DIVERGENCE OF PROJECTIVE STRUCTURES AND LENGTHS OF MEASURED LAMINATIONS

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Abstract. Given a complex structure, we investigate diverging sequences of projective structures on the fixed complex structure in terms of Thurston's parametrization. In particular, we will give a geometric proof to the theorem by Kapovich stating that as the projective structures on a fixed complex structure diverge so do their monodromies. In course of arguments, we extend the concept of realization of laminations for $\mathrm{PSL}(2,\mathbf{C})$-representations of surface groups.

1. Introduction

Throughout this paper, Σ_g denotes an oriented differentiable surface of genus $g > 1$ and P_g the space of $\mathbf{C}\mathrm{P}^1$-structures (or, projective structures) on Σ_g. Here, a $\mathbf{C}\mathrm{P}^1$-structure on a surface is a structure modelled on $(\mathbf{C}\mathrm{P}^1, \mathrm{PSL}(2,\mathbf{C}))$. (Here $\mathrm{PSL}(2,\mathbf{C})$ is identified with the projective automorphism group of $\mathbf{C}\mathrm{P}^1$.) As a Möbius transformation is holomorphic, each $\mathbf{C}\mathrm{P}^1$-structure determines its underlying complex structure.

There are two parametrizations of P_g which are often quoted. One of them is by the bundle $\pi : Q_g \to T_g$ of holomorphic quadratic differentials on Riemann surfaces over Teichmüller space T_g. The canonical projection π represents the correspondence from each $\mathbf{C}\mathrm{P}^1$-structure to its underlying complex structure. As for the detail about this parametrization, see Hejhal [H], for example. The other parametrization was introduced by Thurston, that says every $\mathbf{C}\mathrm{P}^1$-structure is obtained from some hyperbolic structure by grafting a measured lamination to it: P_g is parametrized by $T_g \times \mathcal{ML}$, where $\mathcal{ML}$ stands for the space of measured laminations. As for a description of grafting and this parametrization, see [KT] or [Tg].

In this paper, we will investigate $\mathbf{C}\mathrm{P}^1$-structures on a fixed complex structure in terms of Thurston's parametrization. In particular, we will give a geometric proof for the fact, originally proved by Kapovich [Ka], that the monodromies diverge as the $\mathbf{C}\mathrm{P}^1$-structures diverge on a fixed complex structure.

The idea is to generalize the concept of realization of measured laminations in hyperbolic 3-manifolds to that for $\mathrm{PSL}(2,\mathbf{C})$-representations of the fundamental group $\pi_1(\Sigma_g)$ which is not necessarily discrete, and then observe that the lengths of measured laminations are locally bounded in the space of $\mathrm{PSL}(2,\mathbf{C})$-representations,

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while the length of some measured lamination diverges as $\mathbf{C}\mathrm{P}^1$-structures on the fixed complex structure diverge.

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2. Pleated surfaces and measured laminations for representations

First, we recall the monodromy of a $\mathbf{C}\mathrm{P}^1$-structure. Given a $\mathbf{C}\mathrm{P}^1$-structure on Σ_g, we take its *developing map*: beginning with a local chart of the $\mathbf{C}\mathrm{P}^1$-structure, we take its analytic continuation along curves and have a multivalued meromorphic function (meromorphic with respect to the underlying complex structure of the $\mathbf{C}\mathrm{P}^1$-structure). The values over a point differ from each other by post-compositions of Möbius transformations, which are determined by the closed curves along which the analytic continuation is carried out. Thus this procedure yields a homomorphism $\pi_1\Sigma_g \to \mathrm{PSL}(2,\mathbf{C})$, which is determined up to conjugations by Möbius transformations (depending on the local chart to begin the analytic continuation with). This homomorphism is called a *monodromy* of the $\mathbf{C}\mathrm{P}^1$-structure, or the *monodromy representation of* $\pi_1\Sigma_g$.

