Complex dynamics in several variables

Lectures by John Smillie
Notes by Gregery T. Buzzard

Contents

Abstract

These notes are the outgrowth of a series of lectures given at MSRI in January 1995 at the beginning of the special semester in complex dynamics and hyperbolic geometry. In these notes, the primary aim is to motivate the study of complex dynamics in two variables and to introduce the major ideas in the field. Hence the treatment of the subject is mostly expository.

1 Introduction

These notes are the outgrowth of a series of lectures given at MSRI in January 1995 at the beginning of the special semester in complex dynamics and hyperbolic geometry. The goal of these lectures was to provide an introduction to the relevant ideas and problems of complex dynamics in several variables and to provide a foundation for further study and research. There were parallel sessions in complex dynamics in one variable, given by John Hubbard, and in hyperbolic geometry, given by James Cannon, and notes for those series should also be available.

In these notes, the primary aim is to motivate the study of complex dynamics in two variables and to introduce the major ideas in the field. Hence the treatment of the subject is mostly expository.

2 Motivation

The study of complex dynamics in several variables can be motivated in at least two natural ways. The first is by analogy with the fruitful study of complex dynamics in one variable. However, since this latter subject is covered in detail in the parallel notes of John Hubbard, we focus here on the motivation coming from the study of real dynamics.

A classical problem in the study of real dynamics is the n-body problem, which was studied by Poincaré. For instance, we can think of n planets moving in space. For each planet, there are 3 coordinates giving the position and 3 coordinates giving the velocity, so that the state of the system is determined by a total of $6n$ real variables. The evolution of the system is governed by Newton's laws, which can be expressed as a first order ordinary differential equation. In fact, the state of the system at any time determines the entire future and past evolution of the system.

To make this a bit more precise, set $k = 6n$. Then the behavior of the n planets is modeled by a differential equation

$$(\dot{x}_1, \dots, \dot{x}_k) = F(x_1, \dots, x_k)$$

Research at MSRI is supported in part by NSF grant DMS-9022140

for some $F : \mathbb{R}^k \to \mathbb{R}^k$. Here $\dot{x}$ denotes the derivative of x with respect to t.

From the elementary theory of ODE's, we know that this system has a unique solution $t \mapsto \phi_t(x_1, \ldots, x_k)$ satisfying $\dot{\phi} = F(\phi)$ and $\phi_0(x_1, \ldots, x_k) = (x_1, \ldots, x_k)$.

For purposes of studying dynamics, we would like to be able to say something about the evolution of this system over time, given some initial data. That is, given $p \in \mathbb{R}^k$, we would like to be able to say something about $\phi_t(p)$ as t varies. For instance, a typical question might be the following.

Question 2.1 *Given $p = (x_1, \ldots, x_k)$, is $\{\phi_t(p) : t \geq 0\}$ bounded?*

Unfortunately, the usual answer to such a question is "I don't know." Nevertheless, it is possible to say something useful about related questions, at least in some settings. For instance, one related problem is the following.

Problem 2.2 *Describe the set*

$$K^+ := \{p \in \mathbb{R}^k : \{\phi_t(p) : t \geq 0\} \text{ is bounded}\}.$$

Although this question is less precise and gives less specific information than the original, it can still tell us quite a bit about the behavior of the system.

3 Iteration of maps

In the above discussion, we have been taking the approach of fixing a point $p \in \mathbb{R}^k$ and following the evolution of the system over time starting from this point. An alternative approach is to think of all possible starting points evolving simultaneously, then taking a snapshot of the result at some particular instant in time.

To make this more precise, assume that the solution $\phi_t(p)$ exists for all time t and all $p \in \mathbb{R}^k$. In this case, for fixed t, the map $\phi_t : \mathbb{R}^k \to \mathbb{R}^k$ is a diffeomorphism of $\mathbb{R}^k$ and satisfies the group property

$$\phi_{s+t} = \phi_s \circ \phi_t$$

for any s and t.

In order to make our study more tractable, we make two simplifications.

Simplification 1: Choose some number $\alpha > 0$, which we will call the *period*, and define $f = \phi_\alpha$. Then f is a diffeomorphism of $\mathbb{R}^k$, and given $p \in \mathbb{R}^k$, the group property of ϕ implies that

$$\phi_{n\alpha}(p) = \phi_\alpha \circ \cdots \circ \phi_\alpha(p) = f^n(p).$$

That is, studying the behavior of f under iteration is equivalent to studying the behavior of ϕ at regularly spaced time intervals.

Simplification 2: Set $k = 2$. Although this simplification means that we can no longer directly relate our model to the original physical problem, the ideas and techniques involved in studying such a simpler model are still rich enough to shed some light on the more realistic cases. In fact, there are interesting questions in celestial mechanics which reduce to questions about two dimensional diffeomorphisms, but here we are focusing on the mathematical model rather than on the physical system.

Finally, we make the following definition.

Definition 3.1 *Given $p \in \mathbb{R}^2$, set*

$$\begin{aligned} \mathcal{O}^+(p) &:= \{f^n(p) : n \geq 0\}, \\ \mathcal{O}^-(p) &:= \{f^n(p) : n \leq 0\}, \\ \mathcal{O}(p) &:= \{f^n(p) : n \in \mathbb{Z}\}. \end{aligned}$$

With these simplifications and this definition, we can further reformulate the question from the previous section as follows.

Problem 3.2 *Given a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, describe the sets*

$$\begin{aligned} K^+ &:= \{p \in \mathbb{R}^2 : \mathcal{O}^+(p) \text{ is bounded}\}, \\ K^- &:= \{p \in \mathbb{R}^2 : \mathcal{O}^-(p) \text{ is bounded}\}, \\ K &:= \{p \in \mathbb{R}^2 : \mathcal{O}(p) \text{ is bounded}\}, \end{aligned}$$

For future reference, note that $K = K^+ \cap K^-$.

4 Regular versus chaotic behavior

For the moment, we will make no attempt to define rigorously what we mean by regular or chaotic. Intuitively, one should think of regular behavior as being very predictable and as relatively insensitive to small changes in the system or initial conditions. On the other hand, chaotic behavior is in some sense random and can change drastically with only slight changes in the system or initial conditions. Here is a relevant quote from Poincaré on chaotic behavior:

> *A very small cause, which escapes us, determines a considerable effect which we cannot ignore, and we say that this effect is due to chance.*

We next give some examples to illustrate both kinds of behavior, starting with regular behavior. First we make some definitions.

Definition 4.1 *A point $p \in \mathbb{R}^2$ is a* ***periodic point*** *if $f^n(p) = p$ for some $n \geq 1$. The smallest such n is the* ***period*** *of p.*

Definition 4.2 *A periodic point p is* ***hyperbolic*** *if $(Df^n)(p)$ has no eigenvalues on the unit circle.*

Definition 4.3 *If p is a hyperbolic periodic point and both eigenvalues are inside the unit circle, then p is called a* ***sink****.*

Definition 4.4 *If p is a hyperbolic periodic point, then the set*

$$W^s(p) = \{q \in \mathbb{R}^2 : d(f^n q, f^n p) \to 0 \ \text{ as } \ n \to \infty\}$$

is called the ***basin of attraction*** *of p if p is a sink, and the* ***stable manifold*** *of p otherwise. Here d is a distance function. If f is a diffeomorphism and p is not a sink, then the* ***unstable manifold*** *of p is defined by replacing f by f^{-1} in the above definition.*

Fact: When p is a sink, $W^s(p)$ is an open set containing p.

A sink gives a prime example of regular behavior. Starting with any point q in the basin of attraction of a sink p, the forward orbit of q is asymptotic to the (periodic) orbit of p. Since the basin is open, this will also be true for any point q' near enough to q. Hence we see the characteristics of predictability and stability mentioned in relation to regular behavior.

For an example of chaotic behavior, we turn to a differential equation studied by Cartwright and Littlewood in 1940. This is the system

$$\ddot{y} - k(1-y^2)\,\dot{y} + y = b\cos(t).$$

Introducing the variable $x = \dot{y}$, we can write this as a first-order system

$$\begin{aligned} \dot{y} &= x \\ \dot{x} &= g(x, y, t), \end{aligned}$$

where g is a function satisfying $g(x, y, t+2\pi) = g(x, y, t)$. This system has a solution ϕ_t as before with $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ a diffeomorphism. Although the full group property does not hold for ϕ since g depends on t, we still have $\phi_{s+t} = \phi_s \circ \phi_t$ whenever $s = 2\pi n$ and $t = 2\pi m$ for integers n and m. Hence we can again study the behavior of this system by studying the iterates of the diffeomorphism $f = \phi_{2\pi}$.

Rather than study this system itself, we follow the historical development of the subject and turn to a more easily understood example of chaotic behavior which was motivated by this system of Cartwright and Littlewood: the Smale horseshoe.

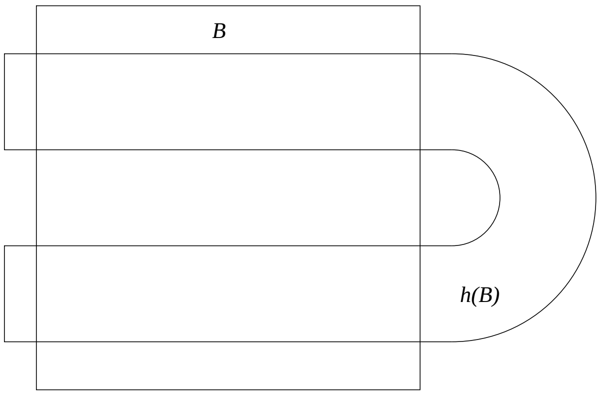

Figure 1: The image of B under the horseshoe map h.

