ELLIPTIC BOUNDARY VALUE PROBLEMS FOR THE INHOMOGENEOUS LAPLACE EQUATION ON BOUNDED DOMAINS

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Abstract. Elliptic estimates in Hardy classes are proved on domains with minimally smooth boundary. The methodology is different from the original methods of Chang/Krantz/Stein.

1. Introduction

Let D be a bounded Lipschitz domain in $\mathbb{R}^n$. In [CKS], Chang, Krantz and Stein introduced certain real variable Hardy spaces. They defined two Hardy spaces $h^p_r(D)$ and $h^p_z(D)$, $0 < p \leq 1$. We say that a distribution f lies in $h^p_r(D)$ if it has an extension $E(f) \in h^p(\mathbb{R}^n)$; and we say that a distribution g lies in $h^p_z(D)$ if there is an extension $E_z(g) \in h^p(\mathbb{R}^n)$ so that $E_z(f) = 0$ on $\mathbb{R}^n \setminus \overline{D}$. Here $h^p(\mathbb{R}^n)$ is a local version, due to Goldberg [G], of the classical Hardy spaces.

Since D is bounded, it is known from Miyachi [M] that the Hardy space $h^p_r(D)$ can be identified with the subspace of distributions $f \in \mathcal{D}(D)$ such that the radial maximal function $f^+(x) = f^+_{\phi,D}(x) \in L^p(D)$. Here

$$f^+(x) = \sup\left\{\left|\int_D \phi_t(x-y)f(y)dy\right| : 0 < t < \delta(x)\right\},$$

where $\delta(x) = \mathrm{dis}(x, D^c)$ with $D^c = \mathbb{R}^n \setminus D$ and ϕ is a fixed function such that $\phi \in C^\infty_0(B^n)$ (B^n the unit ball in $\mathbb{R}^n$), $\phi \geq 0$, and $\int_{\mathbb{R}^n} \phi(x)dx = 1$. We set $\phi_t(x) = t^{-n}\phi(x/t)$.

Let $G(f)$ be the solution of the Dirichlet boundary problem:

$$\Delta u = f \quad \text{in} \quad D, \quad \text{and} \quad u\Big|_{\partial D} = 0; \tag{1.1}$$

also let $N(f)$ be the solution of the Neumann boundary value problem:

$$\Delta u = f \quad \text{in} \quad D, \quad \text{and} \quad \frac{\partial u}{\partial \nu}\Big|_{\partial D} = 0, \tag{1.2}$$

Date: August 2, 1995.

Krantz's research is supported in part by NSF Grant DMS-9022140 during residence at MSRI. Li is partially supported by NSF Grant DMS–9500758.

where $\nu(x)$ denotes the outward unit normal to the boundary ∂D at x, and $\frac{\partial}{\partial\nu} = \nabla\cdot\nu$.

In [CKS], Chang, Krantz and Stein extended a classical theorem in $\mathbb{R}^n$ to a smoothly bounded domain in $\mathbb{R}^n$ and they proved the following theorem.

THEOREM 1.1. *Let D be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary. Let $0 < p \le 1$. Then*

$$\left\|\frac{\partial^2 G(f)}{\partial x_i \partial x_j}\right\|_{h^p_r(D)} + \left\|\frac{\partial^2 N(f)}{\partial x_i \partial x_j}\right\|_{h^p_r(D)} \le C_p(D)\|f\|_{h^p_z(D)}.$$

The question of what is the minimum smoothness condition on ∂D so that the above theorem remains true when D is a bounded domain in $\mathbb{R}^n$ is still open. Certainly one may ask: Does the smoothness of ∂D in Theorem 1.1 depends on p? In [CKS], it was conjectured that Theorem 1.1 remains true if D has $C^{1/p}$ boundary.

The Dirichlet problem for the inhomogeneous Laplace equation (1.1) has been studied by many authors (see, for example, [AND], [GT], [FKP], [LiM], and [Ken2]). Fix $0 \le \alpha < \infty$ and $1 < p < \infty$. When D is a smoothly bounded (C^∞) domain in $\mathbb{R}^n$, Calderòn-Zygmund theory shows (see [AND]) that if $f \in W^p_\alpha(D)$ (the Sobolev space), then there exists a unique u which solves (1.1) with

$$\|u\|_{W^p_{2+\alpha}(D)} \le C\|f\|_{W^p_\alpha(D)}, \quad 1 < p < \infty. \tag{1.3}$$

In [Dah], B. Dahlberg constructed a bounded Lipschitz domain $D \subset \mathbb{R}^2$, and an $f \in C^\infty(\overline{D})$ so that the regularity (1.3) fails when $\alpha = 0$. In [JeK], D. Jerison and C. Kenig constructed a bounded domain in $\mathbb{R}^2$ with C^1 boundary and a function $f \in C^\infty(\overline{D})$ so that (1.3) fails for $p = 1$ with $\alpha = 0$. Since $C^\infty(\overline{D}) \subset h^1_z(D)$, we see that the aforementioned conjecture is not true when $p = 1$.

¿From the definition of p-atom (see [CKS]), one can see that the necessary order of cancellation in an atom depends not only on p but also on n. Since the dual of H^p is $\dot{\Lambda}_{n(1/p-1)}$ (the homogeneous Zygmund class), it seems that a reasonable necessary requirement on the smoothness of ∂D so that Theorem 1.1 remains true will be that the boundary is $C^{n(1/p-1)}$ when $p < n/(n+1)$.

The primary purpose of the present paper is to prove that this last suggested necessary condition is also sufficient. We will work on domains with at least C^2 boundary. As we shall see, our results will be o interest for p small: $0 < p \le n/(n+2)$. We shall leave the case when D has C^q boundary with $1 < q \le 2$ for a future paper.

For $0 < p < \infty$ and any $\epsilon > 0$, we let

$$\alpha(p,\epsilon) = \max\{2+\epsilon, n(1/p-1)+\epsilon\}. \tag{1.4}$$

The first theorem we propose to prove is:

THEOREM 1.2. *Let $0 < p < \infty$ and let D be a bounded domain in $\mathbb{R}^n$ with $C^{\alpha(p,\epsilon)}$ boundary. Then*

$$\left\| \frac{\partial^2 G(f)}{\partial x_i \partial x_j} \right\|_{h^p_r(D)} + \left\| \frac{\partial^2 N(f)}{\partial x_i \partial x_j} \right\|_{h^p_r(D)} \leq C_{p,n,\epsilon}(D) \|f\|_{h^p_z(D)}$$

for any $\epsilon > 0$.

Note 1: From the proof of Theorem 1.2, it may be seen that we need such an $\epsilon > 0$ only when $n(1/p - 1)$ is an integer.

We shall show by example that Theorem 1.2 fails for some domain D having only $C^{n(1/p-1)-\epsilon}$ boundary.

The proof of Theorem 1.1 given by Chang, Krantz and Stein is based on mapping D to the model domain $\mathbb{R}^n_+$. Our approach for proving Theorem 1.2 will be based instead on the machinery connected with the fundamental properties of the Green's function for $-\Delta$ in D. The properties of the Green's function have, historically, played a crucial role in solving the Laplace equation. M. Grüter and K.-O. Widman [GW] as well as E. Fabes and W. Stroock [FaS] studied the Green's function in a Lipschitz domain in $\mathbb{R}^n$. They gave the basic estimates for the Green's function and its first derivative. In order to prove Theorem 1.2, we need the asymptotic behavior of G and its *higher derivatives* near the boundary. The secondary purpose of the present paper is to estimate the Green's function and its derivatives of all orders (see Theorem 2.2.) We believe that the properties of the Green's function that are derived in this paper will be useful in other contexts as well.

