') = \int_{\mathbb{R}^N} f(x_0, x')e^{-2\pi i x'\cdot\xi'}\,dx'.$$

We denote the inverse tangential Fourier transform of a function $g(x_0, \xi')$ by $\breve{g}(x_0, x')$.

For any $s \in \mathbb{R}$ the Sobolev space $W^s(\mathbb{R}^{N+1})$ can be defined via the Fourier transform. Indeed, we set

$$W^s(\mathbb{R}^{N+1}) = \Big\{f \in L^2(\mathbb{R}^{N+1}) : \int_{\mathbb{R}^{N+1}} (1 + |\xi|^2)^s |\mathcal{F}f(\xi)|^2\,d\xi < \infty\Big\}.$$

We will also consider the Sobolev spaces $W^s(b\Omega)$ defined on the boundary of our domain, $s \in \mathbb{R}$. In the case $\Omega = \mathbb{R}^{N+1}_+$, $W^s(b\Omega)$ is just the classical Sobolev space on $\mathbb{R}^N$. In the case of a smoothly bounded domain Ω, the Sobolev space can be defined by fixing a smooth atlas $\{\chi_j\}$ on $\partial\Omega$, letting ϕ_j be a partition of unity subordinate to this atlas, and defining the Sobolev norm of a function f on $b\Omega$ as

$$\|f\|^2_{W^s(b\Omega)} = \sum_j \|\phi_j f \circ {\chi_j}^{-1}\|^2_{W^s(\mathbb{R}^N)}.$$

Of course this norm is highly non-intrinsic, but a different choice of an atlas gives rise to an equivalent norm.

On the half space $\mathbb{R}^{N+1}_+$, and on a bounded domain Ω, we consider the space of q-forms with coefficients in W^s. We denote such spaces by $W^s_q(\mathbb{R}^{N+1}_+)$ and $W^s_q(\Omega)$ respectively. Fix the standard basis for q-forms: $\{dx^I\}$, where $I = (i_1, \ldots, i_q)$ is an increasing multi-index, i.e. $0 \le i_1 < \cdots < i_q \le N$. The Sobolev inner product on the space of q-forms in the case s a non-negative integer is given by

$$\langle\phi, \psi\rangle_s = \sum_I \langle\phi_I, \psi_I\rangle_s = \sum_I \sum_{|\alpha|\le s} \int \frac{\partial^\alpha\phi_I}{\partial x^\alpha}\overline{\frac{\partial^\alpha\psi_I}{\partial x^\alpha}}dV(x),$$

where the integral is taken over all of space, i.e. either on $\mathbb{R}^{N+1}_+$ or on Ω. In the case of general s, for $\phi \in \bigwedge^q$, $\phi = \sum_I \phi_I dx^I$, we define the norm of ϕ by

$$\|\phi\|^2_{W^s} = \sum_I \|\phi_I\|^2_{W^s}.$$

Of course W^s is a Hilbert space when equipped with the foregoing inner product. (In particular, W^s can be identified with its own dual in a natural way.)

Until further notice, we use D_j to denote the partial derivative in the j^{th} variable, and if $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N)$ is a multi-index then we let D^α denote the corresponding differential monomial.

2. Formulation of the Problem and Statement of the Main Results

Given the operator d,

$$d : \bigwedge^q \longrightarrow \bigwedge^{q+1},$$

we think of it as a densely defined operator on the space W^1_q of q-forms with coefficients in W^1, both in the case of the half space $\mathbb{R}^{N+1}_+$ and also in the case of a smoothly bounded domain Ω. Let d^* denote the W^1-Hilbert space adjoint of d. It is a densely defined (unbounded) operator

$$d^* : W^1_{q+1} \longrightarrow W^1_q.$$

We shall study the boundary value problem

$$\begin{cases} (dd^* + d^*d)\phi = \alpha & \text{on } \mathbb{R}^{N+1}_+ \text{ (resp. on } \Omega) \\ \phi \in \operatorname{dom} d^* \\ d\phi \in \operatorname{dom} d^* \end{cases},$$

for $\alpha \in W^s_q(\Omega)$; we shall prove existence and regularity theorems, both in the case of the half space and of a smoothly bounded domain, for q-forms, $q = 0, 1, \ldots, N+1$. The conditions ϕ, $d\phi \in \operatorname{dom} d^*$ are ultimately expressed as boundary conditions.

Our main results are the following.

Proposition 2.1. *Let Ω denote a smoothly bounded domain, or the special domain $\mathbb{R}^{N+1}_+$. Let $q = 0, 1, \ldots, N$. Then we have*

$$\operatorname{dom} d^* \cap \bigwedge\nolimits_0^{q+1}(\overline{\Omega}) = \Big\{ \phi \in \bigwedge\nolimits_0^{q+1}(\overline{\Omega}) : \nabla_{\vec{n}} \phi \lfloor \vec{n} \big|_{b\Omega} = 0 \Big\}.$$

Here $\nabla_{\vec{n}}$ denotes the covariant differentiation of a form in the normal direction, and "$\lfloor$" the contraction operator between a form and a vector field.

The next result shows a striking difference with the classical case. We begin with the case of $\mathbb{R}^{N+1}_+$. Here, and in the rest of the paper, we denote by $\triangle'$ the Laplace operator defined on the boundary of the half space:

$$\triangle' = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}.$$

Proposition 2.2. *Let d' denote the formal adjoint of d. Then on* $\operatorname{dom} d^*$ *we have*

$$d^* = d' + \mathcal{K},$$

where $\mathcal{K}$ is an operator sending $(q+1)$-forms to q-forms. The operator $\mathcal{K}$ is the solution operator of the following boundary value problem

$$\begin{cases} (-\triangle+I)(\mathcal{K}\phi)=0 & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial}{\partial x_0}(\mathcal{K}\phi)=(-\triangle'+I)\left(\phi\lfloor\left(\dfrac{\partial}{\partial x_0}\right)\right) & \text{on } \mathbb{R}^N=b\mathbb{R}^{N+1}. \end{cases}$$

Explicitly, if $\phi\in\bigwedge^{q+1}(\overline{\mathbb{R}^{N+1}_+})$, $\phi=\sum_{|I|=q+1}\phi_I\,dx^I$, then

$$(\mathcal{K}\phi)_I\hat{\ }(x_0,\xi')=-\sqrt{1+|2\pi\xi'|^2}e^{\sqrt{1+|2\pi\xi'|^2}x_0}\hat{\phi}_{0I}(0,\xi').$$

In the case of a smoothly bounded domain we have the following:

Proposition 2.3. *Let Ω be a smoothly bounded domain. Then, on* $\operatorname{dom} d^*$*,*

$$d^*=d'+\mathcal{K}_\Omega,$$

where d' is the formal adjoint of d, and $\mathcal{K}_\Omega$ is an operator sending $(q+1)$-forms to q-forms. The components of $\mathcal{K}\phi$ are solutions of the following boundary value problems

$$\begin{cases} (-\triangle+I)(\mathcal{K}_\Omega\phi)_I=0 & \text{on } \Omega \\ \dfrac{\partial}{\partial n}(\mathcal{K}_\Omega\phi)_I=T_2\phi\lfloor\vec{n} & \text{on } b\Omega \end{cases},$$

where T_2 is a second order tangential differential operator on forms (to be defined in detail below).

At this point we fix the following notation that we will use throughout the entire paper.

Definition 2.4. On the half space $\mathbb{R}^{N+1}_+$ we set

$$G=\mathcal{K}d+d\mathcal{K},$$

and for a smoothly bounded domain Ω we set

$$G_\Omega=\mathcal{K}_\Omega d+d\mathcal{K}_\Omega.$$

Notice that our boundary value problems become

$$\begin{cases} (-\triangle+G)\phi=\alpha & \text{on } \mathbb{R}^{N+1}_+ \\ \phi\in\operatorname{dom} d^* \\ d\phi\in\operatorname{dom} d^* \end{cases}, \tag{2.1}$$

and

$$\begin{cases} (-\triangle+G_\Omega)\phi=\alpha & \text{on } \Omega \\ \phi\in\operatorname{dom} d^* \\ d\phi\in\operatorname{dom} d^* \end{cases}. \tag{2.2}$$

We are now ready to state our results about existence and regularity of the boundary value problem.

THEOREM 2.5. *Let $s > 1/2$. Let $\alpha \in W^s_q(\mathbb{R}^{N+1}_+)$, $\operatorname{supp} \alpha \subseteq \{x : |x| < R\}$. Let $N \geq 4$, Then there exists a unique tempered distribution ϕ solution of the boundary value problem (2.1) such that*

$$\|\phi\|_{s+2} \leq c\|\alpha\|_s,$$

where $c = c(s, R) > 0$ does not depend on α, nor on ϕ.

If $N = 2, 3$ and $\int \alpha_I \, dx = 0$, and if $N = 1$ and $\int x^\beta \alpha_I \, dx = 0$ when $|\beta| \leq 1$, then the conclusion as above still holds.

THEOREM 2.6. *Consider the boundary value problem (2.2). Let $s > 1/2$. Then there exists a finite dimensional subspace (the harmonic space) $\mathcal{H}_q$ of $\bigwedge^q(\overline{\Omega})$ and a constant $c = c_s > 0$ such that if $\alpha \in W^s_q(\Omega)$ is orthogonal (in the W^1-sense) to $\mathcal{H}_q$, then the boundary value problem (2.2) has a unique solution ϕ orthogonal to $\mathcal{H}_q$ such that*

$$\|\phi\|_{s+2} \leq c\|\alpha\|_s.$$

Remark. If $s < 1$, by saying that α is orthogonal in the W^1-sense we mean that α is W^s-limit of smooth forms that are orthogonal in W^1 to $\mathcal{H}_q$.

Finally we have

THEOREM 2.7. *Let Ω be a smoothly bounded domain in $\mathbb{R}^{N+1}$. Let $W^1_q(\Omega)$ denote the 1-Sobolev space of q-forms. Then we have the strong orthogonal decomposition*

$$W^1_q = dd^*(W^1_q) \bigoplus d^*d(W^1_q) \bigoplus \mathcal{H}_q\,,$$

where $\mathcal{H}_q$ is a finite dimensional subspace.

THE PROBLEM ON THE HALF SPACE

3. The Operator d^* on 1-Forms and Its Domain

The aim of this section is to determine $\operatorname{dom} d^* \cap \bigwedge^1(\overline{\mathbb{R}^{N+1}_+})$, and to give an explicit expression for d^*.

Easy computations show that the formal adjoint d' of d with respect to the inner product of W^1 is the same as the formal adjoint in the classical L^2-case. Therefore, for $\phi \in \bigwedge^1_0(\mathbb{R}^{N+1}_+)$, $\phi = \sum_{j=0}^N \phi_j \, dx^j$, we see that

$$d'\phi = -\sum_{j=0}^{N} D_j \phi_j \equiv -\operatorname{div} \phi.$$

It is not hard to see that the specific form of d' is independent of which order s of Sobolev inner product we use.

Next we want to compute the Hilbert space adjoint d^* of d. Recall that W^s_q will be the closure of the smooth q-forms $\bigwedge^q_0(\overline{\mathbb{R}^{N+1}_+})$ with compact support in $\overline{\mathbb{R}^{N+1}_+}$ with respect to the norm

$$\|\phi\|_s \equiv \sum_{\substack{|\alpha| \le s \\ |I| = q}} \int_{\mathbb{R}^{N+1}_+} |D^\alpha \phi_I|^2 \, dV < \infty.$$

Let $u \in \bigwedge^0$, $\phi \in \sum_{i=0}^N \phi_i \, dx^i$ have compact support in $\overline{\mathbb{R}^{N+1}_+}$ (that is, the support is compact but not necessarily disjoint from the boundary). Then

$$\begin{aligned} \langle du, \phi \rangle_s &= \sum_{j=0}^{N} \langle D_j u, \phi_j \rangle_1 \\ &= \sum_{k=0}^{N} \left(\sum_{j=1}^{N} \int_{\mathbb{R}^{N+1}_+} D_j D_k u \overline{D_k \phi_j} \, dV + \int_{\mathbb{R}^{N+1}_+} D_0 (D_k u) \overline{D_k \phi_0} \, dV \right). \end{aligned}$$

The first sum inside the parentheses in the last line equals

$$-\sum_{j=1}^{N} \int_{\mathbb{R}^{N+1}_+} D_k u \, \overline{D_j D_k \phi_j} \, dV$$

since u and ϕ_j have compact support and the derivation is in the tangential directions (i.e. the directions $x_1, \ldots, x_N$). The second term in parentheses equals (with $x' = (x_1, \ldots, x_N)$)

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(D_k u \overline{D_k \phi_0} \Big|_{x_0=0}^{x_0=\infty} - \int_0^\infty D_k u \overline{D_0 D_k \phi_0} \, dx_0 \right) dx' \\ &\qquad = -\int_{\mathbb{R}^N} D_k u \overline{D_k \phi_0} \Big|_{x_0=0} dx' - \int_{\mathbb{R}^{N+1}_+} D_k u \overline{D_0 D_k \phi_0} \, dV(x). \end{aligned}$$

Therefore, combining the last several lines, we find that

$$\begin{aligned}\langle du,\phi\rangle_1 &= \sum_{k=0}^{N}\left(-\sum_{j=0}^{N}\int_{\mathbb{R}^{N+1}_+} D_k u\overline{D_j D_k\phi_j} - \int_{\mathbb{R}^N} D_k u\overline{D_k\phi_0}\bigg|_{x_0=0} dx'\right)\\ &= \langle u, d'\phi\rangle_1 - \int_{\mathbb{R}^N} u\overline{\phi_0}\bigg|_{x_0=0} dx' - \sum_{k=1}^{N}\int_{\mathbb{R}^N} D_k u\overline{D_k\phi_0}\bigg|_{x_0=0} dx'.\end{aligned}$$

Now we can determine the domain of d^* (we must do this before we can calculate the operator d^* itself). Recall that, by the definition of Hilbert space adjoint, $\phi\in\operatorname{dom} d^*$ if and only if there is a number c_ϕ such that for all $u\in\bigwedge^0$ we have

$$\Big|\langle du,\phi\rangle_1\Big| \le c_\phi\|u\|_1.$$

Proposition 3.1. *The* 1*-form* ϕ *lies in* $\Big(\operatorname{dom} d^*\Big)\cap\bigwedge^1(\overline{\mathbb{R}^{N+1}_+})$ *if and only if*

$$D_0\phi_0\bigg|_{x_0=0} \equiv 0.$$

Proof. We begin by setting

$$\mathcal{D} = \operatorname{dom} d^*\cap {\bigwedge}^1_0(\overline{\mathbb{R}^{N+1}_+})$$

and

$$\mathcal{E} = \left\{\phi\in {\bigwedge}^1_0(\overline{\mathbb{R}^{N+1}_+}) : \frac{\partial}{\partial x_0}\phi_0(0,x')\equiv 0\right\}.$$

We first show that $\mathcal{E}\subseteq\mathcal{D}$. Let $\phi\in\mathcal{E}$. We have already computed that

$$\begin{aligned}\langle du,\phi\rangle_1 &= \langle u,d'\phi\rangle_1 - \int_{\mathbb{R}^N} u\overline{\phi_0}\bigg|_{x_0=0} dx' - \sum_{k=1}^{N}\int_{\mathbb{R}^N} D_k u\overline{D_k\phi_0}\bigg|_{x_0=0} dx'\\ &= \langle u,d'\phi\rangle_1 + \int_{\mathbb{R}^N} u\bigg|_{x_0=0}(\triangle'\phi_0-\phi_0)\bigg|_{x_0=0} dx'.\end{aligned}$$

Clearly the mapping

$$u\longmapsto\langle u,d'\phi\rangle_1$$

is bounded on $W^1(\mathbb{R}^{N+1}_+)$, and so

$$|\langle du,\phi\rangle_1| \le c_\phi\Big(\|u\|_{W^1(\mathbb{R}^{N+1}_+)} + \|u|_{x_0=0}\|_{L^2(\mathbb{R}^N)}\Big).$$

(Note that the bound here depends on ϕ, but that is acceptable.) Now, recall the standard trace theorems for Sobolev spaces (see either [LIM] or [KR1]). Then

$$\Big\|u|_{x_0=0}\Big\|_{L^2(\mathbb{R}^N)} \le \|u\|_{W^t(\mathbb{R}^{N+1}_+)},$$

for $t>1/2$, and in particular for $t=1$. This establishes the containment $\mathcal{E}\subseteq\mathcal{D}$.

Next we establish that $\mathcal{D} \subset \mathcal{E}$. Seeking a contradiction, we suppose that $\phi \in \mathcal{D} \equiv \operatorname{dom} d^* \cap \bigwedge_0^1(\mathbb{R}_+^{N+1})$ and that $D_0\phi_0(0,x')$ is not identically zero. We may suppose that ϕ_0 is real and that

$$D_0\phi_0(0,x') \geq 1 \qquad \text{when} \quad |x'| \leq 2$$

(just multiply by a suitable constant and scale).

Now set

$$u_\epsilon(x_0,x') = (x_0+\epsilon)^{3/4}\chi(x_0,x'),$$

where χ is a cutoff function with support in $\{|x_0| \leq 2\} \times \{|x'| \leq 2\}$ and such that $\chi \equiv 1$ in the set $\{|x_0| \leq 1\} \times \{|x'| \leq 1\}$.

We claim that there is a constant $C > 0$, independent of ϵ, such that

$$(i) \qquad \|u_\epsilon\|_{W^s(\mathbb{R}_+^{N+1})} \leq C \qquad \forall \epsilon,\ 0 < \epsilon \leq \epsilon_0\,,$$

and

$$(ii) \qquad \int_{\mathbb{R}^N} D_0 u_\epsilon \overline{D_0\phi_0}\bigg|_{x_0=0} dx' \sim \frac{1}{\epsilon^{1/4}} \qquad \text{as} \quad \epsilon \to 0.$$

Once these two facts are established, it is clear that it is not possible to find a constant such that the mapping $u \longmapsto \langle du, \phi\rangle_1$ is bounded on $W^1(\mathbb{R}_+^{N+1})$, and thus ϕ cannot be in $\operatorname{dom} d^*$.

Fact (ii) follows since

$$D_0 u_\epsilon(x_0,x')\bigg|_{x_0=0} = \left(\frac{3}{4}\epsilon^{-1/4}\chi(0,x') + \epsilon^{3/4}D_0\chi(0,x')\right).$$

In order to prove (i) we need only check the size of

$$\int_{\mathbb{R}_+^{N+1}} |D_0 u_\epsilon|^2 dV.$$

But

$$D_0 u_\epsilon(x_0,x') = \frac{3}{4}\,\frac{1}{(x_0+\epsilon)^{1/4}}\,\chi(x_0,x') + \Phi_\epsilon(x_0,x'),$$

where Φ_ϵ lies in $C_0^\infty(\mathbb{R}_+^{N+1})$ and is uniformly bounded in ϵ. Thus

$$\begin{aligned}\int_{\mathbb{R}_+^{N+1}} |D_0 u_\epsilon|^2\, dV \quad &\leq \quad C\int_0^1 \frac{1}{(x_0+\epsilon)^{1/2}}\, dx_0 + C' \\ &\leq \quad C,\end{aligned}$$

independent of ϵ as $\epsilon \to 0$. This establishes (i). $\square$

The operators $\mathcal{K}$ and d^*. Our next task is to determine the explicit expression for the operator d^*. We will learn that, contrary to the classical case, the operator d^* when restricted to $\mathrm{dom}\,(d^*) \cap \bigwedge^1$ *does not* coincide with d'. Once we have the expression for d^* then we are in a position to formulate the boundary value problem in the case $q = 1$. The fundamental calculation is this:

Proposition 3.2. *If* $\phi \in \mathrm{dom}\, d^* \cap \bigwedge_0^1(\overline{\mathbb{R}^{N+1}_+})$, $\phi = \sum_{j=0}^N \phi_j \, dx^j$, *then*

$$d^*\phi = -\mathrm{div}\,\phi + \int_{\mathbb{R}^N} \frac{\partial}{\partial x_0}\left(e^{-\sqrt{1+4\pi^2|\xi'|^2}x_0}\right)\widehat{\phi_0}(0,\xi')e^{2\pi i x'\cdot\xi'}\,d\xi'$$

Proof. We need to determine the operator d^* that satisfies

$$\langle df, \phi\rangle_1 = \langle f, d^*\phi\rangle_1.$$

The spirit of the calculation that follows is to rewrite the expressions that involve the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_s$ in terms of expressions that involve only the L^2 inner product.

Now

$$\begin{aligned}
\langle df, \phi\rangle_1 \quad &= \quad \int_{\mathbb{R}^{N+1}_+} \sum_{j=0}^N \frac{\partial f}{\partial x_j}\overline{\phi_j}\,dV + \int_{\mathbb{R}^{N+1}_+} \sum_{j,k=0}^N \frac{\partial^2 f}{\partial x_j \partial x_k}\overline{\frac{\partial \phi_j}{\partial x_k}}\,dV \\
&\overset{\text{(parts)}}{=} \quad \langle f, -\,\mathrm{div}\,\phi\rangle_1 - \int_{\mathbb{R}^N} f(0,x')\overline{\phi_0(0,x')}\,dx' \\
&\qquad + \int_{\mathbb{R}^N} f(0,x')\overline{\sum_{k=1}^N \frac{\partial^2}{\partial x_k^2}\phi_0(0,x')}\,dx'.
\end{aligned}$$

Of course we have used the boundary condition that occurs in our characterization of $\mathrm{dom}\, d^*$.

Next we write

$$d^*\phi = -\,\mathrm{div}\,\phi + u,$$

and we wish to determine the explicit expression for $u \equiv \mathcal{K}\phi$, i.e. the formula for the operator $\mathcal{K}$ mapping 1-forms into functions, such that $d^* = d' + \mathcal{K}$. Then we have

$$\begin{aligned}
\langle df, \phi\rangle_1 &= \langle f, -\,\mathrm{div}\,\phi\rangle_1 + \langle f, u\rangle_1 \\
&= \langle f, -\,\mathrm{div}\,\phi\rangle_1 + \int_{\mathbb{R}^{N+1}_+} f\overline{u}\,dV \\
&\qquad - \int_{\mathbb{R}^{N+1}_+} f\overline{\triangle u}\,dV - \int_{\mathbb{R}^N} f(0,x')\overline{\frac{\partial u}{\partial x_0}(0,x')}\,dx'.
\end{aligned}$$

Here we have integrated by parts.

Comparing the two last calculations we see that the function u must satisfy the following equation:

$$-\int_{\mathbb{R}^N} f(0,x')\overline{\phi_0(0,x')}\,dx' + \int_{\mathbb{R}^N} f(0,x')\overline{\sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2}\phi_0(0,x')\,dx'}$$
$$= \int_{\mathbb{R}^{N+1}_+} f\overline{u}\,dV - \int_{\mathbb{R}^{N+1}_+} f\overline{\triangle u}\,dV - \int_{\mathbb{R}^N} f(0,x')\overline{\frac{\partial u}{\partial x_0}(0,x')}\,dx'.$$

This equality must hold for all $f \in C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$. By choosing functions supported away from the boundary we obtain a differential equation on $\mathbb{R}^{N+1}_+$; inserting this in the equation above gives rise to a boundary condition. Thus the function $u \in W^1(\mathbb{R}^{N+1}_+)$ must be a solution of the following boundary value problem:

$$\begin{cases} -\triangle\, u + u = 0 & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial u}{\partial x_0}(0,\cdot) = -\triangle'\,\phi_0(0,\cdot) + \phi_0(0,\cdot) & \text{on } \{0\}\times\mathbb{R}^N \end{cases}.$$

Using the partial Fourier transform with respect to x', we find that the equation $\triangle u - u = 0$ becomes

$$\frac{\partial^2}{\partial x_0^2}\widehat{u}(x_0,\xi') - (1+|2\pi\xi'|^2)\widehat{u}(x_0,\xi') = 0, \qquad \text{with } \ \widehat{u} \in L^2(\mathbb{R}^{N+1}_+).$$

For fixed ξ', this is an ordinary differential equation in x_0 whose solution is

$$\widehat{u}(x_0,\xi') = a(\xi')e^{\sqrt{1+|2\pi\xi'|^2}x_0} + b(\xi')e^{-\sqrt{1+|2\pi\xi'|^2}x_0}.$$

Since $\widehat{u} \in L^2(\mathbb{R}^{N+1}_+)$, it must be that $a(\xi') \equiv 0$. Thus

$$\hat{u}(x_0,\xi') = b(\xi')e^{-\sqrt{1+|2\pi\xi'|^2}x_0} \tag{3.1}$$

and the solution to the above boundary value problem is

$$\mathcal{K}\phi(x_0,x') = \int_{\mathbb{R}^N} b(\xi')e^{-\sqrt{1+|2\pi\xi'|^2}x_0}e^{2\pi i\xi'\cdot x'}\,d\xi'.$$

Now we determine $b(\xi')$ in such a way that the boundary conditions are satisfied. We compute the derivative of $\hat{u}(x_0,\xi)$, as given by (3.1), with respect to x_0, and evaluate at $x_0 = 0$. We find that

$$b(\xi')\sqrt{1+|2\pi\xi'|^2} = -|2\pi\xi'|^2\widehat{\phi_0}(0,\xi') - \widehat{\phi_0}(0,\xi'),$$

that is,

$$b(\xi') = -\sqrt{1+|2\pi\xi'|^2}\widehat{\phi_0}(0,\xi').$$

Therefore the Hilbert space adjoint of d (and this has been the main point of these calculations) is

$$\begin{aligned} d^*\phi &\equiv -\operatorname{div}\phi + \mathcal{K}\phi \\ &= -\operatorname{div}\phi - \int_{\mathbb{R}^N} \sqrt{1+|2\pi\xi'|^2}e^{-\sqrt{1+|2\pi\xi'|^2}x_0}\widehat{\phi_0}(0,\xi')e^{2\pi i\xi'\cdot x'}\,dx' \\ &\equiv -\operatorname{div}\phi + \Big(K_{x_0} * \phi_0(0,\,\cdot\,)\Big)(x'); \end{aligned}$$

here the convolution is in $\mathbb{R}^N$ with respect to the variable x', and

$$K_{x_0}(x') = \left(\frac{\partial}{\partial x_0}e^{-\sqrt{1+|2\pi\xi'|^2}x_0}\right)^{\vee}(x')$$

is the convolution kernel. □

At this point, following the classical methodology of Hodge theory, we want to study the problem (2.1):

$$\begin{cases} \Big(dd^* + d^*d\Big)u = f \\ u, du \in \operatorname{dom} d^* \end{cases}.$$

According to our calculations this problem becomes

$$\begin{cases} -\bigtriangleup u + \displaystyle\int_{\mathbb{R}^N}\left(\frac{\partial}{\partial x_0}e^{-\sqrt{1+|2\pi\xi'|^2}x_0}\right)\widehat{\frac{\partial u}{\partial x_0}}(0,x')e^{2\pi i\xi'\cdot x'}\,dx' = f & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial^2}{\partial x_0^2}u(0,x') = 0 & \text{on } \mathbb{R}^N \end{cases}. \tag{3.2}$$

At the conclusion of these calculations we want to remark that the boundary value problems that appear in the case of higher degree forms is treated in Section 6.

Recall, from Definition 2.4, that

$$\begin{aligned} Gu(x_0,x') &= \mathcal{K}\left(\frac{\partial u}{\partial x_0}\right)(x_0,x') \\ &= \int_{\mathbb{R}^N}\left(\frac{\partial}{\partial x_0}e^{-\sqrt{1+|2\pi\xi'|^2}x_0}\right)\widehat{\frac{\partial u}{\partial x_0}}(0,x')e^{2\pi i\xi'\cdot x'}\,dx. \end{aligned} \tag{3.3}$$

Now we define $\tilde{\mathcal{K}}$ by setting $\mathcal{K} \equiv \tilde{\mathcal{K}} \circ \gamma$, where $\gamma = \gamma_0$ denotes the standard *trace* operator of restriction to the boundary.

Notice that the kernel $\widehat{K_{x_0}}(\xi')$ is $-\sqrt{1+|2\pi\xi'|^2}e^{-\sqrt{1+|2\pi\xi'|^2}}x_0$, and thus is quite similar to the normal derivative of the Fourier transform of the Poisson kernel. But actually the operator $\tilde{\mathcal{K}}$ is more regular than the Poisson integral itself. The higher degree of regularity of $\tilde{\mathcal{K}}$ is seen immediately by noticing that the symbol of the convolution kernel K_{x_0}, that is $-\sqrt{1+|2\pi\xi'|^2}e^{-\sqrt{1+|2\pi\xi'|^2}x_0}$, is smooth in ξ' for all ξ'. We record here a few simple properties of the operators $\tilde{\mathcal{K}}$ and G. We denote by

$\mathcal{S}(\mathbb{R}^N)$ the space of Schwartz functions on $\mathbb{R}^N$, and by $\mathcal{S}(\overline{\mathbb{R}^{N+1}_+})$ the space of restriction to $\mathbb{R}^{N+1}_+$ of Schwartz functions on $\mathbb{R}^{N+1}$. We have the following proposition.

Proposition 3.3. *We have that*

$$\widetilde{\mathcal{K}} : \mathcal{S}(\mathbb{R}^N) \longrightarrow \mathcal{S}(\overline{\mathbb{R}^{N+1}_+})$$

continuously. Moreover

$$\widetilde{\mathcal{K}} : W^{r+1/2}(\mathbb{R}^N) \longrightarrow W^r(\mathbb{R}^{N+1}_+),$$

continuously for $r \geq 0$. *In addition,*

$$G : W^{r+2}(\mathbb{R}^{N+1}) \longrightarrow W^r(\mathbb{R}^{N+1}_+),$$

and

$$G : \mathcal{S}(\overline{\mathbb{R}^{N+1}_+}) \longrightarrow \mathcal{S}(\overline{\mathbb{R}^{N+1}_+}),$$

continuously.

Proof. Recall that $G = \widetilde{\mathcal{K}} \circ \gamma \circ (\partial/\partial x_0)$, so it suffices to prove the corresponding statements for $\widetilde{\mathcal{K}}$.

Let $v \in W^{r+1/2}(\mathbb{R}^N)$. Then

$$\widehat{\widetilde{\mathcal{K}}v}(x_0, \xi') = -\sqrt{1 + |2\pi\xi'|^2} e^{-\sqrt{1+|2\pi\xi'|^2}} \hat{v}(\xi').$$

It suffices to consider the case r an integer, and use interpolation to obtain the general case. For $r \in \mathbb{N}$ we have

$$\begin{aligned}
\|\widetilde{\mathcal{K}}v\|^2_{W^r(\mathbb{R}^{N+1}_+)} &\leq \sum_{j=0}^{r} \int_{\mathbb{R}^{N+1}_+} (1 + |\xi'|^2)^{r-j} \left| \frac{\partial^j \widehat{\widetilde{\mathcal{K}}v}}{\partial x_0^j}(x_0, \xi') \right|^2 d\xi' \, dx_0 \\
&\leq c \int_{\mathbb{R}^{N+1}_+} (1 + |\xi'|^2)^{r+1} |\hat{v}(\xi')|^2 e^{-\sqrt{1+|2\pi\xi'|^2} x_0} \, dx_0 \, d\xi' \\
&\leq c \int_{\mathbb{R}^N} (1 + |\xi'|^2)^{r+1/2} |\hat{v}(\xi')|^2 \, d\xi' \\
&\leq c\|v\|^2_{W^{r+1/2}(\mathbb{R}^N)}.
\end{aligned}$$

To show that $\widetilde{\mathcal{K}}$ is continuous between the (two different) Schwartz spaces, we notice that

$$\begin{aligned}
&\left| x_0^\ell (2\pi\xi')^\alpha \frac{\partial}{\partial x_0^\ell} \frac{\partial}{\partial \xi'^\alpha} \widehat{\widetilde{\mathcal{K}}v}(x_0, \xi') \right| \\
&\quad \leq \left| x_0^{\ell+|\alpha|} (1 + |2\pi\xi'|^2)^{\frac{\ell+1}{2}+|\alpha|} e^{-\sqrt{1+|2\pi\xi'|^2} x_0} \hat{v}(\xi') \right| \\
&\quad \leq c \left\{ \sup_{x_0 \geq 0} \left[x_0^{\ell+|\alpha|} e^{-x_0} \right] \right\} \sup_{|\xi'|} (1 + |2\pi\xi'|)^{\frac{\ell+1}{2}+|\alpha'|} |\hat{v}(\xi')|,
\end{aligned}$$

which is finite for $v \in \mathcal{S}(\mathbb{R}^N)$. □

We conclude this section with some remarks.

REMARK 3.4. We have that

$$-\phi_0 = \sigma(d',\, dx_0)\,\phi,$$

where $\sigma(d',\, dx_0)$ is the symbol of d' in the normal direction. With this notation,

$$\langle du, \phi\rangle_1 = \langle u, d'\phi\rangle_1 + \int_{b\mathbb{R}^{N+1}_+} \langle u, \sigma(d',\, dx_0)\phi\rangle_1\, dx',$$

where $b\mathbb{R}^{N+1}$ is the boundary of the half space.

Indeed, using the standard definition (see, for instance, [KR1]) of symbol, we select a testing function ρ such that $\rho(0,x') = 0$ and $d\rho = dx_0$ (for instance, $\rho(x) = x_0$ locally will do). Then

$$\begin{aligned} \sigma(d',\, dx_0) &= d'(\rho\phi)\Big|_{x=(0,x')} \\ &= -\sum_{j=0}^{N} D_j(\rho\phi_j)\Big|_{x_0=0} \\ &= -\phi_0. \end{aligned}$$

Thus we may write, for $\phi = \sum_{|I|=q} \phi_I\, dx^I$,

$$\langle du, \phi\rangle_1 = \langle u, d'\phi\rangle_1 + \int_{b\mathbb{R}^{N+1}_+} \langle u, \sigma(d',\, dx_0)\phi\rangle_1\, dx'.$$

In the next Proposition we describe the domain of d^* in the topology of $W^s(\mathbb{R}^{N+1}_+)$ in case $s \in \mathbb{N}$. Essentially the same computations as in the case $s = 1$ prove the following result.

Proposition 3.5. *Denote by d^* the Hilbert space adjoint of d in the topology of $W^s(\mathbb{R}^{N+1}_+)$ for $s = 1, 2, \ldots$. Let $\phi \in \bigwedge^1_0(\overline{\mathbb{R}^{N+1}_+})$. Then ϕ lies in $\big(\mathrm{dom}\, d^*\big) \cap \bigwedge^1(\overline{\mathbb{R}^{N+1}_+})$ if and only if*

$$D_0^s\phi_0\Big|_{x_0=0} \equiv \frac{\partial^s}{\partial x_0^s}\phi_0\Big|_{x_0=0} \equiv 0.$$

Proof. We only indicate the changes in the proof for the case $s = 1$. We have that

$$\langle du, \phi\rangle_s = \langle u, d'\phi\rangle_s - \sum_{j=0}^{s-1} \sum_{|\alpha'|\le s-j} \int_{b\mathbb{R}^{N+1}_+} D_0^j D^{\alpha'} u \overline{D_0^j D^{\alpha'}\phi_0}\, dx'.$$

Clearly the mapping

$$u \longmapsto \langle u, d'\phi\rangle_s$$

is bounded on $W^s(\mathbb{R}^{N+1}_+)$. Then, using the trace theorem, we want to show that for any j, $0 \leq j \leq s-1$, and $|\alpha'| \leq s-j$, the mapping

$$u \longmapsto I_j \equiv \int_{b\mathbb{R}^{N+1}_+} D_0^j D^{\alpha'} u \, \overline{D_0^j D^{\alpha'} \phi_0} \, dx'$$

is bounded in the $W^s(\mathbb{R}^{N+1}_+)$ norm. This establishes the containment $\mathcal{E} \subseteq \mathcal{D}$.