5 The horseshoe map and symbolic dynamics

The horseshoe map was first conceived by Smale as a way of capturing many of the features of the Cartwright-Littlewood map in a system which is easily understood.

For our purposes, the horseshoe map, h, is defined first on a square B in the plane with sides parallel to the axes. First we apply a linear map which stretches the square in the x-direction and contracts it in the y-direction. Then we take the right edge of the resulting rectangle and lift it up and around to form a horseshoe shape. The map h is then defined on B by placing this horseshoe over the original square B so that $B \cap h(B)$ consists of 2 horizontal strips in B. See figure 1.

We can extend h to a diffeomorphism of $\mathbb{R}^2$ in many ways. We do that here as follows. First partition $\mathbb{R}^2 - B$ into 4 regions by using the lines $y = x$ and $y = -x$ as boundaries. Denote the union of the two regions above and below B by B^+ and the union of the two regions to the left and right of B by B^-, as in figure 2. Then we can extend h to a diffeomorphism of $\mathbb{R}^2$ in such a way that $h(B^-) \subseteq B^-$. In this situation, points in B^+ can be mapped to any of the 3 regions B^+, B, or B^-, points in B can be mapped to either B or B^-, and points in B^- must be mapped to B^-. Further, we require that points in B^- go to ∞ under iteration, and we require analogous conditions on f^{-1}. Note in particular that points which leave B do not return and that $K \subseteq B$.

It is not hard to see that in this case, we have

$$K^- \cap B = B \cap hB \cap h^2B \cap \cdots .$$

In fact, if we look at the image of the two strips $B \cap hB$ and intersect with B, then the resulting set consists of 4 strips; each of the original two strips is subdivided into two smaller strips. Continuing this process, we see that $K^- \cap B$ is simply the set product of an interval and a Cantor set.

In fact, a simple argument shows that h has a fixed point p in the lower left corner of B, and that the unstable manifold of p is dense in the set $K^- \cap B$ and the stable

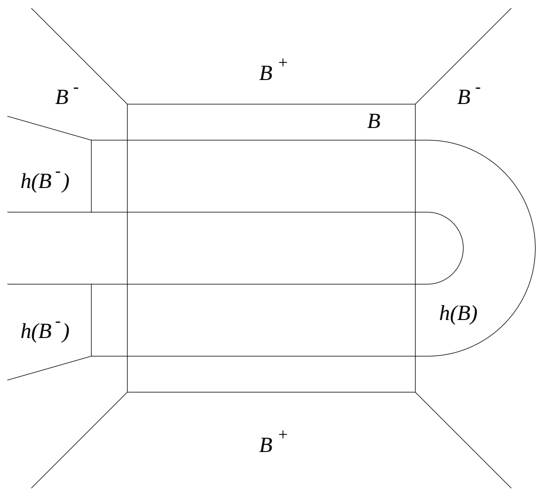

Figure 2: The sets B, B^+, and B^-.

manfold of p is dense in $K^+ \cap B$. The complicated structure of the stable and unstable manifolds plays an important role in the behavior of the horseshoe map.

We can describe the chaotic behavior of the horseshoe using *symbolic dynamics*. The idea of this procedure is to translate from the dynamics of h restricted to K into the dynamics of a shift map on bi-infinite sequences of symbols.

To do this, first label the 2 components of $B \cap hB$ with H_0 and H_1. Then to a point $p \in K$, we associate a bi-infinite sequence of 0's and 1's using the map

$$\psi : p \mapsto s = (\ldots, s_1, s_0, s_{-1}, \ldots),$$

where

$$s_j = \begin{cases} 0 & \text{if } h^j(p) \in H_0 \\ 1 & \text{if } h^j(p) \in H_1. \end{cases}$$

We can put a metric on the space of bi-infinite sequences of 0's and 1's by

$$d(s, s') = \sum_{j=-\infty}^{\infty} |s_j - s'_j|\, 2^{-|j|}.$$

It is not hard to show that the metric space thus obtained is compact and that the map ψ given above produces a homeomorphism between K and this space of sequences.

Moreover, the definition of $\psi(p)$ implies that if σ is the left-shift map defined on bi-infinite sequences, then $\psi(h(p)) = \sigma(\psi(p))$.

Here are a couple of simple exercises which illustrate the power of using symbolic dynamics.

Exercise 5.1 *Show that periodic points are dense in K. Hint: Periodic points correspond to periodic sequences.*

Exercise 5.2 *Show that there are periodic points of all periods.*

6 Hénon maps

The horseshoe map was one motivating example for what are known as Axiom A diffeomorphisms. These received a great deal of attention in the 60's and 70's. Much current work focuses either on how Axiom A fails, as in the work of Newhouse, or on how some Axiom A ideas can be applied in new settings, as in the work of Benedicks and Carleson [BC] or Benedicks and Young [BY]. For more information and further references, the book by Ruelle [R] provides a fairly gentle introduction, while the books by Palis and de Melo [PD], Shub [S], and Palis and Takens [PT] are more advanced. See also the paper by Yoccoz in [Y].

Although there are still some interesting open questions about Axiom A diffeomorphisms and related subjects, much of the study in the post Axiom A era has centered around the study of the so-called Hénon map. This is actually a family of diffeomorphisms $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f_{a,b}(x, y) = (-x^2 + a - by, x)$$

for $b \neq 0$. These maps arise from a simplification of a simplification of a map describing turbulent fluid flow.

We can get some idea of the behavior of the map $f_{a,b}$ and the ways in which it relates to the horseshoe map by considering the image of a large box B under $f_{a,b}$. For simplicity, we write f for $f_{a,b}$. From figure 3, we see that for some values of a and b, the Hénon map f is quite reminiscent of the horseshoe map h.

Since the map f is polynomial in x and y, we can also think of x and y as being complex-valued. In this case, $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a holomorphic diffeomorphism of $\mathbb{C}^2$. This is also in some sense a change in the map f, but all of the dynamics of f restricted to $\mathbb{R}^2$ are contained in the dynamics of the maps on $\mathbb{C}^2$, so we can still learn about the original map by studying it on this larger domain.

We next make a few observations about f. First, note that f is the composition $f = f_3 \circ f_2 \circ f_1$ of the three maps

$$\begin{aligned} f_1(x, y) &= (x, by) \\ f_2(x, y) &= (-y, x) \\ f_3(x, y) &= (x + (-y^2 + a), y) \end{aligned}$$

For $0 < b < 1$, the maps f_1 and $f_2 f_1$ are depicted in figures 4 and 5, while f is depicted in figure 3 with some $a > 0$.

From the composition of these functions, we can easily see that f has constant Jacobian determinant $\det(DF) = b$. Moreover, when $b = 0$, f reduces to a quadratic polynomial on $\mathbb{C}$.

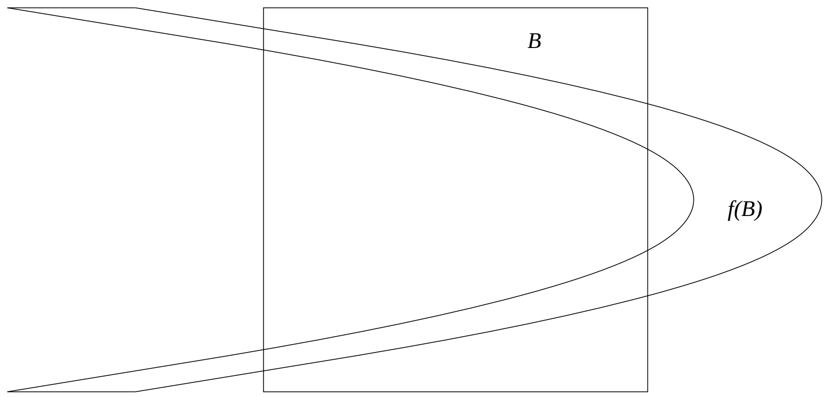

Figure 3: A square B and its image $f(B)$ for some parameter values a and b.

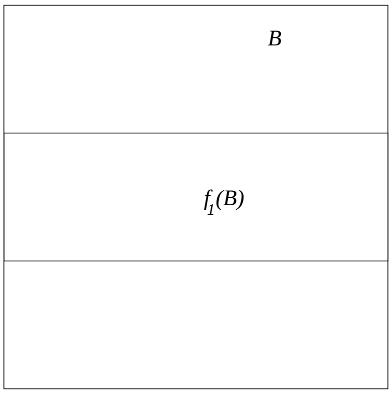

Figure 4: $f_1(B)$

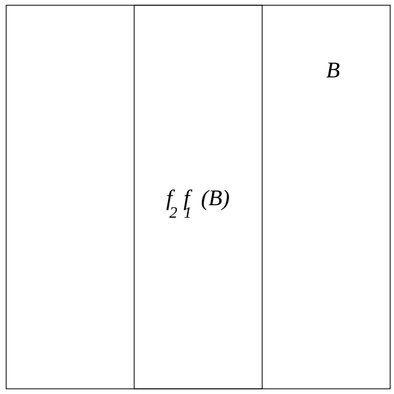

Figure 5: $f_2 f_1(B)$

A simple argument shows that there is an $R = R(a, b)$ such that if we define the three sets

$$\begin{aligned} B &= \{|x| < R, |y| < R\}, \\ B^+ &= \{|y| > R, |y| > |x|\}, \\ B^- &= \{|x| > R, |x| > |y|\}, \end{aligned}$$

then we have the same dynamical relations as for the corresponding sets for the horseshoe map. That is, points in B^+ can be mapped to B^+, B, or B^-, points in B can be mapped to B or B^-, and points in B^- must be mapped to B^-.

As we did for the horseshoe, we can define K^+ to be the set of points with bounded forward orbit, K^- to be the set of points with bounded backward orbit, and K to be the intersection of these two sets.