Note 2: From our proof of Theorem 1.2, we have that

$$\frac{\partial^2 G(f)}{\partial x_i \partial x_j} = \Gamma_2(f) + U_2(f),$$

where Γ_2 is bounded from $h^p_z(D)$ to $h^p_r(D)$ for D with Lipschitz boundary. The operator $U_2 : h^p_z(D) \to h^p(D)$ is bounded, where $h^p(D)$ is the subspace of $h^p_r(D)$ consisting of all harmonic functions on D.

¿From our estimates on the higher derivatives of the Green's function, we are able to consider the elliptic boundary value problems (1.1) and (1.2) with f lying in the Hardy-Sobolev space. For this purpose, let us now introduce the definition of Hardy-Sobolev space.

Let k be a non-negative integer and $0 < p < \infty$. We let $h^{k,p}_z(D)$ denote the space of all measurable functions f with the weak derivative $\nabla^\ell f \in h^p_z(D)$ for all $0 \leq \ell \leq k$; here $\nabla^k f$ denotes all k^{th} derivatives of f. We say that a measurable function f on D belongs to $h^{k,p}_r(D)$ if $\nabla^\ell f \in h^p_r(D)$ for all $0 \leq \ell \leq k$. It is obvious that $h^{k,p}_z(D) \subset h^{k,p}_r(D)$. We shall prove the following:

THEOREM 1.3. *Let $0 < p \le 1$ and let k be a non-negative integer with $k \le n(1/p-1)$. Let D be a bounded domain in $\mathbb{R}^n$ with $C^{3+[n(1/p-1)]}$ boundary. Then*

$$\left\|\frac{\partial^2 G(f)}{\partial x_i \partial x_j}\right\|_{h_r^{k,p}(D)} + \left\|\frac{\partial^2 N(f)}{\partial x_i \partial x_j}\right\|_{h_r^{k,p}(D)} \le C_{p,k,n}(D)\|f\|_{h_z^{k,p}(D)}.$$

The paper is organized as follows. In Section 2, we estimate the behavior of the higher derivatives of the Green's function near the boundary. The results in this section are the key to the rest of the paper. In Section 3, we use the atomic decomposition theorem in [CKS] to reduce the proof of Theorem 1.2 to the study of a single atom. In Section 4, we prove Theorems 1.2 and 1.3 for the Dirichlet boundary value problem. In Section 5, Theorems 1.2 and 1.3 for the Neumann boundary problem are proved and the 'Green's function' for the Neumann boundary value problem is studied. In Section 6, we shall construct examples that show that Theorem 1.2 is reasonably sharp.

The authors wish to thank Jiaping Wang for a useful conversation on the Green's function.

2. Estimate derivatives of Green function

It is well-known that the fundamental solution for the Laplacian Δ in $\mathbb{R}^n$ is

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} -(n-2)^{-1}\omega_n^{-1}|x-y|^{2-n}, & \text{if } n > 2 \\ \frac{1}{2\pi}\log|x-y|, & \text{if } n = 2 \end{cases}$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$ (see [KR]).

Let D be a bounded C^1 domain in $\mathbb{R}^n$. For each $y \in D$, we let $U(\cdot, y)$ be the solution of the Dirichlet problem:

$$\Delta U(\cdot, y) = 0 \text{ in } D, \quad U(x,y) = \Gamma(y-x), \text{ for } x \in \partial D. \tag{2.1}$$

Then the Green's function for Δ on D is

$$G(x,y) = \Gamma(x-y) - U(x,y). \tag{2.2}$$

The Dirichlet problem (1.1) has a unique solution

$$u(x) = G(f)(x) = \int_D G(x,y)f(y)\,dy. \tag{2.3}$$

Now let $V(\cdot, y)$ be the solution of the following Neumann problem:

$$\Delta V = \mathrm{Vol}(D)^{-1} \quad \text{in} \quad D, \quad \frac{\partial}{\partial \nu}V = \frac{\partial}{\partial \nu}\Gamma(y-\cdot) \quad \text{on} \quad \partial D. \tag{2.4}$$

We let

$$N(x,y) = \Gamma(x-y) - V(x,y) \tag{2.5}$$

It is easy to show that the Neumann problem (1.2) has a unique solution, up to an additive constant, given by the formula

$$u(x) = N(f)(x) = \int_D N(x,y)f(y)dy. \tag{2.6}$$

The main purpose of this section is to study the basic properties of $U(x,y)$. We shall derive information about the asymptotic behavior of $U(x,y)$ when x, y are near the boundary ∂D. For convenience, we will always assume that $n > 2$ (the case $n = 2$ is similar, but the details of the formula are different.) Similar results for $V(x,y)$ will be obtained in Section 5.

We will need the following proposition that is due to M. Grüter and K.-O. Widman [GW].

Proposition 2.1. *Let D be a bounded domain in $\mathbb{R}^n$ satisfying the uniform exterior sphere condition (i.e., each boundary point of D has an exterior osculating sphere of uniform size). Then the Poisson kernel*

$$P(x,y) = \frac{\partial G(x,y)}{\partial \nu(y)}$$

satisfies the estimate:

$$0 \le P(x,y) \le \frac{C\delta(x)}{|x-y|^n} \tag{2.7}$$

for all $x \in D$ and $y \in \partial D$. [Note that, on a C^2 domain, these last two expressions are known to be comparable—see [KR]].

The main result of this section is to prove the following theorem about U.

THEOREM 2.2. *Let $k_0 \ge 2$ be an integer. Let D be a bounded domain with $C^{k_0,\epsilon}$ boundary. Then for any multi-indices α and β with $|\alpha| = k \le k_0$ and $|\beta| = \ell \le k_0$, we have*

$$\left|\frac{\partial^{k+\ell}U(x,y)}{\partial x^\beta \partial y^\alpha}\right| \le \frac{C_{k_0,\epsilon}}{(|x-y|+\delta(y))^{n+k+\ell-2}} \tag{2.8}$$

for all $x,\ y \in D$ and any $\epsilon > 0$.

Proof. If we write $f(t) = -t^{1-n/2}$ for all $t \ge 0$, then $\Gamma(x-y) = f(|x-y|^2)$. Now

$$\frac{\partial^k U(x,y)}{\partial y^\alpha} = \int_{\partial D} P(x,z)\frac{\partial^k \Gamma(z-y)}{\partial y^\alpha}\, d\sigma(z), \tag{2.9}$$

where $P(x,z)$ is the Poisson kernel satisfying (2.7) for all $x \in D$ and $z \in \partial D$ (since D has at least C^2 boundary) Thus

$$\frac{\partial^k U(x,y)}{\partial y^\alpha} = \frac{\partial^k \Gamma(x-y)}{\partial y^\alpha}\ , \qquad x \in \partial D,\ y \in D. \tag{2.10}$$

For any fixed x_0, $y \in D$, we let $R = |x_0 - y|/2$. If $R < 16\delta(y)$, then (2.8) holds with $(x, y) = (x_0, y)$ by the maximum principle and Proposition 2.1. We now assume that $R \geq 16\delta(y)$, and we let $\epsilon_0 = \delta(x_0)^{n+\ell+k}\delta(y)^{n+2k}$. For any multi-index α with $|\alpha| = k$, we consider the function

$$f_{\alpha,y}(|x-y|^2+\epsilon_0) = \frac{\partial^k f(|x-z|^2+\epsilon_0)}{\partial z^\alpha}\bigg|_{z=y}.$$

Since $U(\cdot, y)$ is harmonic, we have

$$\begin{aligned}
&\frac{\partial^{\ell+k}U(x,y)}{\partial x^\beta \partial y^\alpha} - \frac{\partial^{|\beta|} f_{\alpha,y}(|x-y|^2+\epsilon_0)}{\partial x^\beta} \\
&= \int_{\partial D} \frac{\partial^\ell P(x,z)}{\partial x^\beta}\Big(\frac{\partial^k \Gamma(y-z)}{\partial y^\alpha} - f_{\alpha,y}(|y-z|^2+\epsilon_0)\Big)\, d\sigma(z) \\
&\quad -\frac{\partial^\ell}{\partial x^\beta}\int_D G(x,z)\Delta_z f_{\alpha,y}(|y-z|^2+\epsilon_0)\, dz \\
&\equiv J_1(x,y) + J_2(x,y)
\end{aligned}$$

Notice that the term involving integration over the interior comes from Green's identity—since the integrand has a singularity.

