HARDY CLASSES, INTEGRAL OPERATORS, AND DUALITY ON SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. The authors study Hardy spaces, of arbitrary order, on a space of homogeneous type. This extends earlier work that treated only H^p for p near 1. Applications are given to the boundedness of certain singular integral operators, especially on domains in complex space.

1. INTRODUCTION

Function spaces play a significant role in harmonic analysis and partial differential equations. The integral operators that form a bridge between function spaces and partial differential equations are the Calderón-Zygmund operators. It is well known that Calderón-Zygmund operators are bounded on the Lebesgue space $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. It is also known that the Calderón-Zygmund operators are not bounded on $L^p(\mathbb{R}^n)$ for any $0 < p \leq 1$. It is natural to ask what are the substitutes for $L^p(\mathbb{R}^n)$ when p is in this range; this circle of ideas has received considerable attention in harmonic analysis during the past thirty years (see, for example, [COI], [CHR2], [COW1, 2], [FS], [KRA2] and [STE], etc.).

We now understand that the best substitutes for the L^p spaces are the atomic Hardy spaces. Of course the holomorphic Hardy spaces on a domain in $\mathbb{C}^n$ and the real variable harmonic Hardy spaces on $\mathbb{R}^N$ are, at least on a formal level, quite different. It is natural to wish to find a way to connect them. With this end in view, the abstract Hardy spaces on a space of homogeneous type have been introduced and studied by Coifman and Weiss [COW1, 2] and others. For the case $0 < p \leq 1$, the atomic Hardy spaces $H^p(X)$ on a space of homogeneous type were introduced in [COI] and [COW1, 2]. With their definition of $H^p(X)$, it cannot be guaranteed that the Calderón-Zygmund operators are bounded on $H^p(X)$ when $p < 1/2$—even when X is the real line.

One of the main purposes of this paper is to find a natural way to define Hardy spaces $H^p(X)$ on a space of homogeneous type with $0 < p < 1$ in such a way that singular integrals will be bounded on all of these Hardy spaces. This will extend the work of Coifman and Weiss in [COW1], [COW2]. In order to achieve this goal we shall have to address several important ancillary issues: how to define higher order moment

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conditions on an arbitrary space of homogeneous type, how to define analogues of smooth functions in that setting, and how to define analogues of polynomials or other "testing functions".

The second purpose of this paper is to establish duality theorems for the Hardy spaces we have defined. The study of singular integrals acting on L^p spaces on a space of homogeneous type has attracted many authors; one significant recent study includes the adaptation of $T(1)$ or $T(b)$ theorems to $L^p(X)$ on a space of homogeneous type X (see [CW1, 2], [CHR1, 2] and references therein.)

The third purpose of this paper is to use some of those ideas to study the boundedness and compactness and other properties of some generalized Toeplitz operators (including commutators of a singular integral operator and a multiplication operator) on function spaces on a space of homogeneous type. In particular, we shall extend the results in [CG], [CRW], [JAN] and [LI] from $\mathbb{R}^n$ or S^{2n-1} to much more general settings.

The paper is organized as follows: In Section 2, we recall and prove some preliminary results, and introduce some notation and definitions. The atomic Hardy spaces $H^p(X)$ when $p < 1$ are defined and the duality theorem is stated and proved in Section 3. In Section 4, we introduce several examples which fit our models. In Section 5, we prove boundedness of singular integral operators on H^p. In Section 6, we study the boundedness of some generalized Toeplitz operators on H^p or L^p. The compactness of commutators on L^p and $H^p(X)$ is studied in Section 7. Finally, in Section 8, we shall apply some of the theorems we proved in the previous sections to prove analogous theorems for holomorphic Hardy space on some domains in $\mathbb{C}^n$.

2. Preliminaries

Let X be a locally compact Hausdorff space. A *homogeneous structure* on X consists of a positive regular Borel measure μ on X and a family $\{B(x,r) : x \in X, r > 0\}$ of basic open subsets of X such that for some constants $c > 1$ and $K > 1$ we have

(1) $x \in B(x,r)$ for all $x \in X$ and every $r > 0$;
(2) If $x \in X$ and $0 < r_1 \leq r_2$, then $B(x,r_1) \subset B(x,r_2)$;
(3) $0 < \mu(B(x,r)) < \infty$ for all $x \in X$ and all $r > 0$;
(4) $X = \cup_{r>0} B(x,r)$ for some (and hence every) $x \in X$;
(5) $\mu(B(x,cr)) \leq K\mu(B(x,r))$ for all $x \in X$ and all $r > 0$.
(6) If $B(x_1,r_1) \cap B(x_2,r_2) \neq \emptyset$ and $r_1 \geq r_2$, then $B(x_2,r_2) \subset B(x_1,cr_1)$;

We say that X is a *space of homogeneous type* if X is a locally compact Hausdorff space having a *homogeneous structure*. Following Christ [CHR1], we assume from now on that

$$\mu(\{x\}) = 0 \tag{2.1}$$

for all $x \in X$.

If X is a space of homogeneous type, then one may define a quasi-distance on X as follows: If $x, y \in X$, then we let

$$d(x,y) = \inf\{t : y \in B(x,t), \text{ and } x \in B(y,t)\} \tag{2.2}$$

It is clear that $d(x,y) = d(y,x)$, and $d(x,y) = 0$ if and only if $x = y$. From the so-called "doubling property" (5), we have that $d(x,z) \leq C(c,K)(d(x,y) + d(y,z))$. Therefore $d(\,\cdot\,,\,\cdot\,)$ is a quasi-metric on X. Coifman and Weiss refer to this quasi-metric as the "measure distance".

For certain purposes, it is useful to choose a quasi-metric such that the measure of a ball $B(x,r)$ associated to the quasi-metric is comparable to a fixed power r^γ of its radius. Follows Theorem 3 in [MS1], , we have the following result:

Lemma 2.1. *Let (X,μ) be a space of homogeneous type. For any positive number γ, there is a quasi-metric d_γ on X such that if*

$$B_\gamma(x_0,r) = \{x \in X : d_\gamma(x,x_0) < r\},$$

then

$$\mu(B_\gamma(x_0,r)) \approx r^\gamma. \tag{2.3}$$

and with this quasimetric, we have

$$\int_{X\setminus B(x_0,t)} \mu(B(x,t))^{-s} d\mu(x) \leq C_s \mu(B(x_0,t))^{-s+1} \tag{2.4}$$

for all $s > 1$.

Now we define the maximum mean oscillation on balls with fixed radius r as follows:

$$M(r,f) = \sup_{x\in X} \left\{ \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - m_B(f)|\, d\mu \right\}. \tag{2.5}$$

[Here $m_B(f)$ is the mean value of f on the ball B.]

Definition 2.2. Let (X,d,μ) be a space of homogeneous type (the homogeneous structure is given by the quasi-metric d.) Let $f \in L^1_{\text{loc}}(X)$. We say that $f \in BMO(X)$ if

$$\|f\|_{BMO} \equiv \|f\|_* = \sup_{0<r<\infty} M(r,f) < \infty.$$

We say that $f \in VMO(X)$ if $f \in BMO(X)$ and

$$\lim_{r\to 0^+} M(r,f) = 0.$$

Now we may define the atomic H^1 space as follows.

Definition 2.3. Let (X, d, μ) be a space of homogeneous type. Let $a \in L^\infty(X)$. We say a is an *atom* (or a 1-atom) if there is a ball B such that $\mathrm{supp}(a) \subset B$ and

$$\text{(i)} \quad |a(x)| \leq 1/\mu(B) \quad ; \quad \text{(ii)} \quad \int_B a(x)\, d\mu = 0.$$

Then

$$H^1(X) = \Big\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j \text{ are atoms and } \{\lambda_j\}_{j=1}^{\infty} \in \ell^1,\ \lambda_j \geq 0 \Big\} \tag{2.6}$$

with norm

$$\|u\|_{H^1} = \inf \Big\{ \sum_{j=1}^{\infty} \lambda_j : u = \sum_{j=1}^{\infty} \lambda_j a_j \Big\}. \tag{2.7}$$

The following result was proved in [COW2] and later in [MS2]:

THEOREM 2.4. *Let X be a space of homogeneous type. Then*

(i) $\Big[H^1(X)\Big]^* = BMO(X)$;
(ii) $\Big[VMO(X)\Big]^* = H^1(X)$.

The question of how to define atomic $H^p(X)$ spaces when $0 < p < 1$ is more complex. For the case when p is very closed to 1, it was treated by Coifman and Weiss [COW1,2] (also see [MS2]). Let a be a bounded function on X with support in some ball $B = B(x_0, r)$. We say that a is a *p-atom* in the sense of Coifman and Weiss if

$$\text{(i)} \quad |a(x)| \leq \mu(B)^{-1/p} \quad ; \quad \text{(ii)} \quad \int_X a(x)\, d\mu(x) = 0.$$

The atoms of Coifman and Weiss are natural for values of p that are close to 1 (where the elementary mean value zero property suffices for the purpose of studying singular integrals); when the value of p is small, then the definition of Hardy space requires a higher order moment condition and is unworkable on an arbitrary space of homogeneous type.

In order to define a more natural p-atom when p is small, we need to develop appropriate machinery to produce the necessary moment conditions for p-atoms. With a view to using Campanato-Morrey theory (see [KRA2]), one essential question is therefore how to define polynomials on a space of homogeneous type. Let us begin by taking a new look at the problem on $\mathbb{R}^n$, where the traditional method is to exploit the canonical coordinate system to define monomials and polynomials.

We shall begin by replacing the usual polynomials by a family of testing functions that have certain of the functorial properties of polynomials.

Definition 2.5. Let X be a space of homogeneous type. We say that there is a family $\mathcal{P}$ of 'testing polynomials' on X if there is a positive integer-valued function

$n(i,x)$ of the non-negative integer i and the point $x \in X$ $(n(0,x)=1)$ and families of functions

$$\mathcal{P}(x_0) = \{1\} \cup \{\mathbf{p}_{ij}(x,x_0) : j = 1, \cdots, n(i,x_0), i = 1, 2, \cdots\}.$$

[It is helpful to think of i as the degree of the monomial $\mathbf{p}_{ij}$.] We let $\mathcal{P} = \cup_{x_0 \in X} \mathcal{P}(x_0)$. In practice we will assume that $\mathcal{P}$ contains the constant functions.

Let k be a positive integer and $x_0 \in X$. We say that $p(x,x_0)$ is a *polynomial* associated to x_0, of *degree* not exceeding k, if there are constants c_{ij}, $0 \leq i \leq k, 1 \leq j \leq n(i,x_0)$ with

$$p(x,x_0) = \sum_{i=0}^{k} \sum_{j=1}^{n(i,x_0)} c_{ij} \mathbf{p}_{ij}(x,x_0), \tag{2.8}$$

with each $p_{ij} \in \mathcal{P}(x_0)$. We will usually consider constant functions to be polynomials.

Let (X,d,μ) be a space of homogeneous type. We shall use the notation $(X,d,\mu,\mathcal{P})$ to indicate the existence of the family of 'testing polynomials'.

EXAMPLE 1. *Let $X = \mathbb{R}^N$, $d(x,y) = |x-y|$, and let μ be ordinary N-dimensional Lebesgue measure. Then we let $n(i,x_0) = \binom{N+i-1}{i-1}$ and $n(0,x) = 1$ for every x as usual. Finally, $p_{ij}(x,x_0) = p_{ij}(x)$ is defined as follows: $p_{i1}(x) = x_1^i$, $p_{i2}(x) = x_1^{i-1}x_2, \cdots, p_{i\,n(i)}(x) = x_N^i$ (note here that we are simply enumerating all the monomials of degree i).* □

Definition 2.6. Let (X,d,μ) be a space of homogeneous type. Let $1 \leq q \leq \infty$, and $0 < \alpha < \infty$. We shall say that a locally integrable function ϕ belongs to $L(\alpha,q)(X)$ if

$$\|\phi\|_{\alpha,q} = \sup_B \left\{ \mu(B)^{-\alpha} \left[\inf \int_B |\phi(x) - m_B(\phi)|^q \frac{d\mu(x)}{\mu(B)} \right]^{1/q} \right\} < \infty. \tag{2.9}$$

Here the supremum is taken over all balls B in X.

We say that $\phi \in \mathrm{Lip}_\alpha(X)$, $\alpha > 0$, if

$$\|f\|_{\mathrm{Lip}_\alpha} = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)^\beta} : x,y \in X, x \neq y \right\} < \infty. \tag{2.10}$$

The following proposition is due to Macás and Segovia [MS1].

Proposition 2.7. *Let (X,d,μ) be a space of homogeneous type. Let $1 \leq q \leq \infty$ and $0 < \alpha < \infty$. If $\phi \in L(\alpha,q)$, then there is a function $\psi(x)$ such that*

(i) $\phi(x) = \psi(x)$ for almost all $x \in X$;   and

(ii) $|\psi(x) - \phi(x)| \leq C\|\phi\|_{\alpha,q}\mu(B)^\alpha$ where B is any ball containing both x and y.

Note: If the α is very large, then the space $L(\alpha, q)$ may contain only constants. This is a saturation theorem in the sense of Favard. For example, if $X = \mathbb{R}^n$ and $d(x, y) = |x - y|$ and μ is the Lebesgue measure, then $L(\alpha, q) = \mathbb{C}$ if $\alpha > 1/n$.

In what follows, we shall identify the functions ϕ and ψ in the last proposition without further comment.