Now we review quickly the two parametrization of P_g mentioned in the introduction.

First we recall the parametrization by the bundle of quadratic differentials $\pi : Q_g \to T_g$. Given a $\mathbf{C}\mathrm{P}^1$-structure, we take its developing map f. Then its Schwarzian derivative, defined by $(f''/f')^2 - 1/2(f''/f')'$, is a quadratic differential holomorphic with respect to the underlying complex structure. (For geometric meanings of Schwarzian derivatives see Thurston [Th2].) Therefore, we have a Riemann surface $X \in T_g$ and a holomorphic quadratic differential on X. Conversely, it is known that given a Riemann surface X and a holomorphic quadratic differential q there is a $\mathbf{C}\mathrm{P}^1$-structure on X whose Schwarzian derivative of the developing map is q. Note that for each point $X \in T_g$, the space of $\mathbf{C}\mathrm{P}^1$-structures on X is the fiber $Q_g(X)$ of $\pi : Q_g \to T_g$ over X, which is complex $3g-3$-dimensional Banach space.

On the other hand, Thurston's parametrization theorem is as follows.

Theorem 2.1 (Thurston, unpublished). *The space of* $\mathbf{C}\mathrm{P}^1$*-structures on* Σ_g *is parametrized by* $T_g \times \mathcal{ML}$*: every* $\mathbf{C}\mathrm{P}^1$*-structure is obtained by grafting a measured lamination to a hyperbolic surface.*

As for a full proof of this parametrization theorem, see section 2 of [KT],or, for a short discription, section 2 of [Tg]. His idea is as follows: Given a hyperbolic surface X and a measured geodesic lamination λ, first, embed the universal cover $\mathbf{H}^2$ of X into $\mathbf{H}^3$. Then, lift λ to $\mathbf{H}^2$ and then bend $\mathbf{H}^2$ along the lift of λ so that the bending measure is the measure of λ. This bent structure can be "pushed forward" to the sphere at infinity to determin a $\mathbf{C}\mathrm{P}^1$-structure.

Note that the projection from $T_g \times \mathcal{ML}$ to its first coordinate T_g is *not* the correspondence between $\mathbf{C}\mathrm{P}^1$-structures and their underlying complex structures. On the contrary, the underlying complex structure of any $\mathbf{C}\mathrm{P}^1$-structure represented by $(X,\mu) \in T_g \times \mathcal{ML}$, differs from X, unless $\mu = 0$.

Remark. In relation to this parametrization, Labourie [L] gave a new parametrization for P_g.

Now we define the pleated maps and the lengths of measured laminations for $\mathrm{PSL}(2,\mathbf{C})$-representations of $\pi_1\Sigma_g$. We begin with "pleated surfaces". It is defined in a parallel way to that of 3-manifolds. .

Definition 2.2. Let $\phi : \pi_1\Sigma_g \to \mathrm{PSL}(2,\mathbf{C})$ be a representation. A *pleated mapping* for ϕ is a continuous mapping $f : \mathbf{H}^2 \to \mathbf{H}^3$ with the following properties:

(1) there exists a Fuchsian group Γ acting on $\mathbf{H}^2$ isomorphic to $\pi_1\Sigma_g$ such that $f \circ \gamma = \phi(\gamma) \circ f$ for all $\gamma \in \Gamma$.
(2) for every $z \in \mathbf{H}^2$, there is an open interval of hyperbolic line containing z which is mapped by f to a straight line segment.
(3) f maps every geodesic segment in $\mathbf{H}^2$ to a rectifiable arc in $\mathbf{H}^3$ with the same length.

This is the same as the usual definition of pleated surfaces when $\phi(\pi_1\Sigma_g)$ is discrete. We shall call the image of a pleated map a *pleated surface*.

A typical example of a pleated surface is the "bent surface" defining a $\mathbf{C}\mathrm{P}^1$-structure in Thurston's geometric parametrization theorem. It is a pleated map for the monodromy of the $\mathbf{C}\mathrm{P}^1$-structure.(see [EM], [KT] and [Tg]).