Since ∂D is $C^{k_0,\epsilon}$ we have (from the maximum principle) that

$$\left|\frac{\partial^{|\gamma|}U(x,z)}{\partial z^\gamma}\right| = \left|\frac{\partial^{|\gamma|}U(z,x)}{\partial z^\gamma}\right| \leq C_{\gamma,\epsilon}\|\Gamma(x-\cdot)\|_{C^{|\gamma|}(\partial D)} \leq C_{n,\gamma}\delta(x)^{-|\gamma|-n+2}$$

for all $|\gamma| \leq k_0$. Moreover, since again $U(x, \cdot)$ is harmonic, we have

$$\left|\frac{\partial^{|\gamma|+|\beta|}G(x,z)}{\partial x^\beta \partial z^\gamma}\right| \leq C_{n,\gamma,\epsilon,\beta}\Big(|x-z|^{-n-|\gamma|-|\beta|+2} + \delta(x)^{-|\gamma|-|\beta|-n+2}\Big) \tag{2.11}$$

for all $|\gamma| \leq k_0$ and any β.

Since $P(\cdot, z)$ (for $z \in \partial D$) is harmonic, we have

$$\left|\frac{\partial^\ell}{\partial x^\beta}P(x,z)\right| \leq C_\ell \delta(x)^{-n-\ell+1}$$

for all $z \in \partial D$. Thus

$$\begin{aligned}
|J_1(x_0,y)| &\leq C_{k,\ell}\int_{\partial D}\delta(x_0)^{-\ell-n}\frac{\delta(x_0)^{\ell+k+n}\delta(y)^{n+2k}}{(|z-y|^2+\epsilon_0)^{(n+k-1)/2}}\, d\sigma(z) \\
&\leq C_{k,\ell}\int_{\partial D}\frac{\delta(y)^{2k+n}}{\delta(y)^{n+k-1}}\, d\sigma(z) \\
&\leq C_{k,\ell}.
\end{aligned}$$

Also, for $R = |x_0 - y|/2 > 0$ fixed,

$$\begin{aligned} |J_2(x_0,y)| &\le \left| \frac{\partial^\ell}{\partial x^\beta} \int_{D\setminus B(y,R)} G(x,z)\Delta_z f_{\alpha,y}(|y-z|^2+\epsilon_0)\,dz \Big|_{x=x_0} \right| \\ &\quad + \left| \int_{D\cap B(y,R)} \frac{\partial^\ell G(x_0,z)}{\partial x^\beta} \Delta_z f_{\alpha,y}(|y-z|^2+\epsilon_0)\,dz \right| \\ &\equiv J_{21}(x_0,y) + J_{22}(x_0,y). \end{aligned}$$

Now

$$\begin{aligned} J_{21}(x_0,y) &\le \left| \frac{\partial^\ell}{\partial x^\beta} \int_{D\setminus B(y,R)} \Gamma(x-z)\Delta_z f_{\alpha,y}(|y-z|^2+\epsilon_0)\,dz \Big|_{x=x_0} \right| \\ &\quad + \int_{D\setminus B(y,R)} \epsilon_0 \delta(x_0)^{-n-\ell+2}(|z-x_0|+R)^{-n-k}\,dz \\ &\le J_{211} + C_{k,\ell}|x_0-y|^{-n-k-\ell+2}, \end{aligned}$$

where J_{211} is defined by the last inequality. We see that

$$\begin{aligned} J_{211}(x_0,y) &= \left| \frac{\partial^\ell}{\partial x^\beta} \int_{D\setminus B(y,R)} \Gamma(x-z)\Delta_z f_{\alpha,y}(|y-z|^2+\epsilon_0)\,dz \Big|_{x=x_0} \right| \\ &\le \left| \frac{\partial^\ell}{\partial x^\beta} \int_{\partial(D\setminus B(y,R))} \Gamma(x-z)D_{\nu(z)} f_{\alpha,y}(|y-z|^2+\epsilon_0)\,d\sigma(z) \Big|_{x=x_0} \right| \\ &\quad + \left| \frac{\partial^\ell}{\partial x^\beta} \int_{\partial(D\setminus B(y,R))} D_{\nu(z)}\Gamma(x-z) f_{\alpha,y}(|y-z|^2+\epsilon_0)\,d\sigma(z)|_{x=x_0} \right| \\ &\quad + \left| \frac{\partial^\ell f_{\alpha,y}(x,y)}{\partial x^\beta} \Big|_{x=x_0} \right| \\ &\le C_{k,\ell}|x_0-y|^{-n-k-\ell+2} \end{aligned}$$

by standard arguments (since $k \le k_0$, ∂D is $C^{k_0,\epsilon}$ with $k_0 \ge 2$, $\epsilon > 0$ and by the fact that $|\nu(z)\cdot(x-z)| \le C(|x-z|^2 + \delta(x)$ for all $z \in \partial D$ and $x \in D$). [Note that the last term in the penultimate line comes from Green's theorem.]

Now we consider $J_{22}(x_0, y)$. Let $\Omega = D \cap B(y, R)$. After applying the divergence theorem many times, we have

$$\begin{aligned}
&J_{22}(x_0, y)\\
= &\ \Big|\int_\Omega \frac{\partial^\ell G(x_0, z)}{\partial x^\beta} \Delta_z f_{\alpha,y}(|y-z|^2+\epsilon_0)dz\Big|\\
= &\ \Big|\int_\Omega \frac{\partial^\ell G(x_0, z)}{\partial x^\beta} \frac{\partial^k}{\partial z^\alpha} \Delta_z f(|z-y|^2+\epsilon_0)dz\Big|\\
\le &\ \Big|\int_\Omega \frac{\partial^{\ell+k} G(x_0, z)}{\partial x^\beta \partial z^\alpha} \Delta_z f(|z-y|^2+\epsilon_0)dz\Big|\\
&+\Big|\int_{\partial\Omega} \sum_{|\gamma|\le k} |(\nu(z)^\beta|\Big|\frac{\partial^{\ell+|\gamma|} G(x_0, z)}{\partial x^\beta \partial z^\gamma}\Big|\Big|\frac{\partial^{k-|\gamma|}}{\partial z^{\alpha-\gamma}} \Delta_z f(|z-y|^2+\epsilon_0)\Big|d\sigma(z)\\
\equiv &\ J_{221}(x_0, y) + J_{222}(x_0, y).
\end{aligned}$$

Now, since $\Delta_z f(|y-z|^2) = 0$ for all $z \ne y$, we have

$$\Big|\Delta_z f(|y-z|^2+\epsilon_0)\Big| = \frac{n(n-2)\,\epsilon_0}{(|z-y|^2+\epsilon_0)^{n/2}} \le C_{n,k} \frac{\delta(x_0)^{n+k+\ell}\delta(y)^{2k+n}}{|z-y|^n} \tag{2.12.}$$