As to the relationship between $L(\alpha, q)$ and $\mathrm{Lip}_\alpha(X)$, we prove the following proposition:

Proposition 2.8. *Let (X, d_γ, μ) be a space of homogeneous type. Let $1 \le q \le \infty$ and $0 < \alpha < \infty$. Then $\phi \in L(\alpha, q)$ if and only if $\phi \in Lip_{\gamma\alpha}(X)$.*

Proof. Suppose that $\phi \in \mathrm{Lip}_{\gamma\alpha}(X)$. Let B be any ball in X. Then we have

$$\begin{aligned}
&\frac{1}{\mu(B)} \int_B |\phi(x) - m_B(\phi)|^q \, d\mu(x) \\
\le \ & \|\phi\|^q_{\mathrm{Lip}_{\gamma\alpha}} \int_B \left(\frac{1}{\mu(B)} \int_B d_\gamma(x, y)^{\gamma\alpha} \, d\mu(y) \right)^q \frac{d\mu(x)}{\mu(B)} \\
\le \ & \|\phi\|^q_{\mathrm{Lip}_{\gamma\alpha}} \int_B \left(\frac{1}{\mu(B)} \int_B r^{\gamma\alpha} \, d\mu(y) \right)^q \frac{d\mu(x)}{\mu(B)} \\
\le \ & \|\phi\|^q_{\mathrm{Lip}_{\gamma\alpha}} r^{\gamma\alpha q}
\end{aligned}$$

Therefore

$$\left(\frac{1}{\mu(B)} \int_B |\phi(x) - m_B(\phi)|^q \, d\mu(x) \right)^{1/q} \le \|\phi\|_{\mathrm{Lip}_{\gamma\alpha}} r^{\gamma\alpha} \le C \|\phi\|_{\mathrm{Lip}_{\gamma\alpha}} \mu(B)^\alpha.$$

Therefore $f \in L(\alpha, q)(X)$.

Conversely, let $\phi \in L(\alpha, q)$. We will show that $\phi \in \mathrm{Lip}_{\gamma\alpha}(X)$. By Proposition 2.8, we have

$$|\phi(x) - \phi(y)| \le C \|\phi\|_{\alpha,q} \mu(B)^\alpha$$

for any ball $B(x_0, r)$ containing x and y. In particular, we have

$$|\phi(x) - \phi(y)| \le C \|\phi\|_{\alpha,q} \mu(B(x, c d_\gamma(x, y))^\alpha \le C^\alpha \|\phi\|_{\alpha,q} K^\alpha d_\gamma(x, y)^{\gamma\alpha}$$

This implies that $\phi \in \mathrm{Lip}_{\gamma\alpha}(X)$. Therefore the proof of the proposition is complete. ∎

Next we shall define smooth functions on X by comparing them with our testing functions above.

For $\gamma > 0$ fixed, let β_γ be the supremum of positive numbers t_γ such that

$$|d_\gamma(x, z) - d_\gamma(y, z)| \le C(K, c) r^{1 - t_\gamma} d_\gamma(x, y)^{t_\gamma}$$

for all $x, y \in X$ with $d_\gamma(x, z) < r$ and $d_\gamma(y, z) < r$ and $r > 0$. In order to define smooth functions based on the family of testing polynomials, it is natural to first assume that $\mathcal{P} \subset \mathrm{Lip}_{\beta_\gamma}(X)$. Thus, we shall give the following definition.

Definition 2.9. Let $(X, d_\gamma, \mu, \mathcal{P})$ be a space of homogeneous type. Fix a number $1 \le q \le \infty$. For $0 < \alpha < \gamma^{-1}\beta_\gamma$ and a nonnegative integer k, we say that a locally integrable function ϕ belongs to $L(\alpha, k, q)$ if $B = B(x_0, r)$ and

$$\|\phi\|_{\alpha,k,q} = \sup_B \left\{ \left[\frac{1}{\mu(B)^{\alpha+k/\gamma}} \inf_p \left[\int_B |\phi(x) - p(x, x_0)|^q \frac{d\mu(x)}{\mu(B)} \right]^{1/q} \right\} < \infty. \tag{2.11}$$

Here the infimum is taken over all testing polynomials p centered at x_0 of degree not exceeding k, and the supremum is taken over all balls $B = B(x_0, r) \subseteq X$.

Proposition 2.10. *Let $(X, d_\gamma, \mu, \mathcal{P})$ be a space of homogeneous type. Let $1 \le q \le \infty$ and $0 < \alpha < \infty$. Then $L(\alpha, 0, q) = L(\alpha, q)$.*

Proof. It is obvious that

$$\|\phi\|_{\alpha,0,q} \le \|\phi\|_{\alpha,q}.$$

The opposite inequality follows from the following standard fact.

$$\left(\int_{B(x_0,r)} |\phi - m_B(\phi)|^q \frac{d\mu}{\mu(B)} \right)^{1/q} \le 2 \inf \left\{ \left(\int_B |\phi - c|^q \frac{d\mu}{\mu(B)} \right)^{1/q} : c \in \mathbb{C} \right\}$$

and the proof of the proposition. ▯

From the definition of $L(\alpha, k, q)(X)$, it is clear that

$$L(\alpha, k, q_2)(X) \subset L(\alpha, k, q_1)(X)$$

if $q_2 \ge q_1$. We shall show in the next lemma that the converse is true as well.

Lemma 2.11. *Let $(X, d_\gamma, \mu, \mathcal{P})$ be a space of homogeneous type. Then we have $L(\alpha, k, 1)(X) = L(\alpha, k, \infty)(X)$ for all $0 < \alpha < \gamma^{-1}\beta_\gamma$.*

Let us begin by recalling the following covering lemma. We will use it in the proof of Lemma 2.11 and in later sections as well.

Lemma 2.12. *Let (X, d_γ, μ) be a space of homogeneous type and E be a compact subset of X. Suppose that there is a family $\mathcal{F}$ of balls which cover E. Then there is a sequence $\{B(x_k, r_k)\}$ of disjoint balls such that E is covered by the family of balls $\{B(x_k, 5cr_k)\}$. Moreover, there is a constant C such that for any point $x \in X$ there are at most C balls $B(x_k, 5cr_k)$ that contain x (that is, the covering $\{B(x_k, 5cr_k)\}$ has valence C).*

For details on the covering lemma, see [COW1], [COW2], or [CHR1].

Proof of Lemma 2.11: It suffices to prove that for any $f \in L(\alpha, k, 1)(X)$ we have $f \in L(\alpha, k, \infty)$. In fact we show that there is a function $g \in \mathrm{Lip}_{\alpha\gamma}$ such that $g = f$ for a.e. $x \in X$ and for any ball $B(x_0, r)$ there is a testing polynomial of degree not exceeding k so that

$$|g(x) - p(x, x_0)| \le C\|f\|_{L(\alpha,k,1)}\mu(B)^{\alpha+k/\gamma} \tag{2.12}$$

for all $x \in B(x_0, r)$.

By Propositions 2.7 and 2.8, there is a function $g \in \operatorname{Lip}_{\gamma\alpha}(X)$ such that $f(x) = g(x)$ for a.e. $x \in X$ and

$$|g(x) - g(y)| \le C\|f\|_{\alpha,0,q} d_\gamma(x,y)^{\gamma\alpha}, \quad x, y \in X.$$

[Note here that we use only the result of 2.7 for the $L(\alpha, 0, 1)$ space—that is, only the case $k = 0$.] Therefore, it suffices to prove that this g satisfies (2.12).

Since $f \in L(\alpha, k, 1)$, we know that $g \in L(\alpha, k, 1)$. For any ball $B(x_0, r)$ in X, there is a testing polynomial $p_1(x, x_0)$ of degree less than or equal to k such that

$$\int_B |g(x) - p_1(x, x_0)| \frac{d\mu}{\mu(B)} \le C\|f\|_{\alpha,k,1}\mu(B)^{\alpha+k/\gamma}.$$

Applying the facts that $g \in \operatorname{Lip}_{\alpha\gamma}(X)$ and $p_1(x, x_0) \in \operatorname{Lip}_{\beta_\gamma}$ (by assumption) and Proposition 2.7 again, we see that

$$|g(x) - p_1(x, x_0) - g(x_0) + p_1(x_0, x_0)| \le C\|f\|_{\alpha,k,1}\mu(B(x_0, d_\gamma(x, x_0))^{\alpha+k/\gamma}$$

for all $x \in B(x_0, r)$.

Now we choose

$$p(x, x_0) = p_1(x, x_0) + g(x_0) - p_1(x_0, x_0).$$

Then

$$|g(x) - p(x, x_0)| \le C\mu(B)^{\alpha+k/\gamma}$$

for $x \in B(x_0, r)$ and $r > 0$. Here we have used the covering lemma; the constant depends only on k, and is independent of x_0.

We have completed the proof of (2.12), and thus of Lemma 2.11. $\square$

3. Hardy Spaces with $p < 1$

We begin with the definition of atom:

Definition 3.1. Fix a positive number n. Let $(X, d_n, \mu, \mathcal{P})$ be a space of homogeneous type ($d_n = d_\gamma$ when $\gamma = n$ in Lemma 2.1.) Let $0 < p \le 1$. Then a bounded measurable function a defined on X is said to be a *p-atom* if there is a ball $B = B(x_0, r)$ such that

(i) $\operatorname{supp}(a) \subset B$; (ii) $|a(x)| \le 1/\mu(B)^{1/p}$;

(iii) For any testing polynomial $p(x, x_0)$ of degree less than or equal to $[n(1/p-1)]$, we have

$$\int_B a(x)p(x, x_0)\, d\mu(x) = 0$$

where $[x]$ is the greatest integer that does not exceed x.

Now we are ready to define the real Hardy spaces on X when $p \le 1$.

Definition 3.2. Let $(X, d_n, \mu, \mathcal{P})$ be a space of homogeneous type. Let $0 < p \leq 1$. A measurable function f is said to belong to $H^p(X)$ if there are a sequence $\{\lambda_k\} \in \ell^p$ of non-negative numbers and a sequence of p-atoms $\{a_k\}$ such that

$$f = \sum_{k=1}^{\infty} \lambda_k a_k$$

in the sense of distributions (when $p = 1$ we may actually take the convergence to be in L^1 but when $p < 1$ we must use the distribution topology). We let

$$\|f\|_{H^p} = \inf \left\{ \sum_{k=1}^{\infty} \lambda_k^p : f = \sum \lambda_k a_k \right\}$$

For simplicity, we shall use the following notation. For each $0 < p \leq 1$ and positive number n, we let

$$\alpha(p, n) = 1/p - 1 - \frac{1}{n}\Big[n(1/p - 1)\Big].$$

The main purpose of this section is to prove the following theorem.

THEOREM 3.3. *Fix a positive number n. Let $(X, d_n, \mu, \mathcal{P})$ be a space of homogeneous type. Let $0 < p \leq 1$ be such that $n\alpha(p, n) \leq \beta_n$. Then*

$$\Big[H^p(X)\Big]^* = L\left(\alpha(p, n), [n(1/p - 1)], q\right)(X)$$

for all $1 \leq q \leq \infty$.

We shall break the proof of Theorem 3.3 up into several lemmas. The dimension parameter n should be considered fixed once and for all.

Lemma 3.4. *Let $(X, d_n, \mu, \mathcal{P})$ be a space of homogeneous type. Let $0 < p \leq 1$ be such that $n\alpha(p, n) \leq \beta_n$ Then*

$$L(\alpha(p, n), [n(1/p - 1)], q)(X) \subset \Big[H^p(X)\Big]^*$$

for all $1 \leq q \leq \infty$.

Proof. It suffices to prove the case $q = 1$. Let $f \in L(\alpha(p, n), [n(1/p - 1)], 1)(X)$. Then, for any $u \in H^p(X)$ with

$$u = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{supp}(a_j) \subset B_j,$$

we have for some $p(x, x_j) \in \mathcal{P}_{[n(1/p-1)]}(x_j)$ (the space of all testing polynomials, associated to x_j, that have degree less than or equal $[n(1/p-1)]$) that

$$\begin{aligned}
&\left|\int_X f \sum_{j=1}^k \lambda_j a_j \, d\mu\right| \\
&= \left|\sum_{j=1}^k \lambda_j \int_{B_j} (f(x) - p(x, x_j) a_j(x) \, d\mu(x)\right| \\
&\leq \sum_{j=1}^k \lambda_j \int_{B_j} |f(x) - p(x, x_j)||a_j(x)| \, d\mu(x) \\
&\leq \sum_{j=1}^k \lambda_j \mu(B_j)^{1-1/p} \int_{B_j} |f(x) - p(x, x_j)| \frac{d\mu}{\mu(B_j)} \\
&\leq C\|f\|_{\alpha(p,n),[n(1/p-1)],1} \sum_{j=1}^k \lambda_j \mu(B_j)^{1-1/p} \mu(B_j)^{\alpha(p,n)+\frac{1}{n}[n(1/p-1)]} \\
&\leq C\|f\|_{\alpha(p,n),[n(1/p-1)],1} \sum_{j=1}^k \lambda_j \\
&\leq C\|f\|_{\alpha(p,n),[n(1/p-1)],1} \|u\|_{H^p}.
\end{aligned}$$

Therefore, letting $k \to \infty$, we have

$$\left|\int_X f u \, d\mu\right| \leq C\|f\|_{L(\alpha(p,n),[n(1/p-1)],1)} \|u\|_{H^p}$$

This completes the proof of the lemma. ▯

To prove the converse, we need the following notation and lemmas.