In order to establish that $\mathcal{D} \subset \mathcal{E}$, suppose that $\phi \in \mathcal{D} \equiv \operatorname{dom} d^* \cap \bigwedge_0^1(\mathbb{R}^{N+1}_+)$ and that

$$D_0^s \phi_0(0, x') \equiv \frac{\partial^s}{\partial x_0^s} \phi_0(0, x')$$

is not identically zero. Setting

$$u_\epsilon(x_0, x') = x_0^{s-1}(x_0 + \epsilon)^{3/4} \chi(x_0, x'),$$

and arguing as before, it is easy to see that

$$(i) \qquad \|u_\epsilon\|_{W^s(\mathbb{R}^{N+1}_+)} \leq C \qquad \forall \epsilon, \ 0 < \epsilon \leq \epsilon_0$$

and

$$(ii) \qquad \sum_{j=0}^{s} \sum_{|\alpha'| \leq s-j} \int_{\mathbb{R}^N} D_0^j D^{\alpha'} u_\epsilon \overline{D_0^j D^{\alpha'} \phi_0} \bigg|_{x_0=0} dx' \ \sim \ \frac{1}{\epsilon^{1/4}} \qquad \text{as} \ \ \epsilon \to 0.$$

These two facts show that ϕ cannot be in $\operatorname{dom} d^*$. □

Finally, we also would like to observe that the boundary value problems (2.1) and (2.2) are in fact pseudodifferential boundary value problems. This kind of boundary value problems has been studied by several authors (see [BDM3], [ESK], [GRU], and [RESC]), as we indicate in the next section.

4. Boutet de Monvel-Type Analysis of the Boundary Value Problem

The boundary value problems (2.1) and (2.2) can be seen as an instance of a very general theory developed by Boutet de Monvel and several other authors. In this section we give a short summary of this theory and of its applications to our problem. Such a general approach mainly produces results of local character in which the data are often assumed to be more regular then we can afford. This is why we actually choose a different, and more direct, approach to solving our boundary value problems.

For maximum generality, let Ω be either a smoothly bounded domain in $\mathbb{R}^{N+1}$ or a half space in $\mathbb{R}^{N+1}$. Define $\mathcal{A}$ to be the system

$$\mathcal{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} : \begin{pmatrix} C_0^\infty(\overline{\Omega}) \\ C_0^\infty(b\Omega)^M \end{pmatrix} \longrightarrow \begin{pmatrix} C^\infty(\overline{\Omega}) \\ C^\infty(b\Omega)^{M'} \end{pmatrix}. \tag{4.1}$$

The components of this formula are as follows:

1) The numbers M, M' are non-negative integers, $C_0^\infty(b\Omega)^M$ is the Cartesian product of M copies of $C_0^\infty(b\Omega)$, and $C^\infty(b\Omega)^{M'}$ is the Cartesian product of M'copies of $C^\infty(b\Omega)$;

2) The operator P_Ω is a pseudodifferential operator taking functions defined on Ω into functions defined on Ω. [For more on pseudodifferential operators, see [KR1].] We will require P_Ω to have the "transmission property" (to be defined below);

3) The operator G is a "singular Green operator." It is defined on functions on Ω, taking values in the set of functions defined on Ω;

4) The operator K is a "Poisson operator". It takes (M-tuples of) functions on the boundary to functions on the domain;

5) The operator T is a "trace operator". Let γ_ℓ be the classical trace operator of order ℓ,

$$\begin{array}{ccc} u & \longmapsto & \left.\dfrac{\partial^\ell u}{\partial x_0^\ell}\right|_{x_0=0} \\ C_0^\infty(\overline{\Omega}) & \longrightarrow & C_0^\infty(b\Omega). \end{array}$$

By definition, a trace operator T is the composition of a pseudodifferential operator S' on the boundary and some restriction operator γ_ℓ: $T = S' \circ \gamma_\ell$.

6) The operator S (to repeat) is a pseudodifferential operator on the boundary.

Notice that our problem

$$\begin{cases} (-\bigtriangleup + G)u = f & \text{on } \mathbb{R}_+^{N+1} \\ \dfrac{\partial^2 u}{\partial x_0^2}(0, x') = 0 & \text{on } \{0\} \times \mathbb{R}^N \end{cases} .$$

is a special case of the above setup, with $M = 0, M' = 1$. We would take $K = 0$, $S = 0$ and consider

$$\mathcal{A} = \begin{pmatrix} P_\Omega + G \\ T \end{pmatrix} : C_0^\infty(\overline{\Omega}) \longmapsto \begin{pmatrix} C^\infty(\overline{\Omega}) \\ C^\infty(b\Omega) \end{pmatrix}.$$

We shall construct a parametrix $\mathcal{R}$ for this operator which will have the form

$$\mathcal{R} = (Q \quad K) : \begin{pmatrix} C_0^\infty(\overline{\Omega}) \\ C_0^\infty(b\Omega) \end{pmatrix} \longrightarrow C^\infty(\overline{\Omega}).$$

Note here that $M = 1$ and $M' = 0$.

The classical boundary value problems (such as the Dirichlet or Neumann problems) have the form

$$\mathcal{A} = \begin{pmatrix} D \\ T \end{pmatrix} : C_0^\infty(\overline{\Omega}) \longrightarrow \begin{pmatrix} C^\infty(\overline{\Omega}) \\ C^\infty(b\Omega)^{M'} \end{pmatrix},$$

where D is a differential operator and M' is the number of boundary conditions.

Definition 4.1. A pseudodifferential operator P of order d, defined on $\mathbb{R}^{N+1}$, is said to have the *transmission property* on Ω if the mapping

$$u \longmapsto RPEu \stackrel{\text{def}}{=} P_\Omega u$$

maps $W^s(\Omega)$ continuously to $W^{s-d}(\Omega)$ for all $s > -1/2$. Here E is the operator given by extending a function on $\overline{\Omega}$ to all of $\mathbb{R}^N$ by setting it equal to zero on the complement. Also R is the operator of restriction to Ω.

For our purposes, it is enough to note that any partial differential operator with smooth coefficients has the transmission property. This follows by inspection.

Definition 4.2. A *Poisson operator* K of order d is an operator defined by the formula

$$(Kv)(x_0, x') = \int_{\mathbb{R}^N} e^{2\pi i x'\cdot\xi'}\tilde{k}(x_0, x', \xi')\widehat{v}(\xi')\, d\xi',$$

where the symbol $\tilde{k}$ satisfies the estimates for $|\xi'| \geq 1$ given by

$$\left\| x_0^\ell D_0^{\ell'} D_{x'}^\beta D_{\xi'}^\alpha \tilde{k}(\,\cdot\,, x', \xi')\right\|_{L^2(\mathbb{R}_+)} \leq c(x')\Big(1 + |\xi'|^2\Big)^{(1/2)(d-1/2-\ell-\ell'-|\alpha|)}.$$

Here $c(x')$ is a continuous function on $\mathbb{R}^N$ that depends on $\ell, \ell', \alpha, \beta$.

One can show that a Poisson operator K of order d is continuous as a mapping from $W^s_{\text{comp}}(\mathbb{R}^N)$ to $W^{s-d+1/2}_{\text{loc}}(\overline{\mathbb{R}^{N+1}_+})$. By $W^s_{\text{comp}}(\Omega)$ we mean those elements of $W^s(\mathbb{R}^N)$ with compact support in Ω.

It is a standard fact that the Poisson integral for the upper half space is a Poisson operator, according to the above definition, of order 0; that is it maps $W^s_{\text{comp}}(\mathbb{R}^N)$ (where $\mathbb{R}^N = \partial\mathbb{R}^{N+1}$) to $W^{s+1/2}_{\text{loc}}(\overline{\mathbb{R}^{N+1}_+})$.

Definition 4.3. A *trace operator* of order $d \in \mathbb{R}$ and of **class** $r \in \mathbb{N}$ is an operator of the form

$$Tu = \left(\sum_{0\leq\ell\leq r-1} S_\ell\gamma_\ell\right)u,$$

where γ_ℓ is the standard trace operator of order ℓ and each S_ℓ is a pseudodifferential operator on $\mathbb{R}^N$ of order $d - \ell$.

The notion of "class" is a natural artifact of dealing with restriction operators in the context of Sobolev spaces. Recall that the classical restriction theorems sending elements of $W^s(\mathbb{R}^{N+1}_+)$ to elements of $W^{s-1/2}(\mathbb{R}^N)$ are only valid when $s > 1/2$. The idea of class addresses the necessary lower bound for s in theorems such as this.

Definition 4.4. A *singular Green operator* of *order* $d \in \mathbb{R}$ and *class* $r \in \mathbb{N}$ is an operator of the form

$$G = \sum_{0\leq\ell\leq r-1} K_\ell\gamma_\ell,$$

where each K_ℓ is a Poisson operator of order $d-\ell$ and each γ_ℓ is a standard trace operator of order ℓ.

It can be shown that a singular Green operator of order d and class r is a continuous operator

$$G : W^s_{\rm comp}(\overline{\mathbb{R}^{N+1}_+}) \longrightarrow W^{s-d}_{\rm loc}(\overline{\mathbb{R}^{N+1}_+})$$

as long as $s > r - 1/2$.

Definition 4.5. Let $\mathcal{A}$ be a system as in (4.1). We define the associated *boundary system* by

$$\mathbf{A} = \begin{pmatrix} p_\Omega(0,x',D_0,\xi') + g(x',D_0,\xi') & k(x',D_0,\xi') \\ t(x',D_0,\xi') & s(x',\xi') \end{pmatrix}.$$

Notice that, in this definition, we fix $x', \xi' \in \mathbb{R}^N$, and the symbol p_Ω of P_Ω is taken at the point $(0,x') \in b\mathbb{R}^{N+1}_+$. Finally, all the operators in the display act only in the D_0 slot.

Further observe that

$$p_\Omega(0,x',D_0,\xi') + g(x',D_0,\xi') : \mathcal{S}(\overline{\mathbb{R}_+}) \longrightarrow \mathcal{S}(\overline{\mathbb{R}^{N+1}_+})$$

and that

$$\begin{array}{rcl} k(x',D_0,\xi') : \mathbb{C} & \longrightarrow & \mathcal{S}(\overline{\mathbb{R}^{N+1}_+}) \\ a & \longmapsto & ak(x',x_0,\xi'). \end{array}$$

Furthermore, the boundary symbol operator of a trace operator is an operator

$$t(x',D_0,\xi')u = \sum_{0\le\ell\le r-1} s_\ell(x',\xi')\gamma_\ell u$$

that maps $\mathcal{S}(\overline{\mathbb{R}_+})$ into $\mathbb{C}$.

A system of the type (4.1) is called *elliptic* if there exists a second system $\mathcal{R}$ of type (4.1) such that

$$\mathcal{R}\mathcal{A} + I \qquad \text{and} \qquad \mathcal{A}\mathcal{R} + I$$

are negligible operators; here "negligible" means that all terms in the system send distributions with compact support into C^∞ functions. In the case of the half space this definition must be considered as "local", that is fixing a compact set in $\overline{\mathbb{R}^{N+1}_+}$.

Now we have

THEOREM 4.6 ([BDM3, Theorem 5.1]). *A system a of type (4.1) is elliptic if and only if both its boundary and its interior symbols are invertible.*

In the case of (2.1),

$$\mathcal{A} = \begin{pmatrix} -\triangle + G \\ T \end{pmatrix},$$

where $T \stackrel{\text{def}}{=} (I - \triangle')^{-1/2}\gamma_2$. Then we return to our analysis of the problem

$$\begin{cases} -\triangle u + Gu = f & \text{on } \mathbb{R}^{N+1}_+ \\ Tu = 0 & \text{on } \{0\} \times \mathbb{R}^N \end{cases} \tag{4.2}$$

Here, for convenience, we have replaced our usual boundary operator $(\partial^2/\partial x_0^2)$ with the (equivalent) operator T.

Our aim is to show that this is a pseudodifferential boundary value problem of type (4.1) and to apply Theorem 4.6 to our system.

First observe that $-\triangle$, being a partial differential operator of order 2 with smooth coefficients, certainly possesses the transmission property.

Next, let us analyze G. Recall that $G = \widetilde{\mathcal{K}}\gamma_1$, where $\widetilde{\mathcal{K}}$ has symbol

$$K_{x_0}(\xi') \equiv k(x_0, \xi') = -\sqrt{1 + |2\pi\xi'|^2} e^{-\sqrt{1+|2\pi\xi'|^2} x_0}.$$

In order to see that G is a singular Green operator, we need to estimate

$$\left\| x_0^\ell D_0^{\ell'} D_{\xi'}^\alpha k(\,\cdot\,, \xi') \right\|^2_{L^2_{x_0}(\mathbb{R}_+)} = \int_0^\infty \left| x_0^\ell D_{\xi'}^\alpha \left[\left(-\sqrt{1 + 4\pi^2|\xi'|^2} \right)^{\ell'+1} e^{-\sqrt{1+4\pi^2|\xi'|^2} x_0} \right] \right|^2 dx_0$$

The calculations made in Proposition 3.3 showed that $\widetilde{\mathcal{K}}$ has order 1. In fact $\widetilde{\mathcal{K}}$ is essentially a single derivative of the Poisson operator (which has order zero) so the assertion is plausible. It follows that

$$\text{order}\,(G) = \text{order}\,(\widetilde{\mathcal{K}}) + \text{order}\,(\gamma_1) = 2$$

and

$$\text{class}\,(G) = \text{order}\,(\gamma_1) + 1 = 2.$$

Next we observe that T is a trace operator of order 1 and class 3. Indeed, $T \stackrel{\text{def}}{=} (I - \triangle'')^{-1/2}\gamma_2$ so that

$$\text{order}\,(T) = \text{order}\,(I - \triangle')^{-1/2} + \text{order}\,(\gamma_2) = -1 + 2 = 1$$

and

$$\text{class}\,(T) = \text{order}\,(\gamma_2) + 1 = 3.$$

The interior symbol of this system is just the symbol of $-\triangle$ which is plainly invertible. The boundary symbol is

$$\begin{pmatrix} \left(|2\pi\xi'|^2 - \frac{\partial^2}{\partial x_0^2}\right) + \left(\frac{\partial}{\partial x_0} e^{-\sqrt{1+|2\pi\xi'|^2} x_0}\right)\gamma_1 \\ (1 + |\xi'|^2)^{-1/2}\gamma_2 \end{pmatrix} : \mathcal{S}(\overline{\mathbb{R}_+}) \longrightarrow \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}_+}) \\ \mathbb{C} \end{pmatrix}.$$

The demand that this system be invertible means that the pseudodifferential system

$$\begin{cases} v''(x_0) - \left(\dfrac{\partial}{\partial x_0} e^{-\sqrt{1+4\pi^2|\xi'|^2}x_0}\right) v'(0) - |2\pi\xi'|^2 v(x_0) = \psi & \text{on } \mathbb{R} \\ (1+|2\pi\xi'|^2)^{-1/2} v''(0) = \alpha \end{cases}$$

has one and only one solution $v \in \mathcal{S}(\overline{\mathbb{R}_+})$ for all $\psi \in \mathcal{S}(\overline{\mathbb{R}_+})$ and $\alpha \in \mathbb{C}$ and for $|\xi'| \geq c > 0$ fixed.

Straightforward and unenlightening calculations now prove the following result.

Proposition 4.7. *For $|\xi'| \geq c > 0$ fixed, the preceding system of equations has a unique solution for every choice of $(\psi, \alpha) \in \mathcal{S}(\overline{\mathbb{R}_+}) \times \mathbb{C}$.*

As a consequence we have:

THEOREM 4.8. *Consider the boundary value problem given by (4.2):*

$$\begin{cases} (-\triangle + G)u = f & \text{on } \mathbb{R}^{N+1}_+ \\ Tu = g & \text{on } \{0\} \times \mathbb{R}^N \end{cases}.$$

Let $s > 5/2$, and suppose that $\operatorname{supp} f$, $\operatorname{supp} g \subseteq B(0,R)$, for some $R > 0$. Moreover, let $\eta \in C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$. Then there exists a finite dimensional subspace $\mathcal{L}$ of $W^s_{comp}(\mathbb{R}^{N+1}_+) \times W^{s-1/2}_{comp}(\mathbb{R}^N)$ such that if $(f,g) \in \mathcal{L}^\perp$ then the problem (4.2) has a unique solution u satisfying

$$\|\eta u\|_{W^{s+2}(\mathbb{R}^{N+1}_+)} \leq C\Big(\|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|g\|_{W^{s-1/2}(\mathbb{R}^N)}\Big).$$

The constant C in the above estimate depends on R, s, the test function η, and on the spatial dimension N.

Our aim is to refine this theorem in order to allow $s > 1/2$ (instead of $s > 5/2$) and to be more precise about the local-global aspects of the problem. Moreover, we do so by determining the solution explicitly, and by describing the compatibility conditions.

5. The Explicit Solution in the Case of Functions

Our goal in the present section is to prove theorems about existence and regularity of the solution of the boundary value problem. We shall determine the solution of our boundary value problem explicitly. This will allow us to give precise estimates.

Thus we concentrate on the problem

$$\begin{cases} -\triangle u + Gu = f & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial^2 u}{\partial x_0^2}(0,\cdot) = 0 & \text{on } \mathbb{R}^N \end{cases} \tag{5.1}$$

for $u \in W^r(\mathbb{R}^{N+1}_+)$, where $r > 5/2$. Notice that we need this limitation on r in order to guarantee that $\partial^2 u/\partial x_0^2(0,\cdot)$ makes sense and belongs to $L^2(\mathbb{R}^N)$ (at least). Here, for $x_0 > 0$ and $\xi' \in \mathbb{R}^N$, we have

$$\widehat{Gu}(x_0,\xi') = -\sqrt{1+|2\pi\xi'|^2}\, e^{-\sqrt{1+|2\pi\xi'|^2}x_0} \frac{\partial \widehat{u}}{\partial x_0}(0,\xi').$$

Recall that we write $\widehat{w}(\xi')$ to denote the partial Fourier transform of w in $\mathbb{R}^N$, with respect to the x'-variables.

For greater flexibility we will solve the boundary value problem

$$\begin{cases} -\triangle u + Gu = f & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial^2 u}{\partial x_0^2}(0,\cdot) = h & \text{on } \mathbb{R}^N \end{cases} \tag{5.2}$$

In what follows we shall denote by $\mathcal{G}$ the classical Green's function for the Laplacian on $\mathbb{R}^{N+1}_+$; also $\mathcal{P}$ denotes the standard Poisson operator on $\mathbb{R}^{N+1}_+$. Recall (see [GAR]) that, for $N \geq 2$,

$$\mathcal{G}(x,y) = \frac{1}{(N-1)\omega_{N+1}} \Bigg[\frac{1}{\Big((x_0-y_0)^2 + |x'-y'|^2\Big)^{(N-1)/2}} - \frac{1}{\Big((x_0+y_0)^2 + |x'-y'|^2\Big)^{(N-1)/2}} \Bigg],$$

where ω_{N+1} denotes the surface measure of the N-dimensional unit sphere in $\mathbb{R}^{N+1}$. If $N = 1$ then

$$\mathcal{G}(x,y) = \frac{1}{2\pi} \log \frac{(x_0+y_0)^2 + (x_1-y_1)^2}{(x_0-y_0)^2 + (x_1-y_1)^2}.$$

Notice that

$$\widehat{\mathcal{G}f}(x_0,\xi') = \frac{1}{2}\frac{1}{|2\pi\xi'|} \int_0^\infty \left(e^{-|2\pi\xi'|\cdot|x_0-y_0|} - e^{-|2\pi\xi'|(x_0+y_0)} \right) \hat{f}(y_0,\xi')\, dy_0. \tag{5.3}$$

Moreover, we shall adopt the following notation:

$$m_1(\xi') = \frac{1+|2\pi\xi'|^2 + |2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}}{1+2|2\pi\xi'|^2 + |2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}}, \tag{5.4}$$

and

$$m_2(\xi') = \left[\frac{|2\pi\xi'| + \sqrt{1+|2\pi\xi'|^2}}{1+2|2\pi\xi'|^2 + |2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}} \right] = \frac{1}{\sqrt{1+|2\pi\xi'|^2}} m_1(\xi'). \tag{5.5}$$

We now construct the solution to the boundary value problem explicitly.

Proposition 5.1. *If u is a solution of the boundary value problem (5.2) in $W^r(\mathbb{R}^{N+1}_+)$ with $r > 5/2$, then u is given by the following formula*

$$\begin{aligned} u(x_0, x') &= \mathcal{G}f(x_0, x') + \mathcal{G}\Big(\int_{\mathbb{R}^N} e^{-\sqrt{1+|2\pi\xi'|^2}x_0} m_1(\xi') \\ &\quad \times \Big[|2\pi\xi'| \int_0^{+\infty} e^{-|2\pi\xi'|y_0} \hat{f}(y_0, \xi')\, dy_0 - \hat{h}(\xi') - \hat{f}(0, \xi') \Big] e^{2\pi i x'\cdot\xi'}\, d\xi' \Big) \\ &\quad + \mathcal{P}\left(\frac{\hat{h}(\xi') + \hat{f}(0, \xi')}{1 + 2|2\pi\xi'|^2 + |2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}} \right)^{\vee} \\ &\quad + \mathcal{P}\left(\frac{m_1(\xi')}{|2\pi\xi'|} \Big(\hat{h}(\xi') + \int_0^{+\infty} \hat{f}(y_0, \xi') e^{-|2\pi\xi'|y_0}\, dy_0 \Big) \right)^{\vee} \end{aligned} \tag{5.6}$$

Proof. Our first task is to reduce the order of the operator appearing in the boundary condition by using the equation on the domain. Thus the problem (5.2) is equivalent to

$$\begin{cases} -\triangle u + Gu = f & \text{on } \mathbb{R}^{N+1}_+ \\ -\triangle' u(0, \cdot) = f(0, \cdot) + h - Gu(0, \cdot) & \text{on } \mathbb{R}^N \end{cases}$$

for $u \in W^r(\mathbb{R}^{N+1}_+)$ with $r > 5/2$, where $\triangle' \stackrel{\text{def}}{=} \sum_{j=1}^N \partial^2/\partial x_j^2$. The problem (5.2) can be rewritten as

$$\begin{cases} -\triangle u(x_0, x') = F(x_0, x') & \text{on } \mathbb{R}^{N+1}_+ \\ u(0, x') = g(x') & \text{on } \mathbb{R}^N \end{cases}$$

(again for $u \in W^r(\mathbb{R}^{N+1}_+)$, $r > 5/2$) where we have set

$$F(x_0, x') = f(x_0, x') + \int_{\mathbb{R}^N} \sqrt{1+|2\pi\xi'|^2} e^{-\sqrt{1+|2\pi\xi'|^2}x_0} \frac{\partial \hat{u}}{\partial x_0}(0, \xi') e^{2\pi i x'\cdot\xi'}\, d\xi', \tag{5.7}$$

and

$$g(x') = \mathcal{N}\big(h + f(0, \cdot)\big)(x') + \mathcal{N}\left(\Big(\sqrt{1+|2\pi\xi'|^2} \frac{\partial \hat{u}}{\partial x_0}(0, \cdot) \Big)^{\vee} (0, x') \right), \tag{5.8}$$

with $\mathcal{N}$ being the Newtonian potential in $\mathbb{R}^N$. By the classical theory we then obtain that the solution we seek can be written as

$$\begin{aligned} u(x_0, x') &= \int_{\mathbb{R}^{N+1}_+} \mathcal{G}(x_0, y) F(y)\, dy + \int_{\mathbb{R}^N} P_{x_0}(x', y') g(y')\, dy' \\ &\equiv (\mathcal{G}F)(x_0, x') + \mathcal{P}g(x_0, x'). \end{aligned}$$

Here P_{x_0} denote the Poisson kernel in $\mathbb{R}^N$. Now we need to isolate u on the left hand side (note that F, g are defined in terms of u). In order to do this we compute the derivative with respect to x_0 of the above equation, take the partial Fourier transform, and evaluate at $x_0 = 0$. Using definitions (5.7) and (5.8) we have

$$\begin{aligned}
\left.\frac{\partial \hat{u}}{\partial x_0}\right|_{x_0=0} &= \left.\left[\left(\frac{\partial(\mathcal{G}F)}{\partial x_0}\right)^{\widehat{}} + \left(\frac{\partial \mathcal{P}g}{\partial x_0}\right)^{\widehat{}}\right]\right|_{x_0=0} \\
&= \int_0^{+\infty} (\mathcal{P}F)\widehat{\ }(y_0,\xi')\,dy_0 - |2\pi\xi'|\hat{g}(\xi') \\
&= \int_0^{+\infty} e^{-|2\pi\xi'|y_0}\,\widehat{F}(y_0,\xi')\,dy_0 - |2\pi\xi'|\hat{g}(\xi) \\
&= \int_0^{+\infty} e^{-|2\pi\xi'|y_0} \hat{f}(y_0,\xi')\,dy_0 \\
&\qquad \sqrt{1+|2\pi\xi'|^2} + \frac{\partial \hat{u}}{\partial x_0}(0,\xi')\int_0^{+\infty} e^{-(\sqrt{1+|2\pi\xi'|^2}+|2\pi\xi'|)y_0}\,dy_0 \\
&\qquad -|2\pi\xi'|\left[\frac{\hat{h}(\xi')}{|2\pi\xi'|^2} + \frac{\hat{f}(0,\xi')}{|2\pi\xi'|^2} + \frac{\sqrt{1+|2\pi\xi'|^2}(\partial\hat{u}/\partial x_0)(0,\xi')}{|2\pi\xi'|^2}\right] \\
&= \int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0 - \frac{\hat{h}(\xi')}{|2\pi\xi'|} - \frac{\hat{f}(0,\xi')}{|2\pi\xi'|} \\
&\qquad + \left[\frac{\sqrt{1+|2\pi\xi'|^2}}{\sqrt{1+|2\pi\xi'|^2}+|2\pi\xi'|} - \frac{\sqrt{1+|2\pi\xi'|^2}}{|2\pi\xi'|}\right]\frac{\partial \hat{u}}{\partial x_0}(0,\xi').
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\partial \hat{u}}{\partial x_0}(0,\xi')\left[1 - \frac{\sqrt{1+|2\pi\xi'|^2}}{\sqrt{1+|2\pi\xi'|^2}+|2\pi\xi'|} + \frac{\sqrt{1+|2\pi\xi'|^2}}{|2\pi\xi'|}\right] \\
&\qquad\qquad = \int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0 - \frac{\hat{h}(\xi')}{|2\pi\xi'|} - \frac{\hat{f}(0,\xi')}{|2\pi\xi'|},
\end{aligned}$$

that is,

$$\frac{\partial \hat{u}}{\partial x_0}(0,\xi') = |2\pi\xi'|m_2(\xi')\left\{\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0 - \frac{\hat{h}(\xi')}{|2\pi\xi'|} - \frac{\hat{f}(0,\xi')}{|2\pi\xi'|}\right\}. \tag{5.9}$$

Now we have the expression for $\partial u/\partial x_0$ at $x_0 = 0$. Notice that, for $t > 0$,

$$\left\|\frac{\partial u}{\partial x_0}(0,\cdot)\right\|_{W^t(\mathbb{R}^N)} \le C\Big(\|f\|_{W^{t+1/2}(\mathbb{R}^{N+1}_+)} + \|h\|_{W^{t-1}(\mathbb{R}^N)}\Big),$$

so that our calculations make sense so far. Substituting equation (5.9) into (5.7) and (5.8), and recalling that $m_1 = \sqrt{1+|2\pi\xi'|^2}m_2$, we find that

$$
\begin{aligned}
F(x_0,x') &= f(x_0,x') \\
&+ \int_{\mathbb{R}^N} |2\pi\xi'| m_1(\xi') e^{-\sqrt{1+|2\pi\xi'|^2}} \left(\int_0^{+\infty} e^{-|2\pi\xi'|y_0} \hat{f}(y_0,\xi')\, dy_0 - \frac{\hat{h}(\xi')}{|2\pi\xi'|} - \frac{\hat{f}(x_0,\xi')}{|2\pi\xi'|} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
g(x') &= \mathcal{N}\big(h + f(0,\cdot)\big) \\
&\quad + \mathcal{N}\bigg(\Big(m_1(\xi')|2\pi\xi'| \Big\{ \int_0^{+\infty} e^{-|2\pi\xi'|y_0} \hat{f}(y_0,\xi')\, dy_0 - \frac{\hat{h}(\xi')}{|2\pi\xi'|} - \frac{\hat{f}(0,\xi')}{|2\pi\xi'|} \Big\} \Big)^{\check{}} \bigg) \\
&= \mathcal{N}\big(h + f(0,\cdot)\big) + \mathcal{N}\bigg(\Big(|2\pi\xi'| m_1(\xi') \int_0^{+\infty} e^{-|2\pi\xi'|y_0} \hat{f}(y_0,\xi')\, dy_0 \Big)^{\check{}} \bigg) \\
&\quad - \mathcal{N}\bigg(\Big(m_1\hat{h} + m_1\hat{f}(0,\cdot) \Big)^{\check{}} \bigg).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
u(x_0,x') &= \mathcal{G}F(x_0,x') + \mathcal{P}g(x') \\
&= \mathcal{G}f(x_0,x') + \mathcal{G}\bigg(\int_{\mathbb{R}^N} e^{-\sqrt{1+|2\pi\xi'|^2}x_0} m_1(\xi') \\
&\qquad \times \Big[|2\pi\xi'| \int_0^{+\infty} e^{-|2\pi\xi'|y_0} \hat{f}(y_0,\xi')\, dy_0 - \hat{h}(\xi') - \hat{f}(0,\xi') \Big] e^{2\pi i x'\cdot\xi'}\, d\xi' \bigg) \\
&\quad + \mathcal{P}\left(\frac{\hat{h}(\xi') + \hat{f}(0,\xi')}{1 + 2|2\pi\xi'|^2 + |2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}} \right)^{\check{}} \\
&\quad + \mathcal{P}\left(\frac{m_1(\xi')}{|2\pi\xi'|} \int_0^{+\infty} \hat{f}(y_0,\xi') e^{-|2\pi\xi'|y_0}\, dy_0 \right)^{\check{}}. \qquad (5.10)
\end{aligned}
$$

This gives the explicit expression for the solution u. □

The *a priori* **Estimate.** In this part we are going to prove the *a priori* estimate for the solution u; then we will turn to the question of existence. Now we need the following two lemmas.