Moreover, we have

$$\Big|D_z^\gamma \Delta_z f(|y-z|^2+\epsilon_0)\Big| \le C_{n,\gamma} \frac{\delta(x_0)^{k+\ell+n}\delta(y)^{n+2k}}{(|z-y|+\epsilon_0)^{n+|\gamma|}}$$

Applying (2.12), we find that

$$\begin{aligned}
&J_{222}(x_0, y)\\
\le &\ C_{\ell,k} \int_{\partial\Omega} \delta(x_0)^{-\ell-k-n+2} \frac{\delta(x_0)^{\ell+n+k}\delta(y)^{2k+n}}{\delta(y)^{n+k}} d\sigma(z)\\
\le &\ C_{n,k,\ell}.
\end{aligned}$$

Finally, we estimate $J_{221}(x_0, y)$. Observe that

$$\begin{aligned}
&J_{221}(x_0, y)\\
= &\ \Big|\int_\Omega \frac{\partial^{k+\ell} G(x_0, z)}{\partial x^\beta \partial z^\alpha} \Delta_z f(|y-z|^2+\epsilon_0)dz\Big|\\
\le &\ \int_\Omega C_{k,n}\delta(x_0)^{-k-\ell-n+2} \frac{\delta(x_0)^{k+n+\ell}\delta(y)^{n+2k}}{(|y-z|+\epsilon_0)^n} dz\\
\le &\ C_{n,k}\, \epsilon_0^{\frac{1}{n+k+\ell}} \left(\log\frac{1}{\epsilon_0}\right)\\
\le &\ C_{n,k}.
\end{aligned}$$

Combining the above estimates, we have

$$\left|\frac{\partial^{k+\ell}U(x_0,y)}{\partial x^\beta \partial y^\alpha} - \frac{\partial^\ell f_\alpha(|x_0-y|^2+\epsilon_0)}{\partial x^\beta}\right| \le C_{n,k}(\delta(y)+|x-y|)^{-n-k-\ell+2}$$

Using this estimate and the definition of $f_{\alpha,y}(x_0)$, we conclude that the proof of Theorem 2.2 is complete. ▯

Corollary 2.3. *Let $\lambda > 2$ be a non-integer and let D be a bounded domain with λ boundary. Then for all $0 \le k \le [\lambda]$ and any cube Q in $\mathbb{R}^n$ with $2Q \subset D$, we have*

$$\left|\frac{\partial^2 U(x,y)}{\partial x_i \partial x_j} - \sum_{|\alpha|\le k} \frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^2 U(x,x_0)}{\partial x_i \partial x_j}(y-x_0)^\alpha\right| \le C_{p,n,\lambda}\frac{\delta^{k(\lambda)}}{|x-x_0|^{n+k(\lambda)}} \tag{2.13}$$

for all $x \in D \setminus 2Q$ and $y, x_0 \in Q$ and $|Q| = \delta^n$. Where $k(\lambda) = k+1$ if $k < [\lambda]$, and $k(\lambda) = \lambda$ if $k = [\lambda]$

Proof. This follows directly from Theorem 2.2 (with a suitable modification for fractional derivatives) and from Taylor's theorem.

3. Reduction of Theorem 1.2

Let f be a measurable function on D and let λ and q be positive numbers. Let us recall the following definition for a p-atom. Let a be a bounded function in $\mathbb{R}^n$; we say that a is an $h^p(\mathbb{R}^n)$ atom if a is supported in a cube Q with $(\int_{\mathbb{R}^n} |a(x)|^2 dx)^{1/2} \le |Q|^{1/2-1/p}$ and either **(i)** $|Q| > 1$ or **(ii)** $|Q| \le 1$ and, for each $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| \le [n(1/p-1)]$, we have

$$\int_Q x^\alpha a(x)dx = 0.$$

where all $\alpha_i \ge 0$ and $|\alpha| = \sum_{i=1}^n \alpha_i$.

In order to prove Theorem 1.2, we recall the following theorem that is formulated from the statement of Theorem 3.2 in [CKS] and its proof.

THEOREM 3.1. *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $0 < p \le 1$. Then $f \in h^p_z(D)$ if and only if f has an atomic decomposition, with $h^p(\mathbb{R}^n)$ atoms whose supports lie in D, i.e.,*

$$f = \sum_{Q\subset D} \lambda_Q a_Q$$

with $2Q \cap \partial D = \emptyset$ if the diameter of Q is small (i.e. ≤ 1), and

$$\sum_{Q\subset D} |\lambda_Q|^p < \infty.$$

By Theorem 3.1, it is easy to verify that Theorem 1.2 holds if and only if it holds for f an h^p-atom. More precisely, the proof of Theorem 1.2 can be reduced to proving the following theorem:

THEOREM 3.2. *Let $0 < p \le 1$. Let D be a bounded domain in $\mathbb{R}^n$ with $C^{\alpha(p,\epsilon)}$ boundary. Then*

$$\left\| \frac{\partial^2 G(a)}{\partial x_i \partial x_j} \right\|_{h^p_r(D)} + \left\| \frac{\partial^2 N(a)}{\partial x_i \partial x_j} \right\|_{h^p_r(D)} \le C_{p,n,\epsilon}(D).$$

holds for all p-atoms a with support Q satisfying $2Q \subset D$.

We shall separate the proof of Theorem 3.2 and Theorem 1.3 into two cases, which are given in Sections 4 and 5 respectively.

4. The Dirichlet Problem

In this section, we shall prove Theorem 3.2 with the Dirichlet boundary condition. In other words, we shall prove Theorem 4.1.

THEOREM 4.1. *Let $0 < p \le 1$ and let D be a bounded domain in $\mathbb{R}^n$ with $C^{\alpha(p,\epsilon)}$ boundary. Then*

$$\left\| \frac{\partial^2 G(a)}{\partial x_i \partial x_j} \right\|_{h^p_r(D)} \le C_{p,n,\epsilon}(D). \tag{4.1}$$

holds for all p-atoms with support Q satisfying $3Q \subset D$ and for any $\epsilon > 0$.

Since

$$G(a)(x) = \int_D \Big(\Gamma(x-y) - U(y,x) \Big) a(y)\, dy,$$

it is clear that

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} \int_D \Gamma(y-x) a(y)\, dy \right\|_{h^p_r(D)} \le C_{p,n}$$

from the results in [ST1] for Hardy spaces in $\mathbb{R}^n$. Let a be an h^p-atom. Without loss of generality, we may assume that $0 \in D$ and

$$\mathrm{supp}(a) \subset \{ y \in D : |y| < \delta \} = B(0,\delta) \subset D.$$

If the atom a is supported in a cube $Q \subset D$ with diameter greater than 1, then $a \in L^2(\mathbb{R}^n)$. Then we have that (4.1) holds by using a modified version of the following result of Jerison and Kenig in [JeK] and Gilbarg and Trudinger [GT].

THEOREM 4.2. *Let D be a bounded domain in $\mathbb{R}^n$ with C^2 boundary. If u is a solution of (1.1), then*

$$\|u\|_{W^p_{2+\alpha}(D)} \le C_{p,\alpha} \|f\|_{W^p_\alpha(D)}.$$

for all $1 < p < \infty$ and $-1 < \alpha \le 0$.

Note, in passing, that there is certainly an analogous version of Theorem 4.2 for the Neumann problem.