Now we define the lengths of measured laminations for $\mathrm{PSL}(2,\mathbf{C})$-representations of $\pi_1\Sigma_g$.

Definition 2.3. Let $\phi : \pi_1\Sigma_g \to \mathrm{PSL}(2,\mathbf{C})$ be a non-elementary representation and let $f : \mathbf{H}^2 \to \mathbf{H}^3$ be a pleated map equivariant with respect to some Fuchsian group Γ isotopic to $\pi_1\Sigma_g$ acting on $\mathbf{H}^2$ and $\phi(\pi_1\Sigma_g)$. Such a mapping determines a hyperbolic structure on Σ_g. Then let $l_f(\mu)$ be the total mass of the product of the transverse measure of μ and the length along the lines of the support of μ. The length $l_\phi(\mu)$ is the infimum of $l_f(\mu)$ where the infimum is taken over all Fuchsian groups Γ isotopic to $\pi_1\Sigma_g$ and pleated maps f.

Remark. As we will see in Lemma 2.5, for any non-elementary $\mathrm{PSL}(2,\mathbf{C})$-representation of Σ_g, there exist at least one pleated map as above.

As Definition 2.2, when the $\mathrm{PSL}(2,\mathbf{C})$-representation of $\pi_1\Sigma_g$ is discrete, the above definition coincides with that of usual length of a measured lamination. Also, given a homotopy class of an equivariant map, the uniqueness of the realization of a lamination holds by the same reason as in the discrete case.

The length function is a function of two variables: measured laminations and representations. We might as well expect the continuity of the length function with respect to the both variables. (In fact, for a simple closed curve, the length is continuous with respect to $\mathrm{PSL}(2,\mathbf{C})$-representation of $\pi_1\Sigma_g$.) However, we do not need that strong property for our purpose. What we need is local boundedness (Lemma 2.5 bellow).

Kapovich ([Ka]) has already mentioned the concept of pleated maps and showed the existence for any non-elementary $\mathrm{SL}(2,\mathbf{C})$-representation of $\pi_1\Sigma_g$ briefly. We use these examples of pleated maps observed by Kapovich.

Here, for convenience, we will exibit somewhat detailed argument about them.

The following decomposition is the key.

Theorem 2.4 ([Ka]). *For any non-elementary representation*

$$\phi : \pi_1\Sigma_g \to \mathrm{PSL}(2, \mathbf{C})$$

there exists a pants decomposition $\Sigma_g = P_1 \cup ... \cup P_{2g-2}$ *such that*

(1) $\phi(\pi_1(P_i))$ *is non-elementary and*
(2) $\phi(\gamma)$ *is loxodromic for each boundary curve* γ *of* P_i *for* $i = 1, ..., 2g-2$.

Lemma 2.5. *The length function of measured laminations is locally bounded in the space of non-elementary* $\mathrm{PSL}(2, \mathbf{C})$*-representations of* $\pi_1\Sigma_g$.

Proof. Given a non-elementary representation $\phi : \pi_1\Sigma_g \to \mathrm{PSL}(2, \mathbf{C})$ take a pants decomposition $\Sigma_g = P_1 \cup ... \cup P_{2g-2}$ as in the above theorem. Decompose each pants into two ideal triangles by adding leaves spiraling around the boundaries.

Take a continuous equivariant map $f : \mathbf{H}^2 \to \mathbf{H}^3$ (with respect to any Fuchsian group isotopic to Σ_g and $\phi(\Sigma_g)$) and homotope f so that f sends each leaf of the ideal triangulation onto a straight line, then straighten the map in each ideal triangle. This is possible because by Theorem 2.4 the restriction of ϕ to the fundamental group of each pants is isomorphic. Thus we have a pleated surface with pleating locus the geodesics defining the ideal triangulation.