Lemma 5.2. *There exists a constant c such that, for $|\alpha| = 2$, and $t \geq 0$,*

$$\|D^\alpha \mathcal{G} w\|_{W^t(\mathbb{R}^{N+1}_+)} \leq c\|w\|_{W^t(\mathbb{R}^{N+1}_+)}.$$

Proof. For a function w defined on $\mathbb{R}^{N+1}_+$, let $w_{\rm odd}$ denote the odd extension (in the x_0 variable) to $\mathbb{R}^{N+1}$, and $\mathcal{N}_{N+1}$ the Newtonian potential in $\mathbb{R}^{N+1}$. Let $\mathcal{F}$ denote the

Fourier transform in $\mathbb{R}^{N+1}$. Observe that

$$(\mathcal{G}w)_{\mathrm{odd}} = \mathcal{N}_{N+1} w_{\mathrm{odd}},$$

so that we have

$$\mathcal{F}\left(\frac{\partial^2 (\mathcal{G}w)_{\mathrm{odd}}}{\partial x_i x_j}\right)(\xi) = \frac{\xi_i \xi_j}{|\xi|^2}\mathcal{F}(w_{\mathrm{odd}})(\xi),$$

and the estimate for $t = 0$ follows. When $t = 1, 2, \dots$ notice that, for $x \in \mathbb{R}^{N+1}_+$,

$$\left(\frac{\partial^2 (\mathcal{G}w)}{\partial x_0^2}\right)_{\mathrm{odd}}(x) = -w - \triangle' \mathcal{G}w(x).$$

Then it suffices to consider the case when the derivative is of degree at most 1 in the x_0-direction. This takes care of the fact that w_{odd} is not differentiable in the x_0-direction. Now the estimate follows easily for t an integer. Interpolation gives the result for all $t \geq 0$. □

The proof of the next lemma is easy:

Lemma 5.3. *Let w be defined on $\mathbb{R}^N$, $|\alpha| = 2$, and $t \geq 0$. Then there exists a constant $C > 0$ such that*

$$\|D^\alpha \mathcal{P} w\|^2_{W^t(\mathbb{R}^{N+1}_+)} \leq C \int_{\mathbb{R}^N} |2\pi\xi'|^3 (1 + |2\pi\xi'|)^{2+t} |\hat{w}(\xi')|^2 \, d\xi'.$$

THEOREM 5.4. *Let $f \in W^s(\mathbb{R}^{N+1}_+)$, with $s > 1/2$. If the boundary value problem (5.2) admits the solution u given by Proposition 5.1, then it satisfies the estimate*

$$\|u\|_{W^{s+2}(\mathbb{R}^{N+1}_+)} \leq C\left\{\|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|h\|_{W^{s-1/2}(\mathbb{R}^N)} + \|u\|_{W^1(\mathbb{R}^{N+1}_+)}\right\}. \tag{5.11}$$

Proof. In order to obtain *a priori* estimates we use the following standard fact (see [LIM] Theorem 9.7). If $s \geq 0$, then

$$\|u\|_{W^{s+2}(\mathbb{R}^{N+1}_+)} \leq c\left(\sum_{|\alpha|=2}\left\|\frac{\partial^{|\alpha|} u}{\partial x^\alpha}\right\|_{W^s(\mathbb{R}^{N+1}_+)} + \|u\|_{W^1(\mathbb{R}^{N+1}_+)}\right).$$

Now recall the expression for the solution given by (5.10)

$$\begin{aligned}
u(x_0,x') &= \mathcal{G}f(x_0,x') + \mathcal{G}\Big(\int_{\mathbb{R}^N} e^{-\sqrt{1+|2\pi\xi'|^2}x_0} m_1(\xi') \\
&\quad \times \Big[|2\pi\xi'|\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0 - \hat{h}(\xi') - \hat{f}(0,\xi')\Big] e^{2\pi i x'\cdot\xi'}\,d\xi'\Big) \\
&\qquad + \mathcal{P}\left(\frac{\hat{h}(\xi')+\hat{f}(0,\xi')}{1+2|2\pi\xi'|^2+|2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}}\right)^{\vee} \\
&\qquad + \mathcal{P}\left(\frac{m_1(\xi')}{|2\pi\xi'|}\int_0^{+\infty}\hat{f}(y_0,\xi')e^{-|2\pi\xi'|y_0}\,dy_0\right)^{\vee} \\
&\equiv \mathcal{G}f + \mathcal{G}f_1 + \mathcal{P}g_1 + \mathcal{P}g_2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u\|_{W^{s+2}(\mathbb{R}^{N+1}_+)} \le c\|u\|_{W^1(\mathbb{R}^{N+1}_+)} &+ c\sum_{|\alpha|=2}\left(\left\|\frac{\partial^{|\alpha|}\mathcal{G}f}{\partial x^\alpha}\right\|_{W^s(\mathbb{R}^{N+1}_+)}\right. \\
&\left. + \left\|\frac{\partial^{|\alpha|}\mathcal{G}f_1}{\partial x^\alpha}\right\|_{W^s(\mathbb{R}^{N+1}_+)} + \left\|\frac{\partial^{|\alpha|}\mathcal{P}g_1}{\partial x^\alpha}\right\|_{W^s(\mathbb{R}^{N+1}_+)} + \left\|\frac{\partial^{|\alpha|}\mathcal{P}g_2}{\partial x^\alpha}\right\|_{W^s(\mathbb{R}^{N+1}_+)}\right)
\end{aligned}$$

Lemma 5.2 implies that, for $|\alpha|=2$,

$$\|D^\alpha \mathcal{G}f\|_{W^t(\mathbb{R}^{N+1}_+)} \le c\|f\|_{W^t(\mathbb{R}^{N+1}_+)},$$

and

$$\|D^\alpha \mathcal{G}f_1\|_{W^t(\mathbb{R}^{N+1}_+)} \le c\|f_1\|_{W^t(\mathbb{R}^{N+1}_+)}\,.$$

Notice that $f_1 = \widetilde{\mathcal{K}}(h_1)$, where

$$\widehat{h_1}(\xi') = m_2(\xi')\left[|2\pi\xi'|\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0 - \hat{h}(\xi') - \hat{f}(0,\xi')\right].$$

Then, for $s>1/2$,

$$\begin{aligned}
\|f_1\|_{W^s(\mathbb{R}^{N+1}_+)} &\le c\|h_1\|_{W^{s+1/2}(\mathbb{R}^N)} \\
&= c\Big\{\int_{\mathbb{R}^N}(1+|\xi'|^2)^{s+1/2}|m_2(\xi')|^2 \\
&\qquad \times \left||2\pi\xi'|\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0 - \hat{h}(\xi') - \hat{f}(0,\xi')\right|^2 d\xi'\Big\}^{1/2} \\
&\le c\left(\|h\|_{W^{s-1/2}(\mathbb{R}^N)} + \|f(0,\cdot)\|_{W^{s-1/2}(\mathbb{R}^N)} + \|f\|_{W^s(\mathbb{R}^{N+1}_+)}\right) \\
&\le c\Big(\|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|h\|_{W^{s-1/2}(\mathbb{R}^N)}\Big).
\end{aligned}$$

Hence it suffices to consider the $W^t(\mathbb{R}^{N+1}_+)$ norm of any second derivative of $\mathcal{P}g_1$ and $\mathcal{P}g_2$. For $s > 1/2$, by Lemma 5.3 we have

$$\begin{aligned}
&\left\| D^\alpha \mathcal{P}\left(\frac{\hat h(\xi') + \hat f(0,\xi')}{1 + 2|2\pi\xi'|^2 + |2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}}\right)\right\|_{W^s(\mathbb{R}^{N+1}_+)} \\
&\leq c\left\{\int_{\mathbb{R}^N} |2\pi\xi'|^3(1+|2\pi\xi'|^2)^{s-2}\left(|\hat h(\xi')|^2 + |\hat f(0,\xi')|^2\right) d\xi'\right\}^{1/2} \\
&\leq c\Big\{\|f(0,\cdot)\|_{W^{s-1/2}(\mathbb{R}^N)} + \|h\|_{W^{s-1/2}(\mathbb{R}^N)}\Big\} \\
&\leq c\Big\{\|f\|_{W^s(\mathbb{R}^{N+1}_+)}\|h\|_{W^{s-1/2}(\mathbb{R}^N)}\Big\},
\end{aligned}$$

Moreover, recalling the definition of $m_1(\xi')$ (see (5.4)), we have

$$\begin{aligned}
&\left\| D^\alpha \mathcal{P}\left(\frac{m_1(\xi')}{|2\pi\xi'|}\int_0^{+\infty} \hat f(y_0,\xi')e^{-|2\pi\xi'|y_0}\,dy_0\right)\right\|_{W^s(\mathbb{R}^{N+1}_+)} \\
&\leq c\left\{\int_{\mathbb{R}^N} |2\pi\xi'|^3(1+|2\pi\xi'|^2)^s \frac{1}{|2\pi\xi'|^2}\left|\int_0^{+\infty} \hat f(y_0,\xi')e^{-|2\pi\xi'|y_0}\,dy_0\right|^2 d\xi'\right\}^{1/2} \\
&\leq c\left\{\int_{\mathbb{R}^N} (1+|2\pi\xi'|^2)^s \int_0^{+\infty} |\hat f(y_0,\xi')|^2\,dy_0\,d\xi'\right\}^{1/2} \\
&= c\|f\|_{W^s(\mathbb{R}^{N+1}_+)}.
\end{aligned}$$

This proves the *a priori* estimate. □

The existence Theorem. Now we turn to the existence statement. If we prove that the function u defined by (5.10) belongs to $W^1(\mathbb{R}^{N+1}_+)$ then, by the estimate (5.11), the equation (5.10) defines the unique solution of the boundary value problem. The solution belongs to $W^r(\mathbb{R}^{N+1}_+)$ with $r > 5/2$.

THEOREM 5.5. *If $f \in W^s \cap L^1(\mathbb{R}^{N+1}_+)$, with $s > 1/2$, then the boundary value problem (5.1) has a unique solution u, and this satisfies*

$$\|u\|_{W^{s+2}(\mathbb{R}^{N+1}_+)} \leq C\left\{\|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|f\|_{L^1(\mathbb{R}^{N+1}_+)} + \|h\|_{W^{s-1/2}(\mathbb{R}^N)}\right\}$$

if $N \geq 4$. If $N = 2, 3$ and f satisfies

$$\int f(x)\,dx = 0, \qquad\qquad f \in L^1(|x|dx, \mathbb{R}^{N+1}_+),$$

then there exists a unique solution u that satisfies the estimate

$$\|u\|_{W^{s+2}(\mathbb{R}^{N+1}_+)} \leq C\left\{\|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|f\|_{L^1(|x|dx,\mathbb{R}^{N+1}_+)} + \|h\|_{W^{s-1/2}(\mathbb{R}^N)}\right\}.$$

If $N = 1$, we assume that f is such that

$$\int f(x)\,dx = 0, \qquad \int x_i f(x)\,dx = 0,\ i = 0,1 \qquad f \in L^1(|x|^2 dx, \mathbb{R}^2).$$

Then there exists a unique solution u such that

$$\|u\|_{W^{s+2}(\mathbb{R}^{N+1}_+)} \leq C\left\{\|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|f\|_{L^1(|x|^2 dx,\mathbb{R}^{N+1}_+)} + \|h\|_{W^{s-1/2}(\mathbb{R}^N)}\right\}.$$

Proof. By Theorem 5.4 it suffices to show that

$$\|u\|_{W^1(\mathbb{R}^{N+1}_+)} \leq c\left\{\|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|f\|_{L^1(\mathbb{R}^{N+1}_+)} + \|h\|_{W^{s-1/2}(\mathbb{R}^N)}\right\},$$

when $N \geq 4$, or the corresponding estimate when $N = 1, 2, 3$, with f satisfying the stated conditions. Notice that

$$\|u\|_{W^1(\mathbb{R}^{N+1}_+)} \leq \|u\|_{L^2(\mathbb{R}^{N+1}_+)} + \sum_{i=0}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\mathbb{R}^{N+1}_+)}.$$

Recall that (5.10) gives the function u as

$$u = \mathcal{G}f + \mathcal{G}f_1 + \mathcal{P}g_1 + \mathcal{P}g_2.$$

We calculate the partial Fourier transform of u using the fact that

$$\left(\mathcal{G}\left[\left(e^{-\sqrt{1+|2\pi\xi'|^2}x_0}\right)^{\vee}\right]\right)^{\wedge}(x_0,\xi') = e^{-|2\pi\xi'|x_0} - e^{-\sqrt{1+|2\pi\xi'|^2}x_0}.$$

[Note here that $^\vee$ and $^\wedge$ do not cancel, since the Green's potential $\mathcal{G}$ occurs in between the two operations. Reference line (5.3).] We obtain

$$\begin{aligned}
\hat{u}(x_0,\xi') &= \widehat{\mathcal{G}f}(x_0,\xi') + \widehat{\mathcal{G}f_1}(x_0,\xi') + \widehat{\mathcal{P}g_1}(x_0,\xi') + \widehat{\mathcal{P}g_2}(x_0,\xi') \\
&= \widehat{\mathcal{G}f}(x_0,\xi') + m_1(\xi')\left[e^{-|2\pi\xi'|x_0} - e^{-\sqrt{1+|2\pi\xi'|^2}x_0}\right] \\
&\qquad\qquad \times\left[|2\pi\xi'|\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0 - \hat{h}(\xi') - \hat{f}(0,\xi')\right] \\
&\qquad + \frac{e^{-|2\pi\xi'|x_0}\left(\hat{h}(\xi') + \hat{f}(0,\xi')\right)}{1 + 2|2\pi\xi'|^2 + |2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}} \\
&\qquad + e^{-|2\pi\xi'|x_0}\frac{m_1(\xi')}{|2\pi\xi'|}\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0\,.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
\left(\frac{\partial \hat{u}}{\partial x_0}\right)(x_0,\xi') &= \left(\frac{\partial}{\partial x_0}\widehat{\mathcal{G}f}\right)(x_0,\xi') \\
&\quad + m_1(\xi')\left[\sqrt{1+|2\pi\xi'|^2}e^{-\sqrt{1+|2\pi\xi'|^2}x_0} - |2\pi\xi'|e^{-|2\pi\xi'|x_0}\right] \\
&\qquad \times\left[|2\pi\xi'|\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0 - \hat{h}(\xi') - \hat{f}(0,\xi')\right] \\
&\quad - \frac{|2\pi\xi'|e^{-|2\pi\xi'|x_0}\Big(\hat{h}(\xi')+\hat{f}(0,\xi')\Big)}{1+2|2\pi\xi'|^2+|2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}} \\
&\quad - e^{-|2\pi\xi'|x_0}m_1(\xi')\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0\,.
\end{aligned}
$$

Then $\|u\|_{W^1(\mathbb{R}^{N+1}_+)}$ will be estimated once we estimate $\|\hat{u}\|_{L^2(\mathbb{R}^{N+1}_+)}$, $\Big\||2\pi\xi'|\hat{u}\Big\|_{L^2(\mathbb{R}^{N+1}_+)}$, and $\|(\partial u/\partial x_0)\hat{\ }\|_{L^2(\mathbb{R}^{N+1}_+)}$.

We begin by studying the terms that arise from $\mathcal{G}f$. We extend f to all of $\mathbb{R}^{N+1}$ as an odd function $f_{\rm odd}$ of x_0 as we did in the proof of Lemma 5.2. We obtain

$$
\|\mathcal{G}f\|_{L^2(\mathbb{R}^{N+1}_+)} = \frac{1}{\sqrt{2}}\left\|\frac{1}{|2\pi\xi'|^2}\mathcal{F}\Big(\mathcal{N}f_{\rm odd}\Big)\right\|_{L^2(\mathbb{R}^{N+1})}
$$

and

$$
\left\|\frac{\partial}{\partial x_i}\mathcal{G}f\right\|_{L^2(\mathbb{R}^{N+1}_+)} = \frac{1}{\sqrt{2}}\left\|\frac{\partial}{\partial x_i}\mathcal{F}f_{\rm odd}\right\|_{L^2(\mathbb{R}^{N+1})}.
$$

Case $N \geq 4$. We have

$$
\begin{aligned}
\|\mathcal{G}f\|_{L^2(\mathbb{R}^{N+1}_+)} &= c\cdot\|\mathcal{F}(\mathcal{N}_{N+1}f_{\rm odd})\|_{L^2(\mathbb{R}^{N+1})} \\
&\leq c\left\{\int_{|2\pi\xi|\geq 1}\frac{|\mathcal{F}f_{\rm odd}(\xi)|^2}{|2\pi\xi|^4}\,d\xi\right\}^{1/2} + c\left\{\int_{|2\pi\xi|\leq 1}\frac{|\mathcal{F}f_{\rm odd}(\xi)|^2}{|2\pi\xi|^4}\,d\xi\right\}^{1/2} \\
&\leq c\left\{\|f\|_{L^2(\mathbb{R}^{N+1})} + \left(\int_{|2\pi\xi|\leq 1}|2\pi\xi|^{-4}\,d\xi\right)^{1/2}\sup_{\xi\in\mathbb{R}^{N+1}}|\mathcal{F}f_{\rm odd}(\xi)|\right\} \\
&\leq c\Big\{\|f\|_{L^2(\mathbb{R}^{N+1})} + \|f\|_{L^1(\mathbb{R}^{N+1})}\Big\}\,.
\end{aligned}
$$

Moreover,

$$\begin{aligned}\left\|\frac{\partial \mathcal{G}f}{\partial x_i}\right\|_{L^2(\mathbb{R}^{N+1})} &\le \left\{\int_{\mathbb{R}^{N+1}} \frac{|\mathcal{F}f_{\rm odd}(\xi)|^2}{|2\pi\xi|^2}\, d\xi\right\}^{1/2}\\ &\le c\left\{\int_{|2\pi\xi|\le 1} \frac{|\mathcal{F}f_{\rm odd}(\xi)|^2}{|2\pi\xi|^2}\, d\xi\right\}^{1/2} + c\left\{\int_{|2\pi\xi|\ge 1} \frac{|\mathcal{F}f_{\rm odd}(\xi)|^2}{|2\pi\xi|^2}\, d\xi\right\}^{1/2}\\ &\le c\left\{\|f\|_{L^2(\mathbb{R}^{N+1})} + \|f\|_{L^1(\mathbb{R}^{N+1})}\right\}.\end{aligned}$$

Now writing $\hat{u}(x_0,\xi') = \widehat{\mathcal{G}f}(x_0,\xi') + B(x_0,\xi')$, i.e. setting

$$B = \widehat{\mathcal{G}f_1} + \widehat{\mathcal{P}g_1} + \widehat{\mathcal{P}g_2}\,, \tag{5.12}$$

we must estimate the terms $\|B\|_{L^2(\mathbb{R}^{N+1}_+)}$, $\big\||2\pi\xi'|B\big\|_{L^2(\mathbb{R}^{N+1}_+)}$, and $\|(\partial/\partial x_0)B\|_{L^2(\mathbb{R}^{N+1}_+)}$. We have

$$\begin{aligned}\|B\|_{L^2(\mathbb{R}^{N+1}_+)} &\le c\left\{\int_{|2\pi\xi|\le 1}\int_0^{+\infty} |B(x_0,\xi')|^2\, dx_0\, d\xi'\right\}^{1/2}\\ &\qquad + c\left\{\int_{|2\pi\xi|\ge 1}\int_0^{+\infty} |B(x_0,\xi')|^2\, dx_0\, d\xi'\right\}^{1/2}.\end{aligned}$$

We begin by estimating the integral on the set where $|2\pi\xi'| \le 1$. Notice that, if $N \ge 4$, then

$$\begin{aligned}|B(x_0,\xi')| &\le C\Bigg\{e^{-|2\pi\xi'|x_0}|\hat{h}(\xi') - \hat{f}(0,\xi')|\left|\frac{1}{1+2|2\pi\xi'|^2 + |2\pi\xi'|\sqrt{1+|2\pi\xi'|^2}} - m_1(\xi')\right|\\ &\qquad + e^{-|2\pi\xi'|x_0}|2\pi\xi'|^{-1}\left|\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')dy_0\right|\Bigg\}\\ &\le C\Bigg\{e^{-|2\pi\xi'|x_0}\Big(\hat{h}(\xi') + \hat{f}(y_0,\xi')\Big)|2\pi\xi'| + \frac{e^{-|2\pi\xi'|x_0}}{|2\pi\xi'|}\int_0^{+\infty} |\hat{f}(y_0,\xi')|\, dy_0\Bigg\}.\end{aligned}$$

From this, the restriction theorem, and the fact that

$$\int_0^{+\infty} |\hat{f}(y_0,\xi')|\, dy_0 \le \int_0^{+\infty}\int_{\mathbb{R}^N} |f(y_0,x')|\, dx'\, dy_0 = \|f\|_{L^1(\mathbb{R}^{N+1}_+)}, \tag{5.13}$$

we obtain that

$$
\begin{aligned}
&\left\{\int_{|2\pi\xi'|\le 1}\int_0^{+\infty} |B(x_0,\xi')|^2\,dx_0\,d\xi'\right\}^{1/2} \\
&\le C\left\{\|h\|_{L^2(\mathbb{R}^N)} + \|f(0,\xi')\|_{L^2(\mathbb{R}^N)} + \left(\int_{|2\pi\xi'|\le 1}\frac{1}{|2\pi\xi'|^3}\Big(\int_0^{+\infty}|\hat{f}(y_0,\xi')|\,dy_0\Big)^2\,d\xi'\right)^{1/2}\right\} \\
&\le C\Big\{\|h\|_{L^2(\mathbb{R}^N)} + \|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|f\|_{L^1(\mathbb{R}^{N+1}_+)}\Big\}.
\end{aligned}
$$

To study the integral for $|2\pi\xi'|\ge 1$ we first notice that

$$
\begin{aligned}
\int_0^{+\infty}\left(e^{-|2\pi\xi'|x_0} - e^{-\sqrt{1+|2\pi\xi'|^2}x_0}\right)^2\,dx_0 &= \left(2|2\pi\xi'|(|2\pi\xi'| + \sqrt{1+|2\pi\xi'|^2})^3\right)^{-1} \\
&= O(|2\pi\xi'|^{-4}) \qquad \text{as } |\xi'|\to\infty.
\end{aligned}
$$

Using this fact, Schwarz's inequality, and the restriction theorem, we see that the integral we want to estimate is less than or equal to a constant times

$$
\begin{aligned}
&\left\{\int_{|2\pi\xi'|\ge 1}\frac{1}{|2\pi\xi'|^4}\left[\left||2\pi\xi'|\int_0^{+\infty}e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0\right|^2 + |\hat{h}(\xi')|^2 + |\hat{f}(0,\xi')|^2\right]\,d\xi'\right\}^{1/2} \\
&\quad\le\quad c\left\{\Big(\int_{|2\pi\xi'|\ge 1}\frac{1}{|2\pi\xi'|^3}\int_0^{+\infty}|\hat{f}(y_0,\xi')|^2\,dy_0\,d\xi'\Big)^{1/2} + \|h\|_{L^2(\mathbb{R}^N)} + \|f(0,\cdot)\|_{L^2(\mathbb{R}^N)}\right\} \\
&\quad\le\quad c\left\{\|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|h\|_{L^2(\mathbb{R}^N)}\right\}.
\end{aligned}
$$

In the same way we obtain that

$$
\begin{aligned}
&\Big\| |2\pi\xi'| B \Big\|_{L^2(\mathbb{R}^{N+1}_+)} \\
&\leq \left\{ \int_{|2\pi\xi'|\leq 1} \int_0^{+\infty} |2\pi\xi'|^2 |B(y_0,\xi')|^2 \, dy_0 \, d\xi' \right\}^{1/2} \\
&\qquad + \left\{ \int_{|2\pi\xi'|\geq 1} \int_0^{+\infty} |2\pi\xi'|^2 |B(y_0,\xi')|^2 \, dy_0 \, d\xi' \right\}^{1/2} \\
&\leq c \Bigg\{ \|f(0,\cdot)\|_{L^2(\mathbb{R}^N)} + \|h\|_{L^2(\mathbb{R}^N)} + \left(\int_{|2\pi\xi'|\leq 1} |2\pi\xi'|^{-1} \, d\xi' \right)^{1/2} \|f\|_{L^1(\mathbb{R}^N)} \\
&\qquad + \left(\int_{|2\pi\xi'|\geq 1} \left| \int_0^{+\infty} e^{-|2\pi\xi'|y_0} \hat{f}(y_0,\xi') \, dy_0 \right|^2 d\xi' \right)^{1/2} \\
&\qquad + \left(\int_{|2\pi\xi'|\geq 1} \frac{|\hat{h}(\xi')|^2 + |\hat{f}(0,\xi')|^2}{|2\pi\xi'|^2} \, d\xi' \right)^{1/2} \Bigg\} \\
&\leq c \left\{ \|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|f\|_{L^1(\mathbb{R}^{N+1}_+)} + \|h\|_{L^2(\mathbb{R}^N)} \right\},
\end{aligned}
$$

by the Schwarz inequality and the restriction theorem.

In order to estimate $\|\partial B/\partial x_0\|_{L^2(\mathbb{R}^{N+1}_+)}$ we observe that

$$
\begin{aligned}
2\int_0^{+\infty} &\Big(|2\pi\xi'| e^{-|2\pi\xi'|x_0} - \sqrt{1+|2\pi\xi'|^2} e^{-\sqrt{1+|2\pi\xi'|^2} x_0} \Big)^2 dx_0 \\
&= (|2\pi\xi'| + \sqrt{1+|2\pi\xi'|^2})^{-3} = O(|2\pi\xi'|^{-3})
\end{aligned}
$$

as $|\xi'| \to \infty$. Using this fact and (5.13) we see that

$$
\begin{aligned}
\left\| \frac{\partial B}{\partial x_0} \right\|_{L^2(\mathbb{R}^{N+1}_+)} &\leq \left\{ \int_{|2\pi\xi'|\leq 1} \int_0^{+\infty} dy_0 \, d\xi' \right\}^{1/2} + \left\{ \int_{|2\pi\xi'|\geq 1} \int_0^{+\infty} dy_0 \, d\xi' \right\}^{1/2} \\
&\leq c \Bigg\{ \left(\int_{\mathbb{R}^N} |\hat{f}(0,\xi')|^2 \, d\xi' \right)^{1/2} + \left(\int_{\mathbb{R}^N} |\hat{h}(\xi')|^2 \, d\xi' \right)^{1/2} \\
&\qquad + \left(\int_{|2\pi\xi'|\leq 1} |2\pi\xi'|^{-1} \, d\xi' \right)^{1/2} \|f\|_{L^1(\mathbb{R}^{N+1}_+)} \\
&\qquad + \left(\int_{|2\pi\xi'|\geq 1} |2\pi\xi'|^{-1} \left| \int_0^{+\infty} e^{-|2\pi\xi'|y_0} \hat{f}(y_0,\xi') \, dy_0 \right|^2 d\xi' \right)^{1/2} \Bigg\} \\
&\leq c \Big\{ \|f\|_{W^{1/2}(\mathbb{R}^{N+1}_+)} + \|f\|_{L^1(\mathbb{R}^{N+1}_+)} + \|h\|_{L^2(\mathbb{R}^N)} \Big\}.
\end{aligned}
$$

This concludes the estimate in the case $N \geq 4$.

Case $N < 4$**.** Now we must consider the lower dimensional cases. Again, we first estimate $\mathcal{G}f$:

$$\begin{aligned}\|\mathcal{G}f\|_{L^2(\mathbb{R}^{N+1}_+)} &= \left\{\int_{|2\pi\xi|\le 1} \frac{1}{|2\pi\xi'|^4}|\mathcal{F}f_{\mathrm{odd}}(\xi)|^2\,d\xi + \int_{|2\pi\xi|>1} \frac{1}{|2\pi\xi'|^4}|\mathcal{F}f_{\mathrm{odd}}(\xi)|^2\,d\xi\right\}^{1/2} \\ &\le c\left\{\left(\int_{|2\pi\xi|\le 1} |2\pi\xi'|^{-4}|\mathcal{F}f_{\mathrm{odd}}(\xi)|^2\,d\xi\right)^{1/2} + \|f\|_{L^2(\mathbb{R}^{N+1})}\right\}.\end{aligned}$$

If $N < 4$ then we need some conditions on f. Assume that $\int_{\mathbb{R}^{N+1}_+} f(x)\,dx = 0$. Then $\mathcal{F}f_{\mathrm{odd}}(0) = 0$, and if $N = 2, 3$ then

$$\begin{aligned}&\left(\int_{|2\pi\xi|\le 1} \frac{1}{|2\pi\xi|^4}|\mathcal{F}f_{\mathrm{odd}}(\xi)|^2\,d\xi\right)^{1/2} \\ &\quad= \left(\int_{|2\pi\xi|\le 1} \frac{1}{|2\pi\xi|^2}\frac{|\mathcal{F}f_{\mathrm{odd}}(\xi) - \mathcal{F}f_{\mathrm{odd}}(0)|^2}{|2\pi\xi|^2}\,d\xi\right)^{1/2} \\ &\quad\le \left(\int_{|2\pi\xi|\le 1} |2\pi\xi|^{-2}\,d\xi\right)^{1/2} \sup_{\xi\in\mathbb{R}^{N+1}} |\mathrm{grad}(\mathcal{F}f_{\mathrm{odd}})(\xi)| \\ &\quad\le c\int_{\mathbb{R}^{N+1}_+} |x|\cdot|f(x)|\,dx.\end{aligned}$$

If $N = 1$ we need even stronger conditions (because the logarithmic potential is poorly behaved at infinity), that is $\mathcal{F}f_{\mathrm{odd}}(0) = 0$, and $\mathrm{grad}(\mathcal{F}f_{\mathrm{odd}})(0) = 0$, corresponding to

$$\int_{\mathbb{R}^2_+} f(x)\,dx = 0; \quad \int_{\mathbb{R}^2_+} x_i f(x)\,dx = 0,\ i = 1, 2; \quad \int_{\mathbb{R}^2_+} |x|^2 f(x)\,dx < \infty.$$

Then, for $N = 1$,

$$\|\mathcal{G}f\|_{\mathbb{R}^2_+} \le c\left\{\|f\|_{L^2(\mathbb{R}^2_+)} + \int_{\mathbb{R}^2_+} |x|^2 f(x)\,dx\right\}.$$

The estimate for B has to be modified only in the part relative to the set where $|2\pi\xi'| \le 1$:

$$\begin{aligned}\|B\|_{L^2(\mathbb{R}^{N+1}_+)} &\le \left(\int_{|2\pi\xi'|\le 1}\int_0^{+\infty} |B(y_0,\xi')|^2\,dy_0\,d\xi'\right)^{1/2} + \|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|h\|_{L^2(\mathbb{R}^N)} \\ &\le c\left\{\|f\|_{W^s(\mathbb{R}^{N+1}_+)} + \|h\|_{L^2(\mathbb{R}^N)}\right. \\ &\qquad \left. + \int_{|2\pi\xi'|\le 1} |2\pi\xi'|^{-3}\left|\int_0^{+\infty} \hat{f}(y_0,\xi')e^{-|2\pi\xi'|y_0}\,dy_0\right|^2\right\}^{1/2}.\end{aligned}$$

With the above assumption we can estimate the last integral with

$$\left\{\int_{|2\pi\xi'|\le 1} |2\pi\xi'|^{-1}\left| \int_0^{+\infty} y_0 \hat{f}(y_0,\xi') \frac{e^{-|2\pi\xi'|y_0}-1}{|2\pi\xi'|y_0}\,dy_0 \right.\right.$$
$$\left.\left. + \int_0^{+\infty} \frac{\hat{f}(y_0,\xi')-\hat{f}(y_0,0)}{|2\pi\xi'|}\,dy_0\right|^2 d\xi'\right\}^{1/2} \le c\int_{\mathbb{R}^{N+1}_+} |x| f(x)\,dx,$$

if $N = 2, 3$. If $N = 1$ the estimate follows in a similar fashion. □

6. Analysis of the Problem on the Half Space for q-Forms

In this section we consider the space of q-forms, $q \ge 1$, with coefficients in $W^1(\mathbb{R}^{N+1}_+)$. Throughout this section, we denote this space of forms by $\bigwedge^q$, that is, we do not write explicitly the index for the Sobolev space $W^1(\mathbb{R}^{N+1}_+)$, this being fixed once and for all.

On the space $\bigwedge^q$ we select the basis $\{dx^I\}$, where $I = (i_1, \dots i_q)$ is a strictly increasing multi-index. For a q-form ϕ,

$$\phi = \sum_I \phi_I dx^I$$

we have that

$$\begin{aligned} d\phi &= \sum_I D_j\phi_I dx^j \wedge dx^I \\ &= \sum_K \left(\sum_{I,j} D_j\phi_I \varepsilon^K_{jI}\right) dx^K, \end{aligned}$$

where K has $q+1$ entries, and

$$\varepsilon^K_{jI} = \begin{cases} 0 & \text{if} \quad K \ne jI \text{ as sets} \\ \pm 1 & \text{if} \quad K = jI \text{ as sets} \end{cases}$$

and the sign is chosen according to the sign of the permutation that puts jI in increasing order.

Lemma 6.1. *The formal adjoint d' of d, $d' : \bigwedge^{q+1} \longrightarrow \bigwedge^q$ with respect to the inner product in the Sobolev space, has the following expression:*

$$d'\psi = \sum_{|I|=q} \left(\sum_{\substack{|K|=q+1 \\ j=0,\dots,N}} \varepsilon^K_{jI} D_j \psi_K \right) dx^I.$$

Proof. Let $\phi \in \bigwedge^q$, $\psi \in \bigwedge^{q+1}$, $\phi = \sum_I \phi_I dx^I$, $\psi = \sum_K \psi_k dx^K$, both with compact support in $\mathbb{R}_+^{N+1}$. Then we have

$$\begin{aligned}
\langle d\phi, \psi, \rangle_1 &= \sum_{\substack{|I|=q,|K|=q+1\\ j=0,\dots,N}} \Big\langle D_j\phi_I dx^j \wedge dx^I, \psi_K dx^K \Big\rangle_1 \\
&= \sum_{I,K,j} \varepsilon^K_{jI} \Big\langle D_j\phi_I, \psi_K \Big\rangle_1 \\
&= \sum_{I,K,j} \left[\sum_{k=0}^N \varepsilon^K_{jI} \Big\langle D_k D_j \phi_I, D_k\psi_K \Big\rangle_0 + \varepsilon^K_{jI} \langle D_j\phi_I, \psi_K\rangle_0 \right] \\
&= \sum_{I,K,j} \left[\sum_{k=0}^N -\varepsilon^K_{jI} \Big\langle D_k\phi_I, D_k D_j\psi_K \Big\rangle_0 + \varepsilon^K_{jI} \langle D_j\phi_I, \psi_K\rangle_0 \right] \\
&= \sum_I \Bigg[\sum_{k=0}^N \Big\langle D_k\phi_I, D_k\Big(- \sum_{\substack{|K|=q+1\\ j=0,\dots,N}} \varepsilon^K_{jI} D_j\psi_K \Big)\Big\rangle_0 \\
&\qquad + \langle \phi_I, \Big(- \sum_{\substack{|K|=q+1\\ j=0,\dots,N}} \varepsilon^K_{jI} D_j\psi_K \Big)\rangle_0 \Bigg]. \quad \square
\end{aligned}$$

Lemma 6.2. *Let d^* be the adjoint of d in the $W^1(\mathbb{R}_+^{N+1})$ norm. Then*

$$\operatorname{dom} d^* \cap \bigwedge^{q+1}(\overline{\Omega}) = \Big\{ \psi \in \bigwedge^{q+1}(\overline{\Omega}) : D_0\psi_{0J}|_{x_0=0}, \quad |J| = q \Big\}.$$

That is, if $\psi = \sum_{|K|=q+1} \psi_K dx^K$, then $\psi \in \operatorname{dom} d^*$ amounts to a condition on the coefficients ψ_K with index $K = (0, \dots, k_N) \equiv 0J$ only, and it requires that

$$D_0\psi_{0J}|_{x_0=0} = 0\,.$$

Proof. Let $\phi \in \bigwedge^q$, $\psi \in \bigwedge^{q+1}$. Then

$$
\begin{aligned}
\langle d\phi, \psi\rangle_1 &= \sum_{|K|=q+1} \Big\langle \sum_{\substack{|I|=q\\ j=0,\dots,N}} D_j\phi_I, \psi_K \Big\rangle_1 \\
&= \sum_K \Bigg[\sum_{a=0,\dots,N} \Big\langle D_a \Big(\sum_{I,j} \varepsilon^K_{I,j} D_j\phi_I \Big), D_a\psi_K \Big\rangle_0 + \Big\langle \sum_{I,j} \varepsilon^K_{I,j} D_j\phi_I, \psi_K \Big\rangle_0 \Bigg] \\
&= \sum_K \Bigg[\sum_{a=0,\dots,N} \sum_{I,j} \Big(\varepsilon^K_{I,j} \Big\langle D_a\phi_I, D_aD_j\psi_K \Big\rangle_0 - \varepsilon^K_{0I} \int_{\mathbb{R}^N} D_a\phi_I \overline{D_a\psi_K}\Big|_{x_0=0} \Big) \\
&\quad + \sum_{I,j} \Big(-\varepsilon^K_{jI} \Big\langle \phi_I, D_j\psi_K \Big\rangle_0 - \varepsilon^K_{0I} \int_{\mathbb{R}^N} \phi_I \overline{\psi_K}\Big|_{x_0=0} \Big) \Bigg] \\
&= \sum_I \Bigg[\sum_{a=0,\dots,N} \Big\langle D_a\phi_I, D_a \Big(-\sum_{K,j} \varepsilon^K_{jI} D_j\psi_K \Big) \Big\rangle_0 + \Big\langle \phi_I, -\sum_{K,j} \varepsilon^K_{jI} D_j\psi_K \Big\rangle_0 \Bigg] \\
&\quad - \sum_{K,I,j} \Bigg[\sum_{a=0,\dots,N} \varepsilon^K_{0I} \int_{\mathbb{R}^N} D_a\phi_I \overline{D_a\psi_K}\Big|_{x_0=0} + \varepsilon^K_{0I} \int_{\mathbb{R}^N} \phi_I \overline{\psi_K}\Big|_{x_0=0} \Bigg].
\end{aligned}
$$

Following the usual pattern we see that the only terms that we cannot control with the W^1 norm of ψ are

$$
\varepsilon^K_{0I} \int_{\mathbb{R}^N} D_0\phi_I \overline{D_0\psi_K}\Big|_{x_0=0} \qquad |K| = q+1, \quad |I| = q.
$$

Since ϕ was arbitrary, it must be $D_0\psi_K|_{x_0=0} = 0$ when $K = 0I$. $\square$

Next we want to determine the expression for the Hilbert space adjoint d^* when acting on $[\operatorname{dom} d^* \cap W^1_{q+1}]$. We have the following result.

Proposition 6.3. *The adjoint of d in the W^1-inner product is given by $d^* = d' + \mathcal{K}$ where, for $\psi = \sum_{|K|=q+1} \psi_K dx^K$,*

$$
\begin{aligned}
\mathcal{K}\psi &= \sum_{|I|=q} (\mathcal{K}\psi)_I \, dx^I \\
&= \sum_{\substack{|I|=q\\ I\not\ni 0}} \Big(K_{x_0} * \psi_{0I}(0, \cdot) \Big)(x')\, dx^I.
\end{aligned}
$$

Before proving the proposition we make a few remarks.

REMARK 6.4.

1. Notice that $\mathcal{K}$ acts diagonally, in the sense that $(\mathcal{K}\psi)_I$ depends only on ψ_{0I}.
2. Only the terms of the form ψ_{0I}, i.e. terms for which $\psi\lfloor\partial/\partial x_0 \neq 0$, contribute to $\mathcal{K}\psi$.

3. $\mathcal{K}\psi$ has only purely "tangential" components, i.e. $(\mathcal{K}\psi)_I = 0$ if $0 \in I$.