Now we assume that a is a classical atom with support in a cube Q with small diameter with $2Q \cap \partial D = \emptyset$. Let the center of Q be $x_0 = 0$. For each $x \in D$, we let

$$H(a)(x) = \frac{\partial^2}{\partial x_i \partial x_j} \int_D U(x,y) a(y) dy.$$

Since $U(x,y)$ is harmonic in both x and y, we have that $H(x)$ is harmonic in x. We choose $\phi \in C_0^\infty(B(0,1))$ to be a non-negative radial function such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Let $x \in D$ and $d(x) = \operatorname{dis}(x, D^c)$. Since $H(a)$ is harmonic and ϕ is radial, the mean value property of harmonic function shows that

$$H(a)^+(x) = \sup_{0<t<d(x)} \left| \int_D H(a)(y)\phi_t(x-y)dy \right| = |H(a)(x)|,$$

where $\phi_t = t^{-n}\phi(t^{-1}x)$. Now

$$\int_D H(a)^+(x)^p dx = \int_{B(0,4\delta)} |H(a)(x)|^p dx + \int_{D \setminus B(0,4\delta)} |H(a)(x)|^p dx \equiv I_1(a) + I_2(a).$$

By (1.3), we have

$$\begin{aligned}
I_1(a) &\leq \left(\int_{B(0,4\delta)} |H(a)(x)|^2 dx \right)^{p/2} \left(\int_{B(0,4\delta)} dx \right)^{(2-p)/2} \\
&\leq C^p \left(\int_{B(0,4\delta)} |a(x)|^2 dx \right)^{p/2} |B(0,4\delta)|^{(2-p)/2} \\
&\leq C^p |B(0,4\delta)|^{(-2/p+1)p/2} |B(0,4\delta)|^{(2-p)/2} \\
&= C^p.
\end{aligned}$$

Next we estimate $I_2(a)$. By Corollary 2.3, we have

$$\left| \frac{\partial^2 U(x,y)}{\partial x_i \partial x_j} - \sum_{|\alpha| \leq n_p} \frac{\partial^\alpha}{\partial y^\alpha} \frac{\partial^2 U(x,x_0)}{\partial x_i \partial x_j} (y-x_0)^\alpha \right| \leq C_p \frac{\delta^{n(1/p-1)+\epsilon/2}}{|x-x_0|^{n+n(1/p-1)+\epsilon/2}}$$

for all $x \in D \setminus 4B(x_0, \delta)$ and $y \in B(x_0, 2\delta) \subset D$, where $n_p = [n(1/p-1)]$. Thus, since the center of Q is $x_0 = 0$, for each $x \in D \setminus B(0, 4\delta)$ we have

$$\begin{aligned}
|H(a)(x)| &= \Big|\int_Q \frac{\partial^2 U(x,y)}{\partial x_i \partial x_j} a(y)\, dy\Big| \\
&= \Big|\int_Q \Big(\frac{\partial^2 U(x,y)}{\partial x_i \partial x_j} - \sum_{|\alpha|\le n_p} \frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^2 U(x,0)}{\partial x_i \partial x_j} y^\alpha\Big) a(y)\, dy\Big| \\
&\le C_{p,\epsilon}\int_Q \frac{\delta^{n(1/p-1)+\epsilon/2}}{|x-x_0|^{n+n(1/p-1)+\epsilon/2}} |a(y)|\, dy \\
&\le C_{p,\epsilon}|Q|^{1-1/p}\frac{\delta^{n(1/p-1)+\epsilon/2}}{|x|^{n+n(1/p-1)+\epsilon/2}} \\
&= C_{p,\epsilon}\frac{\delta^{\epsilon/2}}{|x|^{n/p+\epsilon/2}}.
\end{aligned}$$

Therefore, if $D \subseteq B(0, d_0)$, then

$$\begin{aligned}
\int_{D\setminus B(0,4\delta)} |H(a)(x)|^p dx &\le C_{p,\epsilon}^p \int_{D\setminus B(0,4\delta)} \frac{\delta^{p\epsilon/2}}{|x|^{n+p\epsilon/2}} dx \\
&\le C_p^p \int_{4\delta}^{d_0} C_n \frac{\delta^{p\epsilon/2}}{t^{1+p\epsilon/2}} dt \\
&= C_p^p \frac{2}{p\epsilon}\frac{\delta^{p\epsilon/2}}{(4\delta)^{p\epsilon/2}} \\
&= C(p, n, d_0).
\end{aligned}$$

The proof of Theorem 4.1 is thus complete. $\square$

THEOREM 4.3. *Let $0 < p < \infty$ and let $k \le n(1/p-1)$ be a non-negative integer. Let D be a bounded domain in $\mathbb{R}^n$ with $C^{2+n(1/p-1)+\epsilon}$ boundary for some $\epsilon > 0$. Then*

$$\left\|\frac{\partial^2 G(f)}{\partial x_i \partial x_j}\right\|_{h_r^{k,p}(D)} \le C_{k,p,\epsilon}\|f\|_{h_z^{k,p}(D)}. \tag{4.2}$$

for any $\epsilon > 0$.

Proof. Observe that

$$G(f)(x) = \int_D \Gamma(x-y) f(y) dy - \int_D U(x,y) f(y) dy = \Gamma(f)(x) + U(f)(x)$$

Since f has compact support, it is obvious that

$$\left\|\nabla^2 \Gamma(f)\right\|_{h_r^p(D)} \le C_{k,p,\epsilon}\|f\|_{h_z^{k,p}(D)}. \tag{4.3}$$

Next we prove that

$$\left\|\nabla^2 U(f)\right\|_{h_r^p(D)} \le C_{k,p,\epsilon}\|f\|_{h_z^{k,p}(D)}. \tag{4.4}$$

Now $\nabla^\ell U(a)(x)$ is harmonic for any non-negative integer ℓ, so it suffices to show that $\|\nabla^\ell U(a)\|_{L^p(D)} \le C_p$ for all $0 \le \ell \le k+2$. We shall prove the case $\ell = k+2$; the other cases are even easier. In order to do this, we need the following Sobolev embedding theorem which is a special case of Theorem 2 in [BB] and can also be deduced from Theorem 2 in [HPW]:

$$h_z^{k,p}(D) \subset h_z^{pn/(n-pk)}(D) \tag{4.5}$$

and the embedding is continuous. By Theorem 3.1 and the fact that $\nabla^\ell U(f)$ is harmonic, it suffices to prove that

$$\left\| \frac{\partial^{k+2} U(a)}{\partial x^\alpha} \right\|_{L^p(D)} \le C_{k,p,\epsilon}$$

for any $pn/(n-pk)$–atom with support Q and $3Q \cap \partial D = \emptyset$ (since $k \le n(1/p-1)$, hence $pn/(n-pk) \le 1$.)

Notice that

$$\frac{\partial^{k+2} U(a)(x)}{\partial x^\alpha} = \int_\Omega \frac{\partial^{k+2} U(x,y)}{\partial x^\alpha} a(y)dy.$$

Let x_0 be the center of Q. Then

$$\begin{aligned}
&\int_{2Q} \frac{\partial^{k+2} U(a)}{\partial x^\alpha}(x)|^p dx \\
&\le \; C_{p,k,n,\epsilon} \int_{2Q} \delta(x)^{-kp-np} |Q|^{-(n-pk)/n+p} dx \\
&\le \; C_{p,k,n,\epsilon} \int_{2Q} \delta(x_0)^{-kp-np} \delta(x_0)^{-n+pk+pn} dx \\
&= \; C_{p,k,n,\epsilon}.
\end{aligned}$$

We set

$$\ell_p = [n(n-pk)/np - n)] = [n/p - k - n], \quad \eta(k) = 1 \text{ if } k > 0, \quad \eta(0) = \epsilon.$$