When a is large enough, so that the tip of $f_{a,b}(B)$ is completely outside B, then $f_{a,b}$ "is" a horseshoe. By this we mean that $B \cap f_{a,b}$ has two components, and hence we can use symbolic dynamics exactly as before to show that the dynamics of f

restricted to K are exactly the same as in the horseshoe case. The general proof of this result is due to Hubbard and Oberste-Vorth [HO]. See also [DN].

Example 6.1 *To compare the dynamics of f in the real and complex cases, consider $f_{a,b}$ with a and b real. As an ad hoc definition, let $K_{\mathbf{R}}$ be the set of $p \in \mathbb{R}^2$ with bounded forward and backward orbits, and let $K_{\mathbf{C}}$ be the set of $p \in \mathbb{C}^2$ with bounded forward and backward orbits. Then another result of [HO] is that $K_{\mathbf{C}} = K_{\mathbf{R}}$.*

Thus we already have a mental picture of K for these parameter values. We can also get a picture of K^+ and K^- in the complex case, since we can extend the analogy between f and the horseshoe map by replacing the square B by a bidisk $B = D(R) \times D(R)$ contained in $\mathbb{C}^2$, where $D(R)$ is the disk of radius R centered at 0 in $\mathbb{C}$. In the definitions of B^+ and B^-, we can interpret x and y as complex-valued, in which case the definitions of these sets still make sense. Moreover, the same mapping relations hold among B^+, B^-, and B as before. In this case, $B \cap K^+$ is topologically equivalent to the set product of a Cantor set and a disk, $B \cap K^-$ is equivalent to the product of a disk and a Cantor set, and $B \cap K$ is equivalent to the product of two Cantor sets.

Thesis: A surprising number of properties of the horseshoe (when properly interpreted) hold for general complex Hénon diffeomorphisms.

The "surprising" part of the above thesis is that the horseshoe map was designed to be simple and easily understood, yet it sheds much light on the less immediately accessible Hénon maps.

7 Properties of horseshoe and Hénon maps

We again consider some properties of the horseshoe map in terms of its periodic points. There is another relevant quote from Poincaré about periodic points.

> *What renders these periodic points so precious to us is that they are, so to speak, the only breach through which we might try to penetrate into a stronghold hitherto reputed unassailable.*

As an initial observation, recall that from symbolic dynamics, we know that the periodic points are dense in K. In fact, it is not hard to show that these periodic points are all ***saddle points***; that is, if p has period n, then $(Dh^n)(p)$ has one eigenvalue larger than 1 in modulus, and one smaller. For such a periodic point p, recall the definitions of the stable and unstable manifolds

$$W^s(p) = \{q : d(f^n p, f^n q) \to 0 \text{ as } n \to \infty\}$$
$$W^u(p) = \{q : d(f^n p, f^n q) \to 0 \text{ as } n \to -\infty\}.$$

Exercise 7.1 *For any periodic saddle point of the horseshoe map h, $W^s(p)$ is dense in K^+ and $W^u(p)$ is dense in K^-.*

As a consequence of this exercise, suppose $p \in K^+$, and let $n \in \mathbb{N}$ and $\epsilon > 0$. By exercise 5.2, there is a periodic point q with period n, and by this last exercise, the stable manifold for q comes arbitrarily close to p. In particular, we can find $p' \in W^s(q)$ with $d(p, p') < \epsilon$. Hence in any neighborhood of p, there are points which are asymptotic to a periodic point of any given period. We can contrast this with a point p in the basin of attraction for a sink. In this case, for a small enough neighborhood of p, every point will be asymptotic to the same periodic point.

This example illustrates the striking difference between regular and chaotic behavior. In the case of a sink, the dynamics of the map are relatively insensitive to the precise initial conditions, at least within the basin of attraction. But in the horseshoe case, the dynamics can change dramatically with an arbitrarily small change in the initial condition. In a sense, chaotic behavior occurs throughout K^+.

A second basic example of Axiom A behavior is the solenoid. Take a solid torus in $\mathbb{R}^3$ and map it inside itself so that it wraps around twice. The image of this new set then wraps around 4 times. The solenoid is the set which is the intersection of all the forward images of this map. Moreover, the map extends to a diffeomorphism of $\mathbb{R}^3$ and displays chaotic behavior on the solenoid, which is the attractor for the diffeomorphism.

Example 7.2 *Consider $f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ when a and b are small. Then Hubbard and Oberste-Vorth [HO] show that $f_{a,b}$ has a fixed sink as well as a solenoid, so that it displays both regular and chaotic behavior.*

Note that if q is a sink, then $W^s(q) \subseteq K^+$ is open, and hence $W^s(q) \subseteq \mathrm{int}\, \mathrm{K}^+$. On the interior of K^+, there is no chaos. To see this, suppose $p \in \mathrm{int}\, \mathrm{K}^+$, and choose $\epsilon > 0$ such that $\overline{\mathrm{B}_\epsilon(p)} \subseteq K^+$. A simple argument using the form of f and the definitions of B, B^+, and B^- shows that any point in K^+ must eventually be mapped into B. Hence by compactness, there is an n sufficiently large that $f^n(\overline{\mathrm{B}_\epsilon(p)}) \subseteq B$. Since B is bounded, we see by Cauchy's integral formula that the norm of the derivatives of f^n are uniformly bounded on $\mathrm{B}^\epsilon(p)$ independently of $n \geq 0$. This is incompatible with chaotic behavior. For more information and further references, see [BS3].

As a first attempt at studying sets where chaotic behavior can occur, we make the following definitions.

Definition 7.3 *For a complex Hénon map f, and with K^+ and K^- defined as in problem 3.2, let $J^+ := \partial K^+$ and $J^- := \partial K^-$.*

The following theorem gives an analog of exercise 7.1 in the case of a general complex Hénon mapping, and is contained in [BS1].

THEOREM 7.4 *If p is a periodic saddle point of the Hénon map f, then $W^s(p)$ is dense in J^+, and $W^u(p)$ is dense in J^-.*

It can be shown [BLS2] that a Hénon map f has saddle periodic points of all but finitely many periods, so just as in the argument after exercise 7.1, we see that chaotic behavior occurs throughout J^+, and a similar argument applies to J^- under backward iteration.

8 Dynamically defined measures

Before talking about potential theory proper, we first discuss some measures associated with the horseshoe map h. With notation as in section 5, we define the level-n set of h to be the set $h^{-n}B \cap h^n B$. Since the forward images of B are horizontal strips and the backward images of B are vertical strips, we see that the level-n set consists of 2^{2n} disjoint boxes.

Assertion: For j sufficiently large, the number of fixed points of h^j in a component of the level-n set of h is independent of the component chosen.

In fact, there is a unique probability measure m on K which assigns equal weight to each level-n square, and the above assertion can be rephased in terms of this measure. Let P_k denote the set of $p \in \mathbb{C}^2$ such that $h^k(p) = p$. Then it follows from the above assertion that

$$\frac{1}{2^k} \sum_{p \in P_k} \delta_p \to m \tag{8.1}$$

in the topology of weak convergence.

We can use a similar technique to study the distribution of unstable manifolds. Again we consider the horseshoe map h, and we suppose that p_0 is a fixed saddle point of h and that S is the component of $W^u(p_0) \cap B$ containing p_0. In this case, S is simply a horizontal line segment through p_0. Next, let T be a line segment from the top to the bottom of B so that T is transverse to every horizontal line. See figure 6.

We can define a measure on T using an averaging process as before. This time we average over points in $h^n(S) \cap T$ to obtain a measure m_T^-. Thus we have

$$\frac{1}{2^n} \sum_{p \in h^n(S) \cap T} \delta_p \to m_T^-, \tag{8.2}$$

where again the convergence is in the weak sense. This gives a measure on T which assigns equal weight to each level-n segment; i.e., to each component of $h^n(B) \cap T$.

Note that if T' is another segment like T, then the unstable manifolds of h give a way to transfer the above definition to a measure on T'. That is, given a point

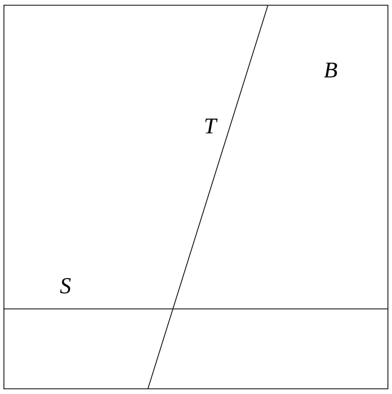

Figure 6: T and S

$p \in h^n(S) \cap T$, we can project along the component of $h^n(S) \cap B$ containing p to obtain a point $p' \in T'$. Using this map we obtain a measure $\phi(m_T^-)$ on T'. It is straightforward to show that this is the same measure as $m_{T'}^-$ obtained by using T' in place of T in (8.2).

Given this equivalence among these measures, we can define m^- to be this collection of equivalent measures, so that m^- is defined on any T. Using an analogous construction with stable manifolds, we can likewise define a measure m^+ defined on "horizontal" segments.

Finally, we can take the product of these two measures to get a measure $m^- \times m^+$ defined on B. Then one can show that this product measure is the same as the measure m defined in (8.1). Hence there are at least two dynamically natural ways to obtain this measure.

9 Potential theory

In the study of dynamics in one variable, there are many tools available coming from classical complex analysis, potential theory, and the theory of quasiconformal mappings. In higher dimensions, not all of these tools are available, but one tool which remains useful is potential theory. In this section we will provide some background for the ways in which this theory can be used to study dynamics.