Proof of 6.3. Notice that we could continue the calculation in the previous proof to obtain (for $\psi \in \operatorname{dom} d^*$)

$$\langle d\phi, \psi\rangle_1 = \langle \phi, d'\psi\rangle_1 + \sum_{\substack{|I|=q \\ I \not\ni 0}} \left[\int_{\mathbb{R}^N} \phi_I \overline{\triangle' \psi_{0I}}\Big|_{x_0=0} - \int_{\mathbb{R}^N} \phi_I \overline{\psi_{0I}}\Big|_{x_0=0} \right]. \tag{6.1}$$

Now, as in the computation in Proposition 3.2, we write

$$d^*\psi = d'\psi + \theta,$$

where $\theta = \mathcal{K}\phi$, and $\mathcal{K}$ is a singular Green's operator mapping $(q+1)$-forms into q-forms. Now

$$\langle \phi, d'\psi\rangle_1 + \langle \phi, \theta\rangle_1 = \langle \phi, d^*\psi\rangle_1 = \langle d\phi, \psi\rangle_1. \tag{6.2}$$

Therefore, (6.1), (6.2) and Green's theorem give that

$$\begin{aligned} &\sum_{\substack{|I|=q \\ I \not\ni 0}} \left[\int_{\mathbb{R}^N} \phi_I \overline{\triangle' \psi_{0I}}\Big|_{x_0=0} - \int_{\mathbb{R}^N} \phi_I \overline{\psi_{0I}}\Big|_{x_0=0} \right] \\ &= \quad \langle \phi, \theta\rangle_1 \\ &= \quad \sum_I \left[\sum_{a=0}^N \langle D_a\phi_I, D_a\theta_I\rangle_0 + \langle \phi_I, \theta_I\rangle_0 \right] \\ &= \quad -\sum_I \langle \phi_I, \triangle\theta_I\rangle_0 - \int_{\mathbb{R}^N} \phi_I \overline{D_0\theta_I}\Big|_{x_0=0} + \langle \phi_I, \theta_I\rangle_0. \end{aligned}$$

Therefore θ must satisfy the following conditions

$$\begin{cases} -\triangle\,\theta_I + \theta_I & = 0 \\ -D_0\theta_I|_{x_0=0} & = \begin{cases} 0 & \text{if } 0 \in I \\ \triangle'\psi_{0I} - \psi_{0I} & \text{if } 0 \notin I \end{cases} \end{cases}$$

Thus

$$\Big(\mathcal{K}\psi\Big)_I = \theta_I = 0 \qquad \text{if } 0 \in I,$$

since the solution of $\triangle u + u = 0$ with the boundary condition $D_0 u(0, x') = 0$ consists of just the zero function. On the other hand, if $I \not\ni 0$ we have that $(\mathcal{K}\psi)_I = \theta_I$ is the solution of

$$\begin{cases} -\triangle\, u + u = 0 & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial u}{\partial x_0}(0, \cdot) = -\triangle'\,\psi_{0I} + \psi_{0I} & \text{on } \mathbb{R}^N \end{cases}.$$

In the proof of Proposition 3.2 , we showed that θ is given by

$$\begin{aligned}\theta_I &= -\int_{\mathbb{R}^N} \sqrt{1+|2\pi\xi'|^2}e^{-\sqrt{1+|2\pi\xi'|^2}x_0}\widehat{\psi_{0I}}(0,\xi')e^{2\pi i\xi'\cdot x'}\,dx' \\ &\equiv \Big(K_{x_0} * \phi_0(0,\,\cdot\,)\Big)(x'). \quad \square\end{aligned}$$

We now return to the main object of our work. We would like to solve the boundary value problem

$$\begin{cases}(dd^*+d^*d)\phi = & \alpha \\ \phi \in \operatorname{dom} d^* \\ d\phi \in \operatorname{dom} d^*\end{cases}$$

for $\alpha \in \bigwedge^q$. Recall that $d^* = d' + \mathcal{K}$. A simple calculation shows that

$$(dd'+d'd)\phi = \sum_{|I|=q} -\triangle\,\phi_I dx^I. \tag{6.3}$$

Thus we need to compute $d\mathcal{K}\phi$ and $\mathcal{K}d\phi$. The next lemma addresses this task.

Lemma 6.5. *Let $\phi, d\phi \in \operatorname{dom} d^*$, with $\phi = \sum_I \phi_I dx^I$. Then we have*

$$d\mathcal{K}\phi = \sum_{|J|=q}\left[\sum_{\substack{|I|=q-1, I\not\ni 0 \\ j=0,\ldots,N}} \varepsilon^J_{jI} D_j K_{x_0} * \Big(\phi_{0I}(0,\cdot)\Big)\right]dx^J,$$

and

$$\mathcal{K}d\phi = \sum_{|I|=q, I\not\ni 0}\left[\sum_{\substack{|J|=q \\ j=0,\ldots,N}} \varepsilon^{0I}_{jJ} K_{x_0} * (D_j\phi_J(0,\cdot))\right]dx^I.$$

Proof. These are just straightforward computations. We have

$$\begin{aligned}d\mathcal{K}\phi &= d\left(\sum_{|I|=q-1, I\not\ni 0} K_{x_0} * \Big(\phi_{0I}(0,\cdot)\Big)dx^I\right) \\ &= \sum_{|I|=q-1, I\not\ni 0}\ \sum_{j=0,\ldots,N} D_j K_{x_0} * \Big(\phi_{0I}(0,\cdot)\Big)dx^j \wedge dx^I \\ &= \sum_{|J|=q}\left[\sum_{\substack{|I|=q-1, I\not\ni 0 \\ j=0,\ldots,N}} \varepsilon^J_{jI} D_j K_{x_0} * \Big(\phi_{0I}(0,\cdot)\Big)\right]dx^J.\end{aligned}$$

This proves the statement for $d\mathcal{K}\phi$. On the other hand,

$$\begin{aligned}
\mathcal{K}d\phi &= \mathcal{K}\left(\sum_{|K|=q+1}\left(\sum_{\substack{|J|=q\\ j=0,\ldots,N}}\varepsilon^{K}_{jJ}D_j\phi_J\right)dx^K\right)\\
&= \sum_{|I|=q,I\not\ni 0} K_{x_0} * \left(\sum_{\substack{|J|=q\\ j=0,\ldots,N}}\varepsilon^{0I}_{jJ}D_j\phi_J(0,\cdot)\right)dx^I\\
&= \sum_{|I|=q,I\not\ni 0}\left[\sum_{\substack{|J|=q\\ j=0,\ldots,N}}\varepsilon^{0I}_{jJ}K_{x_0} * \Big(D_j\phi_J(0,\cdot)\Big)\right]dx^I. \quad \square
\end{aligned}$$

Next we want to compute $(\mathcal{K}d + d\mathcal{K})\phi$. [Note that, in this discussion, the letter K is both a kernel and an index; no confusion should result.]

Proposition 6.6. *Let $\phi \in \bigwedge^q$, with $\phi, d\phi \in \operatorname{dom} d^*$. Set $(\mathcal{K}d + d\mathcal{K})\phi \equiv \beta = \sum_{|K|=q}\beta_K dx^K$. Then*

$$\begin{aligned}
\beta_K &= \frac{\partial}{\partial x_0}K_{x_0} * \Big(\phi_K(0,\cdot)\Big) &&\text{if } K \ni 0\\
\beta_K &= K_{x_0} * \left(\frac{\partial}{\partial x_0}\phi_K(0,\cdot)\right) &&\text{if } K \not\ni 0.
\end{aligned}$$

Remark. Notice that $\mathcal{K}d + d\mathcal{K}$ is a diagonal operator on the space of q-forms $\bigwedge^q$.

Proof of 6.6. Suppose that $K \ni 0$. Then $(\mathcal{K}d\phi)_K = 0$, since $\mathcal{K}$ applied to any form has only tangential components. Moreover, for the same reason, $(d\mathcal{K}\phi)_{0K'}$ can be obtained only by differentiating $(\mathcal{K}\phi)_{K'}$ by D_0, and "wedging" by dx^0. That is,

$$\begin{aligned}
(d\mathcal{K}\phi)_{0K'} &= D_0(\mathcal{K}\phi)_{K'}dx^0 \wedge dx^{K'}\\
&= D_0\Big(K_{x_0} * \Big(\phi_{0K'}(0,\cdot)\Big)dx^0 \wedge d^{K'}.
\end{aligned}$$

Now suppose that $K \not\ni 0$. We use Lemma 6.5 to obtain that

$$\begin{aligned}
d\mathcal{K}\phi + \mathcal{K}d\phi &= \sum_{|K|=q}\left[\sum_{\substack{|I|=q-1,I\not\ni 0\\ j=0,\ldots,N}}\varepsilon^{K}_{jI}D_jK_{x_0} * \Big(\phi_{0I}(0,\cdot)\Big)\right]dx^K\\
&\quad + \sum_{|K|=q}\left[\sum_{\substack{|J|=q\\ j=0,\ldots,N}}\varepsilon^{0K}_{jJ}K_{x_0} * \Big(D_j\phi_{0J}(0,\cdot)\Big)\right]dx^K.
\end{aligned}$$

Thus we need to describe the term on the right hand side of this equation when $K \not\ni 0$. Notice that in this case j cannot be 0 in the first sum, since otherwise

$\varepsilon_{0I}^K = 0$. Then, when $K \not\ni 0$ the coefficient of dx^K in the right hand side above equals

$$K_{x_0} * \Big(D_0\phi_{0K}(0,\cdot)\Big) + \sum_{\substack{|J|=q, J\not\ni 0\\ j=1,\dots,N}} \varepsilon_{0J}^{jK} D_j\bigg(K_{x_0} * \Big(\phi_{0J}(0,\cdot)\Big)\bigg)$$
$$+ \sum_{\substack{|I|=q-1, I\not\ni 0\\ j=1,\dots,N}} \varepsilon_{jI}^{K} D_j\bigg(K_{x_0} * \Big(\phi_{0I}(0,\cdot)\Big)\bigg) \equiv K_{x_0} * \Big(D_0\phi_K(0,\cdot)\Big),$$

since J must be of the form $0I$ with $|I| = q-1$, $I \not\ni 0$, and therefore

$$\varepsilon_{j0I}^{0K} = -\varepsilon_{j0I}^{0K} = -\varepsilon_{jI}^{K}.$$

This concludes the proof. □

Corollary 6.7. *Let $(\mathcal{K}d + d\mathcal{K})\psi = G\psi$. Then $\beta = G\psi$ is a form such that its components solve the boundary value problems*

$$\begin{cases} (-\triangle + I)\beta_K &= 0 \\ \dfrac{\partial \beta_K}{\partial x_0} &= (-\triangle' + I)(d\psi)_{0K} \end{cases} \qquad \text{if } K \not\ni 0,$$

and

$$\begin{cases} (-\triangle + I)\beta_K &= 0 \\ \beta_K &= (-\triangle' + I)\psi_K \end{cases} \qquad \text{if } K \ni 0.$$

Proof. By Proposition 6.3, if $G\psi = \sum_{|K|=q} \beta_K dx^K$ for $K \not\ni 0$, then

$$\beta_K = K_{x_0} * \left(\frac{\partial \psi_K}{\partial x_0}(0,\cdot)\right).$$

By construction

$$(-\triangle + I)\beta_K = 0 \qquad \text{on } \mathbb{R}_+^{N+1},$$

and

$$\begin{aligned} \frac{\partial \beta_K}{\partial x_0} &= (-\triangle' + I)(d\psi)_{0K} \\ &= (-\triangle' + I)\left(\frac{\partial \psi_K}{\partial x_0} + T_1\psi_{K'}\right) \qquad \text{on } b\mathbb{R}_+^{N+1},\ K' \ni 0, \end{aligned}$$

where T_1 is a first order tangential differential operator.

On the other hand, if $K \ni 0$, then $(\mathcal{K}d\psi)_K = 0$. Thus, for $K = 0K'$,

$$\begin{aligned} (G\psi)_K &= (d\mathcal{K}\psi)_K \\ &= \frac{\partial}{\partial x_0}(\mathcal{K}\psi)_{K'}. \end{aligned}$$

Then, on $\mathbb{R}^{N+1}_+$,

$$(-\triangle + I)(G\psi)_K = \frac{\partial}{\partial x_0}(-\triangle + I)(\mathcal{K}\psi)_{K'} = 0;$$

and if $K = 0K'$,

$$\begin{aligned}(G\psi)_K\big|_{b\Omega} &= \frac{\partial}{\partial x_0}(\mathcal{K}\psi)_{K'}\bigg|_{b\Omega} \\ &= (-\triangle' + I)\psi_K\big|_{b\Omega}. \qquad \square\end{aligned}$$

Before analyzing the boundary value problem we need one more result; that is, we wish to make explicit the boundary condition $d\phi \in \operatorname{dom} d^*$.

Lemma 6.8. *Let $\phi \in \bigwedge^q$, and let $\phi \in \operatorname{dom} d^*$. Then $d\phi \in \operatorname{dom} d^*$ if and only if*

$$\frac{\partial^2 \phi_K}{\partial x_0^2}(0, \cdot) = 0 \qquad \text{for } K \not\ni 0.$$

Proof. By Lemma 6.2 we have that $d\phi \in \operatorname{dom} d^*$ if and only if, for all multi-indices $K' \not\ni 0$, $|K'| = q$, there holds the equality

$$\frac{\partial}{\partial x_0}\left(\sum_{I,j} \varepsilon^{0K'}_{jI} \frac{\partial \phi_I}{\partial x_j}\right)(0, x') = 0.$$

If $j \neq 0$, then $I \ni 0$ and

$$\frac{\partial}{\partial x_0}\frac{\partial \phi_I}{\partial x_j}(0, \cdot) = \frac{\partial}{\partial x_j}\left(\frac{\partial \phi_I}{\partial x_0}(0, \cdot)\right) = 0,$$

since, for $\phi \in \operatorname{dom} d^*$, we have $\partial/\partial x_0 \phi_i(0, \cdot) = 0$ when $I \ni 0$. Thus we obtain

$$\frac{\partial^2 \phi_I}{\partial x_0^2}(0, \cdot) = 0 \qquad \text{for all } I \not\ni 0,\ |I| = q. \quad \square$$

We finally are able to formulate the boundary value problem.

THEOREM 6.9. *Consider the boundary value problem*

$$\begin{cases} (dd^* + d^*d)\phi = \alpha & \alpha \in \bigwedge^q(\overline{\Omega}) \\ \phi \in \operatorname{dom} d^* \\ d\phi \in \operatorname{dom} d^* \end{cases} \tag{6.4}$$

Let α be a q-form with coefficients in $W^r(\mathbb{R}^{N+1}_+) \cap L^1(\mathbb{R}^{N+1}_+)$, with $r > 1/2$. Then, if $N \geq 4$, there is a unique q-form ϕ with coefficients in $W^{r+2}(\mathbb{R}^{N+1}_+)$ solving the boundary value problem and satisfying the estimate

$$\|\phi\|_{W^{r+2}} \leq c \cdot \left\{\|\alpha\|_{W^r(\mathbb{R}^{N+1}_+)} + \|\alpha\|_{L^1(\mathbb{R}^{N+1}_+)}\right\}.$$

If $N = 2, 3$ we need to further require that

$$\int \alpha(x)\, dx = 0 \ , \qquad\qquad \alpha \in L^1(|x|dx, \mathbb{R}^{N+1}_+),$$

so that the solution ϕ satisfies the estimate

$$\|\phi\|_{W^{r+2}} \le c \cdot \left\{ \|\alpha\|_{W^r(\mathbb{R}^{N+1}_+)} + \|\alpha\|_{L^1(|x|dx,\mathbb{R}^{N+1}_+)} \right\}.$$

If $N = 1$ we need to require that

$$\int \alpha(x)\, dx = 0 \ , \qquad \int x_i \alpha(x)\, dx = 0,\ i = 0, 1 \qquad \alpha \in L^1(|x|^2 dx, \mathbb{R}^2);$$

in this case similar estimates hold for the solution ϕ, with the addition of the term $\|\alpha\|_{L^1(|x|^2dx,\mathbb{R}^2)}$ to the right hand side.

By the discussion preceding the statement of the theorem we see that the boundary value problem (6.4) is equivalent to the two scalar problems

$$\begin{cases} - \triangle\, \phi_K + \dfrac{\partial}{\partial x_0}\Big(K_{x_0} * \phi_K|_{x_0=0}\Big) = \alpha_K \\ \dfrac{\partial \phi_K}{\partial x_0}\Big|_{x_0=0} = 0 \end{cases} \qquad 0 \in K, \tag{6.5}$$

and

$$\begin{cases} - \triangle\, \phi_K + \Big(K_{x_0} * \dfrac{\partial \phi_K}{\partial x_0}\Big|_{x_0=0}\Big) = \alpha_K \\ \dfrac{\partial^2 \phi_K}{\partial x_0^2}\Big|_{x_0=0} = 0 \end{cases} \qquad 0 \notin K. \tag{6.6}$$

Thus the problem is reduced to solving two different boundary value problems. The boundary value problem (6.6) has been solved already in the case $q = 0$. The solution of the second boundary value problem is contained in the following theorems. As in the case of functions, for greater flexibility we solve the boundary value problem with non-zero boundary data.

THEOREM 6.10. *Consider the boundary value problem*

$$\begin{cases} - \triangle\, u + \dfrac{\partial}{\partial x_0}\Big(K_{x_0} * u(0, \cdot)\Big) = f & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial u}{\partial x_0}(0, \cdot) = h & \text{on } \mathbb{R}^N \end{cases} \tag{6.7}$$

If there exists a solution u, then it satisfies the a priori *estimate*

$$\|u\|_{W^{r+2}} \le c \cdot \left\{ \|f\|_{W^r(\mathbb{R}^{N+1}_+)} + \|h\|_{W^{r+1/2}(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^{N+1}_+)} \right\},$$

for $r > 1/2$.

Proof. We break the proof into two steps. We first estimate the function u which is the solution of the boundary value problem with zero boundary data. Then we show how to reduce the general case to this special one.

In this proof we let $\mathcal{J}$ denote the Green's function for the Neumann problem for the Laplacian on the half space. Suppose now that u is the solution of the boundary value problem with $h = 0$. Then u can be written as

$$u = \mathcal{J}\left(f - \frac{\partial}{\partial x_0}\Big(K_{x_0} * u(0,\cdot)\Big)\right).$$

Recall that, for $N \geq 2$,

$$\begin{aligned}(\mathcal{J}f)(x_0, x') = \frac{1}{(N-1)\omega_{N+1}} \int_{\mathbb{R}^{N+1}_+} \Bigg\{ & \frac{1}{\big((x_0-y_0)^2 + |x'-y'|^2\big)^{(N-1)/2}} \\ & + \frac{1}{\big((x_0+y_0)^2 + |x'-y'|^2\big)^{(N-1)/2}} \Bigg\} f(y_0, y')\, dy_0\, dy',\end{aligned}$$

and

$$\widehat{\mathcal{J}f}(x_0, \xi') = \frac{1}{2|2\pi\xi'|} \int_0^{+\infty} \Big(e^{-|2\pi\xi'||x_0-y_0|} + e^{-|2\pi\xi'|(x_0+y_0)}\Big) \hat{f}(y_0, \xi')\, dy_0.$$

Then

$$\begin{aligned}\hat{u}(0,\xi') &= \widehat{\mathcal{J}f}(0,\xi') - |2\pi\xi'|^{-1} \int_0^{+\infty} e^{-(|2\pi\xi'| + \sqrt{1+|2\pi\xi'|^2})y_0}\, dy_0\, (1 + |2\pi\xi'|^2)\hat{u}(0,\xi') \\ &= \widehat{\mathcal{J}f}(0,\xi') - \frac{(1+|2\pi\xi'|^2)}{|2\pi\xi'|\Big(|2\pi\xi'| + \sqrt{1+|2\pi\xi'|^2}\Big)} \hat{u}(0,\xi').\end{aligned}$$

Thus, recalling the definition (5.5) of m_2, we have that

$$\hat{u}(0,\xi') = |2\pi\xi'| m_2(\xi') \widehat{\mathcal{J}f}(\xi').$$

Set $F = f - (\partial/\partial x_0)\Big(K_{x_0} * u(0,\cdot)\Big)$. Since $u = \mathcal{J}F$, in order to estimate $\|u\|_{W^{3/2}(\mathbb{R}^{N+1}_+)}$ we can proceed as in the proof of Theorem 5.4 and, using the fact that $\partial^2/\partial x_0^2 = \triangle - \triangle'$, we can reduce to estimating derivatives of u of order not exceeding 2 in the x_0-variable.

Let g_{e} denote the even extension (in the x_0 variable) to all of $\mathbb{R}^{N+1}$ of the function g defined initially on $\mathbb{R}^{N+1}_+$. We have that

$$\left[\Big(\mathcal{J}g\Big)(x_0, x')\right]_{\mathrm{e}} = \Big(\mathcal{N}g_{\mathrm{e}}\Big)(x_0, x')\ ,$$

and

$$\mathcal{F}\left(\frac{\partial}{\partial x_0} K_{x_0}\right)_{\mathrm{e}} = \frac{2(1+|2\pi\xi'|^2)^{3/2}}{1+|2\pi\xi|^2}.$$

Combining all of these facts we obtain:

$$\mathcal{F}u_{\mathrm{e}} = |2\pi\xi|^{-2}\Big(\mathcal{F}f_{\mathrm{e}}(\xi) - \frac{2(1+|2\pi\xi'|^2)^{3/2}}{1+|2\pi\xi|^2}|2\pi\xi'|m_2(\xi')\widehat{\mathcal{J}f}(0,\xi')\Big). \tag{6.8}$$

Thus

$$\begin{aligned}\|u\|_{W^{2+r}(\mathbb{R}^{N+1}_+)} &\le c\Big(\int_{\mathbb{R}^{N+1}}\sum_{i,j=0}^{N}(1+|2\pi\xi_i|^2|2\pi\xi_j|^2)|\mathcal{F}u(\xi)|^2(1+|2\pi\xi'|^2)^r\,d\xi\Big)\\ &\le c\Big(\|u\|_{L^2(\mathbb{R}^{N+1}_+)} + \int_{-\infty}^{+\infty}\int_{|2\pi\xi'|\ge 1}|\mathcal{F}F_{\mathrm{e}}(\xi)|^2|2\pi\xi'|^{2r}\,d\xi'\,d\xi_0\Big),\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}F_{\mathrm{e}}(\xi) &= \mathcal{F}f_{\mathrm{e}}(\xi) - \frac{2(1+|2\pi\xi'|^2)^{3/2}}{1+|2\pi\xi|^2}m_2(\xi')\int_0^{+\infty}e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0\\ &= \mathcal{F}f_{\mathrm{e}}(\xi) - \frac{2(1+|2\pi\xi'|^2)}{1+|2\pi\xi|^2}m_1(\xi')\int_0^{+\infty}e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0.\end{aligned}$$

Therefore, by obvious calculations and the Schwarz inequality,

$$\begin{aligned}\|u\|^2_{W^{2+r}(\mathbb{R}^{N+1}_+)} &\le c\Big(\|u\|^2_{L^2(\mathbb{R}^{N+1}_+)} + \|f\|^2_{W^r(\mathbb{R}^{N+1}_+)} + \int_{|2\pi\xi'|\ge 1}\frac{(1+|2\pi\xi'|^2)^{2+r}}{|2\pi\xi|}\\ &\qquad\times\Big(\int_0^{+\infty}|\hat{f}(y_0,\xi')|^2\,dy_0\Big)\Big(\int_0^{+\infty}\frac{1}{(1+|2\pi\xi'|^2)^2+|2\pi\xi_0|^4}\,d\xi_0\Big)\,d\xi'\Big)\\ &\le c\Big(\|u\|^2_{L^2(\mathbb{R}^{N+1}_+)} + \|f\|^2_{W^r(\mathbb{R}^{N+1}_+)}\Big).\end{aligned}$$

This proves the theorem in the case that the boundary data $h \equiv 0$.

In the general case, let $\mathcal{Q}$ be the operator initially defined on $C_0^\infty(\mathbb{R}^N)$ by

$$(\mathcal{Q}g)\widehat{\ }(x_0,\xi') = -\frac{e^{-\sqrt{1+|2\pi\xi'|^2}x_0}}{\sqrt{1+|2\pi\xi'|^2}}\,\hat{g}(\xi').$$

Notice that $(\partial/\partial x_0)(\mathcal{Q}g)|_{x_0=0} = g$, and that

$$\mathcal{Q}: W^s(\mathbb{R}^N) \longrightarrow W^{s+1/2}(\mathbb{R}^{N+1}_+).$$

We seek an *a priori* estimate for a function u that solves of (6.7). Let G' denote the operator $u \longmapsto (\partial/\partial x_0)\big(K_{x_0} * u(0,\cdot)\big)$. Set $v = u - \mathcal{Q}h$. Then v solves the boundary value problem

$$\begin{cases}(-\triangle + G')v = f + (-\triangle + G')\mathcal{Q}h & \text{on } \mathbb{R}^{N+1}_+\\ \dfrac{\partial v}{\partial x_0}(0,\cdot) = 0 & \text{on } \mathbb{R}^N\end{cases}.$$

Thus for such v we have the usual *a priori* estimate. Notice that

$$\Big[(-\triangle+G')\mathcal{Q}h\Big]\hat{}(\xi')=-\frac{|2\pi\xi'|^2e^{-\sqrt{1+|2\pi\xi'|^2}x_0}}{\sqrt{1+|2\pi\xi'|^2}}\hat{h}(\xi'),$$

so that the operator $(-\triangle+G')\mathcal{Q}$ has the same behavior as the operator $\widetilde{\mathcal{K}}$ studied in Proposition 3.3, so it maps $W^s(\mathbb{R}^N)$ into $W^{s+1/2}(\mathbb{R}^{N+1}_+)$ continuously. Therefore

$$\begin{aligned}\|u\|_{W^{r+2}(\mathbb{R}^{N+1}_+)}&\le\|v\|_{W^{r+2}(\mathbb{R}^{N+1}_+)}+\|\mathcal{Q}h\|_{W^{r+2}(\mathbb{R}^{N+1}_+)}\\&\le c\Big\{\|f\|_{W^r(\mathbb{R}^{N+1}_+)}+\|(-\triangle+G')\mathcal{Q}h\|_{W^r(\mathbb{R}^{N+1}_+)}+\|v\|_{L^2(\mathbb{R}^{N+1}_+)}\\&\qquad+\|\mathcal{Q}h\|_{W^r(\mathbb{R}^{N+1}_+)}\Big\}\\&\le c\Big\{\|f\|_{W^r(\mathbb{R}^{N+1}_+)}+\|h\|_{W^{r+1/2}(\mathbb{R}^N)}+\|u\|_{L^2(\mathbb{R}^{N+1}_+)}\Big\},\end{aligned}$$

since $v=u-\mathcal{Q}h$. This concludes the proof. □

THEOREM 6.11. *Let $N\ge4$. Then for any $f\in L^2(\mathbb{R}^{N+1}_+)\cap L^2(\mathbb{R}^{N+1}_+)$, there exists a unique function u solving the boundary value problem (6.5) and such that*

$$\|u\|_{L^2(\mathbb{R}^{N+1}_+)}\le c\cdot\left\{\|f\|_{L^2(\mathbb{R}^{N+1}_+)}+\|f\|_{L^1(\mathbb{R}^{N+1}_+)}+\|h\|_{W^{1/2}(\mathbb{R}^N)}\right\}.$$

If $N=2,3$ we suppose in addition that f satisfies

$$\int f(x)\,dx=0,\qquad\qquad f\in L^1(|x|dx,\mathbb{R}^{N+1}_+).$$

Then there exists a unique solution f that satisfies the estimate

$$\|u\|_{L^2(\mathbb{R}^{N+1}_+)}\le c\cdot\left\{\|f\|_{L^2(\mathbb{R}^{N+1}_+)}+\|f\|_{L^1(|x|dx,\mathbb{R}^{N+1}_+)}+\|h\|_{W^{1/2}(\mathbb{R}^N)}\right\}.$$

If $N=1$ we suppose that f is such that

$$\int f(x)\,dx=0,\qquad\int x_if(x)\,dx=0,\ i=0,1\qquad f\in L^1(|x|^2dx,\mathbb{R}^2).$$

Then there exists a unique solution u such that

$$\|u\|_{L^2(\mathbb{R}^{N+1}_+)}\le c\cdot\left\{\|f\|_{L^2(\mathbb{R}^{N+1}_+)}+\|f\|_{L^1(|x|^2dx,\mathbb{R}^{N+1}_+)}+\|h\|_{W^{1/2}(\mathbb{R}^N)}\right\}.$$

Proof. Recall that we have found that if a solution u exists, then it must be given by the formula (6.8), i.e.

$$\mathcal{F}u_{\mathrm{e}}(\xi)=|2\pi\xi|^{-2}\left(\mathcal{F}f_{\mathrm{e}}(\xi)-2\frac{(1+|2\pi\xi'|^2)^{3/2}}{1+|2\pi\xi|^2}m_2(\xi')\widehat{F}_1(\xi')\right),$$

where F_1 is given by

$$\widehat{F}_1(\xi')=|2\pi\xi'|\widehat{\mathcal{J}f}(0,\xi')=\int_0^{+\infty}e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0.$$

In order to show that u given by the above formula is indeed a solution, by the estimate in Theorem 6.10 it suffices to show that $u \in L^2(\mathbb{R}^{N+1}_+)$.

Suppose that $N \geq 4$. We have

$$\begin{aligned}
&\|u\|_{L^2(\mathbb{R}^{N+1}_+)} \\
&= \frac{1}{2}\int_{\mathbb{R}^{N+1}} |\mathcal{F}u_{\mathrm{e}}(\xi)|^2\,d\xi \\
&\leq \Big\{\int_{|2\pi\xi'|\leq 1}\int_{-\infty}^{+\infty} |\mathcal{F}f_{\mathrm{e}}(\xi)|^2\,d\xi_0\,d\xi'\Big\}^{1/2} + \Big\{\int_{|2\pi\xi'|\geq 1}\int_{-\infty}^{+\infty} |\mathcal{F}f_{\mathrm{e}}(\xi)|^2\,d\xi_0\,d\xi'\Big\}^{1/2} \\
&\leq c\Big\{\Big(\int_{|2\pi\xi'|\leq 1}\int_{-\infty}^{+\infty} |2\pi\xi|^{-4}\,d\xi_0\,d\xi'\Big)\sup_{\xi\in\mathbb{R}^{N+1}}\Big(|\mathcal{F}f_{\mathrm{e}}(\xi)| + \int_0^{+\infty}|\hat{f}(y_0,\xi')|\,dy_0\Big)^2 \\
&\qquad + \int_{|2\pi\xi'|\geq 1}\int_{-\infty}^{+\infty} |\mathcal{F}f_{\mathrm{e}}(\xi)|^2\,d\xi_0\,d\xi' \\
&\qquad + \int_{|2\pi\xi'|\geq 1}\int_{-\infty}^{+\infty} |2\pi\xi|^{-4}\,d\xi_0\Big|\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\hat{f}(y_0,\xi')\,dy_0\Big|^2\,d\xi'\Big\} \\
&\leq c\Big\{\|f\|_{L^1(\mathbb{R}^{N+1}_+)} + \|f\|_{L^2(\mathbb{R}^{N+1}_+)}\Big\}.
\end{aligned}$$

This proves our statement in the case $N \geq 4$.

Let now $N = 2, 3$, and assume that

$$\int_{\mathbb{R}^{N+1}_+} f(x)\,dx = 0.$$

Notice that this implies that $\int_0^{+\infty}\hat{f}(y_0,0)\,dy_0 = 0$. Therefore,

$$\begin{aligned}
\|u\|^2_{L^2(\mathbb{R}^{N+1}_+)} &\leq \int_{|2\pi\xi|\leq 1}\int_{-\infty}^{+\infty} \frac{|\mathcal{F}f_{\mathrm{e}}(\xi) - \mathcal{F}f_{\mathrm{e}}(0)|^2}{|2\pi\xi|^4}\,d\xi_0\,d\xi' \\
&\quad + \int_{|2\pi\xi|\leq 1}\int_{-\infty}^{+\infty} |2\pi\xi|^{-2}\Big|\int_0^{+\infty} e^{-|2\pi\xi'|y_0}\frac{\hat{f}(y_0,\xi') - \hat{f}(y_0,0)}{|2\pi\xi|}\,dy_0\Big|^2\,d\xi_0\,d\xi' \\
&\quad + \int_{|2\pi\xi|\leq 1}\int_{-\infty}^{+\infty} |2\pi\xi|^{-2}\Big(\int_0^{+\infty}\frac{|e^{-|2\pi\xi'|y_0} - 1|}{|2\pi\xi'|}|\hat{f}(y_0,0)|\,dy_0\Big)^2\,d\xi_0\,d\xi' \\
&\quad + \int_{|2\pi\xi'|\geq 1}\int_{-\infty}^{+\infty}\Big[|\mathcal{F}f_{\mathrm{e}}(\xi)|^2 + \Big(\int_0^{+\infty} e^{-|2\pi\xi'|y_0}|\hat{f}(y_0,\xi')|\,dy_0\Big)^2\Big]\,d\xi_0\,d\xi' \\
&\leq C\int_{|2\pi\xi'|\leq 1}|2\pi\xi'|^{-1}\,d\xi'\Big\{\Big(\int_{\mathbb{R}^{N+1}_+}|x|\cdot|f(x)|\,dx\Big)^2 + \Big(\int_{\mathbb{R}^{N+1}_+}|f(x)|\,dx\Big)^2\Big\} \\
&\quad + C\|f\|_{L^2(\mathbb{R}^{N+1}_+)}.
\end{aligned}$$

Here we have used the following facts:

$$\frac{|\mathcal{F}f_{\mathrm{e}}(\xi)-\mathcal{F}f_{\mathrm{e}}(0)|}{|2\pi\xi|}\leq\frac{1}{2\pi}\sup_{\xi\in\mathbb{R}^{N+1}}|\mathrm{grad}\mathcal{F}f_{\mathrm{e}}(\xi)|\leq\int_{\mathbb{R}^{N+1}_+}|x|\cdot|f(x)|\,dx;$$

$$\int_0^{+\infty}e^{-|2\pi\xi'|y_0}|\hat{f}(y_0,\xi')|\,dy_0\leq\int_0^{+\infty}y_0\int_{\mathbb{R}^N}|f(y_0,y')|\,dy'\,dy_0\leq\int_{\mathbb{R}^{N+1}_+}|y|\cdot|f(y)|\,dy;$$

and

$$\begin{aligned}&\int_{|2\pi\xi'|\geq1}\left(\int_0^{+\infty}e^{-|2\pi\xi'|y_0}|\hat{f}(y_0,\xi')|\,dy_0\right)^2d\xi'\\ \leq\;&\int_{|2\pi\xi'|\geq1}\frac{1}{2|2\pi\xi'|}\int_0^{+\infty}|\hat{f}(y_0,\xi')|^2\,dy_0\,d\xi'\\ \leq\;&\int_{\mathbb{R}^{N+1}_+}|f(x)|^2\,dx.\end{aligned}$$

Finally, in the case $N=1$ we can obtain the same kind of estimate if we require the data f to satisfy the additional stated compatibility conditions.