Then

$$\left| \frac{\partial^{k+2} U(x,y)}{\partial x^\alpha} - \sum_{|\alpha| \le \ell_p} \frac{\partial^\alpha}{\partial y^\alpha} \frac{\partial^{k+2} U(x,x_0)}{\partial x^\alpha} (y-x_0)^\alpha \right| \le C_p \frac{\delta^{(n/p-k)-n+\eta(k)}}{|x-x_0|^{(n/p-k)+\eta(k)+k}}$$

for all $x \in D \setminus 2Q$ and $y \in Q$ as above. Thus, for any $x \in D \setminus 2Q$ we have

$$\begin{aligned}
&\Big|\frac{\partial^{k+2}U(a)(x)}{\partial x^\alpha}\Big| \\
&= \Big|\int_Q \frac{\partial^{k+2}U(x,y)}{\partial x^\alpha}a(y)dy\Big| \\
&= \Big|\int_Q \Big(\frac{\partial^{k+2}U(x,y)}{\partial x^\alpha} - \sum_{|\beta|\le \ell_p} \frac{\partial^{|\beta|}}{\partial y^\beta}\frac{\partial^{k+2}U(x,x_0)}{\partial x^\alpha}(y-x_0)^\beta\Big)a(y)dy\Big| \\
&\le C_{p,k,n,\epsilon}\int_Q \frac{\delta^{(n/p-k)-n+\eta(k)}}{|x-x_0|^{k+(n/p-k)+\eta(k)}}|a(y)|dy \\
&\le C_{p,k,n,\epsilon}|Q|^{1-(n-pk)/pn}\frac{\delta^{(n/p-k)-n+\eta(k)}}{|x-x_0|^{k+(n/p-k)+\eta(k)}} \\
&= C_{p,k,n,\epsilon}\frac{\delta^{\eta(k)}}{|x-x_0|^{k+(n/p-k)+\eta(k)}}
\end{aligned}$$

for any $\epsilon > 0$. Therefore, if $D \subseteq B(0,d_0)$, then

$$\begin{aligned}
&\int_{D\setminus 2Q} |H(a)(x)|^p dx \\
&\le C^p_{p,k,n,\epsilon}\int_{D\setminus 2Q}\frac{\delta^{p\eta(k)}}{|x-x_0|^{pk+n-pk+p\eta(k)}}dx \\
&\le C^p_{p,k,n,\epsilon,d_0}\delta^{p\eta(k)-p\eta(k)} \\
&= C_{p,k,n,\epsilon}.
\end{aligned}$$

The proof of Theorem 4.3 is therefore complete. □

5. The Neumann Problem

In this section, we shall prove Theorems 2.2 and 1.3 with the Neumann boundary condition. More precisely, we shall first prove the following theorem.

THEOREM 5.1. *Let $0 < p \le 1$ and let D be a bounded domain with $C^{\alpha(p,\epsilon)}$ boundary. Then*

$$\Big\|\frac{\partial N(a)}{\partial x_i \partial x_j}\Big\|_{h^p_r(D)} \le C_p(D) \tag{5.1}$$

for all $h^p(\mathbb{R}^n)$ atoms with support $Q \subset 2Q \subset D$.

Proof. Let a be an h^p_z atom with support Q and $|Q| = \delta^n$. If $\delta \ge 1$, then $a \in L^2(\mathbb{R}^n)$. By the version of Theorem 4.2 that holds for the Neumann problem, we have that (5.1) holds.

Now we assume that a is a classical atom with support Q and $2Q \cap \partial D = \emptyset$. In this case, (4.1) follows from the argument of the proof of Theorem 4.1 and the following two results (Theorems 5.2 and 5.3.)

THEOREM 5.2. *Let $k_0 \geq 2$ be any positive integer and let D be a bounded domain with $C^{k_0,\epsilon}$ boundary. Then, for any multi-indices α and β with $|\alpha| = k \leq k_0$ and $|\beta| = \ell \leq k_0$, we have*

$$\left| \frac{\partial^{k+\ell} V(x,y)}{\partial x^\beta \partial y^\alpha} \right| \leq \frac{C_{k_0,n,\epsilon}}{(|x-y| + \delta(y))^{n+k+\ell-2}} \tag{5.2}$$

for all $x,\ y \in D$ any $\epsilon > 0$.

Proof. We write $f(t) = -t^{1-n/2}$ for all $t \geq 0$. Then $\Gamma(x-y) = f(|x-y|^2)$. Let $C_D = \mathrm{Vol}(D)^{-1}$. Then

$$\begin{aligned}
V(x,y) &= \int_{\partial D} P(x,z)V(z,y)d\sigma(z) + \int_D G(x,z)C_D dz \\
&= \int_{\partial D} \left(\frac{\partial \Gamma(x-z)}{\partial \nu(z)} - \frac{\partial U(x,z)}{\partial \nu(z)} \right) V(z,y)d\sigma(z) + \int_D G(x,z)C_D dz \\
&= \int_{\partial D} \frac{\partial \Gamma(x-z)}{d\nu(z)} V(z,y)d\sigma(z) \\
&\quad - \int_{\partial D} \frac{\partial U(x,z)}{\nu(z)} V(z,y)d\sigma(z) + \int_D G(x,z)C_D dz \\
&= I_1(x,y) - I_2(x,y) + I_3(y)
\end{aligned}$$

It is obvious that $I_3(y)$ satisfies (5.2) by using estimate (2.8). Now we consider $I_2(x,y)$. Notice that

$$\begin{aligned}
I_2(x,y) &= \int_{\partial D} \frac{\partial U(x,z)}{\partial \nu(z)}) V(z,y)d\sigma(z) \\
&= \int_{\partial D} U(x,z) \frac{\partial V(z,y)}{\partial \nu(z)} d\sigma(z) \\
&\quad + \int_D \Delta_z U(x,z)V(z,y)dz - \int_D U(x,z)\Delta_z V(z,y)dz \\
&= \int_{\partial D} U(x,z) \frac{\partial \Gamma(z-y)}{\partial \nu(z)} d\sigma(z) + 0 - \int_D U(x,z)C_D dz \\
&= \int_{\partial D} \Gamma(x-z) \frac{\partial \Gamma(z-y)}{\partial \nu(z)} d\sigma(z) - C_D \int_D U(x,z)dz.
\end{aligned}$$

Using arguments similar to those in Section 2, we see that $I_2(x,y)$ satisfies the estimate (5.2).

For any $x_0, y \in D$, if $|x_0 - y| < 4\delta(y)$, then it is easy to prove that (5.2) holds by replacing (x,y) by (x_0,y). Without loss of generality, we may assume that $|x_0 - y| >$

$4\delta(y)$. We consider the term $I_1(x,y)$. If we let

$$f_{x,0}(|z-x|^2+\epsilon_0)=(|x-z|^2+\epsilon_0)^{1-n/2}, \quad \epsilon_0=\delta(x_0)^{n+\ell+k}\delta(y)^{n+2k},$$

then

$$\begin{aligned}
I_1(x,y) &= \int_{\partial D}\frac{\partial\Gamma(x-z)}{d\nu(z)}V(z,y)d\sigma(z)\\
&= \int_{\partial D}\frac{\partial f_{x,0}(|z-x|^2+\epsilon_0)}{\partial\nu(z)}V(z,y)d\sigma(z)\\
&\quad+\int_{\partial D}\Big(\frac{\partial\Gamma(x-z)}{\partial\nu(z)}-\frac{\partial f_{x,0}(|z-x|^2+\epsilon_0)}{\partial\nu(z)}\Big)V(z,y)d\sigma(z)\\
&= I_{11}(x,y)+I_{12}(x,y)
\end{aligned}$$

We have that

$$\begin{aligned}
I_{11}(x,y) &= \int_{\partial D} f_{x,0}(|z-x|^2+\epsilon_0)\frac{\partial V(z,y)}{\partial\nu(z)}\,d\sigma(z)\\
&\quad+\int_D \Delta_z f_{x,0}(|z-x|^2+\epsilon_0)V(z,y)\,dz\\
&\quad-\int_D f_{x,0}(|x-z|^2+\epsilon_0)C_D\,dz.
\end{aligned}$$

By the definition of $f_{x,0}$ and by (2.12), one can easily see that $I_{11}(x_0,y)$ satisfies the estimate (5.2).