To provide some physical motivation for the study of potential theory, consider two electrons moving in $\mathbb{R}^d$, each with a charge of -1. Then the repelling force between them is proportional to $1/r^{d-1}$. If we fix one electron at the origin, then the total work in moving the other electron from the point z_0 to the point z_1 is independent of the path taken and is given by $P(z_1) - P(z_0)$, where P is a potential function which depends on the dimension:

$$\begin{aligned} P(z) &= |z| && \text{if } d=1, \\ P(z) &= \log|z| && \text{if } d=2, \\ P(z) &= -\tfrac{1}{\|z\|^{d-2}} && \text{if } d\geq 3. \end{aligned} \tag{9.1}$$

From the behavior of P at 0 and ∞, we see that if $d \leq 2$, then the amount of work needed to bring a unit charge in from the point at infinity is infinite, while this work is finite for $d \geq 3$. On the other hand, if $d \geq 2$, then the amount of work needed to bring two electrons together is infinite, but for $d = 1$ this work is finite.

We can think of a collection of electrons as a charge, and we can represent charges by measures μ on $\mathbb{R}^d$. Then for $S \subseteq \mathbb{R}^d$, $\mu(S)$ is the amount of charge on S.

Example 9.1 *A unit charge at the point z_0 corresponds to the Dirac delta mass δ_{z_0}.*

By using measures to represent charges, we can use convolution to define potential functions for general charge distributions. That is, given a measure μ on $\mathbb{R}^d$, we define

$$P_\mu(z) = \int_{\mathbf{R^d}} P(z-w)d\mu(w), \tag{9.2}$$

where P is the appropriate potential function from (9.1). Note that this definition agrees with the previous definition of potential functions in the case of point charges. Note also that the assignment $\mu \mapsto P_\mu$ is linear in μ.

In order to be able to use potential functions to study dynamics, we first need to understand a little more about their properties. In particular, we would like to know which functions can be the potential function of a finite measure.

In the case $d = 1$, the definition of P_μ and the triangle inequality imply that potential functions are convex, hence also continuous. We also have that $P_\mu(x) = c|x| + O(1)$, where $c = \mu(\mathbb{R})$. In fact, any function, f, satisfying these two conditions is a potential function of some measure. Hence a natural question is, how do we recover the measure from f?

In particular, given a convex function f of one real variable, we can consider the assignment

$$f \mapsto \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} f\right) dx, \tag{9.3}$$

where the right hand side is interpreted in the sense of distributions. By convexity, this distribution is positive. That is, it assigns a positive number to positive test functions, and a positive distribution is actually a positive measure. Hence we have an explicit correspondence between convex functions and positive measures, and with the additional restriction on the growth of potential functions given in the previous paragraph, we have an explicit correspondence between potential functions and finite positive measures. (The 1/2 in the above formula occurs because we have normalized by dividing by the "volume" of the unit sphere in $\mathbb{R}$, i.e., the volume of the two points 1 and -1.)

In the case $d = 2$, the integral definition of P_μ implies that potential functions satisfy the ***sub-average property***. That is, given a potential function f, any z_0 in the plane, and a disk D centered at z_0, $f(z_0)$ is bounded above by the average of f on ∂D. That is, if σ represents 1-dimensional Legesgue measure normalized so that the unit circle has measure 1, and if r is the radius of D, then

$$f(z_0) \leq \frac{1}{r} \int_{\partial D} f(\zeta) d\sigma(\zeta).$$

Moreover, (9.2) implies that potential functions are ***upper semicontinuous*** (u.s.c.); a real-valued function f is said to be u.s.c. if all of its sub-level sets are open. With these two concepts we can make the following definition.

Definition 9.2 *If f is u.s.c. and satisfies the sub-average property, then f is called* ***subharmonic****.*

Finally, if f is subharmonic and satisfies $f(z) = c \log |z| + O(1)$ for some $c > 0$, then f is said to be a ***potential function***. Just as before, a potential function has the form P_μ for some measure μ.

In fact, if f is subharmonic and C^2, then the Laplacian of f is always positive. This is an analog of the fact that the second derivative of a convex function is positive. If f is subharmonic but not C^2, then Δf is a positive distribution, hence a positive measure. Thus the Laplacian gives us a correspondence between potential functions and finite measures much like that in (9.3):

$$f \mapsto \frac{1}{2\pi}(\Delta f) dx dy,$$

where this is to be interpreted in the sense of distributions and again we have normalized by dividing by the volume of the unit sphere.

Example 9.3 *Applying the above assignment to the function* $\log |z|$ *produces the delta mass δ_0 in the sense of distributions.*

Suppose now that $K \subseteq \mathbb{R}^2 = \mathbb{C}$ is compact and connected and put a charge on K and allow it to distribute evenly throughout K. Then the charge on K is distributed according to a finite positive measure μ, and we would like to know what the equilibrium state is for this system. Thus, we want to know what P_μ looks like.

To use more standard notation we write $G = P_\mu$, and we assume that μ has total charge (mass) equal to 1. Then G satisfies the following properties.

1. G is subharmonic.
2. G is harmonic outside K.
3. $G = \log|z| + O(1)$.
4. G is constant on K.

If G satisfies properties 1 through 3, and also property

4′. $G \equiv 0$ on K,

then we say that G is a Green function for K. If G exists, then it is unique, and in this case we can take the Laplacian of G in the sense of distributions. Thus, we say that

$$\mu_K := \frac{1}{2\pi}\Delta G\, dxdy$$

is the equilibrium measure for K.

Example 9.4 *Let D be the unit disk. Then the Green function for D is*

$$G(z) = \log^+|z|,$$

where $\log^+|z| := \max\{\log|z|, 0\}$, and the equilibrium measure is

$$\mu_D = \frac{1}{2\pi}(\Delta \log^+|z|)dxdy,$$

which is simply arc length measure on ∂D, normalized to have mass 1.

10 Potential theory in one variable dynamics

In this section we discuss some of the ways in which potential theory can be used to understand the dynamics of holomorphic maps of the Riemann sphere. This idea was first introduced by Brolin [Br] and later developed by others in both one and several variables.

For this section, let f be a monic polynomial in one variable of degree $d \geq 2$, and let $K \subseteq \mathbb{C}$ be the set of z such that the forward orbit of z is bounded. Then K has a Green function, and in fact, G_K is given by the formula

$$G_K(z) = \lim_{n\to\infty} \frac{1}{d^n}\log^+|f^n(z)|.$$

It is difficult to understand Brolin's paper without knowing this formula. However, it was in fact first written down by Sibony in his UCLA lecture notes after Brolin's paper had already been written.

It is not hard to show that the limit in the definition of G_K converges uniformly on compact sets, and since each of the functions on the right hand side is subharmonic, the limit is also subharmonic. Moreover, on a given compact set outside of K, each of these functions is harmonic for sufficiently large n, so that the limit is harmonic on the complement of K. Property 3 follows by noting that for $|z|$ large we have $|z|^d/c \leq |f(z)| \leq c|z|^d$ for some $c > 1$, then taking logarithms and dividing by d, then using an inductive argument to bound $|\ \log^+ |f^n(z)|/d^n - \log |z|\ |$ independently of d. Finally, property $4'$ is immediate since $\log^+ |z|$ is bounded for $z \in K$. In fact, G_K has the additional property of being continuous.

Hence we see that G_K really is the Green function for K, and we can define the equilibrium measure

$$\mu := \mu_K = \frac{1}{2\pi}(\Delta G_K) dx dy.$$

The following theorem provides a beautiful relationship between the measure μ and the dynamical properties of f. It says that we can recover μ by taking the average of the point masses at the periodic points of period n and passing to the limit or by taking the average of the point masses at the preimages of any nonexceptional point and passing to the limit. A point p is said to be nonexceptional for a polynomial f if the set $\{f^{-n}(p) : n \geq 0\}$ contains at least 3 points. It is a theorem that there is at most 1 exceptional point for any polynomial.

THEOREM 10.1 (Brolin, Tortrat) *Let f be a monic polynomial of degree d, and let $c \in \mathbb{C}$ be a nonexceptional point. Then*

$$\mu = \lim_{n\to\infty} \frac{1}{d^n} \sum_{z\in A_n} \delta_z,$$

in the sense of weak convergence, where A_n is either

1. The set of z satisfying $f^n(z) = c$, counted with multiplicity,

or

2. The set of z satisfying $f^n(z) = z$, counted with multiplicity.

Proof: We prove only part 1 here. Let

$$\mu_n = \frac{1}{d^n} \sum_{f^n(z)=c} \delta_z.$$

Then we want to show that $\mu_n \to \mu_K$. Since the space of measures with the topology of weak convergence is compact, it suffices to show that if some subsequence of μ_n converges to a measure μ^*, then $\mu^* = \mu_K$. By renaming, we may assume that μ_n converges to μ^*.

We can show $\mu^* = \mu_K$ by showing the convergence of the corresponding potential functions. The potential function for μ_n is

$$
\begin{aligned}
G_n(z) &= \frac{1}{d^n} \sum_{f^n(w)=c} \log|z-w| \\
&= \frac{1}{d^n} \log \left| \prod_{f^n(w)-c=0} (z-w) \right| \\
&= \frac{1}{d^n} \log|f^n(z)-c|.
\end{aligned}
$$

Here the sum and products are taken over the indicated sets with multiplicities, and the last equality follows from the fact that we are simply multiplying all the monomials corresponding to roots of $f^n(z) - c$.

Let $G^*(z) := \lim_{n\to\infty} G_n(z)$. Then G^* is the potential function for μ^*, and

$$G^*(z) = \lim_{n\to\infty} \frac{1}{d^n} \log|f^n(z)-c|,$$

while

$$G_K(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+|f^n(z)|,$$

and we need to show that $G^*(z) = G_K(z)$. If $z \notin K$, then $f^n(z)$ tends to ∞ as n increases, so that $G^*(z) = G_K(z)$ in this case. Since G^* is the potential function for μ^*, it is upper semi-continuous, so it follows that $G^*(z) \geq 0$ for $z \in \partial K$. On the other hand, since $G^* = G$ on the set where $G = \epsilon$, the maximum principle for subharmonic functions implies that $G \leq \epsilon$ on the region enclosed by this set. Letting ϵ tend to 0 shows that $G^* \leq 0$ on K.