This proves the result in the case of the boundary data $h\equiv0$. If $h\neq0$ we proceed as in the proof of Theorem 6.10. Consider $v=u-\mathcal{Q}h$. Then v satisfies the above estimate, and therefore

$$\|u\|_{L^2(\mathbb{R}^{N+1}_+)}\leq\|v\|_{L^2(\mathbb{R}^{N+1}_+)}+\|\mathcal{Q}h\|_{L^2(\mathbb{R}^{N+1}_+)}\leq c\Big\{\|v\|_{L^2(\mathbb{R}^{N+1}_+)}+\|h\|_{L^2(\mathbb{R}^{N+1}_+)}\Big\}.$$

In order to estimate $\|v\|_{L^2(\mathbb{R}^{N+1}_+)}$ we observe that we have to replace f with $f+(\triangle-G')\mathcal{Q}h$ in the computations above, so that, for $N\geq4$ (the case $N<4$ is similar),

$$\begin{aligned}\|v\|_{L^2(\mathbb{R}^{N+1}_+)}&\leq\frac{1}{2}\int_{\mathbb{R}^{N+1}}|\mathcal{F}v_{\mathrm{e}}|^2\,d\xi\\ &\leq C\Big\{\|f\|_{L^2(\mathbb{R}^{N+1}_+)}+\|f\|_{L^1(\mathbb{R}^{N+1}_+)}\Big\}\\ &\quad+C\int_{\mathbb{R}^{N+1}}\frac{1}{|2\pi\xi|^4}\left|\mathcal{F}\Big((\triangle-G')\mathcal{Q}h\Big)_{\mathrm{e}}-\frac{2(1+|2\pi\xi'|^2)}{1+|2\pi\xi|^2}m_1(\xi')\hat{F}_2(\xi')\right|^2d\xi,\end{aligned}$$

where

$$\begin{aligned}\hat{F}_2(\xi')&=-\int_0^{+\infty}e^{-|2\pi\xi'|y_0}e^{-\sqrt{1+|2\pi\xi'|^2}y_0}\frac{|2\pi\xi'|^2}{\sqrt{1+|2\pi\xi'|^2}}\,d\xi_0\hat{h}(\xi')\\ &=-\frac{|2\pi\xi'|^2}{(|2\pi\xi'|+\sqrt{1+|2\pi\xi'|^2})\sqrt{1+|2\pi\xi'|^2}}\hat{h}(\xi');\end{aligned}$$

and

$$\mathcal{F}\Big((\triangle-G')\mathcal{Q}h\Big)_{\mathrm{e}}=\frac{2|2\pi\xi'|^2}{1+|2\pi\xi|^2}\hat{h}(\xi).$$

Therefore we obtain

$$\|v\|_{L^2(\mathbb{R}^{N+1}_+)} \le C\Big(\|f\|_{L^2(\mathbb{R}^{N+1}_+)} + \|f\|_{L^2(\mathbb{R}^{N+1}_+)} + \|h\|_{L^2(\mathbb{R}^{N+1}_+)}.$$

This concludes the proof. □

THE PROBLEM ON A SMOOTHLY BOUNDED DOMAIN

7. Formulation of the Problem on a Smoothly Bounded Domain

Let $\Omega = \{x \in \mathbb{R}^{N+1} : \rho(x) < 0\}$. To avoid pathologies, we assume that ρ is a C^∞ function on $\mathbb{R}^{N+1}$ with the property that $\nabla\rho \neq 0$ at all points of $b\Omega$. Then Ω is a domain with C^∞ boundary (see [KR2] for a protracted discussion of these matters). We shall also assume that Ω is bounded.

It will simplify our calculations if we assume in advance that $(\partial\rho/\partial n) = |\nabla\rho| = 1$ on $b\Omega$. This is easily arranged.

Recall that we defined

$$W^1(\Omega) = \left\{ f \in L^2(\Omega) : \sum_{j=0}^{N} \langle D_j f, D_j f\rangle_0 + \langle f, f\rangle_0 < \infty \right\}.$$

Here the derivative is intended in the sense of distributions,

$$\langle f, g\rangle_0 = \int_\Omega f\overline{g}\, dV,$$

and $D_j = \partial/\partial x_j, j = 0, \ldots, N$. Moreover we have defined the 1-Sobolev space of q-forms $W^1_q(\Omega)$, for $q = 1, \ldots, N+1$, by setting

$$W^1_q(\Omega) = \Big\{ \phi = \sum_{|I|=q} \phi_I dx^I : \phi_I \in W^1(\Omega) \Big\}.$$

For $\phi, \psi \in W^1_q(\Omega)$, their inner product in $W^1_q(\Omega)$ is given by

$$\begin{aligned} \langle \phi, \psi\rangle &= \langle \sum_I \phi_I dx^I, \sum_J \psi_J dx^J\rangle_1 \\ &= \sum_I \langle \phi_I, \psi_I\rangle_1. \end{aligned}$$

Throughout the rest of this entire paper, we shall denote the s-Sobolev norm of a form ψ by $\|\psi\|_s$.

Let

$$d : \bigwedge^q \longrightarrow \bigwedge^{q+1}$$

be defined by

$$\begin{aligned} d\Big(\sum_I \phi_I \, dx^I\Big) &= \sum_I \sum_{j=0}^{N} D_j \phi_I \, dx_j \wedge \, dx^I \\ &= \sum_{|J|=q+1} \Bigg(\sum_{\substack{|I|=q \\ j=0,\dots,N}} \varepsilon^J_{jI} D_j \phi_I \Bigg) dx^J \\ &\equiv \sum_{|J|=q+1} \phi'_J dx^J . \end{aligned}$$

We let d^* be the operator on $\bigwedge^{q+1}$ defined by

$$\langle d\phi, \psi\rangle_1 = \langle \phi, d^*\psi\rangle_1 .$$

Recall that

$$\operatorname{dom} d^* \cap \bigwedge^{q+1} = \Big\{ \psi \in \bigwedge^{q+1} : |\langle d\phi, \psi\rangle_1| \le C_\psi \|\phi\|_1 \Big\}.$$

Our goal is to solve the boundary value problem

$$\begin{cases} (dd^* + d^*d)\phi = \alpha & \text{on } \Omega \\ \phi \in \operatorname{dom} d^* \\ d\phi \in \operatorname{dom} d^* \end{cases} \tag{7.1}$$

for $\alpha \in W^1_q(\Omega)$, $q = 0, \dots, N$, and prove existence and regularity theorems.

We need to analyze both the equation on Ω, and the boundary conditions. Our first goal is to describe $\operatorname{dom} d^*$. Since we have developed some familiarity with this type of calculation, we work directly with q-forms for all q (recall that, in the half space case, we restricted attention at first to functions). We have the following result.

Proposition 7.1. *Let Ω be a smoothly bounded domain in $\mathbb{R}^{N+1}$. Then the Hilbert space adjoint d^* of d, in the W^1 inner product, acting on q-forms, has domain satisfying*

$$\operatorname{dom} d^* \cap \bigwedge^{q+1}(\overline{\Omega}) = \Big\{ \psi \in \bigwedge^{q+1}(\overline{\Omega}) : (\nabla_{\vec{n}}\psi) \lfloor \vec{n} \Big|_{b\Omega} \equiv 0 \Big\}.$$

Here we use the notation $\nabla_X \phi$ to denote the covariant differentiation of the form ϕ in the direction given by the vector field X, and "$\lfloor$" is the standard contraction operation from exterior algebra. Recall that by definition, if $Y_1, \dots, Y_q$ are vector fields, then

$$(\nabla_X \phi) = X\Big(\phi(Y_1, \dots, Y_q)\Big) - \sum_{i=q}^{q} \phi(Y_1, \dots, \nabla_X Y_i, \dots, Y_q);$$

also, in local coordinates $(y_0, \dots, y_n)$,

$$\nabla_X V = \sum_{k=0}^{n} \Big(X(V_k) + \sum_{i,j=0}^{n} \Gamma^k_{ij} X_i V_j \Big) \frac{\partial}{\partial y_k} .$$

Clearly covariant differentiation preserves the *type* of a form.

Moreover,

$$\phi\lfloor V = \sum_{iIJ} \phi_J V_i \varepsilon^J_{iJ} dy^I.$$

Notice that, in the standard coordinates of $\mathbb{R}^{N+1}$, $(\nabla_X \phi)_K = X(\phi_K)$ and

$$\Big(\phi\lfloor(\partial/\partial x_0)\Big)_K = \phi_{0K}$$

if $K \not\ni 0$ and $\Big(\phi\lfloor(\partial/\partial x_0)\Big)_K = 0$ if $K \ni 0$. For these and related notions we refer the reader to [FED].

Observe that, if $\phi = \sum_i \phi_i dx_i$ is a 1-form, then the boundary condition $\nabla_{\vec{n}}\phi\lfloor\vec{n}\Big|_{b\Omega} = 0$ can be written as

$$\sum_{j=0}^{N} \frac{\partial\rho}{\partial x_j}\frac{\partial\phi_i}{\partial n} = 0 \qquad \text{on } b\Omega. \tag{7.2}$$

Proof of 7.1. Let $\phi = \sum_{|I|=q} \phi_I dx^I$, and $\psi = \sum_{|J|=q+1} \psi_I dx^J$. We shall use the following form of Green's theorem:

$$\int_\Omega D_j f \overline{g} = -\int_\Omega f \overline{D_j g} + \int_{b\Omega} f \bar{g} \frac{\partial\rho}{\partial x_j}.$$

Recall that $\vec{n} = (D_0\rho, \ldots, D_N\rho)$ is the normal direction.

Let $d\phi = \sum_{|J|=q+1}\Big(\sum_{|I|=q}\sum_{j=0,\dots,N}\varepsilon^J_{jI}D_j\phi_I\Big)dx^J \equiv \sum_{|J|=q+1}\phi'_J dx^J$. Then

$$\begin{aligned}
\langle d\phi,\psi\rangle_1 &= \sum_{|J|=q+1}\left[\sum_{k=0}^N \langle D_k\phi'_J, D_k\psi_J\rangle_0 + \langle \phi'_J,\psi_J\rangle_0\right]\\
&= \sum_{|J|=q+1}\left[\sum_{k=0}^N \sum_{\substack{|I|=q\\ j=0,\dots,N}} \varepsilon^J_{jI}\left(-\langle D_k\phi_I, D_kD_j\psi_J\rangle_0 + \int_{b\Omega} D_k\phi_I\overline{\psi_J}\frac{\partial\rho}{\partial x_j}\right)\right.\\
&\qquad\left. + \sum_{\substack{|I|=q\\ j=0,\dots,N}} \varepsilon^J_{jI}\left(-\langle \phi_I, D_j\psi_J\rangle_0 + \int_{b\Omega}\phi_I\overline{\psi_J}\frac{\partial\rho}{\partial x_j}\right)\right]\\
&= \sum_{|I|=q}\left[\sum_{k=0}^N \langle D_k\phi_I, D_k\Big(-\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}D_j\psi_J\Big)\rangle_0 + \langle\phi_I,\Big(-\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}D_j\psi_J\Big)\rangle_0\right]\\
&\quad + \sum_{|I|=q}\left[\sum_{k=0}^N\left(\int_{b\Omega} D_k\phi_I\overline{\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}D_k\psi_J\frac{\partial\rho}{\partial x_j}\Big)}\right) + \int_{b\Omega}\phi_I\overline{\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}\psi_J\frac{\partial\rho}{\partial x_j}\Big)}\right]\\
&= \langle\phi, d'\psi\rangle_1\\
&\quad + \sum_{|I|=q}\left[\sum_{k=0}^N\left(\int_{b\Omega} D_k\phi_I\overline{\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}D_k\psi_J\frac{\partial\rho}{\partial x_j}\Big)}\right) + \int_{b\Omega}\phi_I\overline{\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}\psi_J\frac{\partial\rho}{\partial x_j}\Big)}\right].
\end{aligned}$$

Notice that d' in the last equality above is precisely the formal adjoint of d. Recall that $d' = -\mathrm{div}$ on 1-forms. It is clear that

$$|\langle\phi, d'\psi\rangle_1| \le \|\phi\|_1\|\psi\|_2,$$

and that

$$\begin{aligned}
\left|\int_{b\Omega}\phi_I\overline{\Big(-\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}\psi_J\frac{\partial\rho}{\partial x_j}\Big)}\right| &\le c\cdot\|\phi\|_{L^2(b\Omega)}\|\psi\|_{L^2(b\Omega)}\\
&\le c\cdot\|\phi\|_1\|\psi\|_1,
\end{aligned}$$

by the trace theorem.

Next we consider the term

$$\sum_{k=0}^N\int_{b\Omega} D_k\phi_I\overline{\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}D_k\psi_J\frac{\partial\rho}{\partial x_j}\Big)}.$$

For $k = 0, \dots, N$ we decompose the differential operator D_k into normal and tangential components, that is, in a suitable neighborhood of $b\Omega$ we write

$$D_k = Y_k + \frac{\partial\rho}{\partial x_k}\frac{\partial}{\partial n}, \tag{7.3}$$

where Y_k is a tangential vector field, and $\partial/\partial n$ is the unit vector field in the normal direction. Therefore, for each fixed I, $|I| = q$, integration by parts yields that

$$\begin{aligned}\sum_{k=0}^{N}\int_{b\Omega} D_k\phi_I \overline{\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}} \varepsilon^J_{jI} D_k\psi_J \frac{\partial\rho}{\partial x_j}\Big)} &= \sum_{k=0}^{N}\int_{b\Omega} \phi_I \overline{Y^*_k\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}} \varepsilon^J_{jI} D_k\psi_J \frac{\partial\rho}{\partial x_j}\Big)} \\ &\quad + \int_{b\Omega} \frac{\partial\phi_I}{\partial n}\overline{\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}} \varepsilon^J_{jI} \frac{\partial\psi_J}{\partial n}\frac{\partial\rho}{\partial x_j}\Big)} \\ &\equiv I + E.\end{aligned}$$

Now observe that on $b\Omega$, by definition,

$$\sum_{\substack{|J|=q+1\\ j=0,\dots,N}} \varepsilon^J_{jI} \frac{\partial\psi_J}{\partial n}\frac{\partial\rho}{\partial x_j} = \big(\nabla_{\vec{n}}\psi\lfloor\vec{n}\big)_I\Big|_{b\Omega}. \tag{7.4}$$

Using the trace theorem it is easy to see that, for ψ with $C^\infty(\overline{\Omega})$ coefficients,

$$|I| \le C_\psi \|\phi\|_1.$$

Therefore, if $\nabla_{\vec{n}}\psi\lfloor\vec{n}\big|_{b\Omega} = 0$, then $\psi \in \operatorname{dom} d^*$.

Conversely, suppose that $\nabla_{\vec{n}}\psi\lfloor\vec{n}\big|_{b\Omega} \not\equiv 0$. Then, repeating the construction that we used in the case of the half space, we can see that the mapping

$$\phi \mapsto \sum_I \int_{b\Omega} \frac{\partial\phi_I}{\partial n}\overline{\big(\nabla_{\vec{n}}\psi\lfloor\vec{n}\big)_I}$$

cannot be continuous on $W^1_q(\Omega)$. This concludes the proof. □

Notice that, from the previous computation, we obtain that for $\psi \in \operatorname{dom} d^*$ it holds that

$$\begin{aligned}\langle d\phi, \psi\rangle_1 &= \langle \phi, d'\psi\rangle_1 \\ &\quad + \sum_{|I|=q}\Big[-\sum_{k=0}^{N}\int_{b\Omega} \phi_I \overline{\nabla_{Y^*_k}\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}} \varepsilon^J_{jI} D_k\psi_J \frac{\partial\rho}{\partial x_j}\Big)} + \int_{b\Omega} \phi_I \overline{(\psi\lfloor\vec{n})_I}\Big].\end{aligned} \tag{7.5}$$

Next we shall determine an explicit expression for d^*. We set (as in the case of the half space)

$$d^*\phi = d'\phi + \mathcal{K}_\Omega\phi, \tag{7.6}$$

where $\mathcal{K}_\Omega$ is to be determined.

Proposition 7.2. *Let $\psi \in \operatorname{dom} d^* \cap \bigwedge^{q+1}(\overline{\Omega})$. Then the I-component $(\mathcal{K}_\Omega\psi)_I$ of $\mathcal{K}_\Omega\psi$ is the solution of the boundary value problem*

$$\begin{cases} -\bigtriangleup v + v = 0 & \text{on } \Omega \\ \dfrac{\partial v}{\partial n} = \displaystyle\sum_{k=0}^{N}\Big[\nabla_{Y_k^*}\big((\nabla_{Y_k}\psi)\lfloor\vec{n}\big)_I + (\psi\lfloor\vec{n})\Big]_I & \text{on } b\Omega \end{cases}.$$

Proof. By the relation

$$\langle d\phi,\psi\rangle_1 = \langle\phi, d'\psi\rangle_1 + \langle\phi,\mathcal{K}_\Omega\psi\rangle_1,$$

and equation (7.5) we see that for $\psi \in \operatorname{dom} d^* \cap \bigwedge^{q+1}(\overline{\Omega})$,

$$\langle\phi,\mathcal{K}_\Omega\psi\rangle_1 = \sum_{|I|=q}\Big[\sum_{k=0}^{N}\int_{b\Omega}\phi_I\overline{Y_k^*\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}D_k\psi_J\frac{\partial\rho}{\partial x_j}\Big)} + \int_{b\Omega}\phi_I\overline{(\psi\lfloor\vec{n})_I}\Big]. \tag{7.7}$$

Now, by Green's theorem, we see that

$$\begin{aligned}\langle\phi,\mathcal{K}_\Omega\psi\rangle_1 &= \sum_{|I|=q}\Big[\sum_{j=0}^{N}\int_\Omega D_j\phi_I\overline{D_j(\mathcal{K}_\Omega\psi)_I} + \int_\Omega\phi_I\overline{(\mathcal{K}_\Omega\psi)_I}\Big]\\ &= \sum_{|I|=q}\Big[-\int_\Omega\phi_I\overline{\triangle(\mathcal{K}_\Omega\psi)_I} + \int_{b\Omega}\phi_I\overline{\frac{\partial(\mathcal{K}_\Omega\psi)_I}{\partial n}} + \int_\Omega\phi_I\overline{(\mathcal{K}_\Omega\psi)_I}\Big].\end{aligned}$$

Therefore, for each fixed multi-index I, we must have

$$\begin{aligned}&\int_\Omega\phi_I\overline{\big[-\bigtriangleup(\mathcal{K}_\Omega\psi)_I + (\mathcal{K}_\Omega\psi)_I\big]} + \int_{b\Omega}\phi_I\overline{\frac{\partial(\mathcal{K}_\Omega\psi)_I}{\partial n}}\\ &\qquad = \sum_{k=0}^{N}\int_{b\Omega}\phi_I\overline{Y_k^*\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}D_k\psi_J\frac{\partial\rho}{\partial x_j}\Big)} + \int_{b\Omega}\phi_I\overline{(\psi\lfloor\vec{n})_I}.\end{aligned}$$

This implies that

$$\big[-\bigtriangleup(\mathcal{K}_\Omega\psi)_I + (\mathcal{K}_\Omega\psi)_I\big] = 0 \qquad \text{on } \Omega.$$

Now recalling that, since $\psi \in \operatorname{dom} d^*$,

$$\sum_{\substack{|J|=q+1\\ j=0,\dots,N}}\varepsilon^J_{jI}\frac{\partial\rho}{\partial x_j}\frac{\partial\psi_J}{\partial n} = 0 \qquad \text{on } b\Omega\,.$$

By writing $D_k = Y_k + (\partial\rho/\partial x_k)\cdot(\partial/\partial n)$, we have that

$$\begin{aligned}\sum_{k=0}^{N} Y_k^*\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}} \varepsilon_{jI}^{J} D_k\psi_J \frac{\partial\rho}{\partial x_j}\Big) &= \sum_{k=0}^{N} Y_k^*\Big(\sum_{\substack{|J|=q+1\\ j=0,\dots,N}} \varepsilon_{jI}^{J} Y_k\psi_J \frac{\partial\rho}{\partial x_j}\Big)\\ &= \sum_{k=0}^{N} Y_k^*\Big(\nabla_{Y_k}\psi\lfloor\vec{n}\Big)_I\\ &= \sum_{k=0}^{N}\Big[\nabla_{Y_k^*}(\nabla_{Y_k}\psi\lfloor\vec{n})\Big]_I.\end{aligned}$$

From this the result follows. $\square$

Thus $\mathcal{K}_\Omega$ is the solution operator for the preceding elliptic boundary value problem. By standard facts of the theory of elliptic problems (see [HOR1]) we know that $\mathcal{K}_\Omega\psi$ is uniquely determined for each $\psi\in\bigwedge^q(\overline{\Omega})$.

As in the case of the half space, the operator $\mathcal{K}_\Omega$ turns out to be of order 1. In order to prove the following corollary, we need to apply Theorem 4.2.4 in [TRI]. This result deals with Schauder estimates for elliptic boundary value problems formulated in norms of negative order. The reference [TRI] happens to use the language of Triebel-Lizorkin spaces $F^s_{p,q}$. Since we consider only the case $p = q = 2$ we shall denote the spaces $F^s_{2,2}$ simply by F^s. We also recall that, for $s\geq 0$, $F^s = W^s(\Omega)$.

Corollary 7.3. *Let $s > 1/2$. Then there exists $C>0$ such that for all $\psi\in\bigwedge^q(\overline{\Omega})$ we have*

$$\|\mathcal{K}_\Omega\psi\|_{F^{s-1}}\leq C\|\psi\|_s.$$

Proof. Notice that by this result the operator $\mathcal{K}_\Omega$, initially defined on the dense subspace $\bigwedge^q(\overline{\Omega})$ can be extended as a continuous operator from F^{s-1} to $W^s(\Omega)$.

We need only apply standard estimates for elliptic boundary value problems. In order to obtain the sharp regularity result we use the estimates for the negative Sobolev spaces on the boundary (see [TRI], Theorem 4.2.4). Notice that the boundary differential operator acting on the data is of order 2, and tangential. Write T_2 for that operator. Then we have

$$\begin{aligned}\|\mathcal{K}_\Omega\psi\|_{F^{s-1}} &\leq C\Big(\|T_2\psi\|_{W^{s-5/2}(b\Omega)}+\|\psi\|_{W^{s-5/2}(b\Omega)}\Big)\\ &\leq C\|\psi\|_{W^{s-1/2}(b\Omega)}\\ &\leq C\|\psi\|_s.\quad\square\end{aligned}$$

We could, in principle, write down an approximate expression for $\mathcal{K}_\Omega$ by using local coordinates to reduce the problem to the half space situation, where we have calculated $\mathcal{K}_\Omega$ quite explicitly. We forego that option for now.

We can now reformulate our boundary value problem (7.1), recalling that on forms $dd' + d'd = \triangle$:

$$\begin{cases} (-\triangle + G_\Omega)\psi = \alpha & \text{on } \Omega \\ \nabla_{\vec{n}}\psi \lfloor \vec{n} = 0 & \text{on } b\Omega \\ \nabla_{\vec{n}} d\psi \lfloor \vec{n} = 0 & \text{on } b\Omega \end{cases}$$

where we set $G_\Omega = d\mathcal{K}_\Omega + \mathcal{K}_\Omega d$.

8. A Special Coordinate System

In this section we introduce a system of local coordinates near the boundary that we will use in the rest of the paper.

It is a standard fact (see [KR2] for instance) that since $b\Omega$ is smooth and compact there exists a tubular neighborhood U of $b\Omega$ such that for each $x \in U$ there is a unique $\pi(x) \in b\Omega$ which realizes the distance of x from $b\Omega$. The line joining $\pi(x)$ and x is orthogonal to $b\Omega$.

Definition 8.1. Let $\{(U_j, \Phi_j)\}$ be a covering of $b\Omega$ by local coordinates patches. The pair (V_j, Ψ_j), $V_j \subset U$, and $\Psi : V_j \to \mathbb{R}^{N+1}$ is a *Fermi coordinate patch* on U if

$$\pi(x) \in U_j \qquad \text{for all } p \in V_j \subset U,$$

and

$$\Psi_j(p) = \Big(\text{dist}\,(\pi(p)), p\Big), \Phi_j(\pi(p))\Big).$$

Throughout the rest of the paper, $X_0 = \vec{n}$ will denote the vector field defined at each $p \in U$ that is given by

$$\vec{n}_p = \vec{n}_{\pi(p)} = \Big(\frac{\partial \rho}{\partial x_0}(p), \dots, \frac{\partial \rho}{\partial x_n}(p)\Big).$$

We remark that the equality

$$\vec{n}_p = \Big(\frac{\partial \rho}{\partial x_0}(p), \dots, \frac{\partial \rho}{\partial x_n}(p)\Big)$$

in general does not hold for $p \in U \setminus b\Omega$.

On the fixed coordinate patch (U_j, Ψ_j) we also have an orthonormal frame of vector fields $X_0 = (\partial/\partial n), X_1, \dots, X_N$. Notice that $X_1, \dots, X_N$ form an orthonormal frame for the tangential vector fields. We also fix the dual basis of 1-forms, $\omega_0 = dx_0, \omega_1, \dots, \omega_N$. Given any q-form ϕ we write $\phi = \sum_{|I|=q} \omega^I$, where $\omega^I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_q}$, where $I = (i_1, \dots, i_q)$ is a strictly increasing multi-index.

In the rest of the paper we shall *always* assume that the forms are written in terms of this given basis. Notice that, in these coordinates,

$$\begin{aligned} d\psi &= d(\sum_I \psi_I \omega^I) \\ &= \sum_j X_j \psi_I \omega_j \wedge \omega^I + \{0 \text{ order terms in } \psi\} \\ &= \sum_J \Big(\sum_{jI} X_j \psi_I \varepsilon^J_{jI}\Big)\omega^J + \{0 \text{ order terms}\}. \end{aligned}$$

Analogously,

$$d'\psi = \sum_L \Big(\sum_{jI} X'_j \psi_I \varepsilon^I_{jL}\Big)\omega^L + \{0 \text{ order terms}\}.$$

Here X'_k denotes the formal adjoint of the vector field X_k.

Recall that (see [HEL] for instance) we can define the (Christoffel symbol) coefficients Γ^k_{ij} as follows: [HEL]:

$$\nabla_{X_i} X_j = \sum_{k=0}^{N} \Gamma^k_{ij} X_k. \tag{8.1}$$

We remark that $\nabla_{\vec{n}}\vec{n} = 0$, since $\vec{n}$ is, by definition, a unit vector field whose integral curves are lines. Therefore the coefficients Γ^k_{ij} satisfy the following relations

$$\Gamma^k_{ij} = 0 \quad \begin{cases} \text{if } k = 0 & \text{and } i, \text{ or } j = 0 \\ \text{if } i = 0 & \text{and } j = 0 \end{cases} \tag{8.2}$$

This is so because

$$\Gamma^0_{0j} \equiv \langle \nabla_{X_0} X_j, X_0\rangle = X_0\langle X_j, X_0\rangle - \langle X_j, \nabla_{X_0} X_0\rangle = 0,$$

$$\Gamma^k_{00} \equiv \langle \nabla_{X_0} X_0, X_k\rangle = 0,$$

and

$$\Gamma^0_{i0} \equiv \langle \nabla_{X_i} X_0, X_0\rangle = \frac{1}{2} X_i \langle X_0, X_0\rangle = 0.$$

Now we have a lemma about covariant differentiation in the normal direction.

Lemma 8.2. *If* $\phi \in \bigwedge^q(\overline{\Omega})$ *is given by* $\phi_I = \sum_I \phi_I \omega^I$, *then*

$$\nabla_{X_0}\phi = \sum_{|I|=q} \left[\frac{\partial \phi_I}{\partial x_0} + \sum_{|J|=q} \gamma_{IJ}\phi_J\right]\omega^I,$$

where

$$\gamma_{IJ} = 0 \begin{cases} \text{for } J \ni 0 & \text{if } I \not\ni 0 \\ \text{for } J \not\ni 0 & \text{if } I \ni 0 \end{cases}$$

Proof. Fix a Fermi coordinate chart, then for $\phi \in \bigwedge^q(\overline{\Omega})$ supported in a small open set we have

$$\begin{aligned}\Big(\nabla_{X_0}(\sum_J \phi_J\omega^J)\Big)(X^I) &= \sum_J \Big(\nabla_{X_0}(\phi_J\omega^J)\Big)(X^I)\\ &= \sum_J \frac{\partial\phi_J}{\partial x_0}\varepsilon^I_J - \sum_{J,J'}\sum_{s=1,\dots,q}\phi_J\Big((\nabla_{X_0}\omega^{j_s})\wedge\omega^{J'}\varepsilon^J_{j_sJ'}\Big)(X^I).\end{aligned}$$

Moreover,

$$\begin{aligned}\Big(\nabla_{X_0}(\omega^j)\Big)(X_\ell) &= -\omega^j\Big(\nabla_{X_0}X_\ell\Big)\\ &= -\omega^j\Big(\sum_k \Gamma^k_{0\ell}X_k\Big)\\ &= -\Gamma^j_{0\ell}.\end{aligned}$$

Therefore

$$\nabla_{X_0}\omega^{j_s} = -\sum_\ell \Gamma^{j_s}_{0\ell}\,\omega^\ell.$$

Hence

$$\Big(\nabla_{X_0}\phi\Big)(X^I) = \sum_{|J|=q}\left[\frac{\partial\phi_J}{\partial x_0}\varepsilon^I_J - \sum_{\substack{|J'|=q-1\\ s=1,\dots,q}}\sum_{\ell=0,\dots,N}\varepsilon^J_{j_sJ'}(-\Gamma^{j_s}_{0\ell})\varepsilon^I_{\ell J'}\phi_J\right].$$

Thus

$$\nabla_{X_0}\phi = \sum_{|I|=q}\Big(\frac{\partial\phi_I}{\partial x_0} + \sum_{|J|=q}\gamma_{IJ}\phi_J\Big)\omega^I,$$

where

$$\gamma_{IJ} = \sum_{\substack{|J'|=q-1\\ s=1,\dots,q}}\sum_{\ell=0,\dots,N}\varepsilon^J_{j_sJ'}\Gamma^{j_s}_{0\ell}\,\varepsilon^I_{\ell J'}.$$

Recall that (see (8.2)) the symbols Γ^k_{ij} are zero if either $k=0$ and either $i=0$ or $j=0$, or $i=j=0$. Suppose that $I\ni 0$ and that $J\not\ni 0$. Then $\ell=0$ so that $\Gamma^{j_s}_{ij}=0$. If $I\not\ni 0$ and $J\ni 0$, then $j_s=0$ so that $\Gamma^0_{0\ell}=0$. This proves the lemma. $\square$

In the sequel we will also use the following observation on the Laplace operator acting on forms. In general, the Laplacian on forms is defined as $dd'+d'd$. Having chosen the aforementioned basis on the space of q-forms, we see that if $\psi=\sum_{|I|=q}\psi_I\omega^I$ then

$$\triangle\psi = \sum_I\left[\sum_{k=0}^N (X_k'X_k + X_kX_k')\psi_I\right]\omega^I + \{\text{lower order terms in } \psi\}.$$

The tangential Laplacian. Fix a Fermi coordinate patch $\Big(U, \Psi = (x_0, \ldots, x_N)\Big)$. (Notice that $(x_0, \ldots, x_N)$ are not the standard coordinates of $\mathbb{R}^{N+1}$.) Given a function (or a form) u defined on U, we denote by $\tilde{u}$ the function $u \circ \Psi^{-1}$ defined on $\Psi(U)$.

In these coordinates the standard Laplacian of $\mathbb{R}^{N+1}$ has the following form:

$$\begin{aligned}
[\triangle u]^{\sim} &= \sum_{j,k=0}^{N} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_j}\Big(g^{jk}\frac{\partial}{\partial x_k}\tilde{u}\Big) \\
&= \sum_{j,k=1}^{N} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_j}\Big(g^{jk}\frac{\partial}{\partial x_k}\tilde{u}\Big) + \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x_0}\Big(\det g\frac{\partial}{\partial x_0}\tilde{u}\Big) \\
&= \triangle'\tilde{u} + \frac{\partial}{\partial x_0}\log(\sqrt{\det g})\frac{\partial \tilde{u}}{\partial x_0} + \frac{\partial^2 \tilde{u}}{\partial x_0^2} \\
&= [\triangle \tilde{u}] + \epsilon \mathcal{T}_2 \tilde{u} + \frac{\partial}{\partial x_0}(\log\sqrt{\det g})\frac{\partial \tilde{u}}{\partial x_0} \\
&\equiv [\triangle \tilde{u}] + \epsilon \mathcal{L}_2 \tilde{u}, \qquad (8.3)
\end{aligned}$$

where g is the metric matrix, and (g^{jk}) is its inverse. The operator $\triangle_T$ is the Laplacian on the submanifold obtained by fixing x_0, so that

$$\triangle_T = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \epsilon \mathcal{T}_2, \qquad (8.4)$$

where $\mathcal{T}_2$ is a second order tangential differential operator with $C^\infty(\overline{\Omega})$ coefficients, and ϵ can be made arbitrarily small by shrinking U.

Notice that we have obtained that, on U,

$$\triangle = \triangle_T + X_0^2 + \Big(X_0 \log\sqrt{\det g}\Big)X_0. \qquad (8.5)$$

The operator G_Ω on functions. Recall that, by Proposition 7.2 $G_\Omega u$ is the unique solution of the elliptic boundary value problem

$$\begin{cases} \triangle v - v = 0 & \text{on } \Omega \\ \dfrac{\partial v}{\partial n} = \displaystyle\sum_{k=0}^{N} Y_k^*\Big((\nabla_{Y_k} du)\lfloor \vec{n}\Big) + \frac{\partial u}{\partial n} & \text{on } b\Omega \end{cases}.$$

We are interested in making explicit the boundary operator in this case. Notice that $Y_k^* = -Y_k + (0 \text{ order terms})$, so that the boundary equation becomes

$$-\sum_{j=0}^{N} Y_k^2(X_0 u) + T_1 X_0 u + T_2 u,$$

where the T_j's are tangential differential operators of order j. Now notice that

$$\sum_{j=0}^{N} Y_k^2 = \sum_{k=0}^{N} D_k^2 - X_0^2 + L_1$$

where L_1 is a first order differential operator. This follows easily from the fact that, by construction,

$$Y_k = D_k - \langle D_k, \vec{n} \rangle X_0,$$

(see (7.3)). In fact, for a function f,

$$\begin{aligned}
\triangle f &= \sum_{k=0}^{N} D_k^2 f \\
&= \sum_{k=0}^{N} (Y_k + \langle D_k, \vec{n} \rangle)^2 f \\
&= \sum_{k=0}^{N} \Big(Y_k^2 f + Y_k(\langle D_k, \vec{n} \rangle X_0) f + X_0(\langle D_k, \vec{n} \rangle Y_k) f \Big) + X_0^2 f \\
&= \sum_{k=0}^{N} \Big(Y_k^2 f + Y_k(\langle D_k, \vec{n} \rangle) X_0 f \Big) + X_0^2 f \\
&= \sum_{k=0}^{N} Y_k^2 + X_0^2 f + a X_0 f \,,
\end{aligned}$$

since $\sum_{k=0}^{N} \langle D_k, \vec{n} \rangle Y_k = 0$. Therefore the boundary equation in u equals

$$\begin{aligned}
\Big(-\sum_{k=0}^{N} Y_k^2 \Big)(X_0 u) + T_1 X_0 u + T_2 u &= (- \triangle + X_0^2)(X_0 u) + a X_0^2 u + T_1 X_0 u + T_2 u \\
&= (- \triangle_T + b X_0)(X_0 u) + a X_0^2 u + T_1 X_0 u + T_2 u \\
&= - \triangle_T X_0 u + T_1 X_0 u + T_2 u,
\end{aligned}$$

where we have used formula (8.5) and the fact that $du \in \operatorname{dom} d^*$. In local coordinates the boundary equation then becomes

$$\frac{\partial v}{\partial x_0} = - \triangle_T (X_0 u)^{\sim} + \epsilon \mathcal{T}_2 (X_0 u)^2 + \Big[T_1 X_0 u + T_2 u \Big]^{\sim}. \tag{8.6}$$

We conclude this section by introducing a convention that we shall use consistently throughout the rest of the paper. By L_j we denote a generic differential operator of order j with $C^\infty(\overline{\Omega})$ coefficients, while by T_j we denote a differential operator of order j with $C^\infty(\overline{\Omega})$ (defined in a suitable neighborhood of $b\Omega$), that involves only tangential derivatives.