Finally, for convenience, we assume that $n/2$ is a positive integer (otherwise we use $(n+1)/2$ instead). Then

$$\begin{aligned}
&I_{12}(x,y)\\
&= \int_{\partial D}\Big(\frac{\partial\Gamma(x-z)}{d\nu(z)}-\frac{\partial f_{x,0}(|z-x|^2+\epsilon_0)}{\partial\nu(z)}\Big)V(z,y)\,d\sigma(z)\\
&= (n-2)\int_{\partial D}\Big(\frac{\langle\nu(z),z-x\rangle}{|x-z|^n}-\frac{\langle\nu(z),z-x\rangle}{(|x-z|^2+\epsilon_0)^{n/2}}\Big)V(z,y)\,d\sigma(z)\\
&= (n-2)\sum_{m=0}^{n/2-1}\int_{\partial D}\Big(\frac{\epsilon_0\langle\nu(z),z-x\rangle}{|x-z|^{n-2m}(|x-z|^2+\epsilon_0)^{1+m}}\Big)V(z,y)\,d\sigma(z).
\end{aligned}$$

By the definition of ϵ_0, one can easily see that $I_{12}(x_0,y)$ satisfies the desired estimate (5.2). Therefore, combining the above estimates, the proof of Theorem 5.2 is complete. ▯

As a corollary, we have the following result.

THEOREM 5.3. *Let $0 < p \leq 1$ and let D be a bounded domain with $C^{\alpha(p,\epsilon)}$ boundary. Then*

$$\left|\frac{\partial^2 V(x,y)}{\partial x_i \partial x_j} - \sum_{|\alpha|\leq n_p} \frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^2 V(x,x_0)}{\partial x_i \partial x_j}(x-x_0)^\alpha\right| \leq C_{p,n,\epsilon}\frac{\delta^{n(1/p-1)+\gamma(\epsilon)}}{|x-x_0|^{n+n(1/p-1)\gamma(\epsilon)}} \tag{5.3}$$

for all $x \in D \setminus 2Q$ and $y, x_0 \in Q \subset 2Q \subset D$ and any $\epsilon > 0$, where $\gamma(\epsilon) = n(1/p - 1) - [n(1/p-1)] + \epsilon/2 \leq 1$.

In conclusion, combining Theorems 4.1 and 5.1, the proof of Theorem 1.2 is complete. ∎

With the same argument, the Neumann problem in Theorem 1.3 can be proved by using Theorem 5.3 and the argument for proving Theorem 4.3. We leave the details for the interested reader. The proof of Theorem 1.3 is complete. ∎

6. Counterexamples

In this section, we shall give some examples to show that the hypothesis on smoothness of ∂D in Theorem 1.2 is essentially sharp. We first prove the following lemma.

Lemma 6.1. *Let D be a bounded domain in $\mathbb{R}^n$ with C^1 boundary. If (integration against the kernel) $\frac{\partial^2 G}{\partial x_i \partial x_j}$ is bounded from $h^p_z(D)$ to $h^p_r(D)$ then, for any x_0 near ∂D with $r(x_0) = \delta(x_0)/C(n)$, we have*

$$r(x_0)^{pn_p - n(1-p)} \int_{D\setminus B(x_0, r(x_0))} \left|\frac{\partial^{2+n_p} G}{\partial x_i \partial x_j \partial y^\alpha}(x, x_0)\right|^p dx \leq C_p \tag{6.1}$$

for all multi-indices α with $|\alpha| \equiv n_p = [n(1/p-1)] + 1$ and $p < 1/2$.

Proof. Let $x_0 \in D$ be near the boundary and let $r(x_0) = \delta(x_0)/C(n)$ with $C(n) >> 1$ a constant to be chosen so that there is cube $Q(x_0)$ with center at x_0 with $B(x_0, r(x_0)) \subset 2Q(x_0) \subset D$. We consider the function

$$g_{x_0}(x) = \phi_{r(x_0)}(x) = \frac{1}{r(x_0)^n}\phi\left(\frac{x-x_0}{r(x_0)}\right). \tag{6.2}$$

[Here ϕ is a radial bump function as usual.] For each $0 < p << 1$, we define

$$a(x) = r(x_0)^{n_p - n(1/p-1)}\frac{\partial^{n_p} g_{x_0}(x)}{\partial x^\alpha}. \tag{6.3}$$

It is easy to show that a is a p-atom with support in $B(x_0, r(x_0)) \subset 2Q \subset D$. Thus, for all $x \in D \setminus B(x_0, r(x_0))$, since $G(x, \cdot)$ is harmonic in $B(x_0, r(x_0))$, we have

$$\begin{aligned}
&\int_D \frac{\partial^2 G(x,y)}{\partial x_i \partial x_j} a(y)dy \\
&= \pm r(x_0)^{n_p - n(1/p-1)} \int_D \frac{\partial^{2+n_p} G(x,y)}{\partial x_i \partial x_j \partial y^\alpha} g_{x_0}(y)dy \\
&= \pm r(x_0)^{n_p - n(1/p-1)} \frac{\partial^{2+n_p} G(x,y)}{\partial x_i \partial x_j \partial y^\alpha}\bigg|_{y=x_0}
\end{aligned}$$

for all multi-indices α with $|\alpha| = n_p$. Therefore the fact $\frac{\partial^2 G}{\partial x_i \partial x_j}$ maps $h^p_z(D)$ to $h^p_r(D)$ boundedly implies that (6.1) holds, and the proof of Lemma 6.1 is complete. ▯

Let B_2 be the unit disc in $\mathbb{R}^2$ and let ψ be a conformal map from B_2 to some domain $D \subseteq \mathbb{C} \approx \mathbb{R}^2$.. We write $D = \psi(B_2)$, and $\varphi(z) = \psi^{-1}(z)$. Then the Green's function for $-\Delta$ on D is

$$G_D(z,w) = \frac{1}{2\pi} \log \frac{|1 - \varphi(z)\overline{\varphi}(w)|}{|\varphi(z) - \varphi(w)|}$$

for all $z, w \in D$. Now we have

Proposition 6.2. *With notation above,*

$$4\pi \frac{\partial^{2+2_p} G(z,w)}{\partial w^{2_p} \partial z^2} = -\varphi''(z) \frac{\partial^{2_p}}{\partial w^{2_p}} \left(\frac{1}{\varphi(z) - \varphi(w)} \right) + \varphi'(z)^2 \frac{\partial^{2_p}}{\partial w^{2_p}} \left(\frac{1}{(\varphi(z) - \varphi(w))^2} \right)$$

for all $z \neq w$, where $2_p = [2(1/p-1)] + 1$.