Finally, using some knowledge of the possible types of components for the interior of K, it follows that if c is non-exceptional, then the measure μ^* assigns no mass to the interior of K. This implies that G^* is harmonic on K since μ^* is the Laplacian of G^*. Hence both the maximum and minimum principles apply to G^* on K, which implies that $G^* \equiv 0$ on K.

Thus $G^* \equiv G_K$ and hence $\mu^* \equiv \mu_K$ as desired. ■

Remark: This theorem provides an algorithm for drawing a picture of the Julia set, J, for a polynomial f. That is, start with a nonexceptional point c, and compute points on the backward orbits of c. These points will accumulate on the Julia set for f, and by discarding points in the first several backwards iterates of c, we can obtain a reasonably good picture of the Julia set. This algorithm has the disadvantage that these backwards orbits tend to accumulate most heavily on points in J which are easily accessible from infinity. That is, it favors points at which a random walk starting at infinity is most likely to land and avoids points such as inward pointing cusps.

Exercise 10.2 *Since G is harmonic both in the complement of K and in the interior of K, we see that* $\operatorname{supp}\mu \subseteq J$*, where $J = \partial K$ is the Julia set. Show that* $\operatorname{supp}\mu = J$*. Hint: Use the maximum principle.*

Note that an immediate corollary of this exercise and Brolin's theorem is that periodic points are dense in J.

11 Potential theory and dynamics in two variables

In Brolin's theorem, we took the average of point masses distributed over the following two sets:

(a) $\{z : f^n(z) = c\}$

(b) $\{z : f^n(z) = z\}$.

In the setting of polynomial automorphisms of $\mathbb{C}^2$, there are two natural questions motivated by these results.

(i) What happens when we iterate 1-dimensional submanifolds (forwards or backwards)?

(ii) Are periodic points described by some measure μ?

In $\mathbb{C}$, we can loosely describe the construction of the measure μ as first counting the number of points in the set (a) or (b), then using potential theory to describe the location of these points.

Before we consider such a procedure in the case of question (i) for $\mathbb{C}^2$, we first return to the horseshoe map and recall the measure m^- defined in section 8. Suppose that B is defined as in that section, that p is a fixed point for the horseshoe map h, that S is the component of $W^u(p) \cap B$ containing p, and that T is a line segment from the top to the bottom of B as before. Then orient T and S so that these orientations induce the standard orientation on $\mathbb{R}^2$ at the point of intersection of T and S.

Now, apply h to S. Then $h(S)$ and T will intersect in two points, one of which is the original point of intersection, and one of which is new. See figure 7. Because of the form of the horseshoe map, the intersection of $h(S)$ and T at the new point will not induce the standard orientation on $\mathbb{R}^2$ but rather the opposite orientation. In general, we can apply h^n to S, then assign $+1$ to each point of intersection which induces the standard orientation, and -1 to each point which induces the opposite orientation. Unfortunately, the sum of all such points of intersection for a given n will always be 0, so this doesn't give us a way to count these points of intersection.

A second problem with real manifolds is that the number of intersections may change with small perturbations of the map. For instance, if the map is changed so

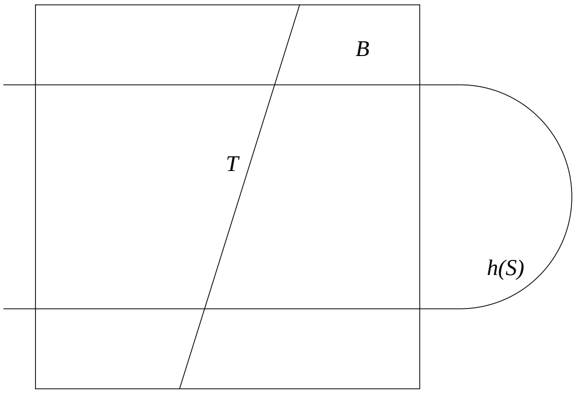

Figure 7: T and $h(S)$

that $h(S)$ is tangent to T and has no other other intersections with T, then for small perturbations g near h, $g(S)$ may intersect T in 0, 1, or 2 points.

Suppose now that B is a bidisk in $\mathbb{C}^2$, that h is a complex horseshoe map, and that T and S are complex submanifolds. In this case, there is a natural orientation on T at any point given by taking a vector v in the tangent space of T over this point and using the set $\{v, iv\}$ to define the orientation at that point. We can use the same procedure on S, then apply h^n as before. In this case, the orientation induced on $\mathbb{C}^2$ by $h^n(S)$ and T is always the same as the standard orientation. Hence assigning $+1$ to such an intersection and taking the sum gives the number of points in $T \cap h^n(S)$.

Additionally, if both S and T are complex manifolds, then the number of intersections, counted with multiplicity, between $h^n(S)$ and T is constant under small perturbations.

Thus in studying question (i), we will use complex 1-dimensional submanifolds.

Recall that in the case of one variable, the Laplacian played a key role by allowing us to relate the potential function G to the measure μ. Here we make an extension of the Laplacian to $\mathbb{C}^2$ in order to achieve a similar goal.

For a function f of two real variables x and y, the exterior derivative of f is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

which is invariant under smooth maps. If we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, then multiplication by i induces the map $(i)^*$ on the cotangent bundle, and this map takes dx to dy and dy to $-dx$. Hence, defining $d^c = (i)^* d$, we have

$$d^c f = \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx,$$

which is invariant under smooth maps preserving the complex structure; i.e., holomorphic maps. Hence dd^c is also invariant under holomorphic maps. Expanding dd^c gives

$$\begin{aligned} dd^c f &= d\left(\frac{\partial f}{\partial x}\right) dy - d\left(\frac{\partial f}{\partial y}\right) dx \\ &= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx dy, \end{aligned}$$

which is nothing but the Laplacian. This shows that the Laplacian, when viewed as a map from functions to 2-forms, is invariant under holomorphic maps, and also shows that this procedure can be carried out in any complex manifold of any dimension. Moreover, it also shows that the property of being subharmonic is invariant under holomorphic maps.

Exercise 11.1 *Let D_r be the disk of radius r centered at 0 in the plane, and compute*

$$\int_{D_r} dd^c \log |z|.$$

Hint: This is equal to

$$\int_{\partial D_r} d^c \log |z|.$$

We next need to extend the idea of subharmonic functions to $\mathbb{C}^2$.

Definition 11.2 *A function $f : \mathbb{C}^2 \to \mathbb{R}$ is **plurisubharmonic** (psh) if h is u.s.c. and if the restriction of h to any 1-dimensional complex line satisfies the subaverage property.*

Intrinsically, an u.s.c. function h is psh if and only if $dd^c h$ is nonnegative, where again we interpret this in the sense of distributions.

In fact, in the above definition we could replace the phrase "complex line" by "complex submanifold" without changing the class of functions, since subharmonic functions are invariant under holomorphic maps. As an example of the usefulness of this and the invariance property of dd^c, suppose that ϕ is a holomorphic embedding of $\mathbb{C}$ into $\mathbb{C}^2$ and that h is smooth and psh on $\mathbb{C}^2$. Then we can either evaluate $dd^c h$ and pull back using ϕ, or we can first pull back and then apply dd^c. In both cases we get the same positive distribution on $\mathbb{C}$.

12 Currents and applications to dynamics

In this section we provide a brief introduction to the theory of currents. A current is simply a linear functional on the space of smooth differential forms, and hence may be viewed as a generalization of measures. That is, a current μ acts on a differential form of a given degree, say $\phi = f_1 dx + f_2 dy$ in the case of a 1-form, to give a complex number $\mu(\phi)$, and this assignment is linear in ϕ. This is a generalization of a measure in the sense that a measure acts on 0-forms (functions) by integrating the function against the measure.

As an example, suppose that $M \subseteq \mathbb{C}^2$ is a submanifold of real dimension n. Then the current of integration associated to M is a current acting on n-forms ϕ, and is simply given by

$$[M](\phi) = \int_M \phi.$$

In this example the linearity is immediate, as is the relationship to measures. Note that in particular, if $p \in \mathbb{C}^2$, then $[p] = \delta_p$, the delta mass at p, acts on 0-forms.

Example 12.1 *Suppose $P : \mathbb{C} \to \mathbb{C}$ is a polynomial having only simple roots, and let R be the set of roots of P. Then $[R]$ is a current acting on 0-forms, and*

$$[R] = \frac{1}{2\pi} dd^c \log |P|.$$

This formula is still true for arbitrary polynomials if we account for multiplicities in constructing $[R]$.

We can extend this last example to the case of polynomials from $\mathbb{C}^2$ to $\mathbb{C}$. This is the content of the next proposition.

PROPOSITION 12.2 *If $P : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial and $V = \{P = 0\}$, then*

$$[V] = \frac{1}{2\pi} dd^c \log |P|,$$

where again $[V]$ is interpreted with weights according to multiplicity. (This is known as the Poincaré-Lelong formula.)

Suppose now that $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a Hénon diffeomorphism, and define

$$G^+(p) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |\pi_1(f^n(p))|$$
$$G^-(p) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |\pi_2(f^{-n}(p))|,$$

where π_j is projection to the jth coordinate. As in the case of the function G defined for a 1-variable polynomial, it is not hard to check that G^+ is psh, is identically 0 on K^+, is pluriharmonic on $\mathbb{C}^2 - K^+$ (i.e., is harmonic on any complex line), and is positive on $\mathbb{C}^2 - K^+$. In analogy with the function G, we say that G^+ is the Green function of K^+. Likewise, G^- is the Green function of K^-.