Definition 3.5. Let $(X, d, \mu, \mathcal{P})$ be a space of homogeneous type. Let a be a measurable function with support in some ball $B = B(x_0, r)$. We say that a is a $(p, 2)$-*atom* if, instead of satisfying the classical size condition $\|a\|_\infty \leq \mu(B)^{1/p}$, the function a instead satisfies $\left(\int_B |a|^2 \, d\mu/\mu(B)\right)^{1/2} \leq 1/\mu(B)^{1/p}$. Let $H^{p,2}(X)$ be the atomic Hardy space created by replacing classical p-atoms with $(p, 2)$-atoms.

Now we shall prove the following simple lemma.

Lemma 3.6. *Let $(X, d_n, \mu, \mathcal{P})$ be a space of homogeneous type. Then $H^p(X) \subset H^{p,2}(X)$ and the embedding is continuous.*

Proof. Let $u \in H^p(X)$. Then there are a sequence of positive numbers $\{\lambda_k\}$ and a sequence of p-atoms $\{a_k\}$ such that

$$u = \sum_k \lambda_k a_k \quad , \quad \|u\|_{H^p}^p \approx \sum_k \lambda_k^p.$$

Since the a_k are p-atoms, they must be $(p,2)$-atoms. Thus $u \in H^{p,2}(X)$ and

$$\|u\|_{H^{p,2}}^p \leq C \sum_k \lambda_k^p \leq C\|u\|_{H^p}^p.$$

Thus $H^p(X) \subset H^{p,2}(X)$ and the embedding is continuous. $\square$

Next we shall prove the following:

Lemma 3.7. *Let $(X, d_n, \mu, \mathcal{P})$ be a space of homogeneous type. Let $0 < p \leq 1$ be such that $n\alpha(p,n) \leq \beta_n$. Then $\left[H^p(X)\right]^* \subset L(\alpha(p,n), [n(1/p-1)], q)(X)$ for all $1 \leq q \leq \infty$.*

Proof. Let $\ell \in \left[H^p(X)\right]^*$. We need to find a function $f \in L(\alpha(p,n), [n(1/p-1)], q)$ such that

$$\ell(u) = \ell_f(u) = \int_X f u \, d\mu$$

for all $u \in H^p(X)$.

Since $H^p(X) \subset H^{p,2}(X)$ and is dense, we may extend ℓ to be a bounded linear functional on $H^{p,2}(X)$. More precisely, each $H^{p,2}$ atom can be taken to be an L^∞ function, so elements of $[H^p]^*$ act naturally on these atoms [the indicated extension of the functional from H^p to $H^{p,2}$ is not valid just by abstract functional analysis]. Alternatively, Coifman and Weiss [COW2] have shown that H^p and $H^{p,2}$ are equivalent spaces, so the point may be taken as moot. All of these matters are laid out in detail in that source.

For each ball $B = B(x_0, c_0)$, ℓ is a bounded linear functional on $L^2(B,p)$, the space of all L^2 functions u with supports on B and satisfying

$$\int_B u(x)p(x,x_0))\, d\mu(x) = 0.$$

Here $p \in \mathcal{P}_{[n(1/p-1)]}(x_0)$. Thus, by elementary Hilbert space theory, there is an $f \in L^2(B)$ (depending on x_0) such that

$$\ell(u) = \int_B f u \, d\mu, \qquad \text{for all } u \in L^2(B,p).$$

Since $x_0 \in X$ and $c_0 > 0$ are arbitrary, there is a function $f \in L^2_{\mathrm{loc}}(X)$ such

$$\ell(a) = \int_X f(x)a(x)\, d\mu(x)$$

for all p-atoms. Therefore, we have

$$\ell(u) = \int_X f(x)u(x)\, d\mu(x)$$

for any u a finite linear combination of p-atoms.

Therefore, by Lemma 2.14, it suffices to show that

$$f \in L(\alpha(p,n), [n(1/p-1)], 1)(X)$$

and

$$\|f\|_{\alpha(p,n),[n(1/p-1)],1} \le C\|\ell\|_{\left[H^p(X)\right]^*}.$$

For any p-atom a with support in $B(x_0, r)$, we have

$$\ell(a) = \ell_f(a) = \int_X f(x)a(x)\ d\mu(x) = \int_B \Big(f(x) - p(x,x_0)\Big)a(x)\ d\mu(x)$$

for all $p(x, x_0) \in \mathcal{P}_{[n(1/p-1)]}(x_0)$. Fix $x_0 \in X$, and $r > 0$. We have

$$\begin{aligned} &\inf\Big\{\int_{B(x_0,r)} |f(x) - p(x,x_0)|\, d\mu(x) : p(x,x_0) \in \mathcal{P}_{[n(1/p-1)]}(x_0)\Big\} \\ = &\ \sup\Big\{\Big|\int_{B(x_0,r)} f(x)g(x)\, d\mu\Big| : \|g\|_\infty \le 1\Big\}, \end{aligned}$$

where the infimum is taken over all testing polynomials associated to the ball $B(x_0, r)$, of degree not exceeding $[n(1/p]-1)]$; also the supremum is taken on all $g \in \mathbf{P}(x_0, p)^\perp$, where

$$\begin{aligned} \mathbf{P}(x_0,p)^\perp = \Big\{ & g \in L^\infty(B(x_0,r)) : \\ & \int_{B(x_0,r)} p(x,x_0)g(x)\ d\mu(x) = 0 \quad \text{for all } p(x,x_0) \in \mathcal{P}_{[n(1/p-1)]}(x_0)\Big\} \end{aligned}$$

We claim that $g \in \mathbf{P}(x_0, p)^\perp$ and $\|g\|_\infty \le 1$ if and only if $a(x) = \mu(B_r(x_0))^{-1/p}g(x)$ is a p-atom supported on $B_r(x_0)$. The proof of this claim is obvious. Thus

$$\begin{aligned} &\mu(B)^{-1/p} \inf_p \{\int_B |f(x) - p(x,x_0)])|\, d\mu(x) : p(x,x_0) \in \mathcal{P}_{[n(1/p-1)]}(x_0)\} \\ = &\ \sup\{\Big|\int_B f(x)\, a(x)\, d\mu(x)\Big| : a \text{ is } p - \text{atom}\} \\ \le &\ \|\ell\|_{\left[(H^p)\right]^*} \end{aligned}$$

Therefore $f \in L(\alpha(p, n), [n(1/p - 1)], 1)(X)$.

Combining Lemmas 3.4 and 3.7, the proof of Theorem 3.3 is complete. ▯

Before ending this section, let us make the following remark about the number β_γ. It is easy to see that $\beta_\gamma = \beta_{\gamma_0}\gamma_0/\gamma$ for any $\gamma_0, \gamma > 0$. Moreover, if $\gamma \ge \gamma_0$ and

$$\gamma_0\alpha - [\gamma_0\alpha] \le \beta_{\gamma_0}, \quad \alpha > 0,$$

then

$$\gamma\alpha - [\gamma\alpha] \le \beta_\gamma$$

since $\gamma[\gamma_0\alpha]/\gamma_0 \ge [\gamma\alpha]$. Therefore, for each $0 < p \le 1$, in order to guarantee that $\gamma\alpha(p, \gamma) \le \beta_\gamma$, we may increase the value of γ by a suitable amount.

4. Some Examples

In this section, we shall introduce several examples of spaces of homogeneous type. In each of these examples, one can see that our definition for the atomic Hardy space when $0 < p \leq 1$ and our duality Theorem 3.3 are natural generalizations of the classical theory.

EXAMPLE 2. *Let $X = \mathbb{R}^N$, equipped with the Euclidean metric $d(x,y) = |x-y|$. It is clear that $d(x,y) \in Lip_1(\mathbb{R}^{2N})$. Let $d\mu = dv$ be the Lebesgue measure on $\mathbb{R}^N$. Then (X, d, μ) is a space of homogeneous type, and $\mu(B(x,r)) \approx r^N$ for any $x \in \mathbb{R}^N$ and $r > 0$.*

Now we define the family of testing polynomials. Let $n(i,x) = \binom{N+i-1}{i-1} = n(i)$. Let $\mathbf{p}_0(x) = 1$, and

$$\mathbf{p}_{i1}(x,x_0) = x_1^i, \ \mathbf{p}_{i2}(x,x_0) = x_1^{i-1}x_2, \ \cdots, \ \mathbf{p}_{in(i)}(x,x_0) = x_N^i$$

for $i = 1, 2, \cdots$. Then

(i) $L(\alpha - [\alpha], [\alpha])(\mathbb{R}^N) = \Lambda_{N\alpha}(\mathbb{R}^N)$, the Zygmund class in $\mathbb{R}^N$, for any $\alpha > 0$;

(ii) a is a p-atom in our sense if and only if it is a p-atom in the classical sense, i.e., there is ball $B = B(x_0, r)$ such that $supp(a) \subset B$, $|a(x)| \leq 1/[v(B)]^{1/p}$ and

$$\int_{\mathbb{R}^N} a(x)x^k dx = 0, \qquad |k| = k_1 + k_2 + \cdots k_N \leq [N(1/p-1)];$$

for all $k = (k_1, \cdots, k_N)$ with $k_i \geq 0$ and $x^k = x_1^{k_1} \cdots x_N^{k_N}$.

(iii) $\left[H^p(\mathbb{R}^n)\right]^* = \Lambda_{n(1/p-1)}(\mathbb{R}^n)$.

Verification of Example: The proof of the results of Example 2 follows from the main theorems of Krantz [KRA1] and Latter [LAT]. $\square$

EXAMPLE 3. *In this example, we take X to be the Heisenberg group $\mathbb{H}_n$. We begin with some notation and definitions.*

The Heisenberg group: $\mathbb{H}_n$ is the Lie group with underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ and group operation ($z = x + iy$, $z' = x' + iy'$):

$$bb' = (x,y,t)(x,'y',t') = (z,t)(z',t') = (z + z', t + t' + 2Im\langle z, z'\rangle).$$

Haar measure on $\mathbb{H}_n$ is, up to a constant multiple, $dv = dxdydt$—Lebesgue measure on $\mathbb{R}^{2n+1}$. For each $c > 0$, we define the dilations on $\mathbb{H}_n$ as follows:

$$c \cdot (x,y,t) = (c\,x, c\,y, c^2 t).$$

The norm of (x,y,t) is defined as:

$$\|(x,y,t)\| = \{t^2 + (|x|^2 + |y|^2)^2\}^{1/4}$$

It is clear that $d((x,y,t),(x',y',t')) \equiv \|(x-x', y-y', t-t')\| \in \mathrm{Lip}_{1/2}(\mathbb{H}_n)$. *Thus* $(\mathbb{H}_n, d, dv)$ *is a space of homogeneous type.*

Moreover,

$$\|c\cdot(x,y,t)\| = |c|\,\|(x,y,t)\|.$$

For each $b \in \mathbb{H}_n$ *and* $r > 0$ *we define the ball*

$$B(b,r) = \{h \in \mathbb{H}_n : \|bh^{-1}\| < r\}, \quad cB(b,r) = B(b,cr)$$

Thus we have

$$v(cB) = c^{2(n+1)}v(B), \quad c > 0.$$

Therefore

$$v\Big(B(g,r)\Big) \approx r^{2(n+1)}, \quad g \in \mathbb{H}_n,\ r > 0.$$

Let $g = (x_1, \cdots, x_n, y_1, \cdots, y_n, t)$. *We say the* $p(g) = p(x,y,t)$ *is a polynomial of degree less than or equal* k *if*

$$p(g) = \sum_{|\alpha|\le n} c_\alpha g^\alpha$$

where $\alpha = (\alpha_1, \cdots, \alpha_n, \alpha_{n+1}, \cdots, \alpha_{2n}, \alpha_0)$ *with* α_0 *and* $\alpha_j, j \ge 1$ *non-negative integers, and*

$$|\alpha| = 2\alpha_0 + \sum_{j=1}^{2n} \alpha_j \le k.$$

Such a function p *is an element of* $\mathcal{P}_0$. *Then we define the family* $\mathcal{P}$ *of testing polynomials on* $\mathbb{H}_n$ *as follows:* $\mathcal{P}_k(x_0) = \mathcal{P}_k(0)$ *and* $\mathcal{P}_k(0)$ *is the set of all polynomials of degree not exceeding* k. *Then* $\mathcal{P} = \cup_{k=0}^{\infty}\mathcal{P}_k(0)$.

Let $X = \mathbb{H}_n$, *the Heisenberg group with metric and measure defined as the above. Moreover, we let the family* $\mathcal{P}$ *of 'testing polynomials' be the set* $\mathcal{P}$ *of all polynomials on* $\mathbb{H}_n$ *with the definition of degree as above. Then*

(i) $L\Big(\alpha - \frac{1}{2(n+1)}[2(n+1)\alpha], [2(n+1)\alpha]\Big)(\mathbb{H}_n) = \tilde{\Lambda}_{(n+1)\alpha}(\mathbb{H}_n)$ *for any* $\alpha > 0$*; this is the non-isotropic Zygmund space;*

(ii) a *is a* p*-atom in our sense if and only if it is a* p*-atom in the "classical" sense, i.e., there is a ball* $B(h,r)$ *in* $\mathbb{H}_n$ *such that supp*$(a) \subset B(h,r)$, $|a(x)| \le 1/v(B(h,r))^{1/p}$ *and*

$$\int_{\mathbb{H}_n} a(x) g^\alpha dv = 0,$$

for all α *and* $|\alpha| = 2\alpha_0 + \sum_{j=1}^{2n} \alpha_j \le [2(n+1)(1/p-1)]$*;*

(iii) $\Big[H^p(\mathbb{H}_n)\Big]^* = \tilde{\Lambda}_{(n+1)(1/p-1)}(\mathbb{H}_n)$.