Proof. We calculate that

$$\begin{aligned}
&4\pi \frac{\partial^{2+2_p} G(z,w)}{\partial w^{2_p} \partial z^2} \\
&= \frac{\partial^{2+2_p}}{\partial z^2 \partial w^{2_p}} (\log|1 - \varphi(z)\overline{\varphi}(w)|^2 - \log|\varphi(z) - \varphi(w)|^2) \\
&= \frac{\partial^{2_p}}{\partial w^{2_p}} \Big(\frac{-\varphi''(z)\overline{\varphi(w)}}{1 - \varphi(z)\overline{\varphi}(w)} - \frac{\varphi'(z)^2 \overline{\varphi(w)}^2}{(1 - \varphi(z)\overline{\varphi}(w))^2} \\
&\quad - \frac{\varphi''(z)}{\varphi(z) - \varphi(w)} + \frac{\varphi'(z)^2}{(\varphi(z) - \varphi(w))^2} \Big) \\
&= -\varphi''(z) \frac{\partial^{2_p}}{\partial w^{2_p}} \left(\frac{1}{\varphi(z) - \varphi(w)} \right) + \varphi'(z)^2 \frac{\partial^{2_p}}{\partial w^{2_p}} \left(\frac{1}{(\varphi(z) - \varphi(w))^2} \right)
\end{aligned}$$

and the proof is complete. ▯

Without loss of generality, we may assume henceforth that $2(1/p-1) > 2$ is an integer.

Proposition 6.3. *Suppose that* $\psi \in C^\beta(\overline{B_2})$ *(the same for* φ*) with* $\beta = 2(1/p-1)-\epsilon$ *(for this last, it is sufficient that* ∂D *be* $C^{2(1/p-1)-\epsilon/2}$*. Assume that* $\frac{\partial^2 G}{\partial x_i \partial x_j}$ *induces a bounded operator from* $h^p_z(D)$ *to* $h^p_r(D)$*. Then*

$$r(x_0)^p |\varphi^{(2_p)}(x_0)|^p C_{\varphi,p,x_0} \le C_p(1+\|\varphi\|_{\Lambda_\beta})^{2_p},$$

where

$$C_{\varphi,p,x_0} \equiv \int_{D\setminus B(x_0,r(x_0))} \left| \frac{2\varphi'(z)^2}{(\varphi(z)-\varphi(x_0))^3} - \frac{\varphi''(z)}{(\varphi(z)-\varphi(x_0))} \right|^p dA(z). \qquad (\dagger)$$

Proof. Now

$$\begin{aligned} &\left|4\pi \frac{\partial^{2+2_p} G(z,w)}{\partial z^2 \partial w^{2_p}}\right| \\ \ge\ & \left| \frac{2\varphi'(z)^2 \varphi^{(2_p)}(w)}{(\varphi(z)-\varphi(w))^3} - \frac{\varphi''(z)\varphi^{(2_p)}(w)}{(\varphi(z)-\varphi(w))^2} \right| \\ & - \sum_{k=1}^{2_p-2} \frac{C_p(1+|\varphi|_{C^{2_p-2}(\overline{D})})^{2_p}}{|\varphi(z)-\varphi(w)|^{3+2_p-k}} - \frac{C_p(1+\|\varphi\|_{\Lambda_\beta(\overline{D})})\delta(w)^{-\epsilon}}{|\varphi(z)-\varphi(w)|^4} \\ \ge\ & \left|\frac{\varphi^{(2_p)}(w)}{|\varphi(z)-\varphi(w)|^2}\right| \left| \frac{2\varphi'(z)^2}{(\varphi(z)-\varphi(w))} - \varphi''(z) \right| \\ & - \frac{C_p}{|\varphi(z)-\varphi(w)|^{2+2_p}} - \frac{C_p}{\delta(w)^\epsilon |\varphi(z)-\varphi(w)|^4}. \end{aligned}$$

We conclude that

$$\begin{aligned} &\left|\frac{\varphi^{(2_p)}(w)}{|\varphi(z)-\varphi(w)|^2}\right| \left| \frac{2\varphi'(z)^2}{(\varphi(z)-\varphi(w))} - \varphi''(z) \right| \\ \le\ & 4\pi \left|\frac{\partial^{2+2_p} G(z,w)}{\partial z^2 \partial w^{2_p}}\right| + \frac{C_p}{|\varphi(z)-\varphi(w)|^{2+2_p}} + \frac{C_p\|\varphi\|_{\Lambda_\beta}}{\delta(w)^\epsilon |\varphi(z)-\varphi(w)|^4}. \end{aligned}$$

Also

$$r(x_0)^p \int_{D\setminus B(x_0,r(x_0))} \frac{1}{|\varphi(z)-\varphi(x_0)|^{p(2+n_p)}}\, dA(z) \le \frac{C_p r(x_0)^p}{r(x_0)^{2+p-2}} = C_p$$

and

$$r(x_0)^p \int_{D\setminus B(x_0,r(x_0))} \frac{1}{\delta(z)^{p\epsilon} |\varphi(z)-\varphi(x_0)|^{4p}}\, dA(z) \le C_p r(x_0)^{p-p\epsilon} \le C_p.$$

Since $2(1/p-1)$ is integer, we have

$$p2_p - 2(1-p) = p(2(1/p-1)+1) - 2(1-p) = p.$$

Combining this and all those estimates with Lemma 6.1 and Propositions 6.2, the proof of the proposition is complete. ∎

Proposition 6.4. *For any $\epsilon > 0$, there is a bounded domain D in $\mathbb{R}^2$ with $C^{2(1/p-1)-\epsilon}$ boundary and the operator induced by $\frac{\partial^2 G}{\partial x_i \partial x_j}$ is not bounded from $h^p_z(D)$ to $h^p_r(D)$.*

Proof. Let

$$\psi(z) = z + \eta(1-z)^{2(1/p-1)-\epsilon}$$

with $0 < \eta = \eta(p) << 1$ sufficiently small, depending on p. Clearly,

$$\psi'(z) = 1 + \eta(2(1/p-1)-\epsilon)(1-z)^{2/p-3-\epsilon}$$

It is easy to see that ψ is a conformal map from B_2 onto $\psi(B_2)$ provided that $0 < \eta = \eta(p)$ is sufficiently small and $\psi \in \Lambda_\beta(B_2)$ where $\beta = 2(1/p-1)-\epsilon$. Let $D = \psi(B_2)$ and $\varphi(w) = \psi^{-1}(w)$. It is clear that

$$C_{\varphi,p,\psi(1-\delta)} > 1/C > 0$$

(refer to equation (†)). Now, since $2_p = 2(1/p-1)+1$, we have

$$\psi^{(2_p)}(z) = c_{p,\epsilon}(1-z)^{-\epsilon-1}$$

where $c_{p,\epsilon} \neq 0$. [Note that superscripts in parentheses denote derivatives.] We let $y_0 = 1-\delta$. Then

$$\delta^p \left|\psi^{(2_p)}(y_0)\right|^p = |c_{p,\epsilon}|^p \delta^p \delta^{-\epsilon p - p} = |c_{p,\epsilon}|^p \delta^{-p\epsilon} \to \infty$$

as $\delta \to 0^+$, since φ has boundary behavior similar to that of ψ. Thus

$$\delta^{-p\epsilon} \le \frac{C_p}{|c_{p,\epsilon}|^p} \delta^p \left|\varphi^{(2_p)}(\psi(1-\delta))\right|.$$

Seeking a contradiction, we suppose that $\frac{\partial^2 G}{\partial x_i \partial x_j}$ induces a bounded operator from $h^p_z(D)$ to $h^p_r(D)$. Then, by Propositions 6.2 and 6.3, we have

$$\delta^{-p\epsilon} \le \frac{C_p}{|c_{p,\epsilon}|^p} \delta^p |\varphi^{(2_p)}(\psi(1-\delta))| < C_{p,\epsilon}(1 + \|\varphi\|_{\Lambda_\beta})^{2_p} < \infty$$

for any $0 < \delta << 1$. This is a contradiction as $\delta \to 0^+$. Thus the operator induced by $\frac{\partial^2 G}{\partial x_i \partial x_j}$ is not bounded from $h^p_z(D)$ to $h^p_r(D)$, and the proof is complete. □

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