Thus, for each non-elementary $\mathrm{PSL}(2, \mathbf{C})$-representation ϕ, the length of any measured lamination is finite. Here, note that when we take an ideal triangulation as above for ϕ, we can take the same one for $\mathrm{PSL}(2, \mathbf{C})$-representations which are sufficiently close to ϕ. As the $\mathrm{PSL}(2, \mathbf{C})$-representations changing continuously near ϕ, we can take hyperbolic structures on Σ_g determined by the pleated maps with the fixed ideal triangulation so that they change continuously. From the continuity of the lengths of measured laminations on surfaces, we have a local upper bound of the lengths of measured laminations by the lengths on the hyperbolic structures. It follows that the lengths for the representations are locally bounded. □

Remark. Let $mon : P_g \to \mathrm{Hom}(\Sigma_g, \mathrm{PSL}(2, \mathbf{C}))$ be the monodromy map, namely, the map mon sends each $\mathbf{C}\mathrm{P}^1$-structure to its monodromy representation. Hejhal [H] showed that mon is a local diffeomorphism. Therefore, the image $mon(P_g)$ is a region of $\mathrm{Hom}(\Sigma_g, \mathrm{PSL}(2, \mathbf{C}))$. Therefore, obviously the lengths of measured laminations are locally bounded on the region $mon(P_g)$ by Theorem 2.1. A theorem by Gallo, Kapovich and Marden [GKM] says that every non-elementary $\mathrm{SL}(2, \mathbf{C})$-representation of Σ_g is obtained as the monodromy of a $\mathbf{C}\mathrm{P}^1$-structure. Therefore, if we would use that result, the local boundedness on the entire space of $\mathbf{C}\mathrm{P}^1$-structures would follow immediately. However, we exibited a direct proof rather than using their theorem.

3. Divergence of projective structures

Our goal is to give a geometric proof for the following:

Theorem 3.1 ([Ka]). *For any compact set* K *of* T_g *and any diverging sequence* $\{q_n\}$ *of* $\mathbf{C}\mathrm{P}^1$*-structures with* $\{\pi(q_n)\} \subset K$, *their monodromies* $mon(q_n)$ *diverge. Here,* $\pi(q_n)$ *stands for the underlying complex structure of* q_n.

The idea is as follows: The extremal length is the square order of the hyperbolic length when hyperbolic structures stay in a compact set of T_g. On the other hand, Theorem 3.2 bellow will show that $l_X(\mu)$ and $E_{gr_\mu(X)}(\mu)$ is in the same order when they are large. Here, $gr_\mu(X)$ denote the Riemann surface obtained by grafting μ to X (namely, the underlying structure of the $\mathbf{C}\mathrm{P}^1$-structure given by Thurston's parameter (X,μ)), $l_X(\mu)$ is the hyperbolic length of μ on X and $E_{gr_\mu(X)}(\mu)$ is the extremal length of μ on the grafted surface $gr_\mu(X)$. Furthermore, by the definition of bendinging, $l_X(\mu)$ is the length of μ for the monodromy of the $\mathbf{C}\mathrm{P}^1$-structure. Therefore, for a diverging sequence $q_n = (X_n, \mu_n)$ as in the assumption of Theorem 3.1, $l_{X_n}(\mu)$ had to grow "too fast" and we can not maintain the monodromy in a bounded set of $\mathrm{PSL}(2,\mathbf{C})$-representation as $n \to \infty$, considering Lemma 2.5.

Theorem 3.2([Tg]). *Let X be a hyperbolic surface and μ be a measured lamination. Let $h : gr_\mu(X) \to X$ be the harmonic map with respect to the hyperbolic metric on X and $\mathcal{E}(h)$ be its energy.*

Then
$$\frac{1}{2} l_X(\mu) \le \frac{1}{2}\frac{l_X(\mu)^2}{E_{gr_\mu(X)}(\mu)} \le \mathcal{E}(h) \le \frac{1}{2} l_X(\mu) + 4\pi(g-1).$$

Recall that in Remark 1 in [Tg], we observed that when $l_X(\mu)$ is very large, that is, when the grafted structure is far from X, grafting is very close to the inverse of the harmonic map h and the pleated surface defining the grafting is close to the image of the harmonic equivariant map for the monodromy representation.