Proof. Conclusion (i) follows from the main theorem in [KRA1] and (ii) is a corollary of Theorem 3.3. ∎

EXAMPLE 4. *Let $\mathcal{U}_n = \{(z,\xi) \in \mathbb{C}^n \times \mathbb{C} : Im\,\xi - |z|^2 > 0\}$ be the Siegel upper half space. Let B_{n+1} be the unit ball in $\mathbb{C}^{n+1}$. It is well known that $\mathcal{U}_n$ and B_n are biholomorphically equivalent.*

The automorphism group of $\mathcal{U}_n$ decomposes into three subgroups via the Iwasawa decomposition; the nilpotent piece of the decomposition acts simply transitively on $\partial\mathcal{U}_n$. Thus we may identify $\partial\mathcal{U}_n$ with that group, and it is the Heisenberg group $\mathbb{H}_n$. By way of the biholomorphism (a generalized Cayley transform), the boundary of the ball (less a point) may be identified with the Heisenberg group. [These matters are treated in detail in [BCK].]

When the boundary of the ball is equipped with these "translations", one has the following results. Let $X = \partial B_{n+1}$. Then

(i) $L(\alpha - \frac{1}{2(n+1)}[2(n+1)\alpha], [2(n+1)\alpha])(\partial B_{n+1}) = \tilde{\Lambda}_{(n+1)\alpha}(\partial B_{n+1})$ for all $\alpha > 0$;

(ii) There are p-atoms on ∂B, which correspond to those on the Heisenberg group;

(iii) $\Big[H^p(\partial B_{n+1})\Big]^* = \tilde{\Lambda}_{(n+1)(1/p-1)}(\partial B_{n+1})$

EXAMPLE 5. *Let $X = \partial\Omega$, where Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then $\partial\Omega$ has the structure of a space of homogeneous type, and it is similar to that for the unit ball. [See [KRA2], [KRA3] for details of this matter.]*

Then results analogous to (i) - (iii) of the last example hold in this context.

EXAMPLE 6. *Our Theorem 3.3 holds when X is a* nilpotent *group discussed in [ROS] and [KRA1] by choosing corresponding family of polynomials, we omit the details here.*

5. Singular Integrals on $H^p(X)$

In this section, we shall study the boundedness of some singular integrals on the Hardy spaces $H^p(X)$. First we recall several definitions from M. Christ [CHR2].

Definition 5.1. Let (X, d, μ) be a space of homogeneous type. A *standard kernel* is a function $K : X \times X \setminus \{x = y\} \to \mathbb{C}$ such that there exist $\epsilon > 0$, and $0 < C < \infty$ satisfying

$$|K(x,y)| \leq \frac{C}{\mu(\lambda(x,y))} \quad \text{for all distinct } x, y \in X; \tag{5.1}$$

here

$$\lambda(x,y) = \mu(B(x, d(x,y))), \tag{5.2}$$

and

$$|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')| \le \left(\frac{d(x,x')}{d(x,y)}\right)^{\epsilon}\frac{C}{\lambda(x,y)} \tag{5.3}$$

whenever $d(x,y)\ge cd(x,x')$.

Definition 5.2. A continuous linear operator $T:\Lambda_\delta\to\Lambda'_\delta$ is said to be *associated* to K if K is locally integrable away from the diagonal and

$$\langle Tf,g\rangle=\int\int K(x,y)f(y)g(x)\,d\mu(y)\,d\mu(x)$$

for all $f,g\in\Lambda_\delta$ whose supports are separated by a positive distance. Here Λ'_δ denotes the dual space of Λ_δ.

Definition 5.3. A singular integral operator T is a continuous linear operator from Λ_δ to Λ'_δ for some $\delta\in(0,\delta_0]$, which is associated to a standard kernel.

The following theorem is due to Coifman and Weiss [COW1]; the basic arguments appear in [CAZ] and [HAS].

THEOREM 5.4. *Any singular integral operator which is bounded on L^2 is also bounded on L^p for all $p\in(1,\infty)$, is weak type (1,1), and is bounded on BMO.*

The celebrated $T(1)$ theorem of David and Journé gives necessary and sufficient conditions to test when a singular integral operator is bounded on L^2. Let T be a singular integral operator. We say that T is *weakly bounded* on $L^2(X)$ if

$$|\langle T\phi,\psi\rangle|\le C\mu(B(x_0,r))$$

for all $\phi,\psi\in\Lambda_\delta(B(x_0,r))$ supported in B and

$$\|\phi\|_{L^2}\|\psi\|_{L^2}\le C\mu(B)$$

for all $x_0\in X$ and $r>0$. It is easy to show that if T is bounded on $L^2(X)$ then T is weakly bounded on L^2. Conversely, we have the following $T(1)$ theorem of David and Journé.

THEOREM 5.5. *Any singular integral operator T is bounded on L^2 if and only the following holds:*

(i) T is weakly bounded on $L^2(X)$;
(ii) $T(1)\in BMO(X)$ and $T^*(1)\in BMO(X)$.

It is known that a singular integral being bounded on L^2 does not imply that it is bounded on L^p for $p\le 1$. The natural question is:

Question 1. *Let T be a singular integral operator which is bounded on $L^2(X)$. Is T bounded from H^p to L^p for all $0<p\le 1$?*

The answer to this query is known to be “yes” when $p \leq 1$ and sufficiently close to 1. That is the following result of [COW2].

THEOREM 5.6. *Any singular integral operator that is bounded on L^2 is also bounded from H^p to L^p for all $p \in (1-\epsilon_1, 1]$ for some positive ϵ_1 depending only on X and the standard kernel K associated to the operator.*

From the proof of the above theorem, one can see that the ϵ_1 depends on the ϵ in the definition of a standard kernel K since the atoms in H^1 have 0-order cancellation. In order to have a theorem like Theorem 5.6 when p is small, one can imagine that we need a condition on K involving a higher order Lipschitz condition that depends on $0 < p < 1$. Therefore we pose the following condition on a standard kernel:

Definition 5.7. Let $(X, d_n, \mu, \mathcal{P})$ be a space of homogeneous type. A (p,k)-*standard kernel* is a function $K : X \times X \setminus \{x = y\} \to \mathbb{C}$ such that there exist $\epsilon > 0$, a positive integer k, and $0 < C < \infty$ satisfying (5.1), $\epsilon \geq 1/k$ and for fixed x and y there are two polynomials $p_1(x', y)$ and $p_2(x', y)$ in x' with degree not exceeding $[n(1/p-1)]$ such that

$$|K(x', y) - p_1(x', y)| + |K(y, x') - p_2(x', y)| \leq \left(\frac{d(x, x')}{d(x, y)}\right)^{\epsilon + \frac{1}{k}[kn(1/p-1)]} \frac{C}{\lambda(x, y)} \quad (5.4)$$

whenever $d(x, y) \leq c d(x, x')$.

Note that, when $[kn(1/p-1)] = 0$, we take $p_1(x', y) = K(x, y)$ and $p_2(x', y) = K(y, x)$.

The main purpose of this section is to prove the following theorem.

THEOREM 5.8. *Let $0 < p \leq 1$. Let T be a singular integral operator with (p,k)-standard kernel which is bounded on L^2. Then T is also bounded from H^p to L^p for all $0 < p \leq 1$.*

Proof. Let T be a singular integral operator with a (p,k)-standard kernel $K(x, y)$. To prove that $T : H^p(X) \to L^p(X)$ is bounded, from the definition of $H^p(X)$, it suffices to prove that

$$\|T(a)\|_{L^p} \leq C_p$$

for all p-atoms on X, where C is a constant independent of a. Let a be a p-atom with support in $B(x_0, r)$ and

$$\|a\|_{L^\infty} \leq 1/\mu(B)^{1/p}, \quad \int_X a(x) p(x, x_0)\, d\mu(x) = 0$$

for all $p(x, x_0) \in \mathcal{P}_{[n(1/p-1)]}(x_0)$. Thus

$$T(a)(x) = \int_X K(x, y) a(y)\, d\mu(y)$$

and

$$\|T(a)\|_{L^2} = \left\| \int_B K(\cdot, y) a(y)\, d\mu(y) \right\|_{L^2} \le C\|a\|_{L^2} = C\mu(B)^{(p-2)/(2p)}.$$

Thus, for $1 \le p < 2$,

$$\begin{aligned}
&\left\| \int_B K(\cdot, y) a(y)\, d\mu(y) \right\|_{L^p} \\
\le\ & \left\| \chi_{cB} \int_B K(\cdot, y) a(y)\, d\mu(y) \right\|_{L^p} + \left\| (1-\chi_{cB}) \int_B K(\cdot, y) a(y)\, d\mu(y) \right\|_{L^p} \\
\le\ & C\mu(cB)^{(2-p)/(2p)} \left\| \int_B K(\cdot, y) a(y)\, d\mu(y) \right\|_{L^2} \\
&\quad + \left\| (1-\chi_{cB}) \int_B K(\cdot, y) a(y)\, d\mu(y) \right\|_{L^p} \\
\le\ & C(c, K) + \left\| (1-\chi_{cB}) \int_B K(\cdot, y) a(y)\, d\mu(y) \right\|_{L^p}.
\end{aligned}$$

Now we consider $x \in X \setminus cB$. We choose a polynomial $p_{x_0}(y, x)$ of degree less than or equal to $[n(1/p-1)]$ such that

$$\begin{aligned}
|K(x, y) - p_{x_0}(y, x)| &\le \left(\frac{d(y, x_0)}{d(y, x)} \right)^{\epsilon + [kn(1/p-1)]/k} \frac{C}{\lambda(y, x)} \\
&\le C \left(\frac{d(y, x_0)}{d(x, x_0)} \right)^{\epsilon + [kn(1/p-1)]/k} \frac{C}{\lambda(x_0, x)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left\| (1-\chi_{cB}) \int_B K(\cdot, y) a(y)\, d\mu(y) \right\|_{L^p}^p \\
\le\ & \int_{X \setminus cB} \left| \int_B (K(x, y) - p_{x_0}(y, x)) a(y)\, d\mu(y) \right|^p d\mu(x) \\
\le\ & C \int_{X \setminus cB} \left[\int_B \left(\frac{d(y, x_0)}{d(x, x_0)} \right)^{\epsilon + [kn(1/p-1)]/k} \frac{C\lambda(x_0, y)^{-1/p}}{\lambda(x_0, x)}\, d\mu(y) \right]^p d\mu(x) \\
\le\ & C \int_{X \setminus cB} \left[\left(\frac{1}{d(x, x_0)} \right)^{\epsilon + [kn(1/p-1)]/k} \frac{C\mu(B)^{1-1/p+\epsilon/n+[kn(1/p-1)]/(kn)}}{\lambda(x_0, x)} \right]^p d\mu(x) \\
\le\ & CC\mu(B)^{p-1+p\epsilon/n+p[kn(1/p-1)]/(kn)} \int_{X \setminus cB} \left(\frac{1}{d(x, x_0)} \right)^{p\epsilon + p[kn(1/p-1)]/k + np} d\mu(x) \\
\le\ & C\mu(B)^{p-1+p\epsilon/n+p[kn(1/p-1)]/(kn)} \mu(B)^{-p\epsilon/n - p[kn(1/p-1)]/(kn) - p+1} \\
\le\ & C_p
\end{aligned}$$

since $p\epsilon + p[kn(1/p-1)]/k + np > n$.

The proof of Theorem 5.8 is complete. ▯

It would be interesting to prove that the integral operator T in Theorem 5.8 is bounded on H^p for $0 < p \leq$. We shall leave the matter for future study.

6. Boundedness of generalized Toeplitz operators

In this section, we shall study the boundedness of some generalized Toeplitz operators which become from a family of singular integral operators (see (6.2)) in the context of the Hardy spaces $H^p(X)$ and the Lebesgue spaces $L^p(X)$ on a space of homogeneous type.

Let $f \in L^2(X, \mu)$. Then we define the multiplication operator M_f with *symbol* f as follows: For any function g on X, we let $M_f(g)(x) = f(x) \cdot g(x), \quad x \in X$. Let T_K be a singular integral operator with a standard kernel $K(x, y)$. Then we denote by $C_f = [M_f, T_K] = M_f T_K - T_K M_f$, the commutator of M_f and T_K. In order to obtain better control over the singular integrals defined in Section 5, we let

$$K^\eta(x,y) = K(z,y) \text{ if } d(x,y) \geq \eta; \quad K^\eta(x,y) = 0 \text{ if } d(x,y) < \eta$$

and

$$T^\eta(g)(x) = \int_X K^\eta(x,y) g(y)\, d\mu(y)$$

and

$$\tilde{T}(f)(x) = \sup_{0<\eta<<1} |T^\eta(f)(x)|.$$

Then, by Theorem 12 in [CHR1], we have:

T is bounded on $L^2(X) \iff \tilde{T}$ is bounded on $L^p(X)$ for $1 < p < \infty$.

We assume that the balls and measure μ satisfying the following condition: There is an $\epsilon_0 > 0$ such that

$$c^{j\epsilon}\mu(B(x,t)) \leq \mu(B(x, c^j t) \leq C(c,K)^j \mu(B(x,t)) \tag{6.1}$$

for all $t > 0$ and $x \in X$. This condition is enough to guarantee (2.4) holds.