9. The Existence Theorem

In this section we study the question of existence for the boundary value problem

$$\begin{cases} (dd^* + d^*d)\phi = \alpha & \text{on } \Omega \\ \phi \in \operatorname{dom} d^* \\ d\phi \in \operatorname{dom} d^* \end{cases}$$

for $\alpha \in \bigwedge^q(\overline{\Omega})$, $q = 0, \ldots, N$, from an abstract Hilbert space point of view. We remark that by Proposition 7.1 the boundary conditions are given by the equations

$$\nabla_{\vec{n}}\phi \lfloor \vec{n} = 0 \qquad \text{on } b\Omega$$

and

$$\nabla_{\vec{n}} d\phi \lfloor \vec{n} = 0 \qquad \text{on } b\Omega.$$

Notice that, using the coordinates and the frame introduced in (8.1), the above equations can be rewritten as

$$X_0\phi_I + L_0\phi = 0 \qquad \text{on } b\Omega \quad \text{if } I \ni 0 \tag{9.1}$$

and

$$X_0^2\phi_I + L_1\phi = 0 \qquad \text{on } b\Omega \quad \text{if } I \not\ni 0, \tag{9.2}$$

where L_j are differential operators of order j in the components of ϕ. We remark that if ϕ is a function then the first boundary equation is empty, and the second one becomes

$$X_0^2\phi = 0 \qquad\qquad \text{on } b\Omega, \tag{9.3}$$

since $\nabla_{\vec{n}}\vec{n} = 0$.

Our development in what follows parallels classical studies such as that which can be found in [FOK]. Let

$$\mathcal{D} \equiv \Big\{\phi \in \bigwedge\nolimits^q(\overline{\Omega}) : \phi\,,\ d\phi \in \operatorname{dom} d^*\Big\}.$$

For $\phi, \psi \in \mathcal{D}$ we define the bilinear form

$$Q(\phi, \psi) = \langle d\phi, d\psi\rangle_1 + \langle d^*\phi, d^*\psi\rangle_1 + \langle \phi, \psi\rangle_1.$$

Our first claim is that $\mathcal{D}$ is dense in W^1_q. We argue as follows. Let ϕ be any q-form. We may assume that ϕ has coefficients smooth up to the boundary. Then it suffices to find a $\psi \in \bigwedge^q(\overline{\Omega})$ of small norm such that $\phi + \psi \in \mathcal{D}$. We use Fermi local coordinates in a tubular neighborhood of $b\Omega$, so that $x = (x_0, x')$ with $x' \in b\Omega$ and $x_0 = \operatorname{dist}(x_0, b\Omega)$ parametrizing the normal direction. Set

$$\psi_1(x') = \nabla_{\vec{n}}\Big(\phi\lfloor\vec{n}\Big)\Big|_{b\Omega},$$

and

$$\psi_2(x') = \nabla_{\vec{n}}\Big(d\phi\lfloor\vec{n}\Big)\Big|_{b\Omega}.$$

Finally, set

$$\psi(x) = \Big(-x_0 dx_0 \wedge \psi_1(x') - \frac{1}{2}x_0^2\psi_2\Big)\,\chi(x_0),$$

where $\chi \in C_0^\infty[-2\epsilon, 2\epsilon]$, $\chi = 1$ in a neighborhood of 0, and $\|\chi\|_1 \leq C\epsilon^{-1/2}$. Then $\|\psi\|_1 < C\epsilon^{1/2}$ and $\phi + \psi \in \mathcal{D}$. That completes the argument.

Now we let $\tilde{\mathcal{D}}$ be the closure of $\mathcal{D}$ in the topology induced by Q. We wish to check that $\tilde{\mathcal{D}}$ is still contained in W_q^1. It is easy to see that d is closed in W_q^1; of course d^* is closed also (since adjoints are always closed). These facts imply that $\tilde{\mathcal{D}}$ is a subspace of W_q^1.

At this point we apply the Friedrichs extension theorem, as in [FOK], to show that there exists a canonical self adjoint operator

$$T : W_q^1 \to \tilde{\mathcal{D}}$$

which is bounded in the W^1 topology, is injective, and such that

$$Q(T\phi, \psi) = \langle \phi, \psi\rangle_1.$$

If we set $F = T^{-1}$, then

$$Q(\phi, \psi) = \langle F\phi, \psi\rangle_1 \qquad\qquad \forall \phi \in \operatorname{dom} F, \psi \in \tilde{\mathcal{D}}.$$

Notice that $F = (d^*d + dd^*) + I$ when restricted to $\mathcal{D}$.

Now we show that the Q-unit ball of $\tilde{\mathcal{D}}$ is compactly embedded in W_q^1 by proving that the Q norm is equivalent to the W^2 norm. Notice that $Q(\phi, \phi) \leq c\|\phi\|_2^2$ since d^* is of order 1. The reverse inequality follows from the next theorem.

THEOREM 9.1. *There exists a constant $C_0 > 0$ such that, for all $\psi \in \tilde{\mathcal{D}}$,*

$$Q(\psi, \psi) \geq C_0 \cdot \|\psi\|_2^2.$$

Assume the theorem for now. Since the Q-unit ball of $\tilde{\mathcal{D}}$ is compactly embedded in $W_q^1(\Omega)$, it follows that T is a compact operator. Notice that if $T\alpha \in \mathcal{D}$, then $T\alpha$ is the unique solution of the boundary value problem

$$\begin{cases} (dd^* + d^*d)\phi + \phi = \quad \alpha \\ \phi \in \operatorname{dom} d^* \\ d\phi \in \operatorname{dom} d^* \end{cases}$$

for $\alpha \in W_q^1$, $q = 0, 1, \ldots, N + 1$.

We now wish to establish conditions for the solvability of

$$\begin{cases} (dd^* + d^*d)\phi = \quad \alpha \\ \phi \in \operatorname{dom} d^* \\ d\phi \in \operatorname{dom} d^* \end{cases} \tag{9.4}$$

Let Q_0 be the bilinear form on $\tilde{\mathcal{D}}$ defined by

$$Q_0(\phi, \psi) = \langle d\phi, d\psi \rangle_1 + \langle d^*\phi, d^*\psi \rangle_1.$$

Thus ϕ is a solution of (9.4) precisely when $\phi \in \mathcal{D}$ and

$$Q_0(\phi, \psi) = \langle \alpha, \psi \rangle_1 \qquad\qquad \forall \psi \in \tilde{\mathcal{D}}.$$

This in turn holds if and only if

$$\begin{aligned} Q(\phi, \psi) &= Q_0(\phi, \psi) + \langle \phi, \psi \rangle_1 \\ &= \langle \alpha, \psi \rangle_1 + \langle \phi, \psi \rangle_1 \\ &= \langle \alpha + \phi, \psi \rangle_1. \end{aligned}$$

By the Friedrichs theorem, we have reduced our situation to solving the equation

$$(F - I)\phi = \alpha \tag{9.5}$$

with $\phi \in \operatorname{dom} F$; i.e., setting $\theta = F\phi$,

$$\theta - T\theta = \alpha.$$

We now apply the standard theory of compact operators to obtain that the above equation has a solution θ for all α orthogonal to the finite dimensional subspace $\mathcal{H}_q \equiv \ker(I - T)$. It is easy to check that $\ker(I - T)$ is exactly the kernel of $F - I$. Thus, for α orthogonal (in the W^1-inner product) to $\ker(I - T)$, we obtain that $\phi = T\theta$ is the solution of the equation (9.5). Notice that, if $\phi \in \mathcal{D}$, then the equation (9.5) reduces to the boundary value problem (9.4). Moreover, $Z \equiv (F - I)(\mathcal{D})$ is a dense subspace of $\mathcal{H}_q^\perp \subseteq W_q^1$, i.e.

$$Z^\perp = \mathcal{H}_q.$$

Let $\beta \in \mathcal{D}$. For $\phi \in \mathcal{D}$ we have

$$\begin{aligned} \langle (F - I)\phi, \beta \rangle_1 &= Q_0(\phi, \beta) \\ &= Q(\phi, \beta) - \langle \phi, \beta \rangle_1 \\ &= \langle \phi, (T - I)\beta \rangle_1. \end{aligned}$$

Thus, if $\beta \in Z^\perp$, then $\langle (F - I)\phi, \beta \rangle_1 = 0$ for all $\phi \in \mathcal{D}$. Since $\mathcal{D}$ is dense in W_q^1, $(T - I)\beta = 0$, that is $\beta \in \mathcal{H}_q$.

Thus we have proved the following theorem:

THEOREM 9.2 (Existence for all q). *The boundary value problem*

$$\begin{cases} (dd^* + d^*d)\phi = \alpha \\ \phi \in \operatorname{dom} d^* \\ d\phi \in \operatorname{dom} d^* \end{cases}$$

has a finite dimensional kernel $\mathcal{H}_q^\perp$ and finite dimensional cokernel. [Note that the space Z is dense in $\mathcal{H}_q^\perp$.] The problem has a solution $\phi \in \mathcal{D}$ for $\alpha \in Z \subseteq \mathcal{H}_q^\perp$.

The coercive estimate. We now prove the fundamental estimate from below for the bilinear form Q.

Proof of 9.1. We begin by noticing that

$$\begin{aligned} Q(\psi,\psi) &= \langle d\psi, d\psi\rangle_1 + \langle d^*\psi, d^*\psi\rangle_1 + \langle \psi,\psi\rangle_1 \\ &= \langle d\psi, d\psi\rangle_1 + \langle d'\psi, d^*\psi\rangle_1 + \langle \mathcal{K}_\Omega\psi, d^*\psi\rangle_1 + \langle \psi,\psi\rangle_1 \\ &= \langle d\psi, d\psi\rangle_1 + \langle d'\psi, d'\psi\rangle_1 + \langle \psi,\psi\rangle_1 + \langle d'\psi, \mathcal{K}_\Omega\psi\rangle_1 + \langle \mathcal{K}_\Omega\psi, d^*\psi\rangle_1. \end{aligned}$$

Our plan is to prove that the following claims hold true.

Claim 1. There exists a constant $C_1 > 0$ such that for all $\psi \in \mathcal{D}$ we have

$$\langle d\psi, d\psi\rangle_1 + \langle d'\psi, d'\psi\rangle_1 + \langle \psi,\psi\rangle_1 \geq C_1\|\psi\|_2^2.$$

Claim 2. For any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that all $\psi \in \mathcal{D}$

$$|\langle \mathcal{K}_\Omega\psi, d^*\psi\rangle_1| \leq \epsilon\|\psi\|_2^2 + C_\epsilon\|d^*\psi\|_1^2,$$

and

$$|\langle d'\psi, \mathcal{K}_\Omega\psi\rangle_1| \leq \epsilon\|\psi\|_2^2 + C_\epsilon(\|d\psi\|_1^2 + \|\psi\|_1^2).$$

Assuming the claims for now, we shall finish the proof. We have that

$$\begin{aligned} Q(\psi,\psi) &= \langle d\psi, d\psi\rangle_1 + \langle d'\psi, d'\psi\rangle_1 + \langle \psi,\psi\rangle_1 + \langle d'\psi, \mathcal{K}_\Omega\psi\rangle_1 + \langle \mathcal{K}_\Omega\psi, d^*\psi\rangle_1 \\ &\geq C_1\|\psi\|_2^2 - |\langle \mathcal{K}_\Omega\psi, d^*\psi\rangle_1| - |\langle d'\psi, \mathcal{K}_\Omega\psi\rangle_1| \\ &\geq (C_1 - 2\epsilon)\|\psi\|_2^2 - C_\epsilon(\|d^*\psi\|_1^2 + \|d\psi\|_1^2 + \|\psi\|_1^2). \end{aligned}$$

Therefore the constant $C_0 = (C_1 - 2\epsilon)/(1 + C_\epsilon)$ does the job. Thus it remains to prove the claims. We begin with Claim 2.

Proof of Claim 2. Recall that, by formula (7.7),

$$\langle \mathcal{K}_\Omega\theta, \phi\rangle_1 = \sum_I \left[\sum_{k=0}^N \int_{b\Omega} (\nabla_{Y_k}(\theta\lfloor\vec{n}))_I \overline{(\nabla_{Y_k}\phi)_I} + \int_{b\Omega} (\theta\lfloor\vec{n})_I \overline{\phi_I}\right].$$

Therefore, by applying the Schwarz inequality for Sobolev spaces on $b\Omega$, and recalling that the Y_k are tangential vector fields, we see that

$$\begin{aligned}
|\langle \mathcal{K}_\Omega \psi, d^*\psi\rangle_1| &\le \sum_I \sum_{k=0}^N \int_{b\Omega} |(\nabla_{Y_k}\psi\lfloor\vec{n})_I(\nabla_{Y_k}(d^*\psi))_I| + \|\psi\|_{L^2(b\Omega)}\|d^*\psi\|_{L^2(b\Omega)} \\
&\le c\sum_{kI} \|\nabla_{Y_k}(\psi\lfloor\vec{n})_I\|_{W^{1/2}(\Omega)}\|\nabla_{Y_k}(d^*\psi)_I\|_{W^{-1/2}(\Omega)} + \|\psi\|_{L^2(b\Omega)}\|d^*\psi\|_{L^2(b\Omega)} \\
&\le c\Big(\|\psi\|_{W^{3/2}(b\Omega)}\|d^*\psi\|_{W^{1/2}(b\Omega)}\Big) \\
&\le c\|\psi\|_2\|d^*\psi\|_1 \\
&\le \epsilon\|\psi\|_2^2 + C_\epsilon\|d^*\psi\|_1^2.
\end{aligned}$$

On the other hand, for $\psi \in \mathcal{D}$, we see that (using "l.o.t." to denote lower order terms)

$$\begin{aligned}
\langle d'\psi, \mathcal{K}_\Omega\psi\rangle_1 &= \sum_I \left[\sum_{k=0}^N \int_{b\Omega} (\nabla_{Y_k}(d'\psi))_I \overline{\Big((\nabla_{Y_k}\psi)\lfloor\vec{n}\Big)_I}\right] + \{\text{l.o.t.}\} \\
&= \sum_{kI}\int_{b\Omega}\Big(\sum_{j\ne 0,J} X_j Y_k \psi_J \varepsilon^J_{jI}\Big)\overline{(\nabla_{Y_k}\psi\lfloor\vec{n})_I} + \{\text{l.o.t.}\} \\
&= -\sum_{kJ}\int_{b\Omega}(Y_k\psi_J)\overline{\Big(\sum_{j\ne 0,I}\varepsilon^J_{jI}X_j(\nabla_{Y_k}\psi\lfloor\vec{n})_I\Big)} + \{\text{l.o.t.}\} \\
&= -\sum_{kJ}\int_{b\Omega}(Y_k\psi_J)\overline{Y_k\Big(\sum_{j\ne 0,I}\varepsilon^J_{jI}X_j(\psi\lfloor\vec{n})_I\Big)} + \{\text{l.o.t.}\},
\end{aligned}$$

where we have used the fact that the term containing $X_0\psi_J$ with $J \ni 0$ can be absorbed in the error terms because of the equation (9.1). Now notice that, for $\psi \in \mathcal{D}$, we have that

$$\begin{aligned}
X_0(\psi\lfloor\vec{n})_I &= \Big(\nabla_{\vec{n}}(\psi\lfloor\vec{n})\Big) + \{0 \text{ order terms}\} \\
&= \Big((\nabla_{\vec{n}}\psi)\lfloor\vec{n}\Big)_I + \{0 \text{ order terms}\} \\
&= \{0 \text{ order terms}\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{kJ}\int_{b\Omega}(Y_k\psi_J)\overline{Y_k\Big(\sum_{j\ne 0,I}\varepsilon^J_{jI}X_j(\psi\lfloor\vec{n})_I\Big)} \\
&\qquad\qquad = \sum_{kJ}\int_{b\Omega}(Y_k\psi_J)\overline{\Big(\nabla_{Y_k}(d(\psi\lfloor\vec{n}))\Big)_J} + \{\text{l.o.t.}\}.
\end{aligned}$$

Now we notice that, using equation (9.1), it follows that for $\psi \in \mathcal{D}$, the coefficients of $d(\psi\lfloor\vec{n})$ are all coefficients of $d\psi$, modulo lower order terms in the components of ψ. Notice also that the lower order terms that we produced in the previous calculation are all $\mathcal{O}(\|\psi\|_{W^1(b\Omega)})$.

Therefore

$$\begin{aligned}
|\langle d'\psi, \mathcal{K}_\Omega \psi\rangle_1| &\le C\Big(\sum_{kJ} \|\nabla_{Y_k}\psi_J\|_{W^{1/2}(b\Omega)} \|\nabla_{Y_k} d(\psi\lfloor\vec{n})\|_{W^{-1/2}(b\Omega)} + \|\psi\|^2_{W^1(b\Omega)}\Big) \\
&\le C\Big(\sum_J \|\psi_J\|_{W^{3/2}(b\Omega)} \|d(\psi\lfloor\vec{n})_J\|_{W^{1/2}(b\Omega)} + \|\psi\|^2_{W^1(b\Omega)}\Big) \\
&\le C\Big(\|\psi\|_{W^{3/2}(b\Omega)} \|d\psi\|_{W^{1/2}(b\Omega)} + \|\psi\|^2_{W^1(b\Omega)}\Big) \\
&\le C(\|\psi\|_2 \|d\psi\|_1 + \|\psi\|^2_{3/2}) \\
&\le \epsilon\|\psi\|_2^2 + C_\epsilon(\|d\psi\|_1^2 + \|\psi\|_1^2).
\end{aligned}$$

This proves Claim 2.

Proof of Claim 1. Fix an open cover $\{U_\ell\}_0^M$ of $\overline{\Omega}$ such that $\overline{U_0} \subseteq \Omega$ and, for $\ell = 1, \ldots, M$, on each U_ℓ we can find an orthonormal frame $\omega_0 = dx_0, \omega_1, \ldots, \omega_N$ for the space of 1-forms. Let $X_0 = \partial/\partial x_0, X_1, \ldots, X_N$ be the dual frame of vector fields. With respect to this basis the Laplacian is not diagonal on the space of q-forms, but it is diagonal in the top order terms.

On U_ℓ we write $\psi = \sum_I \psi_I \omega^I$. Let $\{\eta_\ell^2\}$ be a partition of unity subordinate to the cover $\{U_\ell\}$. In this proof we denote by $\mathcal{E}$ any quantity which is $\mathcal{O}(\|\psi\|_2\|\psi\|_1)$. Let A for the moment denote either operator d or d'. Then notice that

$$\begin{aligned}
\langle A\psi, A\psi\rangle_1 &= \sum_{jI} \langle D_j(A\psi)_I, D_j(A\psi)_I\rangle_0 + \mathcal{E} \\
&= \sum_{jI}\sum_\ell \int_\Omega \eta_\ell^2 (D_jA\psi)_I \overline{(D_jA\psi)_I} + \mathcal{E} \\
&= \sum_{jI}\sum_\ell \int_\Omega D_j\Big(A(\eta_\ell\psi)\Big)_I \overline{D_j\Big(A(\eta_\ell\psi)\Big)_I} + \mathcal{E}.
\end{aligned}$$

Then we see that it suffices to consider forms with support in one of the patches U_ℓ. When integrating by parts, this reduction only produces error terms of the type $\mathcal{E}$ already considered.

Notice that the form $\eta_\ell\psi$ may not belong to $\mathcal{D}$. Nonetheless $\eta_\ell\psi$ satisfies boundary equations of type (9.1) and (9.2), and this is all we need.

Thus, without loss of generality, let $\psi \in \mathcal{D}$ have support in a small open set on which we can write $\psi = \sum_I \psi_I \omega^I$. We have

$$d\psi = \sum_J \Big(\sum_{jI} \varepsilon^J_{jI} X_j \psi_I\Big)\omega^J + \{0 \text{ order terms}\},$$

and

$$d'\psi = \sum_L \Big(\sum_{\ell I} \varepsilon^I_{\ell L} X'_\ell \psi_I\Big)\omega^L + \{0 \text{ order terms}\},$$

where X'_ℓ is the formal adjoint of X_ℓ.

Now notice that there exists a constant C_0 such that for $\theta \in W^1_q$, $\theta = \sum_I \theta_I \omega^I$, we have that

$$\|\theta\|_1^2 \geq C_0 \sum_I \Big(\sum_j \|X_j \theta_I\|_0^2 + \|\theta_I\|_0^2 \Big), \tag{9.6}$$

where the constant C_0 depends only on the choice of $\omega_0, \dots, \omega_N$.

Then,

$$\begin{aligned}
&\sum_{jI} \Big(\|X_j (d\psi)_I\|_0^2 + \|X_j (d'\psi)_I\|_0^2 \Big) \\
&= \sum_J \sum_{pIqI'} \sum_{k=0}^N \int_\Omega (\varepsilon^J_{pI} X_p X_k \psi_I) \overline{(\varepsilon^J_{qI'} X_q X_k \psi_{I'})} \\
&\qquad + \sum_L \sum_{pIqI'} \sum_{k=0}^N \int_\Omega (\varepsilon^I_{pL} X'_p X_k \psi_I) \overline{(\varepsilon^{I'}_{qL} X'_q X_k \psi_{I'})} + \mathcal{E} \\
&= \sum_{kII'} \Big[\sum_{pq} \int_\Omega \Big(\sum_J \varepsilon^J_{pI} \varepsilon^J_{qI'} \Big) (X_p X_k \psi_I) \overline{(X_q X_k \psi_{I'})} \\
&\qquad + \sum_{pq} \int_\Omega \Big(\sum_L \varepsilon^I_{pL} \varepsilon^{I'}_{qL} \Big) (X'_p X_k \psi_I) \overline{(X'_q X_k \psi_{I'})} \Big] + \mathcal{E} \\
&= \sum_{kII'} \Big[\sum_{pq} \int_\Omega \Big(\sum_J \varepsilon^J_{pI} \varepsilon^J_{qI'} \Big) (X_p X_k \psi_I) \overline{(X_q X_k \psi_{I'})} \\
&\qquad + \sum_{pq} \int_\Omega \Big(\sum_L \varepsilon^I_{pL} \varepsilon^{I'}_{qL} \Big) (X_p X_k \psi_I) \overline{(X_q X_k \psi_{I'})} \Big] + \mathcal{E},
\end{aligned} \tag{9.7}$$

where the last equality holds since $X'_\ell = -X_\ell + L_0$, where L_0 is an operator of order 0.

Consider the term

$$\int_\Omega \varepsilon^J_{pI} \varepsilon^J_{qI'} (X_q X_k \psi_I) \overline{(X_p X_k \psi_{I'})} + \int_\Omega \varepsilon^I_{pL} \varepsilon^{I'}_{qL} (X_p X_k \psi_I) \overline{(X_q X_k \psi_{I'})} \equiv I + II. \tag{9.8}$$

We need to distinguish two cases, according to whether $p = q$ or $p \neq q$. Notice that if $p = q$ in (9.8), then

$$I + II = 2 \|X_p X_k \psi_I\|_0^2 + \mathcal{E}, \tag{9.9}$$

since $\varepsilon^J_{qI} \varepsilon^J_{pI'} = \varepsilon^I_{pL} \varepsilon^{I'}_{qL} = 0$ if $I \neq I'$.

Suppose now that $p \neq q$. We show in this case that the term in (9.8) is of type $\mathcal{E}$. Notice that if $p \neq q$ then

$$\varepsilon^J_{qI} \varepsilon^J_{pI} + \varepsilon^I_{pL} \varepsilon^I_{qL} = 0, \tag{9.10}$$

because, if $p < q$, then $\varepsilon^J_{qI} = -\varepsilon^{I'}_{qL}$ and $\varepsilon^J_{pI'} = \varepsilon^I_{pL}$. Then, if $p, q \neq 0$, integration by parts gives rise to no boundary term, and (9.10) shows that

$$I + I\!I = \mathcal{E} \tag{9.11}$$

when $p \neq q$, $p, q \neq 0$.

If $p \neq 0$ and $q = 0$ we distinguish two cases according to whether $k = 0$ or $k \neq 0$.

If $q = 0$ and $k = 0$, then we integrate by parts in $I\!I$. Recall that $\psi \in \mathcal{D}$ implies that for $I' \ni 0$ $\psi_{I'}$ satisfies the boundary equation (9.1). Then we have

$$\begin{aligned} I\!I &= -\int_\Omega \varepsilon^J_{0I}\varepsilon^J_{pI'}(X_0X_pX_0\psi_I)\overline{X_0\psi_{I'}} + \int_{b\Omega} \varepsilon^I_{pL}\varepsilon^{I'}_{0L}(X_pX_0\psi_I)\overline{X_0\psi_{I'}} + \mathcal{E} \\ &= \int_\Omega \varepsilon^J_{0I}\varepsilon^J_{pI'}(X_0X_0\psi_I)\overline{(X_pX_0\psi_{I'})} + \int_{b\Omega}(X_pX_0\psi_I)\overline{L_0\psi_{I'}} + \mathcal{E}. \end{aligned}$$

Now using equation (9.10) again, we see that

$$\begin{aligned} |I + I\!I| &\leq \Big| \int_{b\Omega}(X_pX_0\psi_I)L_0\psi_{I'}\Big| + \mathcal{E} \\ &\leq C\int |X_0\psi_I X_p\psi_{I'}| + \mathcal{E} \\ &\leq C\|X_0\psi_I\|_{W^{1/2}(b\Omega)}\|\psi_I\|_{W^{1/2}(b\Omega)} + \mathcal{E} \\ &= \mathcal{E}, \end{aligned} \tag{9.12}$$

that is $I + I\!I = \mathcal{E}$ in this case.

Finally, suppose that $q = 0$ and $k \neq 0$. Notice that $\varepsilon^J_{0I} = \varepsilon^{I'}_{0L} = 1$, and $\varepsilon^J_{pI'} = -\varepsilon^I_{pL}$ since $I' \ni 0$. Then

$$\begin{aligned} I + I\!I &= -\int_\Omega \varepsilon^I_{pL}(X_0X_k\psi_I)\overline{(X_pX_k\psi_{I'})} + \int_\Omega \varepsilon^I_{pL}(X_pX_k\psi_I)\overline{(X_0X_k\psi_{I'})} + \mathcal{E} \\ &= \int_\Omega \varepsilon^I_{pL}(X_pX_0X_k\psi_I)\overline{X_k\psi_{I'}} + \int_\Omega \varepsilon^I_{pL}(X_pX_k\psi_I)\overline{(X_0X_k\psi_{I'})} + \mathcal{E} \\ &= -\int_\Omega \varepsilon^I_{pL}(X_pX_k\psi_I)\overline{(X_0X_k\psi_{I'})} + \int_{b\Omega} \varepsilon^I_{pL}(X_pX_k\psi_I)\overline{X_k\psi_{I'}} \\ &\qquad + \int_\Omega \varepsilon^I_{pL}(X_pX_k\psi_I)\overline{(X_0X_k\psi_{I'})} + \mathcal{E} \\ &= -\int_{b\Omega} X_k\psi_I\overline{X_k(X_p\psi_{I'}\varepsilon^I_{pL})} + \mathcal{E}. \end{aligned}$$

Hence, using the fact that $k \neq 0$ we have that

$$
\begin{aligned}
|\sum_{pI'} I + II| &\leq \Big| \int_{b\Omega} X_k \psi_I \overline{X_k\Big(\sum_{pI'} X_p \psi_{I'} \varepsilon^I_{pI'}\Big)} \Big| + \mathcal{E} \\
&= \Big| \int_{b\Omega} X_k \psi_I \overline{X_k (d\psi \lfloor \vec{n})_I} \Big| + \mathcal{E} \\
&\leq \|X_k \psi_I\|_{W^{1/2}(b\Omega)} \|X_k (d\psi \lfloor \vec{n})_I\|_{W^{-1/2}(b\Omega)} + \mathcal{E} \\
&\leq C \|\psi\|_{W^{3/2}(b\Omega)} \|d\psi \lfloor \vec{n}\|_{W^{1/2}(b\Omega)} + \mathcal{E} \\
&\leq C \|\psi\|_2 \|d\psi\|_1 + \mathcal{E} \\
&\leq \epsilon \|\psi\|_2^2 + C_\epsilon \|d\psi\|_1^2 + \mathcal{E}. \qquad (9.13)
\end{aligned}
$$

Therefore, collecting (9.9) - (9.13) and substituting them into (9.7), and recalling (9.6), we obtain that

$$
\begin{aligned}
\frac{1}{C_0}(\|d\psi\|_1^2 + \|d'\psi\|_1^2) &\geq \sum_{kI} \Big(\|X_k (d\psi)_I\|_0^2 + \|X_k (d'\psi)_I\|_0^2 \Big) \\
&= \sum_{kII'} \Big[\sum_{pq} I + II \Big] \\
&\geq \sum_{kpI} 2 \|X_p X_k \psi_I\|_0^2 - (\epsilon \|\psi\|_2^2 + C_\epsilon \|d\psi\|_1^2) + \mathcal{E} \\
&= (2-\epsilon) \|\psi\|_2^2 - C_\epsilon \|d\psi\|_1^2 + \mathcal{E} \\
&\geq (2 - 2\epsilon) \|\psi\|_2^2 - C_\epsilon (\|d\psi\|_1^2 + \|\psi\|_1^2).
\end{aligned}
$$

From this Claim 1 follows easily. This proves the theorem. □

10. The Regularity Theorem in the Case of Functions

We now turn to the question of regularity. In this section we are going to prove the regularity result for the case $q = 0$.

THEOREM 10.1. *Let $f \in C^\infty(\overline{\Omega})$ be orthogonal in the W^1 inner product to the space of constant functions. Then there is a unique function $u \in C^\infty(\overline{\Omega})$ that solves the boundary value problem*

$$
\begin{cases}
d^* d\, u = f & \text{on } \Omega \\
du \in \operatorname{dom} d^*
\end{cases}
$$

In other words, u solves the system

$$
\begin{cases}
(-\triangle + G_\Omega) u = f & \text{on } \Omega \\
\dfrac{\partial^2 u}{\partial n^2} = 0 & \text{on } b\Omega
\end{cases}
$$

Moreover, the solution u satisfies the desired (coercive) estimates for each $s > 1/2$; i.e. there exists a $c_s > 0$ such that

$$\|u\|_{s+2} \le c_s\Big(\|f\|_s + \|u\|_0\Big). \tag{10.1}$$

Proof. By the existence theorem, Theorem 9.2, we know that the solution to the above boundary value problem exists for all f orthogonal to the harmonic space $\mathcal{H}_0$. In the forthcoming Theorem 12.2 we shall prove that $\mathcal{H}_0$ reduces to the constants. Now we turn to the question of estimates.

Let u be a solution of the boundary value problem, with $f \in C^\infty(\overline{\Omega})$. Let $s > 1/2$. We wish to estimate $\|u\|_{s+2}$ in terms of $\|f\|_s$. It is clear that it suffices to estimate $\|\eta u\|_{s+2}$ for a given cut-off function η with small support.

We first suppose that $\operatorname{supp}\eta \cap b\Omega = \emptyset$. [The second step will be to assume that $\operatorname{supp}\eta \cap b\Omega \neq \emptyset$.]

The interior estimate. Let $\eta \in C_0^\infty(\Omega)$. Recalling that $f = -\triangle u + G_\Omega u$ we have that

$$\begin{aligned}
\|\eta u\|_{s+2} &\le c\|\triangle(\eta u)\|_s \\
&\le c\Big(\|\triangle(\eta u) + \eta f\|_s + \|\eta f\|_s\Big) \\
&\le c\Big(\|\triangle(\eta u) - \eta\triangle u\|_s + \|\eta G_\Omega u\|_s + \|\eta f\|_s\Big) \\
&\le c\Big(\|\eta_1 u\|_{s+1} + \|\eta G_\Omega u\|_s + \|\eta f\|_s\Big),
\end{aligned} \tag{10.2}$$

where $\eta_1 \equiv 1$ on $\operatorname{supp}\eta$. Next we want to show that $\|\eta G_\Omega u\|_s \le c\|u\|_{s+1}$. Recall that by Proposition 7.2 and equation (8.6) G_Ω is the solution of the boundary value problem

$$\begin{cases} -\triangle v + v = 0 & \text{on } \Omega \\ \dfrac{\partial v}{\partial n} = \displaystyle\sum_{k=0}^{N} \nabla_{Y_k^*}\Big((\nabla_{Y_k} du)\lfloor\vec{n}\Big) + \dfrac{\partial u}{\partial n} & \text{on } b\Omega \end{cases}.$$

Now it is easy to see that (using the Fourier transform for instance)

$$\|\eta G_\Omega u\|_s \le c\|\eta_1 G_\Omega u\|_{s-1} \le c\|G_\Omega u\|_{s-1}.$$

(Here, for $t > 0$, $\|\cdot\|_{-t}$ is the norm in the Sobolev space $W_{-t}(\Omega) \equiv (\overset{\circ}{W}{}^t(\Omega))^*$.) Moreover, by [TRI] Theorem 4.2.4 we have

$$\begin{aligned}
\|G_\Omega u\|_{s-1} &\le c\|T_2\frac{\partial u}{\partial n} - T_2' u\|_{W^{s-5/2}(b\Omega)} \\
&\le c\Big(\Big\|\frac{\partial u}{\partial n}\Big\|_{W^{s-1/2}(b\Omega)} + \|u\|_{W^{s-1/2}(b\Omega)}\Big) \\
&\le c\|u\|_{s+1}.
\end{aligned} \tag{10.3}$$

Thus (10.2) and (10.3) give that

$$\|\eta u\|_{s+2} \le c(\|\eta f\|_s + \|u\|_{s+1}).$$

The boundary estimate. Fix a Fermi coordinate patch

$$\Big(U, \Psi = (x_0, \ldots, x_N)\Big),$$

and let $\eta \in C_0^\infty(U)$.