Note that for n large and $p \notin K$, $|\pi_1 f^n(p)|$ is comparable to the square of $|\pi_2 f^n(p)|$, and hence we may replace $|\pi_1 f^n(p)|$ by $\|f^n(p)\|$ in the formula for G^+, and likewise for G^-.

Again in analogy with the 1-variable case, and using the equivalence between the Laplacian and dd^c outlined earlier, we define

$$\mu^+ = \frac{1}{2\pi} dd^c G^+$$
$$\mu^- = \frac{1}{2\pi} dd^c G^-.$$

Then μ^+ and μ^- are currents supported on $J^+ = \partial K^+$ and $J^- = \partial K^-$, respectively. Moreover, $\mu^\pm$ restrict to measures on complex 1-dimensional submanifolds in the sense that we can pull back $G^\pm$ from the submanifold to an open set in the plane, then take dd^c on this open set.

As an analog of part 1 of Brolin's theorem, we have the following theorem.

THEOREM 12.3 *Let V be the (complex) x-axis in $\mathbb{C}^2$, i.e., the set where π_2 vanishes, and let f be a complex Hénon map. Then*

$$\lim_{n\to\infty} \frac{1}{2^n}[f^{-n}V] = \mu^+.$$

Proof: Note that the set $f^{-n}V$ is the set where the polynomial $\pi_1 f^n$ vanishes. Hence the previous proposition implies that

$$[f^{-n}V] = \frac{1}{2\pi} dd^c \log|\pi_1 f^n|.$$

Passing to the limit and using an argument like that in Brolin's theorem to replace log by $\log^+$, we obtain the theorem. See [BS1] or [FS1] for more details. ■

As a more comprehensive form of this theorem, we have the following.

THEOREM 12.4 *If S is a complex disk and f is a complex Hénon map, then*

$$\lim_{n\to\infty} \frac{1}{2^n}[f^{-n}S] = c\mu^+,$$

where $c = \mu^-[S]$. An analogous statement is true with μ^+ and μ^- interchanged and f^n in place of f^{-n}.

As a corollary, we obtain the following theorem from [BS3].

COROLLARY 12.5 *If p is a periodic saddle point, then $W^u(p)$ is dense in J^-.*

Proof: Replacing f by f^n, we may assume that p is fixed. Let S be a small disk in $W^u(p)$ containing p. Then the forward iterates of S fill out the entire unstable manifold. Moreover, by the previous theorem, the currents associated with these iterates converge to $c\mu^-$ where $c = \mu^+[S]$. If $c \neq 0$, then the corollary is complete since then $W^u(p)$ must be dense in supp $\mu^- = J^-$.

But c cannot be 0, because if it were, then $G^+|S$ would be harmonic, hence identically 0 by the minimum principle since G is nonnegative on S and 0 at p. Hence S would be contained in K, which is impossible since the iterates of S fill out all of $W^u(p)$, which is not bounded. Thus $c \neq 0$, so $W^u(p)$ is dense in J^-. ∎

This corollary gives some indication of why pictures of invariant sets on complex slices in $\mathbb{C}^2$ show essentially the full complexity of the map. If we start with any complex slice which is transverse to the stable manifold of a periodic point p, then the forward iterates of this slice accumulate on the unstable manifold of p, hence on all of J^- by the corollary. All of this structure is then reflected in the original slice, giving rise to sets which are often self-similar and bear a striking resemblance to Julia sets in the plane.

13 Currents and Hénon maps

In this section we continue the study of the currents μ^+ and μ^- in order to obtain more dynamical information.

We first consider this in the context of the horseshoe map. Recall that B is a square in the plane and that we have defined measures m^+ and m^-, and their product measure m in section 8.

In fact, m^+ and m^- generalize to μ^+ and μ^- in the case that the Hénon map is a horseshoe. More explicitly, let D_λ be a family of complex disks in $\mathbb{C}^2$ which intersect $\mathbb{R}^2$ in a horizontal segment in B and such that these segments fill out all of B. Then we can recover μ^+, at least restricted to B, by

$$\mu^+|B = \int [D_\lambda] dm^+(\lambda).$$

In analogy with the construction of m as a product measure using m^+ and m^-, we would like to combine μ^+ and μ^- to obtain a measure μ. Since μ^+ and μ^- are currents, the natural procedure to try is to take $\mu = \mu^+ \wedge \mu^-$. While forming the wedge product is not well-defined for arbitrary currents, it is well-defined in this case using the fact that these currents are obtained by taking dd^c of a continuous psh function and applying a theorem of pluripotential theory. In this way we get a measure μ on $\mathbb{C}^2$.

Definition 13.1 $\mu = \mu^+ \wedge \mu^-$.

We next collect some useful facts about μ.

(1) μ is a probability measure. For a proof of this, see [BS1].

(2) μ is invariant under f. To see this, note that since

$$G^{\pm} = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|f^{\pm n}\|,$$

we have $G^{\pm}(f(p)) = 2^{\pm} G^{\pm}(p)$. Since $\mu^{\pm} = (1/2\pi) dd^c G^{\pm}$, this implies that $f^*(\mu^{\pm}) = 2^{\pm}\mu^{\pm}$, and hence

$$\begin{aligned} f^*(\mu) &= f^*(\mu^+) \wedge f^*(\mu^-) \\ &= 2\mu^+ \wedge \frac{1}{2}\mu^- \\ &= \mu^+ \wedge \mu^- \\ &= \mu. \end{aligned}$$

Definition 13.2 $J = J^+ \cap J^-$.

(3) $\mathrm{supp}(\mu) \subseteq J$. This is a simple consequence of the fact that the support of μ is contained in the intersection of $\mathrm{supp}(\mu^+) = J^+$ and $\mathrm{supp}(\mu^-) = J^-$ and the definition of J.

In order to examine the support of μ more precisely, we turn our attention for a moment to Shilov boundaries. Let X be a subset either of $\mathbb{C}$ or $\mathbb{C}^2$. We say that a set B is a *boundary* for X if B is closed and if for any holomorphic polynomial P we have

$$\max_X |P| = \max_B |P|.$$

With the right conditions, the intersections of any set of boundaries is a again a boundary by a theorem of Shilov, so we can intersect them all to obtain the smallest such boundary. This is called the ***Shilov boundary*** for X.

Example 13.3 *Let $X = D_1 \times D_1$, where D_1 is the unit disk. Then the Shilov boundary for X is $(\partial D_1) \times (\partial D_1)$, while the topological boundary for X is*

$$\partial X = (D_1 \times \partial D_1) \cup (\partial D_1 \times D_1).$$

The following theorem is contained in [BT].

THEOREM 13.4 $\mathrm{supp}(\mu) = \partial_{\mathrm{Shilov}}(K)$.

We have already defined J as the intersection of J^+ and J^-, and the choice of notation is designed to suggest an analogy with the Julia set in one variable. However, in 2 variables, the support of μ is also a natural candidate as a kind of Julia set. Hence we make the following definition.

Definition 13.5 $J^* = \mathrm{supp}(\mu)$.

14 Heteroclinic points and Pesin theory

In the previous section, we discussed some of the formal properties of μ arising from considerations of the definition and of potential theory. In this section we concentrate on the less formal properties of μ and on the relation of μ to periodic points. The philosophy here is that since μ^+ and μ^- describe the distribution of 1-dimensional objects, μ should describe the distribution of 0-dimensional objects.

An example of a question using this philosophy is the following. For a periodic point p, we know that μ^+ describes the distribution of $W^s(p)$ and μ^- describes the distribution of $W^u(p)$. Does μ describe (in some sense) the distribution of intersections $W^s(p) \cap W^u(q)$?

Definition 14.1 *Let p and q be saddle periodic points of a diffeomorphism f. A point in the set $(W^u(p) \cap W^s(q)) - \{p, q\}$ is called a **heteroclinic point**. If $p = q$, then such a point is called a **homoclinic point**.*

Unfortunately, none of the techniques discussed so far allow us to show that there is even one heteroclinic point. That is, it is possible for the stable manifold of p and the unstable manifold of q to have an empty intersection.

To see how this could happen, we first need the fact that for a holomorphic diffeomorphism f of $\mathbb{C}^2$ with a saddle point p, there is an injective holomorphic map $\phi_u : \mathbb{C} \to W^u(p)$ which maps *onto* the unstable manifold, and likewise for the stable manifold.

Now, if π_j represents projection onto the jth coordinate, then $\pi_1\phi_u : \mathbb{C} \to \mathbb{C}$ is an entire function, and as such can have an omitted value. As an example, $\pi_1\phi_u(z)$ could be equal to e^z and hence would omit the value 0. It could happen that there is a second saddle point q such that $W^s(q)$ is the x-axis, in which case $W^u(p) \cap W^s(q) = \emptyset$.

At first glance, it may seem that this contradicts some of our earlier results. It might seem that theorem 12.4 should imply that $W^u(p)$ intersects transversals which cross J^-, but in fact, that statement is a statement about convergence of distributions. Each of the distributions must be evaluated against a test function, and the test function must be positive on an open set. Thus there is still room for $W^u(p)$ and $W^s(p)$ to be disjoint.

Hence, in order to understand more about heteroclinic points, we need a better understanding of the stable and unstable manifolds. One possible approach is to use what is known as Ahlfors' three island theorem. This theorem concerns entire maps $\psi : \mathbb{C} \to \mathbb{C}$. Roughly, it says that if we have n open regions in the plane and consider their inverse images under ψ, then some fixed proportion of them will have a preimage which is compact and which maps injectively under ψ onto the corresponding original region.