Let $T_{j,1}, T_{j,2}$ $(j = 1, \cdots, m)$ are a finite sequence of C-Z type operators. We shall consider a generalized Toeplitz operator:

$$\mathcal{T}_b = \sum_{j=1}^m T_{j,1} M_b T_{j,2}. \tag{6.2}$$

We shall always assume that $T_{j,1}, T_{j,2}$ are bounded on $L^2(X)$.

One of the main purposes of this section is to prove the following theorem.

THEOREM 6.1. *Let (X, d, μ) be a space of homogeneous type. Let $T_{j,i}$ are a sequence of C-Z operators which are bounded on $L^2(X)$. If $g \in L^p(X)$ and $\mathcal{T}_1(g) = 0$, then for any $b \in BMO(X)$, we have $\mathcal{T}_b(g) \in L^p(X)$. Moreover,*

$$\|\mathcal{T}_b(g)\|_{L^p(X)} \leq C_p(\sum_{j=1}^m \|T_{j,1}\|)(\sum_{j=1}^m \|T_{j,2}\|)\|g\|_{L^p}\|b\|_*$$

for all $1 < p < \infty$

By Theorem 5.4, we have immediatly the following corollary.

Corollary 6.2. *Let* (X, d, μ) *be a space of homogeneous type. Let* $T_{j,i}$ *are a sequence of C-Z operators which are bounded on* $L^2(X)$ *and* $\mathcal{T}_1 = 0$. *If* $b \in BMO(X)$, *then* $\mathcal{T}_b$ *is bounded on* $L^p(X)$ *for all* $1 < p < \infty$.

When X is Heisenberg group, the above result was proved by L. Grafakos and X. Li [GrL].

If we choose $m = 2$ and $T_{1,1} = T_{2,2} = I$ and $T_{1,2} = T_{2,1} = T_K$, then $\mathcal{T}_b = C_b$. Thus

Corollary 6.3. *Let* (X, d, μ) *be a space of homogeneous type. Let* T_K *be a C-Z operator which is bounded on* $L^2(X)$. *If* $b \in BMO(X)$, *then* $[M_b, T_K]$ *is bounded on* $L^p(X)$ *for all* $1 < p < \infty$.

For each $f \in L^1_{\text{loc}}(X, \mu)$. Then we define the sharp maximal function on X as follows: For a.e. $x \in X$,

$$f^{\#}(x) = \sup\Big\{\frac{1}{\mu(B)}\int_B |f(y) - f_B|\, d\mu(y) : r > 0\Big\}. \tag{6.3}$$

Also the q-maximal function of f is this:

$$M_q(f)(x) = \sup\Big\{\Big(\frac{1}{\mu(B)}\int_B |f(y)|^q\, d\mu(y)\Big)^{1/q} : r > 0\Big\}. \tag{6.4}$$

Then we have following three lemmas

Lemma 6.4. *Let* (X, d, μ) *be a space of homogeneous type. Let* $f \in L^{p_0}(X, \mu)$ *for some* $1 \le p_0 < p$. *Then* $f \in L^p(X, \mu)$ *if and only if* $f^{\#}(x) \in L^p(X, \mu)$; *and* $f \in L^p(X, \mu)$ *if and only if* $M(f) = M_1(f) \in L^p(X, \mu)$ *for all* $1 < p < \infty$.

The proof of Lemma 6.4 can be found in M. Christ and R. Fefferman [CHF]; Calderón [CAL] for $\Omega = R^n$; H. Aimar and R. Maciás [AIM] for the Hardy-Littlwood maximal function on spaces of homogeneous type. Also see J. O. Strömberg and A. Torchinski [STT] for sharp maximal functions on spaces of homogeneous type.

Lemma 6.5. *(a)* T *is bounded on* $L^2(X) \iff \tilde{T}$ *is bounded on* $L^p(X)$ *for* $1 < p < \infty$.

(b) Let $1 < p < \infty$. *Then* $M_q(f) \in L^p(X, \mu)$ *for all* $1 \le q < p$.

(c) If $f \in BMO(X)$, *then we have* $|f_{2^k B} - f_B| \le K\|f\|_* k$ *and*

$$\sup\Big\{\frac{1}{|B|}\int_B |f(y) - f_B|^p\, d\mu(y) : B = B(x_0, r) \subset X\Big\} < C_p\|f\|_*^p,$$

where $B = B(x_0, r) \subset X$.

Now we are ready to prove Theorem 6.1.

Let $b \in BMO(X)$ have compact support, and $g \in L^p(X)$ with $\mathcal{T}_1(g) = 0$. Then $\mathcal{T}_b(g) \in L^{p_0}(X, \mu)$ for some $1 \le p_0 < p$. By Lemma 6.4, it suffices to prove that $\mathcal{T}_b(g)^\# \in L^p(X, \mu)$ and $||\mathcal{T}_b(g)^\#||_p \le C_p ||b||_* ||g||_p$.

Let $B = B(x, r)$ be any ball in X and $cB = B(x, cr)$. We let

$$\mathcal{X}^1 = \mathcal{X}_{2B}; \quad \mathcal{X}^2 = 1 - \mathcal{X}_{2B}.$$

Since $\mathcal{T}_1(g) = 0$, and so $\mathcal{T}_{b_B}(g) = b_B \mathcal{T}_1(g) = 0$. Thus

$$\mathcal{T}_b(g) = \mathcal{T}_{(b-b_B)\mathcal{X}^1}(g) + \mathcal{T}_{(b-b_B)\mathcal{X}^2}(g) = g_1 + g_2. \tag{6.5}$$

Note that

$$g_1(y) = \mathcal{T}_{(b-b_B)\mathcal{X}^1}(g)(y) = \sum_{j=1}^m T_{j,1}[(b - b_B)\mathcal{X}^1 T_{j,2}(g)](y).$$

Thus for each $1 < q < p$, we can choose $1 < \gamma < \infty$ such that $q\gamma < p$.

$$\begin{aligned}
&\Big(\int_B |g_1(y)|^q d\mu \Big)^{1/q} \\
\le\ & \sum_{j=1}^m C_{q,j} \Big(\int_{2B} |b - b_B|^q |T_{j,2}(g)(y)|^q d\mu(y)^{1/q} \\
\le\ & \sum_{j=1}^m C_{q,\gamma,j} \Big(\int_B |b - b_B|^{q\gamma'} \, d\mu \Big)^{1/(q\gamma')} \Big(\int_{2B} |T_{j,2}(g)|^{q\gamma} \, d\mu \Big)^{1/(q\gamma)} \\
\le\ & \sum_{j=1}^m C_{q,\gamma,j} ||b||_* \mu(B)^{1/(\gamma' q)} M_{q\gamma}(T_{j,2}(g)(x) \mu(2B)^{1/q\gamma} \\
\le\ & ||b||_* \sum_{j=1}^m C_{q,\gamma,j} M_{q\gamma}[T_{j,2}(g)](x) \mu(B(x, r))^{1/(q\gamma) + 1/(q\gamma')} \\
\le\ & ||b||_* \sum_{j=1}^m C_{q,\gamma,j} M_{q\gamma}[T_{j,2}(g)](x) \mu(B)^{1/q}.
\end{aligned}$$

Therefore

$$\int_B |g_1| \frac{d\mu}{\mu(B)} \le \Big(\int_B |g_1|^q \, d\mu)^{1/q} \mu(B) \Big)^{1/q' - 1} \le ||b||_* \sum_{j=1}^m C_{q,\gamma,j} M_{q\gamma}[T_{j,2}(g)](x).$$

Now we consider g_2. We shall prove the following lemma first.

Lemma 6.6. *Let T_K be a C-Z operator with a standard kernel K such that T_K is bounded on $L^2(X)$. Then for any $y \in B = B(x, r)$, we have*

$$|T_K(g\mathcal{X}^2)(y) - T_K(g\mathcal{X}^2)(x)| \le C_{\epsilon,\epsilon_0} M(g)(x) \tag{6.6}$$

and

$$|T_K[(b - b_B)\mathcal{X}^2 g](y) - T_K((b - b_B)\mathcal{X}^2 g](x)| \le C_{\epsilon,\epsilon_0,\gamma} ||b||_* M_\gamma(g)(x) \tag{6.7}$$

where $1 < \gamma < p$.

Proof. The proof of (6.6) is similar and easier than the proof of (6.7). We shall present the proof of (6.7) here. Let $y \in B(x,r)$. Then

$$
\begin{aligned}
&|T_K[(b-b_B)g\mathcal{X}^2](y) - T_K[(b-b_B)g\mathcal{X}^2](x)| \\
= &\ \Big| \int_X (b-b_B)g(z)\mathcal{X}^2(z)(K(y,z)-K(x,z))\,d\mu(z) \Big| \\
\le &\ \int_{X-2B} |b-b_B||g(z)||K(y,z)-K(x,z)|\,d\mu(z) \\
\le &\ \sum_{k=2}^{\infty} \int_{2^kB-2^{k-1}B} |b-b_B||g(z)||K(y,z)-K(x,z)|\,d\mu(z) \\
\le &\ C\sum_{k=2}^{\infty} \int_{2^kB} |b-b_B||g(z)|\,d\mu(z)\mu(B)^{\epsilon}\mu(B(x,2^{k-1}r))^{-1-\epsilon} \\
\le &\ C\sum_{k=2}^{\infty} \Big(\int_{2^kB} \frac{|b-b_B|^{\gamma}\,d\mu}{\mu(2^kB)}\Big)^{1/\gamma'} \Big(\int_{2^kB} \frac{|g(z)|^{\gamma}\,d\mu(z)}{\mu(2^kB)}\Big)^{1/\gamma} \mu(B)^{\epsilon}\mu(B(x,2^{k-1}r))^{-\epsilon} \\
\le &\ C\sum_{k=2}^{\infty} C_{\gamma'}\|b\|_* k M_{\gamma}(g)(x)\mu(B)^{\epsilon}\mu(B(x,2^{k-1}r))^{-\epsilon} \\
\le &\ C_{\gamma}\|b\|_* M_{\gamma}(g)(x)\sum_{k=2}^{\infty} \mu(B(x,r))^{\epsilon} k c^{-(k-2)\epsilon_0\epsilon}\mu(B(x,r))^{-\epsilon} \\
\le &\ C_{\gamma}\|b\|_* M_{\gamma}(g)(x)\sum_{k=2}^{\infty} k c^{-(k-2)\epsilon_0\epsilon} \\
\le &\ CC_{\epsilon,\epsilon_0,\gamma}\|b\|_* M_{\gamma}(g)(x),
\end{aligned}
$$

and the proof of lemma is complete. ▯

Since

$$\frac{1}{\mu(B)}\int_B |g_2(y)-(g_2)_B|\,d\mu(y) \le \frac{2}{\mu(B)}\int_B |g_2(y)-g_2(x)|d\mu(y).$$

and

$$g_2(y) = \sum_{j=1}^{m} T_{j,1}[(b-b_B)\mathcal{X}^2 T_{j,2}(g)](y).$$

For each $y \in B(x,r)$, we have

$$|g_2(y)-g_2(x)| \le \sum_{j=1}^{m} C_{\epsilon,\epsilon_0,\gamma,j}\|b\|_* M_{\gamma}(T_{j,2}(g))(x)$$

Thus

$$\frac{1}{\mu(B)}\int_B |g_2(y)-(g_2)_B|d\mu(y) \le \sum_{j=1}^{m} C_{\epsilon,\epsilon_0,\gamma,j}\|b\|_* M_{\gamma}(T_{j,2}(g))(x)$$

Since $T_{j,i}$ are bounded on $L^p(X)$ and

$$\mathcal{T}_b(g)^{\#}(x) = g_1^{\#}(x) + g_2^{\#}(x)$$

Combining this and the estimation of $g_1^{\#}(x)$ and $g_2^{\#}(x)$, we have

$$\|\mathcal{T}_b(g)^{\#}\|_{L^p} \leq C_p \sum_{j=1}^{m} \|T_{j,1}\|_\infty \|T_{j,2}\|_\infty \|b\|_* \|g\|_{L^p(X)}.$$

By Lemmas 6.4, we have $\mathcal{T}_b(g) \in L^p(X,\mu)$ for all $1 < p < \infty$. So the proof of Theorem 6.1 is complete. ▯

As a direct consequence of Theorem 6.1, we have the following theorem in [CG] and [GL] for X is $\mathbb{R}^n$ or some homogeneous group.

THEOREM 6.7. *Let (X, d, μ) be a space of homogeneous type. If $f \in L^p(X)$ such that $\mathcal{T}_1(f) = 0$, then the linear operator $B_f(g) = \sum_{j=1}^{m} (T_{j,1}^*(g), T_{j,2}(f))$ is bounded from $L^q(X) \to H^1(X)$, where $p, q > 1$ and $1/p + 1/q = 1$ and $(\cdot,\cdot)$ denotes the inner product in $\mathbb{C}^n$.*

Proof. Since

$$\begin{aligned}
\langle \mathcal{T}_b(f), g\rangle &= \sum_{j=1}^{m} \langle T_{j,1}[bT_{j,2}(f)], g\rangle \\
&= \sum_{j=1}^{m} \langle [bT_{j,2}(f)], T_{j,1}^*(g)\rangle \\
&= \sum_{j=1}^{m} \langle b, \overline{T_{j,2}(f)} T_{j,1}^*(g)\rangle \\
&= \langle b, \sum_{j=1}^{m} \overline{T_{j,2}(f)} T_{j,1}^*(g)\rangle \\
&= \langle b, B_f(g)\rangle
\end{aligned}$$

for all $b \in BMO(X)$ with compact support, and by Theorem 6.1, we have

$$|\langle b, B_f(g)\rangle| \leq C_{p,q} (\sum_{j=1}^{n} \|T_{j,1}\|)(\sum_{j=1}^{n} \|T_{j,2}\|) \|b\|_* \|f\|_{L^p} \|g\|_{L^q}$$

Since VMO functions with compact support is in $VMO(X)$ and Theorem 2.4, we have $B_f(g) \in H^1(X)$, and the proof is complete. ▯

The second purpose of this section is to prove the following theorem.