In what follows we denote by G_Ω both the operator defined on Ω and (when restricted to U) the same operator expressed in the local chart and thus defined on $\Psi(U)$. This technical ambiguity should cause no confusion. We will denote by $G_{\mathbb{R}^{N+1}}$ the operator arising from considering the adjoint of d^* in the half space. Finally, given a function $u \in C^\infty(U)$ we denote by $\tilde{u}$ the function

$$u \circ \Psi^{-1} \in C^\infty(\Psi(U)).$$

Now consider $\eta, \eta_1 \in C_0^\infty(U)$ with $\eta_1 = 1$ on the support of η. These functions are chosen to be constant along the normal direction to $b\Omega$ near the boundary. In our local coordinates, ηu satisfies the following boundary condition:

$$\frac{\partial^2}{\partial {x_0}^2}(\widetilde{\eta u}) = 0 \qquad\qquad \text{on } \{x_0 = 0\}.$$

Recall that if v satisfies the boundary value problem

$$\begin{cases} - \triangle\, v + G_{\mathbb{R}_+^{N+1}} v = f & \text{on } \mathbb{R}_+^{N+1} \\ \dfrac{\partial^2 v}{\partial x_0^2} = 0 & \text{on } b\mathbb{R}_+^{N+1} \end{cases}$$

and if v has compact support, then

$$\|v\|_{s+2} \le C\Big\{\|f\|_s \;+\; \|v\|_{s+1}\Big\}, \qquad\qquad s > 1/2.$$

Then

$$\|\widetilde{\eta u}\|_{s+2} \le C\Big\{\Big\|(- \triangle + G_{\mathbb{R}_+^{N+1}})(\widetilde{\eta u})\Big\|_s + \|\widetilde{\eta u}\|_{s+1}\Big\}. \tag{10.4}$$

Our goal is to replace $-\triangle + G_{\mathbb{R}^{N+1}_+}$ with $-\triangle + \epsilon\mathcal{L}_2 + G_\Omega$ modulo error terms that are controlled by lower order norms of ηu. Thus the estimate (10.4) gives that

$$
\begin{aligned}
\|\eta u\|_{s+2} &\leq C\Big\{\|(-\triangle + \epsilon\mathcal{L}_2 + G_\Omega)(\widetilde{\eta u})\|_s + \|\widetilde{\eta u}\|_{s+1} \\
&\qquad + \epsilon\|\mathcal{L}_2(\widetilde{\eta u})\|_s + \Big\|G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta u})\Big[\eta G_\Omega u\Big]^{\sim}\Big\|_s\Big\} \\
&\leq C\Bigg\{\Big\|\tilde{\eta}[(-\triangle + \epsilon\mathcal{L}_2 + G_\Omega)(u)]^{\sim}\Big\|_s + \epsilon\|\eta u\|_{s+2} + \|\eta_1 u\|_{s+1} \\
&\qquad + \Big\|G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta u}) - [\eta G_\Omega u]^{\sim}\Big\|_s\Bigg\} \\
&\leq C\Big\{\Big\| - \tilde{\eta}\triangle \tilde{u} + G_{\mathbb{R}^{N+1}}(\widetilde{\eta u})\Big\|_s + \|\eta_1 u\|_{s+1}\Big\} \\
&\leq C\Big\{\Big\|\tilde{\eta}(-\triangle + \epsilon\mathcal{L}_2 + G_\Omega)(\tilde{u})\Big\|_s + \epsilon\|\eta u\|_{s+2} \\
&\qquad + \Big\|G_{\mathbb{R}^{N+1}}(\widetilde{\eta u}) - \Big[\eta G_\Omega u\Big]^{\sim}\Big\|_s + \|\eta_1 u\|_{s+1}\Big\} \qquad (10.5) \\
&= C\Big\{\|\eta f\|_s + \epsilon\|\eta u\|_{s+2} + \Big\|G_{\mathbb{R}^{N+1}}(\widetilde{\eta u}) - \Big[\eta G_\Omega u\Big]^{\sim}\Big\|_s + \|\eta_1 u\|_{s+1}\Big\}.
\end{aligned}
$$

The function $G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta u})$ solves the boundary value problem

$$
\begin{cases}
(-\triangle + I)\Big(G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta u})\Big) = 0 & \text{on } \mathbb{R}^{N+1}_+ \\
\dfrac{\partial}{\partial x_0}\Big(G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta u})\Big) = \dfrac{\partial}{\partial x_0}(\widetilde{\eta u}) - \triangle'_{\mathbb{R}^N}\dfrac{\partial}{\partial x_0}(\widetilde{\eta u}) & \text{on } b\mathbb{R}^{N+1}_+
\end{cases}.
$$

On the other hand, by (8.6), the function $[\eta G_\Omega u]^{\sim}$, solves the boundary value problem

$$
\begin{cases}
\Big(-\triangle + I + \epsilon\mathcal{L}_2\Big)([\eta G_\Omega u]^{\sim}) = -[2\nabla\eta\cdot\nabla G_\Omega u + \triangle\eta\cdot G_\Omega u]^{\sim} & \text{on } \mathbb{R}^{N+1}_+ \\
\dfrac{\partial}{\partial x_0}\Big([\eta G_\Omega u]^{\sim}\Big) = T_1\dfrac{\partial}{\partial x_0}(\widetilde{\eta u}) - (\triangle'_{\mathbb{R}^{N+1}} - \epsilon\mathcal{L}_2)\Big(\dfrac{\partial}{\partial x_0}(\widetilde{\eta u})\Big) + T_2(\widetilde{\eta u}) & \text{on } b\mathbb{R}^{N+1}_+
\end{cases}.
$$

Let $w \equiv G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta u}) - [\eta G_\Omega u]^{\sim}$. Then w solves the system

$$
\begin{cases}
(\triangle - I)w = \epsilon\mathcal{L}_2\Big([\eta G_\Omega u]^{\sim}\Big) + [2\nabla\eta\cdot\nabla G_\Omega u + (\triangle\eta)G_\Omega u]^{\sim} & \text{on } \mathbb{R}^{N+1}_+ \\
\dfrac{\partial w}{\partial x_0} = \tilde{\eta}\epsilon\mathcal{T}_2\dfrac{\partial\tilde{u}}{\partial x_0} + \tilde{\eta}T_1\dfrac{\partial\tilde{u}}{\partial x_0} + \tilde{\eta}T_2\tilde{u} & \text{on } b\mathbb{R}^{N+1}_+
\end{cases} \qquad (10.6)
$$

The right hand side of the first equation in the system (10.6) can be rewritten as

$$\epsilon\eta' L_2\widetilde{G_\Omega u} + \eta'' L_1\widetilde{G_\Omega u}$$

where the order of L_j is j and $\operatorname{supp}\eta', \operatorname{supp}\eta''$ lie in $\operatorname{supp}\tilde{\eta}$.

Now we apply Theorem 4.2.4 in [TRI]. We obtain that, for any $s > 1/2$,

$$\begin{aligned}
\|w\|_s \quad & \leq \quad C\Big\{\epsilon\Big\|L_2([\eta G_\Omega u]^{\sim})\Big\|_{F^{s-2}} + \Big\|\eta' L_1[G_\Omega u]^{\sim}\Big\|_{F^{s-2}} \\
& \qquad + \epsilon\Big\|\eta \mathcal{T}_2 \frac{\partial u}{\partial n}\Big\|_{W^{s-3/2}(b\Omega)} + \Big\|\eta T_1 \frac{\partial u}{\partial n}\Big\|_{W^{s-3/2}(b\Omega)} + \Big\|\eta T_2 u\Big\|_{W^{s-3/2}(b\Omega)}\Big\} \\
& \leq \quad C\Big\{\epsilon\Big\|\eta G_\Omega u\Big\|_{F^s} + \Big\|\eta'[G_\Omega u]\Big\|_{F^{s-1}} + \epsilon\left\|\eta\frac{\partial u}{\partial n}\right\|_{W^{s+1/2}(b\Omega)} \\
& \qquad + \left\|\eta\frac{\partial u}{\partial n}\right\|_{W^{s-1/2}(b\Omega)} + \Big\|\eta_1 u\Big\|_{W^{s+1/2}(b\Omega)}\Big\} \\
& \overset{G_\Omega u = \triangle u + f}{\leq} \quad C\Big\{\epsilon\Big\|\eta \triangle u\Big\|_s + \epsilon\Big\|\eta f\Big\|_s + \Big\|\eta_1 \triangle u\Big\|_{F^{s-1}} \\
& \qquad + \Big\|\eta_1 f\Big\|_s + \epsilon\Big\|\eta u\Big\|_{s+2} + \Big\|\eta_1 u\Big\|_{s+1}\Big\} \\
& \leq \quad C\Big\{\epsilon\Big\|\eta u\Big\|_{s+2} + \Big\|\eta_1 f\Big\|_s + \Big\|\eta_1 u\Big\|_{s+1}\Big\}. \qquad (10.7)
\end{aligned}$$

Now estimates (10.5) and (10.7) yield that

$$\|\eta u\|_{s+2} \leq C\Big\{\|\eta_1 f\|_s + \epsilon\|\eta u\|_{s+2} + \|\eta_1 u\|_{s+1}\Big\},$$

that is, for $s > 1/2$,

$$\|\eta u\|_{s+2} \leq C\Big\{\|\eta_1 f\|_s + \|\eta_1 u\|_{s+1}\Big\}.$$

This concludes the proof. ∎

11. Estimates for q-Forms

The Regularity Theorem. In this section we prove the estimate for the solution of the boundary value problem in the case of q-forms. We follow the same outline as in the case of functions. For higher degree forms some extra technicalities are needed. The differential operators acting on forms that are involved are not (usually) diagonal, and we need some extra care for the off-diagonal terms.

THEOREM 11.1. *Let $s > 1/2$. Let $\alpha \in W^s_q(\Omega)$, and α orthogonal to the kernel of the boundary value problem. Let $\psi \in \bigwedge^q$ be a solution of*

$$\begin{cases} (dd^* + d^*d)\psi = \alpha & \text{on } \Omega \\ \psi \in \operatorname{dom} d^* \\ d\psi \in \operatorname{dom} d^* \end{cases}$$

There exists $c_s > 0$, independent of ψ, such that

$$\|\psi\|_{s+2} \leq c_s(\|\alpha\|_s + \|\psi\|_{s+1}).$$

Thus we wish to estimate $\|\psi\|_{s+2}$ in terms of $\|\alpha\|_s$, for $s > 1/2$. It suffices to estimates $\eta\psi$ for a cut-off function η with support contained in a small open set. If $\eta \in C_0^\infty(\Omega)$ then the argument used in the case of functions (see the proof of 10.1) applies with no substantial change.

Thus we assume that η is a cut-off function whose support is contained in an open set on which there exists a Fermi coordinate chart. As before, we also suppose that $(\partial/\partial n)\eta = 0$ in a tubular neighborhood of the boundary.

As in the case of q-forms in the half space we write

$$\begin{aligned}(dd^* + d^*d) &= (dd' + d'd) + (d\mathcal{K}_\Omega + \mathcal{K}_\Omega d) \\ &\equiv -\triangle + G_\Omega.\end{aligned}$$

We now proceed as in the function case. We need to estimate $\|\eta\psi\|_{s+2}$. We introduce Fermi local coordinates, and set the stage as in Section 8. Recall that such coordinates have the property that the normal direction is orthogonal to the remaining directions at all points in the coordinate patch (that is, such coordinates "flatten" the boundary.). Given a form ϕ on Ω (i.e. written in global coordinates), we write $\tilde{\phi}$ to indicate the form written in the Fermi coordinates. We adopt the same notation for the operators.

We wish to apply the estimates proved in the half space for the q-forms. In order to do this we notice that $\widetilde{(\eta\psi)}$ is a form defined on the half space, and we wish to check if it satisfies a boundary value problem for which we have favorable estimates. Since ψ and $d\psi$ belong to $\operatorname{dom} d^*$, they satisfy equations (9.1) and (9.2). We need to examine those equations more closely. In order to do this we rewrite equations (9.1) and (9.2) in our local coordinates.

Proposition 11.2. *Let $\psi \in \bigwedge^q(\overline{\Omega})$ be such that both ψ, $d\psi \in \operatorname{dom} d^*$. Let η be a cut-off function as before. Then the form $\widetilde{\eta\psi}$ satisfies the boundary equations*

$$\frac{\partial}{\partial x_0}(\widetilde{\eta\psi})_I + \sum_{\substack{|J|=q \\ J\ni 0}} \gamma_{IJ}(\widetilde{\eta\psi})_J = 0 \qquad \text{if } I \ni 0, \tag{11.1}$$

and,

$$\frac{\partial^2}{\partial x_0^2}(\widetilde{\eta\psi})_I + \sum_{\substack{|K|=q \\ K\ni 0}} T_1\Big((\widetilde{\eta\psi})_K\Big) + \sum_{\substack{|L'|=q \\ L'\not\ni 0}} a_{K'L'}\frac{\partial}{\partial x_0}(\widetilde{\eta\psi})_{L'} + \mathcal{E} = 0 \quad \text{if } I \not\ni 0, \tag{11.2}$$

where $\mathcal{E}$ are $\{0$ order terms$\}$ in the components of ψ.

We stress that we have improved equations (9.1) and (9.2) by separating the "normal" and "tangential" components of the lower order terms in the boundary equations.

Proof. By Lemma 8.2, the condition $\psi \in \operatorname{dom} d^*$, i.e. $\nabla_{\vec{n}}\psi \lfloor \vec{n} = 0$ on $b\Omega$, becomes

$$\frac{\partial}{\partial x_0}(\widetilde{\eta\psi})_I + \sum_{\substack{|J|=q \\ J\ni 0}} \gamma_{IJ}(\widetilde{\eta\psi})_J = 0 \qquad \text{on } \mathbb{R}^N \text{ if } I \ni 0.$$

On the other hand, the second boundary condition gives that

$$\frac{\partial}{\partial x_0}(\widetilde{d\psi})_K + \sum_{\substack{|L|=q+1 \\ L\ni 0}} \gamma_{KL}(\widetilde{d\psi})_L = 0 \qquad \text{on } \mathbb{R}^N \text{ if } K \ni 0,$$

i.e., on $\mathbb{R}^N$ for $K \ni 0$,

$$\frac{\partial}{\partial x_0}\Big(\frac{\partial\tilde{\psi}_{K'}}{\partial x_0}\varepsilon^K_{0K'} + \sum_{\substack{K'' \\ j=1,\dots,N}} \frac{\partial\tilde{\psi}_{K''}}{\partial x_j}\varepsilon^K_{jK''}\Big) + \frac{\partial}{\partial x_0}\Big(\sum_J \psi_J d\omega^J\Big)_J$$
$$+ \sum_{\substack{|L|=q+1 \\ L\ni 0}} \gamma_{KL}\Big(\frac{\partial\tilde{\psi}_{L'}}{\partial x_0}\varepsilon^L_{0L'} + \sum_{\substack{L'' \\ j=1,\dots,N}} X_j\widetilde{\psi_{L''}}\varepsilon^L_{jL''}\Big) + \{0 \text{ order terms}\} = 0.$$

Now we use the first boundary condition to replace the terms of the form $(\partial/\partial x_0)\tilde{\psi}_{K''}$, with $K'' \ni 0$, with 0 order terms in ψ. Hence, the term $(\partial/x_0)\Big(\sum_J \psi_J d\omega^J\Big)_J$ only contributes normal derivatives of tangential components and 0 order terms.

Therefore, for $K' \not\ni 0$, we have

$$\frac{\partial^2}{\partial x_0^2}(\widetilde{\eta\psi})_{K'} + \sum_{K''} T_1(\widetilde{\eta\psi})_{K''}\varepsilon^{0K'}_{jK''} + (0 \text{ order terms}) + \sum_{\substack{|L'|=q \\ L'\not\ni 0}} a_{K'L'}\frac{\partial}{\partial x_0}(\widetilde{\eta\psi})_{L'} = 0.$$

From this identity, equation (11.2) follows. ☐

At this point we adapt the estimates in the half space for the boundary value problems

$$\begin{cases} (-\triangle + G_{\mathbb{R}^{N+1}_+})\phi = \alpha & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial^2\phi_I}{\partial x_0} = 0 & \text{on } \mathbb{R}^N \quad \text{if } I \not\ni 0 \end{cases}, \tag{11.3}$$

and

$$\begin{cases} (-\triangle + G_{\mathbb{R}^{N+1}_+})\phi = \alpha & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial\phi_I}{\partial x_0} = 0 & \text{on } \mathbb{R}^N \quad \text{if } I \ni 0 \end{cases}. \tag{11.4}$$

to the form $\widetilde{\eta\psi}$.

We wish to apply Theorems 5.4 and 6.10 to the problems (11.3) and (11.4) respectively. In both cases the boundary data is estimated by $\|\widetilde{\eta\psi}\|_{s+1}$. Let $\eta_1 \in C_0^\infty(\overline{\Omega})$,

$\eta_1 \equiv 1$on $\operatorname{supp}\eta$. We obtain that

$$\begin{aligned}
\|\widetilde{\eta\psi}\|_{s+2} &\leq c\left(\|\eta_1(-\triangle + G_{\mathbb{R}^{N+1}_+})(\widetilde{\eta\psi})\|_s + \|\widetilde{\eta\psi}\|_{s+1}\right) \\
&\leq c\Big(\Big\|\Big[\eta(-\triangle + G_\Omega)\psi\Big]^{\sim}\Big\|_s + \Big\|\Big[-\eta\triangle\psi\Big]^{\sim} + \triangle(\widetilde{\eta\psi})\Big\|_s \\
&\qquad + \Big\|\eta_1 G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta\psi}) - \eta_1\Big[\eta G_\Omega\psi\Big]^{\sim}\Big\|_s + \|\widetilde{\eta\psi}\|_{s+1}\Big) \\
&\leq c\left(\|\widetilde{\eta\alpha}\|_s + \Big\|G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta\psi}) - \Big[\eta G_\Omega\psi\Big]^{\sim}\Big\|_s + \|\widetilde{\eta\psi}\|_{s+1}\right).
\end{aligned} \tag{11.5}$$

Now we need to estimate $\Big\|\eta_1 G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta\psi}) - \eta_1\Big[\eta G_\Omega\psi\Big]^{\sim}\Big\|_s$ in the equation (11.5) above. We have the following theorem.

THEOREM 11.3. *Let $s > 1/2$. Then, for any $\epsilon > 0$, there exists a $C_\epsilon > 0$ such that*

$$\Big\|\eta_1 G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta\psi}) - \eta_1\Big[\eta G_\Omega\psi\Big]^{\sim}\Big\|_s \leq \epsilon\|\widetilde{\eta\psi}\|_{s+2} + C_\epsilon\|\eta_1\psi\|_s,$$

where $\eta_1 \in C^\infty_0$ and $\eta_1 \equiv 1$ on $\operatorname{supp}\eta$.

Assuming the Theorem for now, we finish the proof of the estimate for $\|\widetilde{\eta\psi}\|_{s+2}$, and therefore the proof of Theorem 11.1. Using Theorem 11.3 and (11.5) above we see that

$$\|\widetilde{\eta\psi}\|_{s+2} \leq c\Big(\|\eta\alpha\|_s + \epsilon\|\eta_1\psi\|_{s+2}\|\eta_1\psi\|_{s+1}\Big),$$

from which we obtain that

$$\|\widetilde{\eta\psi}\|_{s+2} \leq c\Big(\|\eta\alpha\|_s + \|\eta_1\psi\|_{s+1}\Big),$$

which is what we wished to prove. □

Proof of Theorem 11.3. Now we turn to the proof of Theorem 11.3. The proof will be broken up into several lemmas.

The form $G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta\psi}) \equiv \theta = \sum_I \theta_I\omega^I$ is such that its components θ_I satisfy the following boundary value problems

$$\begin{cases} (-\triangle + I)\theta_I = 0 & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial\theta_I}{\partial x_0} = (-\triangle' + I)(d(\widetilde{\eta\psi})_{0I}) & \text{on } \mathbb{R}^N \quad \text{if } I \not\ni 0 \end{cases} \tag{11.6}$$

and

$$\begin{cases} (-\triangle + I)\theta_I = 0 & \text{on } \mathbb{R}^{N+1}_+ \\ \theta_I = (-\triangle' + I)(\widetilde{\eta\psi})_I & \text{on } \mathbb{R}^N \quad \text{if } I \ni 0 \end{cases} \tag{11.7}$$

On the other hand we need to investigate what kind of boundary value problem the form $[\eta G_\Omega\psi]^{\sim}$ satisfies. Recall that $G_\Omega \equiv (\mathcal{K}_\Omega d + d\mathcal{K}_\Omega)$, and that the operator $\mathcal{K}_\Omega$ is the operator on forms that gives the solution of the following boundary value problem for $\theta \in \bigwedge^q$. Namely we have

$$\begin{cases} (-\triangle + I)\theta = 0 & \text{on } \Omega \\ \nabla_{\vec{n}}\theta = \sum_{i=0}^{N} \nabla_{Y_i^*}(\nabla_{Y_i}\phi\lfloor\vec{n}) + \phi\lfloor\vec{n} & \text{on } b\Omega \end{cases}$$

for $\phi \in \bigwedge^{q+1}(\overline{\Omega})$.

Lemma 11.4. *Let $\phi \in \bigwedge^q(\overline{\Omega})$. The form $[\eta\mathcal{K}_\Omega\phi]^\sim \equiv \theta$ is such that its I-component satisfies the following boundary value problem on $\mathbb{R}^{N+1}_+$*

$$\begin{cases} (-\triangle - \epsilon\mathcal{L}_2 + I)\theta_I = \Big(\eta_1 L_1 \widetilde{\mathcal{K}_\Omega\phi}\Big)_I & \text{on } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial\theta_I}{\partial x_0} = -\sum_{|K|=q-1} \tilde{\eta}\gamma_{IK}(\widetilde{\mathcal{K}_\Omega\phi})_K + (E_2\phi)_I & \text{on } \mathbb{R}^N \end{cases}$$

where

$$(E_2\phi)_I = \begin{cases} -\tilde{\triangle}_T\phi_{0I} + \phi_{0I} + (T_1\phi)_I & \text{if } I \not\ni 0 \\ 0 & \text{if } I \ni 0 \end{cases}$$

$\mathcal{L}_2$ is defined by equation (8.3). Moreover, L_1 is a first order differential operator, T_1 is a first order differential operator, both sending $(q+1)$-forms into q-forms.

Proof. It suffices to use Lemma 8.2 and the computation leading to formula (8.6). $\square$

Lemma 11.5. *Let $\phi \in \bigwedge^q(\overline{\Omega})$, and let $I \ni 0$. Then for any $s > 1/2$,*

$$\|[\eta\mathcal{K}_\Omega\phi]_I^\sim\|_s \le c\|\phi\|_s.$$

Proof. Recall (see Corollary 7.3) that $\mathcal{K}_\Omega$ is an operator of order 1. Therefore we direct the reader's attention to the fact that the content of this lemma is that, on the normal components of the form, the operator $\mathcal{K}_\Omega$ is of order 0. (Recall that, in the case of the half space, the operator $\mathcal{K}_\Omega$ was identically zero on normal components.)

It suffices to recall the estimates in the negative norms for elliptic boundary value problems, as in [TRI], Theorem 4.2.4. Using the spaces $F^s_{2,2}(\Omega)$, which we will simply denote by F^s, we have

$$\|[\eta\mathcal{K}_\Omega\phi]_I^\sim\|_s \le c\Big(\|\Big(\eta_1 L_1\widetilde{\mathcal{K}_\Omega\phi}\Big)_I\|_{F^{s-2}} + \sum_{K\ni 0}\|[\eta\mathcal{K}_\Omega\phi]_K^\sim\|_{W^{s-3/2}(b\Omega)}\Big).$$

Let $\epsilon > 0$. By selecting η with support suitably small, we can achieve

$$\|[\eta\mathcal{K}_\Omega\phi]_K^\sim\|_{W^{s-3/2}(b\Omega)} \le \epsilon\|[\eta\mathcal{K}_\Omega\phi]_K^\sim\|_{W^{s-1/2}(b\Omega)}\,.$$

Therefore

$$\begin{aligned}\sum_{I\ni 0}\|[\eta\mathcal{K}_\Omega\phi]_I^\sim\|_s &\le c\|\Big(\eta_1 L_1\widetilde{\mathcal{K}_\Omega\phi}\Big)_I\|_{F^{s-2}} + \epsilon\sum_K\|[\eta\mathcal{K}_\Omega\phi]_K^\sim\|_{W^{s-1/2}(b\Omega)} \\ &\le c\|\Big(\eta_1 L_1\widetilde{\mathcal{K}_\Omega\phi}\Big)_I\|_{F^{s-2}} + \epsilon\sum_K\|[\eta\mathcal{K}_\Omega\phi]_K^\sim\|_{F^s}.\end{aligned}$$

Then, by absorbing the last term on the right hand side over on the left, using the continuity of differential operators on the spaces F^s, and invoking Corollary 7.3, we obtain that

$$\sum_{I\ni 0} \|[\eta\mathcal{K}_\Omega\phi]_I\tilde{}\|_s \le c\|[\eta\mathcal{K}_\Omega\phi]\tilde{}\|_{F^{s-1}(\Omega)} \\ \le c\|\eta_1\phi\|_s.$$

This concludes the proof of the lemma. ◻

Lemma 11.6. *Let $\psi \in \bigwedge^q(\overline{\Omega})$. Then, for $I \ni 0$, $[\eta G_\Omega\psi]_I\tilde{}$ is a solution of the following boundary value problem*

$$\begin{cases}(-\bigtriangleup -\epsilon\mathcal{L}_2 + I)w = \Big[\eta_1 L_1(G_\Omega\psi)\Big]_I^{\sim} & \text{on } \mathbb{R}^{N+1}_+ \\ w = \tilde\eta(-\bigtriangleup_T + I)\widetilde{\psi_I} + \zeta & \text{on } \mathbb{R}^N\end{cases}$$

where $\mathcal{L}_2$ is defined by equation (8.3), ζ is a function satisfying the estimate

$$\|\zeta\|_{W^{s-1/2}(b\Omega)} \le c\|\eta_1\psi\|_{s+1},$$

L_1 is a differential operator of order 1, and $\eta_1 \in C^\infty_0$ with $\eta_1 \equiv 1$ on $\operatorname{supp}\eta$.

Proof. The equation on $\mathbb{R}^{N+1}_+$ is easily seen to be satisfied. Concerning the boundary equation, recall that $G_\Omega = \mathcal{K}_\Omega d + d\mathcal{K}_\Omega$. Then, for $I \ni 0$, $I = 0I'$, the function $\Big[\eta\mathcal{K}_\Omega d\psi\Big]_I^{\sim}|_{\mathbb{R}^N}$ is the restriction to $b\mathbb{R}^{N+1}_+$ of the solution of the boundary value problem in Lemma 11.4 with $\phi = d\psi$. Moreover, for $I = 0I'$, on $\mathbb{R}^N$ we have

$$\begin{aligned}\Big[\eta d\mathcal{K}_\Omega\psi\Big]_I^{\sim} &= \frac{\partial}{\partial x_0}\Big(\eta\mathcal{K}_\Omega\psi\Big)_{I'}^{\sim} + \sum_{\substack{i=1,\dots,N\\ |J|=q-1}} X_i\Big[\eta\mathcal{K}_\Omega\psi\Big]_J^{\sim}\varepsilon^{0I'}_{jJ} + (0 \text{ order terms}) \\ &= -\sum_{K\not\ni 0}\tilde\eta\gamma_{I'K}(\widetilde{\mathcal{K}_\Omega\psi})_K + (-\bigtriangleup_T + I)\psi_{0I'} + \sum_{J\ni 0} T_1\Big[\eta\mathcal{K}_\Omega\psi\Big]_J^{\sim} + (0 \text{ o. t.'s})\,,\end{aligned}$$

where T_1 is a tangential differential operator of order 1. Therefore w satisfies the boundary equation

$$\begin{aligned}w &= \tilde\eta(-\bigtriangleup_T + I)\psi_I - \sum_{K\not\ni 0}\tilde\eta\gamma_{I'K}(\widetilde{\mathcal{K}_\Omega\psi})_K + \Big[\eta\mathcal{K}_\Omega d\psi\Big]_I^{\sim} + \sum_{J\ni 0} T_1\Big[\eta\mathcal{K}_\Omega\psi\Big]_J^{\sim} \\ &\equiv \tilde\eta(-\tilde\bigtriangleup' + I)\psi_I + \zeta.\end{aligned}$$

Now the desired estimate follows from Lemma 11.5. Indeed, recalling that $I \ni 0$, we see that

$$\begin{aligned}
\|\zeta\|_{W^{s-1/2}(b\Omega)} &\le c\Big(\|\widetilde{\mathcal{K}_\Omega\psi}\|_{W^{s-1/2}(b\Omega)} + \|\big[\eta\mathcal{K}_\Omega d\psi\big]_I^{\sim}\|_{W^{s-1/2}(b\Omega)} \\
&\qquad + \sum_{J\ni 0}\|T_1\big[\eta\mathcal{K}_\Omega\psi\big]_J^{\sim}\|_{W^{s-1/2}(b\Omega)}\Big) \\
&\le c\left(\|\widetilde{\mathcal{K}_\Omega\psi}\|_s + \|\big[\eta\mathcal{K}_\Omega d\psi\big]_I\|_s + \sum_{J\ni 0}\|\big[\eta\mathcal{K}_\Omega\psi\big]_J\|_{s+1}\right) \\
&\le c\|\eta_1\psi\|_{s+1}. \quad \square
\end{aligned}$$

Before estimating the term $[\eta G_\Omega\psi]_I^{\sim}$ for $I \not\ni 0$ we need an extra lemma.

Lemma 11.7. *Let $w_1 = \eta(\mathcal{K}_\Omega d\psi)_K^{\sim} - \eta(d\mathcal{K}_\Omega\psi)_K^{\sim}$, with $K \not\ni 0$. Then w_1 solves*

$$\begin{cases}(-\triangle + I)w_1 = L_1(\mathcal{K}_\Omega d\psi) + L_2(\mathcal{K}_\Omega\psi) & \text{on } \Omega\\ \dfrac{\partial w_1}{\partial n} = T_2\psi + L_0(\mathcal{K}_\Omega\psi) & \text{on } \mathbb{R}^N\end{cases},$$

where the L_j's are operators of order j, and T_2 is a tangential differential operator of order 2.

Moreover, there exists $c > 0$ independent of ψ such that

$$\|w_1\|_s \le c\|\psi\|_{s+1}.$$

Proof. Recall that $(\mathcal{K}_\Omega\phi)_K$ satisfies the boundary value problem

$$\begin{cases}(-\triangle + I)\theta = 0\\ \dfrac{\partial\theta}{\partial n} = Y_k^*\Big((\nabla_{Y_k}\phi\lfloor)\vec{n}\Big)_K + (\phi\lfloor\vec{n})_K\end{cases}$$

Therefore,

$$\begin{aligned}
&(-\triangle + I)\Big(\eta\big[(\mathcal{K}_\Omega d\psi)_K - (d\mathcal{K}_\Omega\psi)_K\big]\Big)\\
&= L_1(\mathcal{K}_\Omega d\psi)_K + \eta\Big[\triangle(d\mathcal{K}_\Omega\psi)_K - (d\mathcal{K}_\Omega\psi)_K\Big] + L_1'(d\mathcal{K}_\Omega\psi)\\
&= L_1(\mathcal{K}_\Omega d\psi)_K + L_1'(d\mathcal{K}_\Omega\psi) + L_2(\mathcal{K}_\Omega\psi) + \psi d\Big[(\triangle\mathcal{K}_\Omega\psi)_K - (\mathcal{K}_\Omega\psi)_K\Big]\\
&= L_1(\mathcal{K}_\Omega d\psi)_K + L_1'(d\mathcal{K}_\Omega\psi)_K + L_2'(\mathcal{K}_\Omega\psi)\\
&= L_1(\mathcal{K}_\Omega d\psi)_K + L_2(\mathcal{K}_\Omega\psi).
\end{aligned}$$

Next we analyze the boundary equation. We have

$$\frac{\partial}{\partial n}\Big[\eta(\mathcal{K}_\Omega d\psi)_K - \eta(d\mathcal{K}_\Omega\psi)_K\Big] = \eta\sum_{k=0}^{N} Y_k^*\Big[(\nabla_{Y_k}d\psi)\lfloor\vec{n}\Big]_K - \eta\frac{\partial}{\partial n}(d\mathcal{K}_\Omega\psi)_K\,.$$

Now we want to commute the normal derivative and the operator d in the far right-most term in the equation above. We have

$$\begin{aligned}
\frac{\partial}{\partial n}(d\mathcal{K}_\Omega\psi)_K &= \frac{\partial}{\partial n}\Big(\sum_{\substack{i=1,\dots,N\\ I}} X_i(\mathcal{K}_\Omega\psi)_I\varepsilon^K_{iI}\Big) + \frac{\partial}{\partial n}L_0(\mathcal{K}_\Omega\psi)\\
&= \sum_{\substack{i=1,\dots,N\\ I}} X_i\Big(\frac{\partial}{\partial n}(\mathcal{K}_\Omega\psi)_I\Big)\varepsilon^K_{iI} + L_1'(\mathcal{K}_\Omega\psi) + \frac{\partial}{\partial n}L_0(\mathcal{K}_\Omega\psi)\\
&= \sum_{\substack{i=1,\dots,N\\ I}} X_i\left(\sum_{k=0}^N Y_k^*\Big[(\nabla_{Y_k}\psi)\lfloor\vec{n}\Big]_I\right)\varepsilon^K_{iI} + L_1'(\mathcal{K}_\Omega\psi) + \frac{\partial}{\partial n}L_0(\mathcal{K}_\Omega\psi)\\
&= \sum_{\substack{i=1,\dots,N\\ I}} X_i\left(\sum_{k=0}^N Y_k^* Y_k\psi_{0I}\right)\varepsilon^K_{iI} + T_2\psi + L_1'(\mathcal{K}_\Omega\psi) + \frac{\partial}{\partial n}L_0(\mathcal{K}_\Omega\psi)\\
&= \sum_{k=0}^N Y_k^* Y_k\left(\sum_{\substack{i=1,\dots,N\\ I}} X_i\psi_{0I}\varepsilon^K_{iI}\right) + T_2'\psi + L_1'(\mathcal{K}_\Omega\psi)\\
&\qquad\qquad + L_0\frac{\partial}{\partial n}(\mathcal{K}_\Omega\psi) + L_0'(\mathcal{K}_\Omega\psi)\\
&= \sum_{k=0}^N Y_k^*\Big[(\nabla_{Y_k}d\psi)\lfloor\vec{n}\Big]_K + T_2\psi + L_0(\mathcal{K}_\Omega\psi),
\end{aligned}$$

where we have used the fact that $(\partial/\partial n)\mathcal{K}_\Omega\psi$ on $b\Omega$ equals a second order tangential operator in ψ. Hence,

$$\frac{\partial}{\partial n}\Big[\eta(\mathcal{K}_\Omega d\psi)_K - \eta(d\mathcal{K}_\Omega\psi)_K\Big] = T_2\psi + L_0(\mathcal{K}_\Omega\psi).$$

Finally we prove the estimate. Using [TRI] Theorem 4.2.4 and Corollary 7.3, we have

$$\begin{aligned}
\|w_1\|_s &\le c\Big(\|L_1\mathcal{K}_\Omega d\psi\|_{F^{s-2}} + \|L_2\mathcal{K}_\Omega\psi\|_{F^{s-2}} + \|T_2\psi\|_{W^{s-3/2}(\Omega)} + \|T_1\mathcal{K}_\Omega\psi\|_{W^{s-3/2}(\Omega)}\Big)\\
&\le c\Big(\|\mathcal{K}_\Omega d\psi\|_{F^{s-1}} + \|\mathcal{K}_\Omega\psi\|_s + \|\psi\|_{s+1} + \|\mathcal{K}_\Omega\psi\|_s\Big)\\
&\le c\|\psi\|_{s+1}. \quad \square
\end{aligned}$$

Lemma 11.8. *Let $\psi \in \bigwedge^q$. Then for $I \not\ni 0$, the function $[\eta G_\Omega\psi]_I\tilde{}$ is a solution of the following boundary value problem:*

$$\begin{cases}(-\triangle + \epsilon\mathcal{L}_2 + I)w = \Big[\eta_1 L_1(G_\Omega\psi)\Big]_I^{\sim} & \text{on } \mathbb{R}^{N+1}_+\\ \dfrac{\partial w}{\partial x_0} = (-\triangle_T + I)\dfrac{\partial\widetilde{\psi_{0I}}}{\partial x_0} + \zeta & \text{on } \mathbb{R}^N\end{cases}$$

Here ζ is a function satisfying the estimate

$$\|\zeta\|_{W^{s-3/2}(b\Omega)} \le c\|\eta_1\psi\|_{s+1},$$

and L_1 and η_1 are as in the previous lemma.