If we apply this theorem to the map $\pi_1\phi_u$ giving $W^u(p)$, then we can divide the plane into increasingly more and smaller islands, and we can do this in such a way that at each stage we obtain more of $W^u(p)$ as the injective image of regions in the plane. The result is that we get a picture of $W^u(p)$ which is locally laminar.

Since $W^u(p)$ is dense in J^-, this gives us one possible approach to studying μ^-, and we can use a similar procedure to study μ^+. However, recall that our goal here is to describe heteroclinic points. Thus in order for this approach to apply, we need to be able to get the disks for μ^+ to intersect the disks for μ^-. Unfortunately, we don't get any kind of uniformity in the disks using this approach, so getting this intersection is difficult.

An alternative approach is to use the theory of hyperbolicity in the sense of Oseledec and Pesin. Since μ is a (non-uniformly) hyperbolic measure with respect to this theory, we get that at μ-almost every point of $\mathbb{C}^2$ there are stable and unstable manifolds and that these manifolds are transverse. We can then identify the stable and unstable manifolds obtained using this theory with the disks obtained in the previous non-uniform laminar picture to guarantee that we get intersections between stable and unstable manifolds and hence heteroclinic points. Putting all of this together, we obtain the following theorem, contained in [BLS1].

THEOREM 14.2 *J^* is the closure of*

1. *the set of all periodic saddle points,*
2. *the union of all $W^u(p) \cap W^s(q)$ over all periodic saddles p and q.*

This theorem can be viewed as an analog of the theorem in one variable dynamics that the Julia set is the closure of the repelling periodic points. For this reason, the set J^* is perhaps a better analogue of the Julia set in the two dimensional case than is J.

Recall that $J^* = \partial_{\text{shilov}} K \subseteq \partial K = J$. In the case that f is an Axiom A diffeomorphism, it is a theorem that $J^* = J$. However, it is an interesting open question whether this equality holds in general. If it were the case that $J \neq J^*$, then there would be a saddle periodic point q and another point p such that $p \in \overline{W^s(q)} \cap \overline{W^u(q)}$, but $p \notin \overline{W^s(q) \cap W^u(q)}$.

In fact, using the ideas of Pesin theory, one can get precise information about the number of periodic points of a given period and how their distribution relates to the measure μ. This is contained in the following theorem and corollary, contained in [BLS2].

THEOREM 14.3 *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a complex Hénon map, and let P_n be either the set of fixed points of f^n or the set of saddle points of minimal period n. Then*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{p \in P_n} \delta_p = \mu.$$

For the following corollary, let P_n be the set of saddle points of f of minimal period n, and let $|P_n|$ denote the number of points contained in this set.

COROLLARY 14.4 *There are periodic saddle points of all but finitely many periods. More precisely, we have*

$$\lim_{n\to\infty} \frac{|P_n|}{2^n} = 1.$$

Recall that the horseshoe map had periodic points of all periods, so while we haven't achieved that result for general Hénon maps, we have still obtained a good deal of information about periodic points and heteroclinic points.

15 Topological entropy

Recall that the horseshoe map is topologically equivalent to the shift map on two symbols. One could also ask if it is topologically equivalent to the shift on four symbols. That is, if h is the horseshoe map defined on the square B, then $h(B) \cap h^{-1}(B)$ consists of four components, and we can label these components with four symbols. However, with this labeling scheme, one can check by counting that not all sequences of symbols correspond to an orbit of a point in the way that sequences of two symbols did. In fact, the number of symbol sequences of length 2 corresponding to part of an orbit is 8, while the total number of possible sequences of length 3 is 16. Allowing longer sequences and letting $S(n)$ denote the number of sequences of length n which correspond to part of an orbit, we obtain the formula

$$\lim_{n\to\infty} \frac{1}{n} \log S(n) = \log 2.$$

The number $\log 2$ is the topological entropy of the horseshoe map, and can be defined as the maximum growth rate over all finite partitions. In general, the shift map on N

symbols has entropy $\log N$, and since entropy is a topological invariant, we see that all of these different shift maps are topologically distinct.

In the case of a general Hénon map, we have the following theorem, contained in [Sm].

THEOREM 15.1 *The topological entropy of a complex Hénon map is* $\log 2$.

Topological entropy is a useful idea because it is connected to many different aspects of polynomial automorphisms. It is a measure of area growth and of the growth rate of the number of periodic points, both of which are closely related to the degree of the map as a polynomial. Moreover, it is related to measure theoretic entropy in the sense that for any probability measure ν, the measure theoretic entropy, $h_\nu(f)$, is bounded from above by the topological entropy $h_{\text{top}}(f)$. Moreover, μ is the *unique* measure for which $h_\mu(f) = h_{\text{top}}(f)$.

We can also consider topological entropy for real Hénon maps. In contrast to the theorem above, in this case we have $0 \leq h_{\text{top}}(f_{\mathbf{R}}) \leq \log 2$, and all values are possible. However, one can show that not all values are possible for Axiom A automorphisms, but only logarithms of algebraic numbers [Fr]. Moreover, we also have the following theorem [BLS1].

THEOREM 15.2 *For a real Hénon map $f_{\mathbf{R}}$, the following are equivalent.*

1. $h_{\text{top}}(f_{\mathbf{R}}) = \log 2$.
2. $J^* \subseteq \mathbb{R}^2$.
3. $K \subseteq \mathbb{R}^2$.
4. *All periodic points are real.*

Moreover, these conditions imply that $J = J^*$.

Proof: Condition 1 implies that $f_{\mathbf{R}}$ has a measure μ' of maximal entropy with $\text{supp}(\mu') \subseteq \mathbb{R}^2$. By uniqueness we have $\mu' = \mu^*$, so $\text{supp}(\mu^*) \subseteq \mathbb{R}^2$, thus giving condition 2.

Condition 2 implies that $J^* = \partial_{\text{Shilov}} K$ is contained in $\mathbb{R}^2$, which implies that K is contained in $\mathbb{R}^2$. This gives condition 3, and in fact, since polynomials in $\mathbb{R}^2$ are dense in the set of continuous functions of $\mathbb{R}^2$, this also implies that $\partial_{\text{Shilov}} K = K$, and hence $J^* = K$ and thus $J^* = J$ since $J^* \subseteq J \subseteq K$.

Condition 3 immediately implies condition 4.

Condition 4 together with theorem 14.2 implies that $J^* \subseteq \mathbb{R}^2$, which implies that $\text{supp}(\mu^*) \subseteq \mathbb{R}^2$, which implies condition 1. ■

These conditions are true for the set of real Hénon maps which are horseshoes. We can identify such maps with their parameter values in $\mathbb{R}^2$, in which case the set of horseshoe maps is an open set in $\mathbb{R}^2$. Since topological entropy is continuous for C^∞ diffeomorphisms, we see that maps on the boundary of this set also satisfy the above conditions.

These conditions also apply in the following theorem, from [BS5]. Recall that a homoclinic intersection is an intersection of $W^u(p)$ and $W^s(p)$ at some point $q \neq p$ for some saddle point p. This intersection is a homoclinic tangency if the stable and unstable manifolds are tangent at q, and this is a quadratic tangency if the manifolds have quadratic contact at q.

THEOREM 15.3 *If the previous conditions hold, then*

1. *periodic points are dense in K,*
2. *every periodic point is a saddle with expansion constants bounded below,*
3. *either f is Axiom A or f has a quadratic homoclinic tangency.*

Suppose we have a 1-parameter family of real Hénon maps which starts out as a horseshoe, then passes through a homoclinic tangency, so that as we increase the parameter value, some local pieces of the stable and unstable manifold in $\mathbb{R}^2$ first intersect in 2 points, then at one tangent point, then don't intersect at all. In this case, the intersections of the stable and unstable manifolds move out of $\mathbb{R}^2$, which causes a decrease in topological entropy by theorem 15.2 and the fact that $\log 2$ is the maximum possible entropy for a real Hénon map. Since topological entropy is continuous and is a topological invariant, it follows that we pass through infinitely many conjugacy classes as we change the parameter. That is, for two maps with different topological entropy, there is no homeomorphism which conjugates one to the other. This presents a striking contrast to the horseshoe example in which small changes of the original horseshoe were all conjugate to the shift map on two symbols.

16 Conclusion

We have presented here an overview of some of the major techniques and results in the study of the iteration of polynomial automorphisms of $\mathbb{C}^2$ and have demonstrated some of the influences coming from real and measurable dynamics as well as complex dynamics of one variable.

There has also been a great deal of work in other directions in the study of complex dynamics in several variables, including the study of rational maps on complex projective space by Fornæss, Sibony, Ueda, and others, and the study of holomorphic vector fields by Ahern, Bass, Coomes, Fornæss, Forstneric, Grellier, Meisters, Sibony, Suzuki, and others. There is also work on nonpolynomial automorphisms of $\mathbb{C}^2$ by Buzzard, Fornæss, Sibony, and others, as well as the study of foliations of $\mathbf{P}^2$ by Cano, Camacho, Gomez-Mont, Lins-Neto, Sad, and others. There are many open problems in each of these areas.

References

[A] M. Abate, *Iteration theory of holomorphic maps on taut domains*, Mediterranean Press, Rende, 1990.

[AF] P. Ahern and F. Forstneric, One parameter automorphism groups on $\mathbb{C}^2$, *Complex Variables*, to appear.

[BM] H. Bass and G. Meisters, Polynomial flows in the plane, *Adv. in Math.*, 55 (1985), 173-208.

[B] A. Beardon *Iteration of Rational Functions*, Springer-Verlag, New York, 1991.

[Be] E. Bedford, Iteration of polynomial automorphisms of $\mathbb{C}^2$, *Proc. I.C.M.*, Kyoto, Japan, 1990, 847-858.