THEOREM 6.8. *Let (X, d, μ) be a space of homogeneous type. Let T_K be a singular integral operator with a standard kernel $K(\cdot,\cdot)$. Suppose that T_K is bounded on $L^2(X)$ and that $f \in BMO(X)$. Then we have that C_f is bounded from $H^1(X)$ to $L^1_{\mathrm{loc}}(X)$.*

Proof. From the definition of $H^1(X)$, it suffices to prove that

$$\|C_f(a)\|_{L^1(X_0)} \le C\mu(X_0)\|f\|_* \tag{6.8}$$

for all atoms a and any compact subset X_0 in X. Let a be an atom with support $B = B(x, r)$. This can be done by using a standard computation, we omit the details here. ∎

In order to state and prove our next theorem, we reqire that our testing polynomials $\mathcal{P}$ be closed under the product operation, more precisely, if $p(x_0, x)$ and $q(x_0, x)$ are polynomials of degree at most n and m, then pq is a polynomial of degree at most $n+m$. This is analogous, but not identical to, requiring that our testing polynomials form a graded ring. Along this direction, we have

THEOREM 6.9. *Let $0 < p \le 1$ and let $(X, d, \mu, \mathcal{P})$ be a space of homogeneous type. Further suppose that $\mathcal{P}$ is closed under product operation. Let T_K be a singular integral operator with a p-standard kernel $K(\cdot, \cdot)$. If T_K is bounded on $L^2(X)$ and if $f \in L(\alpha(p, n), [n(1/p - 1)], \infty)$, then we have that C_f is bounded from $H^p(X)$ to $L^p(X)$.*

Proof. By Theorem 5.8, we have $\|T_K(g)\|_{L^p} \le C\|g\|_{H^p}$. Since

$$f \in L(\alpha(p, n), [n(1/p - 1)], \infty),$$

we have $f \in L^\infty(X)$. Thus $M_f T_K(g) \in L^p(X)$. Therefore the proof of Theorem 6.7 is reduced to proving the following theorem.

THEOREM 6.10. *Let $0 < p \le 1$ and let $(X, d, \mu, \mathcal{P})$ be a space of homogeneous type. Let T_K be a singular integral operator with a p-standard kernel $K(\cdot, \cdot)$. If T_K is bounded on $L^2(X)$, and if $f \in L(\alpha(p, n), [n(1/p-1)], \infty)$ then we have that $T_K M_f$ is bounded from $H^p(X)$ to $L^p(X)$.*

Proof. By Theorem 5.8, it suffices to prove that $K(x, y)f(y)$ is a (p, k)-standard kernel. This is a direct computation since $\mathcal{P}$ is closed under the product operation and $f \in L(\alpha(p, n), [n(1/p - 1)], \infty)$. We omit the details. ∎

7. Compactness of Commutators

In this section, we shall study the compactness of the commutator of a singular integral operator and a multiplication operator on $L^p(X)$ with $p > 1$ on a space of homogeneous type.

In order to study the compactness of C_f, we assume that $K \in C(X \times X \setminus \{(x, x) : x \in X\})$. We also assume that the measure μ satisfies the following condition: There is a positive constant $\epsilon_0 < \epsilon/2$ such that

$$|\mu(B(x, r) - \mu(y, r)| \le C\mu(B(x, d(x, y))^{\epsilon_0} \tag{7.1}$$

for some $\epsilon_0 > 0$, all $x, y \in X$ and $d(x, y) \le r < 1$.

The main purpose of this section is to prove the following theorem.

THEOREM 7.1. *Let (X, d, μ) be a space of homogeneous type satisfying (7.1). Let T_K be a singular integral operator with a standard kernel $K(\cdot, \cdot)$. If T_K is bounded on $L^2(X)$, and if $f \in VMO(X)$ then $C_f : L^p(X, \mu) \to L^p(X, \mu)$ is compact for all $1 < p < \infty$.*

Let $UC(X)$ denote the set of all uniformly continuous functions on X; $BUC(X)$ denotes the subspace of all bounded uniformly continuous functions. We begin by stating the following lemmas (their proofs are given later.)

Lemma 7.2. *Let X be a space of homogeneous type. Let $f \in VMO(X, \mu)$. Then for any $\eta > 0$, there is a function $f_\eta \in BUC(X)$ such that*

$$||f_\eta - f||_* < \eta. \tag{7.2}$$

Also there is an ϵ_1, $0 < \epsilon_1 \leq \epsilon_0/2 < 1$, such that

$$|f_\eta(x) - f_\eta(y)| \leq C_\eta \mu(B(x, r(x, y))^{\epsilon_1}. \tag{7.3}$$

For each $0 < \eta << 1$, we let $K^\eta(x, y)$ be a continuous extension of $K(x, y)$ from $X \times X - \{(x, y) : d(x, y) < \eta\}$ to $X \times X$ such that

$$\begin{aligned} K^\eta(x, y) &= K(x, y) \quad \text{if } d(x, y) \geq \eta; \\ |K^\eta(x, y)| &\leq C\mu(B(x, \eta))^{-1} \quad \text{if } d(x, y) < \eta; \\ K^\eta(x, y) &= 0 \quad \text{if } d(x, y) \leq \eta/c. \end{aligned}$$

We shall let T_{K^η} denote the integral operator associated to K_η and we let $C_f^\eta = [M_f, T_{K^\eta}]$. Then we have the following Lemma.

Lemma 7.3. *Let $f \in BUC(X)$ satisfy (7.2). Let T_K be a singular integral operator with a standard kernel K satisfying (7.1) which is bounded on $L^2(X, \mu)$. Then $||C_f - C_f^\eta||_\infty \to 0$, as an operator on $L^p(X, \mu)$, when $\eta \to 0$.*

We shall postpone the proof of Lemmas 7.2 and 7.3. Let us first prove Theorem 7.1.

Proof. Applying Theorem 6.1, we have

$$||C_f(g) - C_{f_\eta}(g)||_p \leq C_p\|g\|_{L^p}||f - f_\eta||_* < \eta. \tag{7.5}$$

Therefore, in order to prove that C_f is compact on $L^p(X, \mu)$, it suffices to prove that C_{f_η} is compact on $L^p(X, \mu)$.

So, for convenience, we may assume that $f \in BUC(X)$ satisfies (7.2) for some constant C and ϵ_0. By Lemma 7.2, it suffices to prove that C_f^η is compact on $L^p(X, \mu)$.

Since $K(x,y) \in C(X \times X \setminus \{(x,x) : x \in X\})$, we have for each $g \in L^p(X)$ that $C_f^\eta(g) \in C(X)$. Moreover, for any $x, y \in X$ with $d(x,y) < 1$, we have

$$\begin{aligned}
&C_f^\eta(g)(x) - C_f^\eta(g)(y) \\
&= f(x)\int_X K^\eta(x,z)g(z)\,d\mu(z) - \int_X K^\eta(x,z)f(z)g(z)\,d\mu(z) \\
&\quad -f(y)\int_X K^\eta(y,z)g(z)\,d\mu(z) + \int_X K^\eta(y,z)f(z)g(z)\,d\mu(z) \\
&= (f(x)-f(y))\int_X K^\eta(x,z)g(z)\,d\mu(z) \\
&\quad +f(y)\int_X (K^\eta(x,z)-K^\eta(y,z))g(z)\,d\mu(z) \\
&\quad +\int_X (K^\eta(y,z)-K^\eta(x,z))f(z)g(z)\,d\mu(z) \\
&= (f(x)-f(y))\int_X K^\eta(x,z)g(z)\,d\mu(z) \\
&\quad +\int_X (K^\eta(y,z)-K^\eta(x,z))(f(z)-f(y))g(z)\,d\mu(z) \\
&= I_1(x,y) + I_2(x,y).
\end{aligned}$$

And

$$\begin{aligned}
|I_1(x,y)| &= |(f(x)-f(y))|(|\int_X K^\eta(x,z)g(z)\,d\mu(z)| \\
&\le C|f(x)-f(y)|\{\int_X |K^\eta(x,z)|^{p'}\,d\mu(z)\}^{1/p'}||g||_p \\
&\le C_{\eta,p'}|f(x)-f(y)|||g||_p.
\end{aligned}$$

Notice that f is bounded. If we let $r = d(x,y) < (1/c^2)\eta$, then

$$\begin{aligned}
|I_2(x,y)| &\le \int_{X-B(x,\eta/c)} |K(y,z)-K(x,z)||(f(z)-f(y)||g(z)|\,d\mu(z) \\
&\le \|f\|_{L^\infty}\int_{X-B(x,\eta/2)} C\mu(B(x,r))^\epsilon \mu(B(x,d(x,z))^{-1-\epsilon}|g(z)|\,d\mu(z) \\
&\le C\|f\|_{L^\infty}\mu(B(x,d(x,y))^\epsilon\left\{\int_{X-B(x,\eta/2)} \mu(B(x,r(x,z))^{-p'(1+\epsilon)}\,d\mu(z)\right\}^{1/p'}||g||_{L^p} \\
&\le C_{\eta,p'}\mu(B(x,d(x,y))^\epsilon||f||_{L^\infty}\|g\|_{L^p}.
\end{aligned}$$

Therefore $\left\{C_f^\eta(U)\right\}$ is an equicontinuous family. Here U is the unit ball in $L^p(X,\mu)$. Therefore the Ascoli/Arzela Theorem shows that $\left\{C_f^\eta\right\}$ is compact on $L^p(X,\mu)$. This completes the proof of Theorem 7.1. $\square$

Next we return to the proofs of Lemmas 7.2 and 7.3.

Proof of Lemma 7.3

Let $g \in L^p(X)$. Then for each $x, y \in X$, we have

$$
\begin{aligned}
& C_f(g)(x) - C_f^\eta(g)(x) \\
= & \; f(x) \int_{B(x,\eta)} K(x,z)g(z)\,d\mu(z) - \int_{B(x,\eta)} K(x,z)f(z)g(z)\,d\mu(z) \\
& -f(x) \int_{B(x,\eta)-B(x,\eta/c)} K^\eta(,z)g(z)\,d\mu(z) + \int_{B(x,\eta)-B(x,\eta/c)} K^\eta(x,z)f(z)g(z)\,d\mu(z) \\
= & \; -\int_{B(x,\eta)} K(x,z)(f(z)-f(x)g(z)\,d\mu(z) \\
& + \int_{B(x,\eta)-B(x,\eta/c)} (K^\eta(x,z)(f(z)-f(x))g(z)\,d\mu(z) \\
= & \; I_1(x) + I_2(x).
\end{aligned}
$$

We first estimate $I_2(x)$, and we may let $\eta \le 1/c^2$. Then

$$
\begin{aligned}
|I_2(x)| & \le \max\{|f(z)-f(x)| : z \in B(x,\eta)\}| \times \\
& \qquad \int_{B(x,\eta)-B(x,\eta/c^2)} C\mu(B(x,\eta/c^2))^{-1}|g(z)|\,d\mu(z)| \\
& \le C \max\{|f(z)-f(x)| : z \in B(x,\eta)\} M(g)(x).
\end{aligned}
$$

Note that

$$
\begin{aligned}
I_1(x) & = |\mathrm{PV} \int_{B(x,\eta)} K(x,z)(f(z)-f(x))g(z)\,d\mu(z)| \\
& \le \int_{B(x,\eta)} C\mu(B(x,d(x,z))^{-1}\mu(B(x,d(x,z))^{\epsilon_0}|g(z)|\,d\mu(z) \\
& \le C\mu(B(x,\eta))^{\epsilon_0/2} \int_{B(x,\eta)} \mu(B(x,d(x,z))^{-1+\epsilon_0/2}|g(z)|\,d\mu(z)
\end{aligned}
$$

Applying Schur's Lemma, we have

$$
\int_X \left\{ \int_{B(x,\eta)} \mu(B(x,r(x,z))^{-1+\epsilon_0/2}|g(z)|\,d\mu(z) \right\}^p d\mu(x) \le C_{\epsilon_0}||g||_p^p.
$$

for all $\eta \le 1$.

Therefore we have

$$
||C_f(g) - C_f^\eta(g)||_p^p \le C^p \mu(B(x,\eta))^{\epsilon_0 p/4}||g||_p^p.
$$

for all $g \in L^p(X,\mu)$.

This implies that

$$
||C_f - C_f^\eta|| \to 0, \qquad \text{as an operator in } L^p(X,\mu), \qquad \text{when } \eta \to 0.
$$

The proof of Lemma 7.3 is complete. □

Finally, we prove Lemma 7.2. In order to achieve this end, let us first prove the following Lemma.

Lemma 7.4. *Let $f \in BMO(X, \mu)$. Then we have*

$$|f_{B(x,r)}| \le C||f||_* \log(C/\mu(B(x,r)), \quad r \le 1; \tag{7.6}$$

and

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)-B(y,r)} |f| \, d\mu(z) \le C_r ||f||_* \mu(B(x, d(x,y))^{\epsilon_0} \tag{7.7}$$

for all $x, y \in X$ and $cr(x,y) \le r \le 1$.