The following fact follows easily from the above inequality (cf. [Tg]).

Corollary 3.3. *For any $X \in T_g$, the mapping $gr_\cdot(X) : \mathcal{ML} \to T_g$ is a proper map. For any $\mu \in \mathcal{ML}$, the mapping $gr_\mu(\cdot) : T_g \to T_g$ is a proper map.*

Proof. The properness of $gr_\mu(\cdot) : T_g \to T_g$ was shown in [Tg]. For the properness of $gr_\cdot(X) : \mathcal{ML} \to T_g$, note that

$$\frac{E_{gr_\mu(X)}(\mu)}{E_X(\mu)} \le \frac{l_X(\mu)}{E_X(\mu)},$$

by Theorem 3.2. When X is fixed and μ tends to infinity, the right term tends to 0, therefore, so is the left term. It follows that the Riemann surface $gr_\mu(X)$ tends to infinity. $\square$

Remark. In fact, in what follows we use only the leftmost inequality in Theorem 3.2. The others was necessary to show that $gr_\mu(\cdot)$ is proper.

Now we can present a geometric proof of Theorem 3.1.

Proof of Theorem 3.1.

Let $(X_n, \mu_n) \in T_g \times \mathcal{ML}$ be the Thurston coordinate of the projective structure q_n and denote the underlying complex structure by $\pi(q_n) = gr_{\mu_n}(X_n)$. Note that in Corollary 3.3 in fact we can show that the mappings are "locally uniformly" proper with respect to each variable. Therefore, when $\pi(q_n)$ stays in K and $\{q_n\}$ tends to infinity, both of $\{X_n\}$ and $\{\mu_n\}$ tend to infinity. Take a sequence ϵ_n converging to 0 such that $\epsilon_n \mu_n$ converges to a non-zero measured lamination μ, taking a subsequence of $\{\mu_n\}$ if necessary.

By Theorem 3.1,

$$\frac{l_{X_n}(\mu_n)^2}{l_{X_n}(\mu_n) + 8\pi(g-1)} \le E_{gr_{\mu_n}(X_n)}(\mu_n) \le l_{X_n}(\mu_n).$$

Multiply each term of the above inequality by ϵ_n^2. Then we have

$$\frac{l_{X_n}(\epsilon_n\mu_n)^2}{l_{X_n}(\mu_n)+8\pi(g-1)} \leq E_{gr_{\mu_n}(X_n)}(\epsilon_n\mu_n) \leq \epsilon_n l_{X_n}(\epsilon_n\mu_n).$$

Note that $l_{X_n}(\epsilon_n\mu_n)$ is the length of $\epsilon_n\mu_n$ for the monodromy representation $mon(q_n)$ because the pleated surface defining the Thurston coordinate for $q_n = (X_n, \mu_n)$ has pleating locus μ_n. Therefore, if the representation $mon(q_n)$ did not diverge, $l_{X_n}(\epsilon_n\mu_n)$ would converge to a non-negative number by Lemma 2.5, taking a subsequence if necessary. Therefore, by the above inequality,

$$\lim_{n\to\infty} E_{gr_{\mu_n}(X_n)}(\epsilon_n\mu_n) = 0$$

However, by taking a subsequence if necessary, we may assume that $\pi(q_n) = gr_{\mu_n}(X_n)$ converges to a point $Y \in T_g$ by the assumption that $\{q_n\}$ stays in the compact set K. Therefore $E_{gr_{\mu_n}(X_n)}(\epsilon_n\mu_n)$ converges to $E_Y(\mu)$, which is a positive number. This is a contradiction. $\square$

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