Proof. As in the proof of Lemma 11.6 we need only check that the boundary equation is satisfied, and that the desired estimate holds.

Notice that, by Lemma 11.4, on the set $\{x_0 = 0\}$, for $I \not\ni 0$, we have

$$\frac{\partial}{\partial x_0}\Big[\eta \mathcal{K}_\Omega d\psi\Big]_I^{\sim} = -\sum_{K\not\ni 0} \tilde{\eta}\tilde{\gamma}_{IK}[\mathcal{K}_\Omega d\psi]_K{}^{\sim} + (E_2\widetilde{d\psi})_I\,,$$

where E_2 is defined as in Lemma 11.4.

On the other hand, if $I \not\ni 0$, on $\mathbb{R}^N$ we have that

$$\begin{aligned}\frac{\partial}{\partial x_0}\Big[\eta d\mathcal{K}_\Omega \psi\Big]_I^{\sim} &= \tilde{\eta}\sum_{\substack{J\not\ni 0\\ i=1,\dots,N}} \frac{\partial}{\partial x_0} X_i(\widetilde{\mathcal{K}_\Omega\psi})_I + \frac{\partial}{\partial x_0}\Big[\sum_J \Big(\eta\mathcal{K}_\Omega\psi\Big)_J^{\sim} d\omega^J\Big]\\ &= -\sum_{\substack{K\not\ni 0\\ i=0,\dots,N}} \tilde{\eta}a_{IK}X_i\Big(\widetilde{\mathcal{K}_\Omega\psi}\Big)_K + [L_0(\widetilde{\mathcal{K}_\Omega\psi})]_I + (T_2\widetilde{\eta\psi})_I\,,\end{aligned}$$

where T_2 is a second order tangential differential operator. Here we have used the fact that $(\partial/\partial x_0)\mathcal{K}_\Omega\psi$ on the set $\{x_0 = 0\}$ equals a second order tangential operator on ψ. Therefore, for $I \not\ni 0$,

$$\begin{aligned}\frac{\partial w}{\partial x_0}\bigg|_{\mathbb{R}^N} &= (-\triangle_T + I)\frac{\partial\tilde{\psi}_{0I}}{\partial x_0}\\ &\qquad -\sum_{K\not\ni 0}\tilde{\eta}\gamma_{IK}\Big[\mathcal{K}_\Omega d\psi\Big]_K^{\sim} - \sum_{\substack{i=1,\dots,N\\ K\not\ni 0}} \eta a_{IK}X_i\Big(\widetilde{\mathcal{K}_\Omega\psi}\Big)_K + [L_0(\widetilde{\mathcal{K}_\Omega\psi})]_I + (T_2\widetilde{\eta\psi})_I\\ &\equiv (-\triangle_T + I)\frac{\partial\tilde{\psi}_{0I}}{\partial x_0} + \zeta.\end{aligned}$$

Hence we need only check the estimate.

Let w_1 be given by Lemma 11.7. Then

$$\zeta = -\sum_{K\not\ni 0}\gamma_{IK}\tilde{\eta}\Big[d\mathcal{K}_\Omega\psi\Big]_K^{\sim} + w_1 - \sum_{\substack{i=1,\dots,N\\ K\not\ni 0}} \eta a_{IK}X_i\Big(\widetilde{\mathcal{K}_\Omega\psi}\Big)_K + [L_0(\widetilde{\mathcal{K}_\Omega\psi})]_I + (T_2\widetilde{\eta\psi})_I.$$

Using the above equality, Lemma 11.5, Lemma 11.7, and Corollary 7.3 we see that

$$\begin{aligned}\|\zeta\|_{W^{s-3/2}(b\Omega)} &\le c\bigg(\sum_{K\not\ni 0}\|\tilde{\eta}_1\Big(d\mathcal{K}_\Omega\psi\Big)_K^{\sim}\|_{W^{s-3/2}(b\Omega)} + \|w_1\|_{W^{s-3/2}(b\Omega)}\\ &\qquad\qquad + \|\eta_1\mathcal{K}_\Omega\psi\|_{W^{s-1/2}(b\Omega)} + \|\widetilde{\eta\psi}\|_{W^{s+1/2}(b\Omega)}\bigg)\\ &\le c\Big(\|\tilde{\eta}_1\widetilde{\mathcal{K}_\Omega\psi}\|_{W^{s-1/2}(b\Omega)} + \|w_1\|_{W^{s-1/2}(b\Omega)} + \|\eta_1\psi\|_{s+1}\Big)\\ &\le c\|\eta_1\psi\|_{s+1}.\end{aligned}$$

This proves the estimates and concludes the proof. $\square$

End of the Proof of 11.3. We are finally able to compare $\left[\eta G_\Omega\psi\right]_I^\sim$ and $\left[G_{\mathbb{R}^{N+1}_+}(\widetilde{\eta\psi})\right]_I$. We begin with the case $I \ni 0$. By (11.7) and Lemma 11.8 it is easy to see that the function w solves the boundary value problem

$$\begin{cases}(-\triangle+I)w = (-\epsilon\mathcal{L}_2)\left[\eta G_\Omega\psi\right]_I^\sim + \eta_1 L_1\left[G_\Omega\psi\right]^\sim & \text{on } \mathbb{R}^{N+1}_+\\ w = (-\epsilon\mathcal{T}_2)(\widetilde{\eta\psi})_I + \eta_1(T_1\psi)_I + \zeta & \text{on } \mathbb{R}^N\end{cases}$$

where ζ is as in Lemma 11.8. Therefore, using Theorem 4.2.4 in [TRI], Lemma 11.8, and Corollary 7.3 we see that

$$\begin{aligned}\|w\|_s &\le c\Big(\epsilon\|\mathcal{L}_2(\eta G_\Omega\psi)_I\|_{F^{s-2}} + \|\eta_1 L_1(G_\Omega\psi)\|_{F^{s-2}} + \|\epsilon\mathcal{T}_2(\widetilde{\eta\psi}) + \eta T_1\psi\|_{W^{s-1/2}(b\Omega)}\\ &\qquad + \|\zeta\|_{W^{s-1/2}(b\Omega)}\Big)\\ &\le c\Big(\epsilon\|\eta G_\Omega\psi\|_s + \|\eta G_\Omega\psi\|_{F^{s-1}} + \|G_{\mathbb{R}^{N+1}_+}(\eta\psi)\|_{F^{s-1}} + \epsilon\|\eta\psi\|_{s+2} + \|\eta_1\psi\|_{s+1}\Big)\\ &\le c\Big(\epsilon\|\eta\psi\|_{s+2} + \|\eta\psi\|_{s+1}\Big),\end{aligned}$$

where we also use the estimate

$$\|\eta G_\Omega\psi - G_\Omega(\eta\psi)\|_s \le c\|\eta_1\psi\|_{s+1}.$$

This follows by writing $G_\Omega = \mathcal{K}_\Omega d + \mathcal{K}_\Omega d$, and noticing that the commutator between $\mathcal{K}_\Omega$ and the multiplication by η is a 0 order operator, since $\eta\mathcal{K}_\Omega\theta - \mathcal{K}_\Omega(\eta\theta)$ solves the boundary value problem

$$\begin{cases}(\triangle - I)w = L_1(\mathcal{K}_\Omega\theta) & \text{on } \Omega\\ \dfrac{\partial w}{\partial n} = T_1\theta & \text{on } b\Omega\end{cases}$$

This is a problem to which we apply Theorem 4.2.4 in [TRI] again.

Now let $I \not\ni 0$. Then the function w solves the boundary value problem

$$\begin{cases}(-\triangle+I)w = (-\epsilon\mathcal{L}_2)\left[\eta G_\Omega\psi\right]_I^\sim + \eta_1 L_1\left[G_\Omega\psi\right]^\sim & \text{on } \mathbb{R}^{N+1}_+\\ \dfrac{\partial w}{\partial x_0} = (\epsilon\mathcal{T}_2)\dfrac{\partial(\widetilde{\eta\psi})_{0I}}{\partial x_0} + \zeta & \text{on } \mathbb{R}^N\end{cases}$$

where ζ is given by Lemma 11.8. Therefore, by Corollary 7.3,

$$\begin{aligned}\|w\|_s &\le \epsilon\|\eta\psi\|_{s+2} + c\left(\|\eta_1 G_\Omega\psi\|_{F^{s-1}(\Omega)} + \|\zeta\|_{W^{s-3/2}(b\Omega)}\right)\\ &\le \epsilon\|\eta\psi\|_{s+2} + c\|\eta_1\psi\|_{s+1}. \quad \square\end{aligned}$$

12. The Decomposition Theorem and Conclusions

Proof of the Main Result. We are now in a position to finish the proof of Theorem 2.6.

Let X be the subspace of $W^{s+2}_q(\Omega)$, $(s > 1/2)$ consisting of the forms ϕ satisfying the boundary conditions ϕ, $d\phi \in \operatorname{dom} d^*$. Observe that X is a closed subspace of

$W_q^{s+2}(\Omega)$, (this is the reason why $s > 1/2$). By the Regularity Theorem 11.1 we know that, on X, the norm

$$\|\psi\|_{s+1} + \|(-\triangle + G_\Omega)\psi\|_s$$

is equivalent to the norm $\|\psi\|_{s+2}$. Let $\mathcal{P}$ be the orthogonal projection of X onto the kernel of $-\triangle + G_\Omega$. Then, by standard functional analysis arguments, (see [ZIE] p.178),

$$\|\psi - \mathcal{P}\psi\|_{s+1} \leq C\| - \triangle\psi + G_\Omega\psi\|_s.$$

Therefore, if ψ is orthogonal to the kernel of $-\triangle + G_\Omega$, the regularity theorem Theorem 11.1 says that

$$\|\psi\|_{s+2} \leq C_s\| - \triangle\psi + G_\Omega\psi\|_s.$$

We now complete the proof of the existence theorem—Theorem 9.2. For $\alpha \in W_q^1 \cap \mathcal{H}_q^\perp$ fixed, we approximate α by $\alpha_m \in Z \equiv (F - I)(\mathcal{D})$. For such α_m's we can find $\phi_m \in \mathcal{D}$ orthogonal to $\mathcal{H}_q$, solutions of the boundary value problem, and such that, for $\alpha \geq 1$,

$$\|\phi\|_{s+2} \leq c_s\|\alpha\|_s\,.$$

Thus the ϕ_m's converge in W_q^{s+2} to a certain ϕ. Clearly ϕ solves the boundary value problem with data α, since the boundary conditions are preserved in the W^{s+2} topology for $s > 1/2$.

Hence, if $\alpha \in W^s(\Omega)$, $(s \geq 1)$ and α is orthogonal to $\ker(-\triangle + G_\Omega)$ in the W^1 inner product, then there exists a unique solution to the problem (2.2)—$\phi \in W^{s+2}(\Omega)$ orthogonal to $\ker(-\triangle + G_\Omega)$ and such that

$$\|\phi\|_{s+2} \leq C_s\|\alpha\|_s.$$

If $\alpha \in W^s(\Omega)$ with $1/2 < s < 1$, then we still have existence and regularity by using a density argument, since $\alpha \in W_q^s(\Omega)$ can be approximated by $\alpha_m \in W_q^1(\Omega)$ in the W^s-topology. This concludes the proof of Theorem 2.6. □

Decomposition of W_q^1. The aim of this part is to prove the following result:

THEOREM 12.1. *Let Ω be a smoothly bounded domain in $\mathbb{R}^{N+1}$. Let $W_q^1(\Omega)$ denote the 1-Sobolev space of q-forms. Then we have the strong orthogonal decomposition*

$$W_q^1 = dd^*(W_q^1) \bigoplus d^*d(W_q^1) \bigoplus \mathcal{H}_q\,,$$

where $\mathcal{H}_q$ is a finite dimensional subspace, and d^* denotes the W_q^1-Hilbert space adjoint of d.

Proof. All of this is standard. If α is orthogonal to $\mathcal{H}_q$ then α belongs to $(dd^* + d^*d)W_q^1$. The fact that $d(W_q^1)$ and $d^*(W_q^1)$ are orthogonal subspaces is also clear since

$$\langle d\phi, d^*\psi\rangle_1 = \langle d^2\phi, \psi\rangle_1 = 0,$$

for all $\phi, \psi \in C^\infty(\overline{\Omega})$. □

In the case $q = 0$ we are able to determine the harmonic space $\mathcal{H}_0$. The rest of the section is devoted to this end.

THEOREM 12.2. *We have that* $\mathcal{H}_0 = \{\text{constants}\}$. *More precisely, the boundary value problem*

$$\begin{cases} d^* du = f & \text{on } \Omega \\ du \in \operatorname{dom} d^* \end{cases} \tag{12.1}$$

has a unique solution u orthogonal (in the $W^1(\Omega)$ inner product) to the constant functions, for each $f \in W^1(\Omega)$ which is also orthogonal to the constants, i.e.

$$\int_\Omega f = 0.$$

This theorem is a consequence of the next theorem. By equation (7.2) we can rewrite the boundary value problem (12.1) as

$$\begin{cases} -\triangle u + G_\Omega u = f & \text{on } \Omega \\ \displaystyle\sum_{j=0}^N n_j \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial x_j}\right) = 0 & \text{on } b\Omega \end{cases}, \tag{12.2}$$

where $\vec{n} = (n_0, \ldots, n_N)$ is the normal vector field. By our choice of $\vec{n}$, and by setting $G_\Omega u = v$ we obtain the new system

$$\begin{cases} -\triangle u + v = f & \text{on } \Omega \\ \dfrac{\partial^2 u}{\partial n^2} = 0 & \text{on } b\Omega \\ \triangle v - v = 0 & \text{on } \Omega \\ \dfrac{\partial v}{\partial n} = \dfrac{\partial u}{\partial n} + \displaystyle\sum_{k=0}^N Y_k^*\Big[(\nabla_{Y_k} du)\lfloor\vec{n}\Big] & \text{on } b\Omega \end{cases}$$

Now assume f to be suffiently regular. By solving the first equation for v and substituting we find that (12.2) finally becomes

$$\begin{cases} \triangle^2 u - \triangle u = f - \triangle f & \text{on } \Omega \\ \dfrac{\partial^2 u}{\partial n^2} = 0 & \text{on } b\Omega \\ \dfrac{\partial}{\partial n} \triangle u - \displaystyle\sum_{k=0}^N Y_k^*\Big[(\nabla_{Y_k} du)\lfloor\vec{n}\Big] - \dfrac{\partial u}{\partial n} = -\dfrac{\partial f}{\partial n} & \text{on } b\Omega \end{cases} \tag{12.3}$$

This last is the problem that we are going to study.

THEOREM 12.3. *The boundary value problem (12.3) is elliptic. It has index 0. The kernel of the system is given by $\{u = \text{constant}\}$ and a (unique) solution exists* *for each piece of data f that satisfies*

$$\int_\Omega f = 0 \ , \qquad\qquad f \in W^r, \ r \geq 2.$$

REMARK 12.4. The condition $f \in W^r$, $r \geq 2$ is necessary in order to guarantee that $\triangle f \in L^2(\Omega)$. In this case the solution u satisfies the estimates

$$\|u\|_{W^{s+4}(\Omega)} \leq C\bigg(\|f - \triangle f\|_{W^s(\Omega)} + \left\|\frac{\partial f}{\partial n}\right\|_{W^{1/2+s}(b\Omega)} + \|u\|_s\bigg),$$

that is, for $s \geq 0$,

$$\|u\|_{W^{s+4}(\Omega)} \leq C\Big(\|f\|_{s+2} + \|u\|_s\Big).$$

This result, when applied to problem (12.1), is not optimal. This is the reason why we had to go through the more complicated computations in the previous sections.

Proof. In order to prove that the boundary value problem is elliptic it suffices to verify the Lopatinski condition as in [HOR1]. This is standard, and since we do not use this fact in what follows, we leave the details to the reader. Thus the boundary value problem (12.3) admits solutions with appropriate estimates modulo a finite dimensional kernel and a finite dimensional cokernel. Our next job is to explicitly determine this kernel and cokernel.

Determination of the Kernel: Suppose that the function u, defined on Ω, satisfies

$$\begin{cases} \triangle(\triangle u) - \triangle u = 0 & \text{on } \Omega \\ \dfrac{\partial^2 u}{\partial n^2} = 0 & \text{on } b\Omega \\ \dfrac{\partial}{\partial n} \triangle u - \displaystyle\sum_{k=0}^{N} Y_k^*\Big[(\nabla_{Y_k} du)\lfloor \vec{n}\Big] - \dfrac{\partial u}{\partial n} = 0 & \text{on } b\Omega \end{cases}$$

We intend to show that u must therefore be constant. This will then imply that the kernel is the one dimensional space of constant functions. Without loss of generality we suppose that all functions are *real valued*.

Using Green's theorem we see that

$$\begin{aligned} 0 &= \int_\Omega u \triangle \Big[\triangle u - u\Big]\, dV \\ &= \int_\Omega |\triangle u|^2 + \int_{b\Omega} u \frac{\partial}{\partial n}(\triangle u) - \int_{b\Omega} \triangle u \frac{\partial u}{\partial n} + \int_\Omega |\mathrm{grad} u|^2 - \int_{b\Omega} u \frac{\partial u}{\partial n}. \end{aligned}$$

Thus we find that

$$\begin{aligned}
&\int_\Omega |\triangle u|^2 + |\mathrm{grad} u|^2 \\
&\quad= \int_{b\Omega} \frac{\partial u}{\partial n}\Big(\triangle u + u\Big) - u\frac{\partial}{\partial n}(\triangle u) \\
&\overset{(2^{\text{nd}} \text{ bdry. cond.})}{=} \int_{b\Omega} \frac{\partial u}{\partial n}\Big(\triangle u + u\Big) - \int_{b\Omega} u\left[\sum_{k=0}^N Y_k^*\Big[(\nabla_{Y_k} du)\lfloor\vec{n}\Big] + \frac{\partial u}{\partial n}\right] \\
&\quad= \int_{b\Omega} \frac{\partial u}{\partial n} \triangle u - \int_{b\Omega} u \sum_{k=0}^N Y_k^*\Big[(\nabla_{Y_k} du)\lfloor\vec{n}\Big]. \qquad (12.4)
\end{aligned}$$

Now we consider the second integral on the right hand side of equation (12.4). We have

$$\begin{aligned}
\int_{b\Omega} \sum_{k=0}^N Y_k^*\Big[(\nabla_{Y_k} du)\lfloor\vec{n}\Big] u &= \sum_k \int_{b\Omega} Y_k u\Big[(\nabla_{Y_k} du\lfloor)\vec{n}\Big] \\
&\overset{(1^{\text{st}} \text{ bdry. cond.})}{=} \sum_k \int_{b\Omega} Y_k u\Big[\frac{\partial(Y_k u)}{\partial n}\Big] \\
&= -\frac{1}{2}\int_{b\Omega} \frac{\partial}{\partial n}|\mathrm{grad} u|^2,
\end{aligned}$$

where we use the first boundary condition, and the simple facts that $(\nabla_{Y_k} du) \lfloor\vec{n} = (\partial/\partial n)Y_k u$, and that $|\mathrm{grad} u|^2 = \sum_k |Y_k u|^2 + |(\partial/\partial n)u|^2$.

Substituting this last equality into (12.4), and using Green's theorem again, we find that

$$\begin{aligned}
\int_\Omega |\triangle u|^2 + |\mathrm{grad} u|^2 &= \int_{b\Omega} \triangle u\frac{\partial u}{\partial n} - \frac{1}{2}\frac{\partial}{\partial n}|\mathrm{grad} u|^2 \\
&= \int_\Omega |\triangle u|^2 + \mathrm{grad} u \cdot \mathrm{grad}(\triangle u) - \frac{1}{2}\triangle(|\mathrm{grad} u|^2).
\end{aligned}$$

Notice that

$$\begin{aligned}
\triangle|\mathrm{grad} u|^2 &= 2\sum_j \left(\triangle\frac{\partial u}{\partial x_j}\right)\frac{\partial u}{\partial x_j} + 2\mathrm{grad}\frac{\partial u}{\partial x_j}\cdot\mathrm{grad}\frac{\partial u}{\partial x_j} \\
&= 2\mathrm{grad}(\triangle u)\cdot\mathrm{grad} u + 2\sum_j \mathrm{grad}\frac{\partial u}{\partial x_j}\cdot\mathrm{grad}\frac{\partial u}{\partial x_j}.
\end{aligned}$$

As a result,

$$\int_\Omega |\triangle u|^2 + |\mathrm{grad} u|^2 = \int_\Omega |\triangle u|^2 - \sum_j \mathrm{grad}\frac{\partial u}{\partial x_j}\cdot\mathrm{grad}\frac{\partial u}{\partial x_j}.$$

That is,

$$\int_\Omega |\mathrm{grad} u|^2 + \sum_j \left|\mathrm{grad}\frac{\partial u}{\partial x_j}\right|^2 = 0.$$

This equality implies that u is a constant.

Determination of the Cokernel: We want to show that if f is such that the system (12.3) admits a solution then

$$\int_\Omega f = 0.$$

Notice that this would imply that Cokernel $\supseteq \Big(\ker(f \mapsto \int_\Omega f)\Big)^\perp$.

We have

$$\begin{aligned}
\int_\Omega \triangle(\triangle u) - \triangle u &= \int_{b\Omega} \frac{\partial}{\partial n}(\triangle u - u) \\
&\overset{(2^{\text{nd}}\text{ bdry. cond.})}{=} \int_{b\Omega} \sum_k Y_k^* \Big[(\nabla_{Y_k} du \lfloor)\vec{n}\Big] - \frac{\partial f}{\partial n} \\
&\overset{(\text{Green on } b\Omega)}{=} -\int_{b\Omega} \frac{\partial f}{\partial n} \\
&= -\int_\Omega \triangle f.
\end{aligned}$$

Now the left hand side in these last equations equals $\int_\Omega (f - \triangle f)$. Therefore

$$\int_\Omega f = 0.$$

Now we examine the dimension of the cokernel. Suppose that

$$(g; v_1, v_2) \in C^\infty(\overline{\Omega}) \times C^\infty(b\Omega) \times C^\infty(b\Omega)$$

and

$$(g; v_1, v_2) \perp \text{Range of}\,(12.3),$$

i.e.

$$\begin{aligned}
\int_\Omega \Big(\triangle^2 u - \triangle u\Big) g + \int_{b\Omega} \left[\sum_{j=0}^N n_j \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial x_j}\right)\right] v_1 \\
+ \int_{b\Omega} \left[\frac{\partial}{\partial n} \triangle u + \sum_{j=1}^N \mathrm{div}^T \left(n_j \nabla^T \frac{\partial u}{\partial n}\right) - \frac{\partial u}{\partial n}\right] v_2 = 0 \quad (12.5)
\end{aligned}$$

for all $u \in C^\infty$. Then we want to show that

$$(g; v_1, v_2) \text{ lies in some one dimensional subspace.}$$

This will finish the proof.

Thus suppose that (12.5) holds for all $u \in C^\infty(\overline{\Omega})$. By taking $u \in C_0^\infty(\Omega)$ and integrating by parts we obtain

$$0 = \int_\Omega g(\triangle^2 u - \triangle u) = \int_\Omega u\Big(\triangle^2 g - \triangle g\Big).$$

This equality implies that $\triangle^2 g - \triangle g = 0$ on Ω.

Next, assume (12.5) and apply Green's theorem to the integral on Ω. It equals

$$\begin{aligned}
&\int_\Omega \triangle g(\triangle - I)u + \int_{b\Omega} g\frac{\partial}{\partial n}(\triangle - I)u - \int_{b\Omega} \frac{\partial g}{\partial n}(\triangle - I)u \\
&= \int_\Omega u(\triangle^2 g - \triangle g) + \int_{b\Omega} \frac{\partial u}{\partial n} \triangle g - \int_{b\Omega} u\frac{\partial}{\partial n} \triangle g \\
&\qquad + \int_{b\Omega} g\frac{\partial}{\partial n} \triangle u - \int_{b\Omega} g\frac{\partial u}{\partial n} - \int_{b\Omega} \triangle u \frac{\partial g}{\partial n} + \int_{b\Omega} u\frac{\partial g}{\partial n}.
\end{aligned}$$

Substituting this last identity into (12.5), and recalling that $\triangle^2 g - \triangle g = 0$ on Ω we find that

$$\begin{aligned}
0 &= \int_{b\Omega} \frac{\partial u}{\partial n} \triangle g - \int_{b\Omega} u\frac{\partial}{\partial n} \triangle g + \int_{b\Omega} g\frac{\partial}{\partial n} \triangle u - \int_{b\Omega} g\frac{\partial u}{\partial n} \\
&\quad - \int_{b\Omega} \triangle u\frac{\partial g}{\partial n} + \int_{b\Omega} u\frac{\partial g}{\partial n} + \int_{b\Omega} \frac{\partial^2 u}{\partial n^2} v_1 \\
&\quad + \int_{b\Omega} v_2\frac{\partial}{\partial n} \triangle u - \sum_k \int_{b\Omega} Y_k^*\Big[(\nabla_{Y_k} du)\lfloor\vec{n}\Big]v_2 - \int_{b\Omega} v_2\frac{\partial u}{\partial n} \\
&= \int_{b\Omega} u\left[\frac{\partial g}{\partial n} - \frac{\partial}{\partial n} \triangle g\right] + \int_{b\Omega} \frac{\partial u}{\partial n}\Big[\triangle g - g - v_2\Big] \\
&\quad + \int_{b\Omega} \frac{\partial^2 u}{\partial n^2} v_1 - \sum_k \int_{b\Omega} Y_k^*\Big[(\nabla_{Y_k} du)\lfloor\vec{n}\Big]v_2 - \int_{b\Omega} \triangle u\frac{\partial g}{\partial n} \\
&\quad + \int_{b\Omega} (g + v_2)\frac{\partial}{\partial n} \triangle u. \qquad (12.6)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
0 = \int_{b\Omega} u\left[\frac{\partial g}{\partial n} - \frac{\partial}{\partial n} \triangle g\right] + \int_{b\Omega} \frac{\partial u}{\partial n}\Big[\triangle g - g - v_2\Big] + \int_{b\Omega} \frac{\partial^2 u}{\partial n^2} v_1 \\
- \sum_k \int_{b\Omega} \Big[\frac{\partial}{\partial n} Y_k u\Big] Y_k v_2 - \int_{b\Omega} \triangle u\frac{\partial g}{\partial n} + \int_{b\Omega} \frac{\partial}{\partial n} \triangle u(g + v_2). \quad (12.7)
\end{aligned}$$

As in the half space case, we write x' to indicate the variable in $b\Omega$. Given any function v on $b\Omega$, we let

$$\tilde{v}(x) = v(\pi(x)) \equiv v(x'),$$

where π is the normal projection defined in a neighborhood of $b\Omega$. The functions $\tilde{v}$ are defined in a (fixed) neighborhood of $b\Omega$. Then we can extend them to all of $\mathbb{R}_+^{N+1}$ by multiplying by a (fixed) cut-off function.

Now we proceed in several steps to calculate the cokernel:

Step 1: Take $u(x)$ to be of the form

$$u(x) = \rho(x)^3 \tilde{u}_1(x)\,,$$

where $\tilde{u}_1 \in C^\infty(b\Omega)$ is generic. Such a choice yields that

$$g + v_2 = 0 \qquad\qquad \text{on } b\Omega.$$

Proof of Step 1: For such a choice of u, equation (12.7) becomes

$$0 = \int_{b\Omega} (g + v_2) \frac{\partial}{\partial n} \triangle \left(\rho^3 \tilde{u}_1\right)$$

Now,

$$\begin{aligned} \frac{\partial}{\partial n} \triangle \left[\rho^3 \tilde{u}_1\right]\Big|_{b\Omega} &= \frac{\partial}{\partial n}\Big[\tilde{u}_1 \triangle \rho^3 + 2\Big(\mathrm{grad}\rho^3 \cdot \mathrm{grad}\tilde{u}_1\Big) + \rho^3 \triangle \tilde{u}_1\Big]\Big|_{b\Omega} \\ &= \left(\frac{\partial}{\partial n} \triangle \rho^3\right) \tilde{u}_1\Big|_{b\Omega} \\ &\equiv 6u_1. \end{aligned}$$

[It is a standard fact that we may assume that $(\partial\rho/\partial n) = |\mathrm{grad}\,\rho| = 1$ on $b\Omega$.] This implies that $g + v_2 = 0$ on $b\Omega$.

Step 2: Take $u = \rho^2(x)\tilde{u}_1(x)$ with $u_1 \in C^\infty(b\Omega)$. This implies that

$$v_1 = \frac{\partial g}{\partial n} \qquad\qquad \text{on } b\Omega.$$

Proof of Step 2: Using Step 1, we see that this particular choice of u gives

$$\begin{aligned} \triangle u\Big|_{b\Omega} &= \triangle\Big(\rho^2 \tilde{u}_1\Big)\Big|_{b\Omega} \\ &= \Big(\tilde{u}_1 \triangle \rho^2 + 2\mathrm{grad}\rho^2 \cdot \mathrm{grad}\tilde{u}_1 + \rho^2 \triangle \tilde{u}_1\Big)\Big|_{b\Omega} \\ &= 2|\mathrm{grad}\rho|^2 \tilde{u}_1\Big|_{b\Omega} \\ &= 2u_1, \end{aligned}$$

while $(\partial^2\rho^2/\partial n) = 2$ on $b\Omega$. Therefore,

$$v_1 = \frac{\partial g}{\partial n} \qquad\qquad \text{on } b\Omega.$$

Hence, putting Steps 1 and 2 together, equation (12.7) becomes

$$0 = \int_{b\Omega} u\left[\frac{\partial g}{\partial n} - \frac{\partial}{\partial n}\triangle g\right] + \int_{b\Omega}\frac{\partial u}{\partial n}\triangle g + \int_{b\Omega}\frac{\partial^2 u}{\partial n^2}\frac{\partial g}{\partial n} + \sum_k \int_{b\Omega}\big[\frac{\partial}{\partial n}Y_k u\big]Y_k\frac{\partial g}{\partial n} - \int_{b\Omega}\triangle u\frac{\partial g}{\partial n} \quad (12.8)$$

for all $u \in C^\infty(\overline{\Omega})$.

Step 3: We have that

$$\int_{b\Omega}\frac{\partial u}{\partial n}\triangle g + \int_{b\Omega}\frac{\partial^2 u}{\partial n^2}\frac{\partial g}{\partial n}\sum_k + \int_{b\Omega}\big[\frac{\partial}{\partial n}Y_k u\big]Y_k\frac{\partial g}{\partial n} - \int_{b\Omega}\triangle u\frac{\partial g}{\partial n} = \int_{b\Omega} u\left[Y_k^*\frac{\partial}{\partial n}D_k g\right] + \int_{b\Omega}\frac{\partial u}{\partial n}\frac{\partial^2 g}{\partial n} \quad (12.9)$$

Proof of Step 3: Recall that for $k = 0,\dots,N$, $Y_k \equiv D_k - n_k(\partial/\partial n)$, and that $\vec{n} = \sum_j n_j D_j$. Therefore, it is easy to see that

$$\frac{\partial^2 u}{\partial n^2}\frac{\partial g}{\partial n} + \sum_k (Y_k g)\Big(\frac{\partial}{\partial n}Y_k u\Big) = \sum_k D_k g\frac{\partial}{\partial n}(D_k u).$$

Thus a repeated application of Green's theorem gives that

$$\begin{aligned}
&\int_{b\Omega}\frac{\partial u}{\partial n}\triangle g + \int_{b\Omega}\frac{\partial^2 u}{\partial n^2}\frac{\partial g}{\partial n} + \sum_k \int_{b\Omega}\big[\frac{\partial}{\partial n}Y_k u\big]Y_k\frac{\partial g}{\partial n} - \int_{b\Omega}\triangle u\frac{\partial g}{\partial n}\\
&= \int_{b\Omega}\frac{\partial u}{\partial n}\triangle g + \sum_k \int_{b\Omega}\big[\frac{\partial}{\partial n}Y_k u\big]Y_k\frac{\partial g}{\partial n} - \int_{b\Omega}\triangle u\frac{\partial g}{\partial n}\\
&= \int_{b\Omega}\frac{\partial u}{\partial n}\triangle g + \sum_k\left[\int_{\Omega}\mathrm{grad} D_k u\cdot \mathrm{grad} D_k u\right] - \int_{\Omega}\triangle u\triangle g\\
&= \int_{b\Omega}\frac{\partial u}{\partial n}\triangle g + \sum_k \int_{b\Omega}\Big(\frac{\partial u}{\partial n}\frac{\partial D_k g}{\partial n} - \int_{b\Omega}\frac{\partial u}{\partial n}\triangle g\\
&= \sum_k \int_{b\Omega} u\Big(Y_k^*\frac{\partial}{\partial n}D_k g\Big) + \sum_k \int_{b\Omega}\frac{\partial u}{\partial n}\Big(n_k\frac{\partial}{\partial n}D_k g\Big)\\
&= \sum_k \int_{b\Omega} u\Big(Y_k^*\frac{\partial}{\partial n}D_k g\Big) + \int_{b\Omega}\frac{\partial u}{\partial n}\frac{\partial^2 g}{\partial n^2}.
\end{aligned}$$

This proves Claim 3.

Now because of Step 3, equality (12.8) becomes

$$\int_{b\Omega} u\left[\frac{\partial g}{\partial n} - \frac{\partial}{\partial n}\triangle g + \sum_k\Big(Y_k^*\frac{\partial}{\partial n}D_k g\Big)\right] + \int_{b\Omega}\frac{\partial u}{\partial n}\frac{\partial^2 g}{\partial n}. \quad (12.10)$$

Thus g must satisfy the system

$$\begin{cases} \triangle^2 g - \triangle g = 0 & \text{on } \Omega \\ \sum_j \dfrac{\partial^2 g}{\partial n^2} = 0 & \text{on } b\Omega \\ \dfrac{\partial g}{\partial n} - \dfrac{\partial}{\partial n} \triangle g + \sum_k Y_k^* \dfrac{\partial}{\partial n} D_k g = 0 & \text{on } b\Omega \end{cases}$$

Using the first boundary equation we can rewrite the second one as in the homogeneous problem that we have studied earlier when we calculated the kernel. We conclude that $g \equiv C$. Therefore the cokernel of the problem is given by

$$(g; v_1, v_2) = (c; 0, -c),$$

which is obviously a one dimensional space.

All of these calculations can be applied to our original problem provided that $f \in W^2(\Omega)$. Using the regularity result we can extend it to all $f \in W^s$, $s > 1/2$ and conclude that has a unique solution u with $\int_\Omega u = 0$, for all $f \in W^s(\Omega)$ with $\int_\Omega f = 0$. $\square$

13. Final Remarks

In the present paper we have laid the foundations for Hodge theory of the standard exterior differential operator d in the Sobolev topology acting on q-forms on a domain in $\mathbb{R}^N$. The theory that have presented here is complete. However, there is much work that remains to be done. For maximum applicability, the Hodge theory on a (compact) manifold needs to be developed. Also the case of all s and all q should be treated in a unified manner.

The ultimate goal of this program is to develop the Hodge theory of the $\overline{\partial}$-Neumann problem in the Sobolev topology on a strongly pseudoconvex domain. We will turn to this task in a subsequent sequence of papers.

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