Proof. Let us prove (7.7) first. By hypothesis (7.3), we have

$$\begin{aligned} & \int_{B(x,r)-B(y,r)} |f| \, d\mu(z) \\ \le \quad & (\int_{B(x,r)} |f(z)|^2 \, d\mu(z))^{1/2} (\int_{B(x,r)-B(y,r)} 1 \, d\mu(z))^{1/2} \\ \le \quad & C_r C \mu(B(x, d(x,y))^{\epsilon_0/2}. \end{aligned}$$

This completes the proof of (7.7). And the proof of (7.6) can be found in [KRL2]. ▯

Proof of Lemma 7.2.

For any $\eta > 0$, since $f \in VMO(X, \mu)$, there is a $\delta(\eta) > 0$ such that

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \, d\mu(y) < \epsilon / C^2$$

for all $r < c\delta(\eta)$. Let

$$f_\eta(x) = f_{B(x,\delta(\eta))}.$$

Then we have

$$\begin{aligned} & |f_\eta(x) - f_\eta(y)| \\ \le \quad & |\frac{1}{\mu(B(x,\delta))} \int_{B(x,\delta)} f(z) - f_{B(y,\delta)} \, d\mu(z)| \\ & \quad + \left| \frac{1}{B(x,\delta)} - \frac{1}{\mu(B(y,\delta)} \right| \int_{B(y,\delta)} f(z) \, d\mu(z)| \\ = \quad & I_1(x,y) + I_2(x,y). \end{aligned}$$

Also, by Lemma 7.4, we have

$$I_1(x,y) \le C_\delta \mu(B(x, d(x,y))^{\epsilon_0/2}.$$

and

$$\begin{aligned} I_2(x,y) \quad \le \quad & |\mu(B(x,\delta) - \mu(B(y,\delta)|||f||_* \frac{|\log \mu(B(y,\delta)|}{\mu(B(x,\delta)} \\ \le \quad & C_\delta \mu(B(x, d(x,y))^{\epsilon_0/2} \end{aligned}$$

for all $x, y \in X$. Therefore f_η satisfies (7.3).

Next we show that (7.2) holds.

Let $B = B(x, r)$ be any ball on X with $r > \delta$, let $\{B(x, \delta) : x \in B\}$ be an open cover of B. By Theorem 1.2 in [COW2], there are pairwise disjoint balls $B(x_j, \delta), j = 1, 2, \cdots, N$ such that $B \subset \cup_{j=1}^N B(x_j, c\delta)$. It is clear that $\cup_{j=1}^N B(x_j, c\delta) \subset B(x, c_0 cr)$, and c_0 is independent of δ. Therefore we have

$$\begin{aligned}
&\frac{1}{\mu(B)}\int_B |f(y) - f_\eta(y) - f_B + f_{\eta B}|\, d\mu(y) \\
&\leq 2\frac{1}{\mu(B)}\int_B |f(y) - f_\eta(y)|\, d\mu(y) \\
&\leq \frac{2}{|B|}\sum_{j=1}^N \int_{B(x_j, c\delta)} |f(y) - f_\eta(y)|\, d\mu(y) \\
&\leq \frac{1}{|B|}\sum_{j=1}^N \Big\{ \int_{B(x_j, c\delta)} |f(y) - f_{B(x_j, c\delta)}(y)|\, d\mu(y) \\
&\quad + \int_{B(x_j, c\delta)} |f_{B(x_j, c\delta)} - f_\eta(y)|\, d\mu(y) \Big\} \\
&\leq \frac{2}{|B|}\sum_{j=1}^N (\epsilon/C)||f||_* \mu(B(x_j, c_0 c\delta)) \\
&\quad + \frac{2}{|B|}\sum_{j=1}^N \int_{B(x_j, c\delta)} \frac{1}{\mu(B(y, \delta))}\int_{B(y, \delta)} |f(z) - f_{B(x_j, c\delta)}|\, d\mu(z)\, d\mu(y) \\
&\leq (\epsilon/C^2)||f||_* \frac{2}{|B|}\mu(B(x, c_0 cr)) + \frac{2}{|B|}\sum_{j=1}^N \mu(B(x_j, c\delta))||f||_* C(\epsilon/C^2) \\
&\leq C||f||_* \epsilon
\end{aligned}$$

where C is a constant independent of δ and so ϵ. Therefore the proof of Lemma 7.2 is complete. $\square$

8. The Case that X is Either $\mathbb{R}^N$ or the Boundary of a Domain in $\mathbb{C}^n$

In this section, we shall give several applications of the theorems we have proved in sections 6 and 7.

The first special case of our theorems in Section 5, 6, and 7 is to $X = \mathbb{R}^N$.

Corollary 8.1. *Let $X = R^N$ with the standard Euclidean metric and Lebesgue measure; and let K be a Calderòn-Zygmund kernel such that T_K is bounded on L^2. Then*

(a) If $f \in BMO(\mathbb{R}^N)$, then C_f is bounded on $L^p(\mathbb{R}^N)$ for all $1 < p < \infty$;
(b) If $f \in BMO(\mathbb{R}^N)$, then C_f is bounded from $H^1(\mathbb{R}^N)$ to $L^1_{\rm loc}(\mathbb{R}^N)$;
(c) If $f \in VMO(\mathbb{R}^N)$, then C_f is compact on $L^p(\mathbb{R}^N)$for all $1 < p < \infty$;

(d) If $f \in \Lambda_{N(1/p-1)}(\mathbb{R}^N)$ and K is a p-standard kernel, then $C_f : H^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is bounded for all $0 < p < 1$.

The result (a) generalizes the sufficient conditions in the main theorem proved by Coifman, Rochbergh and Weiss in [CRW] and S. Janson in [JAN]. The result (c) is a generalization of the main theorem proved by Uchiyama in [UCH].

The converse of the above corollary is also true for some special Calderòn-Zygmund kernels (see [CRW], [JAN] and [UCH] for details.)

Next we shall consider applications of the theorems we obtained in Sections 5, 6 and 7 to X the boundary of a strictly pseudoconvex domain in $\mathbb{C}^n$, a pseudoconvex domain of finite type in $\mathbb{C}^2$, or a convex domain of finite type in $\mathbb{C}^n$.

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary $\partial\Omega$. Let $d\sigma$ denote the Lebesgue surface measure on $\partial\Omega$ and $L^p(\partial\Omega)$ the usual Lebesgue space on $\partial\Omega$ with respect to the measure $d\sigma$. Let $\mathcal{H}^p(\Omega)$ be the holomorphic Hardy spaces defined in [KRA3] or [STE2]. Fatou's theorem [KRA1] shows that, for any $0 < p \leq \infty$, a holomorphic function $f \in \mathcal{H}^p(\Omega)$ has a radial limit at almost all points on $\partial\Omega$. Thus one can identify $\mathcal{H}^p(\Omega)$ as a closed subspace of $L^p(\partial\Omega)$. Let $S : L^2(\partial\Omega) \to \mathcal{H}^2(\Omega)$ be the orthogonal projection via the reproducing kernel $S(z, w)$—the Szegö kernel. For many special instances and classes of Ω, we may identify the operator S as a singular integral operator on $\partial\Omega$; in fact, in many instances $S(z, w)$ is given by a kernel that is C^∞ on $\partial\Omega \times \partial\Omega \setminus D$ (where D is the diagonal of $\partial\Omega \times \partial\Omega$).

First, we consider the case in which Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let $X = \partial\Omega$. We denote by d the usual quasi-metric on $\partial\Omega$ defined in [STE2] or [FEF] or [KRL1]. (In general, the quasi-metric defined on $\partial\Omega$ is not symmetric, but we can define $\hat{d}(x, y) = (1/2)(d(x, y) + d(y, x))$. Then $\hat{d}$ is a symmetric quasi-metric having the same properties as d.) Also, we let the measure $d\mu = d\sigma$ be the usual Lebesgue/Hausdorff surface measure on $\partial\Omega$. Then we have $|B(z, \delta)| = \sigma(B(z, \delta) \approx \delta^n$. It is clear that condition (7.3) is satisfied with this measure. By theorems in [BFG] and [BMS], the Szegö kernel $S(z, w) \in C^\infty(\partial\Omega \times \partial\Omega \setminus \{(x, x) : x \in \partial\Omega\})$ is a p-standard kernel (see the formulation in [KRL1]).

The first main purpose of this section is to prove the following theorem.

THEOREM 8.2. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. With the notation above, let $T_K = T_S$ (S is the Szegö kernel) and let $f \in L^1(\partial\Omega)$. Then*

(i) $f \in BMO(\partial\Omega)$ if and only if $C_f : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is bounded.

(ii) $f \in VMO(\partial\Omega)$ if and only if $C_f : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is compact.

Proof. Since $S(z, w)$ is a standard kernel on $\partial\Omega \times \partial\Omega$ and T_S as a singular integral is the main part of the Szegö projection (In fact, one has $S(f)(z) = 1/2f(z) + cT_S(f)$, a.e. $z \in \partial\Omega$.) it is bounded on $L^2(\partial\Omega)$. Therefore, the sufficient conditions for the commutator C_f to be bounded and compact on $L^p(\partial\Omega)$ for all $1 < p < \infty$ in Theorem 8.2 follow directly from Theorems 6.1 and 7.1.

Now we prove the necessary conditions of Theorem 8.2.

We first consider part (a). We assume that $f \in L^{p_0}(\partial\Omega)$ for some $1 < p_0 << p$ and C_f is bounded on $L^p(\Omega)$ for some $1 < p < \infty$. We shall show $f \in BMO(\partial\Omega)$. Following [FEF], [BFG] or [BMS], we let $\rho(z)$ be a strictly pluri-superharmonic defining function for Ω, and we set

$$\psi(z,w) = \sum_{j=1}^{n} \frac{\partial\rho}{\partial w_j}(w)(z_j - w_j) + (1/2)\sum_{jk} \frac{\partial^2\rho}{\partial w_j \partial w_k}(w)(z_j - w_j)(w_k - w_k).$$

Then there is a positive number $\delta > 0$ such that

$$S(z,w) = F(z,w)\psi(z,w)^{-n} + G(z,w)\log\psi(z,w) \tag{8.3}$$

for all $(z,w) \in R_\delta = \{(z,w) \in \partial\Omega \times \partial\Omega : d(z,w) < \delta\}$, where $F, G \in C^\infty(\partial\Omega \times \partial\Omega)$ and $F(z,z) > 0$ on $\partial\Omega$. Moreover, we have the following lemma proved by the authors [Lemma 5.2, KRL1].

Lemma 8.3. *Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Then for each point $z_0 \in \partial\Omega$, there are holomorphic functions g_j, and C^∞ functions h_j $(j = 1, \cdots, M)$ and a function $E(z,w)$ which is holomorphic in z and $C^{n-1}(R_\delta)$ in w such that the reciprocal of the Szegö kernel has the following decomposition property (for z close to w):*

$$\frac{1}{S(z,w)} = \sum_{j=1}^{M} g_j(z,z_0)h_j(w,z_0) + E(z,w) \tag{8.4}$$

with

$$|g_j(z,z_0)| \le Cd(z,z_0)^{\gamma_j}, \quad |h_j(z,z_0)| \le Cd(z,z_0)^{\eta_j} \tag{8.5}$$

and $\gamma_j + \eta_j \ge n$ and $\eta_j,\ \gamma_j \ge 0$ integers; and

$$|E(z,w)| \le C_M(|z-w|^M + |\psi(z,w)|^{2n-\epsilon}) \tag{8.6}$$

for all $z, w \in B(z_0,\delta)$.

Let $B = B(z_0,\delta)$ be any ball in $\partial\Omega$. If we choose $0 < \epsilon \le n/p_0'$, then for each $z \in B(z_0,\delta)$ we have

$$\left\{ \int_B |\psi(z,w)|^{(n-\epsilon)p'} d\sigma(w) \right\}^{1/p'} \le C\delta^n.$$

Thus we have

$$
\begin{aligned}
&\int_B |f(z)-f_B|d\sigma(z)\\
&= \frac{1}{|B|}\int_B \Big|\int_B (f(w)-f(z))d\sigma(w)\Big|d\sigma(z)\\
&= \frac{1}{|B|}\int_B \Big|\int_B (f(w)-f(z))S(z,w)S(z,w)^{-1}d\sigma(w)\Big|d\sigma(z)\\
&= \frac{1}{|B|}\int_B \Big|\int_B (f(w)-f(z))\Big\{\sum_{j=1}^M g_j(z_0,z)h_j(z_0,w)\\
&\qquad +O(|z-w|^{2n}\delta^n+|\psi(z,w)|^{2n-\epsilon})d\sigma(w)\Big\}\Big|d\sigma(z)\\
&\le \frac{1}{|B|}\int_B |\sum_{j=1}^M C_f(h_j(z_0,\cdot)\mathcal{X}_B)(z)g_j(z_0,z)|d\sigma(z)\\
&\quad +\frac{C}{|B|}\int_B |\int_B |f(w)-f(z)||S(z,w)||z-w|^{2n}\delta^n d\sigma(w)d\sigma(z)\\
&\quad +\frac{C}{|B|}\int_B \int_B |f(w)-f(z)||\psi(z,w)|^{n-\epsilon}d\sigma(w)d\sigma(z)\\
&\le \frac{1}{|B|}\sum_{j+1}^M ||C_f(h_j(z_0,\cdot)\mathcal{X}_B)||_p||g_j(z_0,\cdot)\mathcal{X}_B||_{p'}+C||f||_1|B|\\
&\le C\frac{1}{|B|}\sum_{j=1}^M C||C_f||||h_j(z_0,\cdot)\mathcal{X}_B||_p||g_j(z_0,\cdot)\mathcal{X}_B)||_{p'}+C||f||_{p_0}|B|\\
&\le \frac{1}{|B|}CM||C_f||\delta^{\beta_j}|B|^{1/p}\delta^{\alpha_j}|B|^{1/p'}+C||f||_{p_0}|B|\\
&\le CM||C_f||\delta^n+C||f||_{p_0}|B|\\
&\le C(||f||_{p_0}+||C_f||)|B|.
\end{aligned}
$$

Therefore

$$\frac{1}{|B|}\int_B |f(z)-f_B|d\sigma(z)\le C(||f||_{p_0}+||C_f||).$$

for any ball B in $\partial\Omega$. This proves that $f\in BMO(\partial\Omega)$, and $||f||_*\le C(||f||_{p_0}+||C_f||)$.