[BLS1] E. Bedford, M. Lyubich, and J. Smillie, Polynomial diffeomorphisms of $\mathbb{C}^2$. IV: The measure of maximal entropy and laminar currents, *Invent. Math.*, 112 (1993), 77-125.

[BLS2] E. Bedford, M. Lyubich, and J. Smillie, Distribution of periodic points of polynomial diffeomorphisms of $\mathbb{C}^2$, *Invent. Math*, 114 (1993), 277-288.

[BS1] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbb{C}^2$: Currents, equilibrium measure and hyperbolicity, *Invent. Math.*, 103 (1990), 69-99.

[BS2] E. Bedford and J. Smillie, Fatou-Bieberbach domains arising from polynomial automorphisms, *Indiana Univ. Math. J.*, 40 (1991), 789-792.

[BS3] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbb{C}^2$. II: Stable manifolds and recurrence, *J. AMS*, 4 (1991), 657-679.

[BS4] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbb{C}^2$. III: Ergodicity, exponents and entropy of the equilibrium measure, *Math. Ann.*, 294 (1992), 395-420.

[BS5] E. Bedford and J. Smillie, Quasihyperbolic polynomial diffeomorphisms of $\mathbb{C}^2$, to appear.

[BT] E. Bedford and B.A. Taylor, Fine topology, Šilov boundary, and $(dd^c)^n$, *J. Funct. Anal.*, 72 (1987), no. 2, 225-251.

[BC] M. Benedicks and L. Carleson, The dynamics of the Henon map, *Ann. of Math.*, (2) 133 (1991), no. 1, 73-169.

[BY] M. Benedicks and L.S. Young, Sinai-Bowen-Ruelle measures for certain Henon maps, *Invent. Math.*, 112 (1993), no. 3, 541-576.

[Bo] T. Bousch, Sur quelques problèmes de dynamique holomorphique, Thèse, Université de Paris-Sud, Orsay, 1992.

[Br] H. Brolin, Invariant sets under iteration of rational functions, *Ark. Mat.*, 6 (1965), 103-144.

[Bu1] G. Buzzard, Persistent homoclinic tangencies and infinitely many sinks for automorphisms of $\mathbb{C}^2$, thesis, University of Michigan, 1995.

[Bu2] G. Buzzard, Kupka-Smale theorem for automorphisms of $\mathbb{C}^n$, preprint.

[CG] L. Carleson and T. Gamelin, *Complex Dynamics*, Springer-Verlag, New York, 1993.

[C] B. Coomes, Polynomial flows on $\mathbb{C}^n$, *Trans. Amer. Math. Soc.*, 320 (1990), no. 2, 493-506.

[DN] R. Devaney and Z. Nitecki, Shift automorphisms in the Henon mapping, *Comm. Math. Phys.*, 67 (1979), no. 2, 137-146.

[F] P. Fatou, Substitutions analytiques et équations fonctionelles à deux variables, *Ann. Ecole Norm. Sup.*, 3 XLI (1924), 67-142.

[FG] J.E. Fornæss and E. Gavosto, Existence of generic homoclinic tangencies for Hénon mappings, *J. Geom. Anal.*, 2 (1992), 429-444.

[FGr] J.E. Fornæss and S. Grellier, Exploding orbits of holomorphic Hamiltonian and complex contact structures, *C. R. Acad. Sci. Paris Ser. I Math.*, 319 (1994), no. 6, 553-557.

[FS1] J.E. Fornæss and N. Sibony, Complex Hénon mappings in $\mathbb{C}^2$ and Fatou-Bieberbach domains, *Duke Math. J.*, 65 (1992), 345-380.

[FS2] J.E. Fornæss and N. Sibony, Critically finite rational maps on $\mathbf{P}^2$, *Contemporary Mathematics: The Madison Symposium on Complex Analysis (Madison, WI, 1991)*, 137 (1992), 245-260.

[FS3] J.E. Fornæss and N. Sibony, Complex dynamics in higher dimension I. Complex analytic methods in dynamical systems, *Astérisque*, 222 (1994), 201-231.

[FS4] J.E. Fornæss and N. Sibony, Complex dynamics in higher dimension II, preprint.

[FS5] J.E. Fornæss and N. Sibony, Oka's inequality for currents and applications, preprint.

[FS6] J.E. Fornæss and N. Sibony, Holomorphic symplectomorphisms in $\mathbb{C}^2$, preprint, 1994.

[FS7] J.E. Fornæss and N. Sibony, Holomorphic symplectomorphisms in $\mathbb{C}^{2p}$, preprint, 1994.

[Fo] F. Forstneric, Actions of $(\mathbb{R},+)$ and $(\mathbb{C},+)$ on complex manifolds, *Math. Z.*, to appear.

[Fr] D. Fried, Rationality for isolated expansive sets, *Adv. in Math.*, 65 (1987), 35-38.

[FM] S. Friedland and J. Milnor, Dynamical properties of plane polynomial automorphisms, *Ergodic Theory Dynamical Systems*, 9 (1989) 67-99.

[G] E. Gavosto, Attracting basins in $\mathbf{P}^2$, preprint.

[Gr] M. Gromov, Entropy, homology and semi-algebraic geometry, in Séminaire Bourbaki, *Astérisque*, 145-146 (1987), 225-240.

[H] M. Herman, Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of $\mathbb{C}^n$ near a fixed point, *VIIIth International Congress on Mathematical Physics (Marseille, 1986)*, World Scientific Publ., Singapore, 1987, 138-184.

[Hu] J.H. Hubbard, The Hénon mapping in the complex domain, in *Chaotic Dynamics and Fractals*, ed. M. Barnsley and S. Demko, Academic Press, Orlando, 1986, 101-111.

[HO] J.H. Hubbard and R. Oberste-Vorth, Hénon mappings in the complex domain I: The global topology of dynamical space, *Publ. Sci. I.H.E.S.*, to appear.

[HO] J.H. Hubbard and R. Oberste-Vorth, Hénon mappings in the complex domain II: Projective and inductive limits of polynomials, preprint.

[HP1] J.H. Hubbard and P. Papadopol, Superattractive fixed points in $\mathbb{C}^n$, *Indiana Univ. Math. J.*, 43 (1994), no. 1, 321-365.

[HP2] J.H. Hubbard and P. Papadopol, Newton's method in $\mathbb{C}^2$, (to appear).

[K] T. Kimura, On Fatou-Bieberbach domains in $\mathbb{C}^2$, *J. Fac. Sci. Univ. Tokyo*, 35 (1988), 103-148.

[O] R. Oberste-Vorth, Complex horseshoes (to appear).

[PD] J. Palis and W. de Melo, *Geometric theory of dynamical systems: an introduction*, translated by A.K. Manning, Springer-Verlag, New York, 1982.

[PT] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations : fractal dimensions and infinitely many attractors*, Cambridge University Press, New York, 1993.

[R] D. Ruelle, *Elements of differentiable dynamics and bifurcation theory*, Academic Press, Boston, 1989.

[S] M. Shub, *Global stability of dynamical systems*, Springer-Verlag, New York, 1987.

[Si] J. Silverman, Geometric and arithmetic properties of the Hénon map, *Math. Z.*, 215 (1994), no. 2, 237-250.

[Sm] J. Smillie, The entropy of polynomial diffeomorphisms of $\mathbb{C}^2$, *Ergodic Theory Dynamical Systems*, 10 (1990), 823-827.

[Su1] M. Suzuki, Sur les opérations holomorphes du groupe additif complexe sur l'espace de deux variables complexes, *Ann. Ecole Norm. Sup.*, 10 (1977), 517-546.

[Su2] M. Suzuki, A note on complex Hénon mappings, *Math. J. Toyama Univ.*, 15 (1992), 109-121.

[T] P. Tortrat, Aspects potentialistes de l'itération des polynômes, *Séminaire de Théorie du Potentiel, Paris, No. 8*, (Lect. Notes Math., vol. 1235), Springer-Verlag, Berlin, 1987, 195-209.

[U1] T. Ueda, Local structure of analytic transformations of two complex variables, I, *J. Math. Kyoto Univ.*, 26 (1986) 233-261.

[U2] T. Ueda, Local structure of analytic transformations of two complex variables, II, *J. Math. Kyoto Univ.*, 31 (1991), 695-711.

[U3] T. Ueda, Fatou sets in complex dynamics on projective spaces, *J. Math. Soc. Japan*, 46 (1994), no. 3, 545-555.

[U4] T. Ueda, Complex dynamical systems on projective spaces, *Surikaisekikenkyusho Kokyuroku*, (1992), No. 814 169-186.

[U5] T. Ueda, Critical orbits of holomorphic maps on projective spaces, *J. Geom. Anal.*, to appear.

[Us1] S. Ushiki, On super-stable manifolds of super saddle type Julia sets in $\mathbb{C}^2$, *Surikaisekikenkyusho Kokyuroku*, (1992), No. 814 203-217.

[Us1] S. Ushiki, B ottcher's theorem and super-stable manifolds for multidimensional complex dynamical systems, preprint, Kyoto University, Kyoto, 606-01 Japan.

[VW] A. Verjovsky and H. Wu, Hausdorff dimension of Julia sets of complex Hénon mappings, preprint.

[W] H. Wu, Complex stable manifolds of holomorphic diffeomorphisms, *Indiana Univ. Math. J.*, 42 (1993), no. 4, 1349-1358.

[Y] J.-C. Yoccoz in *Proceedings of the NATO Advanced Study Institute on Real & Complex Dynamical Systems*, Kluwer, to appear.

John Smillie, Department of Mathematics, Cornell University, Ithaca, NY 14853 (smillie@math.cornell.edu)

Gregery T. Buzzard, Department of Mathematics, Indiana University, Bloomington, IN 47405 (gbuzzard@msri.org, gbuzzard@indiana.edu)