Next we prove part (b). Assume that $f\in L^{p_0}(\partial\Omega)$ with $1<p_0<<p<\infty$ and suppose that C_f is compact on $L^p(\partial\Omega)$ for some p. We will show that $f\in VMO(\partial\Omega)$.

Since C_f is compact, we have C_f is bounded on $L^p(\partial\Omega)$. Thus we have $f \in BMO(\partial\Omega)$. Next we show that

$$\frac{1}{|B(z,\delta)|}\int_{B(z,\delta)} |f(w) - f_B| d\sigma(w) \to 0 \tag{8.7}$$

uniformly for all $z \in \partial\Omega$ as $\delta \to 0$.

Suppose (8.7) is not true. Then there are a sequence $z_k \subset \partial\Omega$ and δ_k and $\eta_0 > 0$ such that $\delta_k \to 0$ and

$$\frac{1}{|B_k|}\int_{B_k} |f(w) - f_B| d\sigma(w) \geq \eta_0 > 0 \tag{8.8}$$

for all k where $B_k = B(z_k, \delta_k)$.

For each $1 \leq j \leq M$ and positive integer k, we let

$$\phi_{k,j}(w) = \frac{1}{|B_k|} h_j(z_k, w)\mathcal{X}_{B_k}.$$

Using the same argument as we did a few moments ago, we have

$$\begin{aligned}
&\int_{B_k} |f(z) - f_{B_k}| d\sigma(z) \\
\leq\ & C\int_{B_k}\left|\sum_{j=1}^{M} C_f(\phi_{k,j})(z) g_j(z_k, z)\right| d\sigma(z) \\
&\quad + C\delta_k^{n-\epsilon}\int_{B_k} |f(z) - f_{B_k}| d\sigma(z).
\end{aligned}$$

Therefore, when k is big enough, we have

$$\begin{aligned}
&\int_{B_k} |f(z) - f_{B_k}| d\sigma(z) \\
\leq\ & C\int_{B_k}\left|\sum_{j=1}^{M} C_f(\phi_{k,j})(z) g_j(z_k, z)\right| d\sigma(z) \\
=\ & C\int_{B_k}\left|\sum_{j=1}^{M} C_f(\delta_k^{n/p'-\eta_j}\phi_{k,j})(z)\delta_k^{-n/p'+\eta_j} g_j(z_k, z)\right| d\sigma(z).
\end{aligned}$$

Now we let

$$y_{j,k}(z) = \delta_k^{n/p'-\eta_j}\phi_{k,j}(z)$$

Then

$$\|y_{j,k}\|_{L^p} \leq C$$

and $y_{k,j} \to 0$ weakly on $L^p(\partial\Omega)$ as $k \to \infty$ for all $1 \le j \le M$. In fact, for any smooth function $y(z)$, we have

$$\begin{aligned} &\int_{\partial\Omega} y_{jk}(z)y(z)d\sigma(z) \\ = &\ \frac{1}{|B_k|}\int_{B_k} \delta_k^{n/p'-\eta_j} h_j(z_k,z)y(z)d\sigma(z) \to 0 \end{aligned}$$

as $k \to \infty$.

Since C_f is compact on $L^p(\partial\Omega)$, we have $||C_f(y_{k,j})||_p \to 0$ as $k \to \infty$ for all $1 \le j \le M$. This implies that

$$\begin{aligned} &\frac{1}{|B(z_k,\delta_k)|}\int_{B(z_k,\delta_k)} |f(w) - f_{B(z_k,\delta_k)}|d\sigma(w) \\ \le &\ C\|C_f(y_{j,k})\|_{L^p}\|\delta_k^{-n/p'+\eta_j} g_j \frac{1}{|B_k|}\mathcal{X}_{B_k}\|_{p'} \\ \le &\ C\|C_f(y_{j,k})\|_{L^p}\|\delta_k^{-n/p'+\gamma_j+\eta_j}\frac{1}{|B_k|}\||B_k|^{1/p'} \\ = &\ C\|C_f(y_{j,k})\|_{L^p}\|\delta_k^{-n/p'+\gamma_j+\eta_j-n+n/p'} \\ \le &\ C\|C_f(y_{j,k})\|_{L^p} \to 0. \end{aligned}$$

as $k \to \infty$. This assertion contradicts (8.8). Therefore we have proved that $f \in VMO(\partial\Omega)$. Part (b) follows.

The proof of Theorem 8.2 is complete. ▯

Corollary 8.4. *Let $X = S^{2n-1}$, the unit sphere in $\mathbb{C}^n$ or $\mathbb{H}_n$, and let $K = S$ be the corresponding Cauchy-Szegö kernel. Let $f \in L^{p_0}(X)$ for some $1 < p_0 < \infty$. Then*

(i) $C_f : L^p(X) \to L^p(X)$ is bounded if and only if $f \in BMO(X)$ for $1 < p < \infty$;

(ii) $C_f : L^p(X) \to L^p(X)$ is compact if and only if $f \in VMO(X)$ for $1 < p < \infty$;

(iii) If $f \in BOM(S)$ then $C_f : H^p(S) \to L^1(S)$ is bounded;

(iv) If $0 < p < 1$ and $f \in \Lambda_{n(1/p-1)}(S)$, the non-isotropic Zygmund space, then $C_f : H^p(S) \to L^p(S)$ is bounded.

Part (i) was proved by Feldman and Rochberg in [FER] and part (iv) is related to some results in [LI].

Next we shall consider the case when $X = \partial\Omega$, Ω a pseudoconvex domain of finite type in $\mathbb{C}^2$ or a convex domain of finite type in $\mathbb{C}^n$.

When Ω is a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$, then we shall use a variant of the quasi-metric defined in [NRSW] as formulated in [KRL1] or [MCN1]. When Ω is a smoothly bounded convex domain in $\mathbb{C}^n$, we shall use the quasi-metric introduced by McNeal, for example see [MCN1], [MCS1] and [KRL3]. We shall prove the following theorem:

THEOREM 8.5. *Let Ω be either a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ or a smoothly bounded convex domain of finite type in $\mathbb{C}^n$. Let $K = S$ be the Szegö kernel for Ω. Let $f \in L^{p_0}(\partial\Omega)$ for some $1 < p_0 < \infty$. Then*

(i) $C_f : L^p(X) \to L^p(X)$ is bounded if $f \in BMO(X)$ for $1 < p < \infty$;
(ii) $C_f : L^p(X) \to L^p(X)$ is compact if $f \in VMO(X)$ for $1 < p < \infty$.

In order to prove Theorem 8.5, we need some more notation and some lemmas. Let $-r$ be a smooth defining function for Ω such that, in a small neighborhood of $\partial\Omega$, $r(z)$ is the distance from z to $\partial\Omega$. Let $\nu(z)$ denote the unit inward normal vector to $\partial\Omega$ at $z \in \partial\Omega$. Then we may choose an $\epsilon_0 > 0$ small enough such that

$$z = \pi(z) + r(z)\nu(\pi(z)), \quad \nu(z) = \left(\frac{\partial r}{\partial \overline{z}_1}, \cdots, \frac{\partial r}{\partial \overline{z}_n}, \frac{\partial r}{\partial z_1}, \cdots, \frac{\partial r}{\partial z_n}\right)$$

for all $z \in \Omega_{\epsilon_0}$.

Let $L^2(\Omega)$ be the Lebesgue space on Ω and $A^2(\Omega)$ the subspace of holomorphic functions; we call A^2 the Bergman space. Let $P : L^2(\Omega) \to A^2(\Omega)$ be the orthogonal projection —the Bergman projection with reproducing kernel $K(z, w)$, the Bergman kernel.

For each $a \in C^\infty(\overline{\Omega}_{\epsilon_0})$, we define kernels $C(z, w)$ and $C_\epsilon(z, w)$ on $\partial\Omega \times \partial\Omega$ by

$$C(z, w) = \int_0^{\epsilon_0} a(z + t\nu(z))K(z + t\nu(z), w)dt, \tag{8.9}$$

and

$$C_\epsilon(z, w) = \int_\epsilon^{\epsilon_0} a(z + t\nu(z))K(z + t\nu(z), w).dt \tag{8.10}$$

Following the main estimate in [NRSW] on $K(z, w)$ when Ω is a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ and the estimate on $K(z, w)$ given in [MAN] and [MAS], and also formulations in [KRL2] and [KRL3], we have

Lemma 8.6. *Let Ω be either a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ or a smoothly bounded convex domain in $\mathbb{C}^n$. Then C and C_ϵ as defined above are standard kernels on $\partial\Omega \times \partial\Omega$*

Let Ω be either a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ or a smoothly bounded convex domain in $\mathbb{C}^n$. Then, by the results in [NRSW] for finite type domain in $\mathbb{C}^2$, and those in [MCN3] and [MCS1] for convex domain of finite type in $\mathbb{C}^n$, we have $P : L^p(\partial\Omega) \to \mathcal{H}^p(\Omega)$ is bounded for all $1 < p < \infty$. Let $\epsilon_0 >$ be as the above. Then we define an operator:

$$Q(f)(z) = (P(f)(z + \epsilon_0\nu(z)), \quad z \in \partial\Omega.$$

It is easy to see that $Q : L^p(\Omega) \to C^k(\partial\Omega)$ is bounded for all $1 < p < \infty$ and $0 \le k < \infty$.

Then we have the following lemma proved in [KRL1] and [KRL3].

Lemma 8.7. *Let Ω be either a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ or a smoothly bounded convex domain in $\mathbb{C}^n$. Then*

$$S(f)(z) = A(f)(z) + ES(f)(z) + Q(f), \quad z \in \Omega, \quad f \in L^p(\partial\Omega),$$

where $A = \sum_{j=1}^n r_j I_{C_j}$, and $E = \sum_{j=1}^n [I_{C_j}, M_{r_j}]$ and $C_j(z,w)$ is obtained from (5.1) by replacing a by a_j, $j = 1, \cdots, n$.

Lemma 8.8. *Let Ω be either a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ or a smoothly bounded convex domain in $\mathbb{C}^n$. Let $f \in BMO(\partial\Omega)$ and $g \in C^1(\partial\Omega)$. Then $fg \in BMO(\partial\Omega)$.*

Proof. Let $B = B(z_0, \delta)$ be any ball on $\partial\Omega$. Then we have

$$|g(z) - g(w)| \le ||g||_{C^1}|z - w| \le C\delta^\eta.$$

for all $z, w \in B$ and for some $\eta > 0$, where η is a constant depending only on the type of Ω. Therefore

$$\begin{aligned}
&|f(z)g(z) - (fg)_B| \\
&= |(f(z) - f_B)g(z) + f_B g(z) - (fg)_B| \\
&= |(f(z) - f_B)g(z) + \frac{1}{|B|}\int_B (f(w)g(z) - f(w)g(w))d\sigma(w)| \\
&\le |f(z) - f_B||g(z)| + \frac{1}{|B|}\int_B |f(w)||g(w) - g(z)|d\sigma(w) \\
&\le |f(z) - f_B||g(z)| + C\frac{1}{|B|}\int_B |f(w)|\delta^\eta d\sigma(w) \\
&\le |f(z) - f_B||g(z)| + C\delta^\eta |f_B| \\
&\le |f(z) - f_B||g(z)| + C\delta^\eta ||f||_* \log\frac{1}{|B|} \\
&\le |f(z) - f_B||g(z)| + C_\eta ||f||_*.
\end{aligned}$$

Thus

$$\frac{1}{|B|}\int_B |fg(z) - (fg)_B|d\sigma(z)| \le C||g||_{C^1}||f||_*.$$

So we have

$$||fg||_* \le C||f||_*||g||_{C^1}$$

and the proof of Lemma 8.8 is complete. □

Remark: It should be noted that Stegenga [ST1] has given a characterization, in the real variable setting, of those functions g as in the last lemma that are BMO multipliers.

Now we apply Theorems 6.1, 7.1 and Lemmas 8.7 and 8.8 to see that

$$[A, M_f] = \sum_{j=1} [r_j C_j, M_f] = [C_j, M_{fr_j}]$$

is bounded on $L^p(\partial\Omega)$ if $f \in BMO(\partial\Omega)$; and $[A, M_f]$ is compact on $L^p(\partial\Omega)$ if $f \in VMO(\partial\Omega)$ for all $1 < p < \infty$. [We use here the fact that E and Q are compact operators on L^p for all $1 < p < \infty$.]

Combining the above facts, the proof (i) and (ii) of Theorem 8.5 is complete. ∎

Finally, we make the following remark. One may prove a result similar to (iii) in Corollary 8.4 for a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ (with a different argument from the one in [KRL1]) or for a smoothly bounded convex domain in $\mathbb{C}^n$; one obtains these results as an application of Theorem 6.10.

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