ON LOCAL COEFFICIENTS FOR NON-GENERIC REPRESENTATIONS OF SOME CLASSICAL GROUPS

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Abstract. This paper is concerned with representations of split orthogonal and quasi-split unitary groups over a nonarchimedean local field which are not generic, but which support a unique model of a different kind, the generalized Bessel model. The properties of the Bessel models under induction are studied, and an analogue of Rodier's theorem concerning the induction of Whittaker models is proved for Bessel models which are minimal in a suitable sense. The holomorphicity in the induction parameter of the Bessel functional is established. Last, local coefficients are defined for each irreducible supercuspidal representation which carries a Bessel functional and also for a certain component of each representation parabolically induced from such a supercuspidal.

Introduction. L-functions are a central object of study in representation theory and number theory. Over a global field, one has the Langlands Conjectures, which assert in particular the meromorphic continuation and functional equation of a class of Euler products. Over a local field one has additional conjectures due to Langlands, expressing the Plancherel measure arithmetically as the ratio of certain local L-functions and root numbers.

In many cases these conjectures have been established by Shahidi [**Shab**,**Shac**, **Shad**], following a path laid out by Langlands [**Lana**]. The framework for Shahidi's work is the study of Eisenstein series or their local analogues, induced representations. One knows the continuation of these Eisenstein series due to Langlands [**Lanb**]. Langlands also showed that the constant coefficients of the Eisenstein series may be expressed in terms of local intertwining operators which are almost everywhere quotients of certain L-functions. It remains to study these intertwining operators for the finite set of 'bad' places. If the inducing data is generic, that is, admits a Whittaker model, then Shahidi has succeeded in relating them to local L-functions. Thus the careful study of the Eisenstein series, both local and global, affords a proof of certain of the Langlands conjectures for these L-functions.

The aim of this work is to suggest that the Langlands-Shahidi method may be extended beyond the generic spectrum by the use of other models. The Whittaker model is unique (an irreducible admissible representation admits at most one such model up to scalars). In this paper we study the properties of local representations

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of split orthogonal groups and quasi-split unitary groups which are not generic, but which support a unique model of a different kind, the generalized Bessel model. These models involve a character of a proper subgroup of the unipotent radical of a Borel subgroup, but transform under a reductive group of some, in general non-zero, rank. The uniqueness of the models has been proved by S. Rallis [**Ral**] in the orthogonal case, but as the argument has not yet been written out in full detail in the unitary case we make it a hypothesis throughout the paper.

We first study the properties of Bessel models under induction, and prove an analogue of Rodier's Theorem [**Rodb**] concerning the induction of Whittaker models. Our analogue, Theorem 2.1, states that if one parabolically induces a representation with a Bessel model of minimal rank, or more generally one which is minimal in the sense of Definition 1.5 below, then the induced representation has a unique Bessel model of the same rank and compatible type. In the case of rank 0, we recover Rodier's theorem. To carry out the proof we use Bruhat's extension [**Bru**] of Mackey theory and investigate precisely which double cosets of the appropriate type may support a functional with the desired equivariance property. We show that there is a unique such double coset by an extensive combinatorial argument.

Next, we establish the holomorphicity of the Bessel functional which arises from one which is minimal by parabolic induction of the underlying representation. Our approach is based on Bernstein's theorem [**Ber**], which uses uniqueness to conclude meromorphicity under some regularity hypotheses, and Banks's extension [**Ban**], which allows one to prove holomorphicity as well. We show in Theorem 3.6 that there is a non-zero Bessel functional $\Lambda(\nu,\pi)$, attached to an irreducible admissible representation π of the Levi subgroup M and a parameter ν in the complexified dual of the Lie algebra of the split component of M, which is holomorphic in ν.

If π is supercuspidal and has a Bessel model, or more generally if π is irreducible and carries a Bessel model corresponding to a minimal Bessel model of the supercuspidal from which it is induced, these results allow us to establish the existence of a local coefficient. In the generic case, such a local coefficient was crucial for Shahidi's study of the intertwining operators and of the relation between Plancherel measures and L–functions; see Shahidi [**Shad**]. Let $A(\nu,\pi,w)$ denote the standard intertwining operator attached to inducing data ν,π and Weyl group element representative w (see (3.4) below). We shall prove (cf. Theorem 3.8):

Theorem. *Let π be an irreducible representation of M which is a component of the representation parabolically induced from an irreducible admissible supercuspidal representation ρ of a parabolic subgroup of M. Suppose that π carries a Bessel model corresponding (in the sense of Theorem 2.1) to a minimal Bessel model of ρ. For each $\tilde{w}$ in the Weyl group, choose a representative w for $\tilde{w}$. Then there is a complex number $C(\nu,\pi,w)$ so that*

$$\Lambda(\nu,\pi)=C(\nu,\pi,w)\Lambda(\tilde{w}\nu,\tilde{w}\pi)A(\nu,\pi,w).$$

Moreover, the function $\nu\mapsto C(\nu,\pi,w)$ is meromorphic and depends only on the class of π and the choice of the representative w.

We call $C(\nu,\pi,w)$ the *local coefficient* attached to π, ν, and w. We then show in Corollary 3.9 that the local coefficients behave as expected with respect to the

Langlands decomposition of the intertwining operators. This generalizes a property of the local coefficients introduced by Shahidi [**Shaa**] in the generic case.

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§1 **Preliminaries on Bessel Models.** In this section we recall the notion of a Bessel model following [**Ral**] and [**GPR**], and review some properties of such models. Let F be a nonarchimedean local field of characteristic zero. Let $\mathbf{G}$ be one of the classical groups $SO_{2r+1}, U_{2r+1}, U_{2r}$, or SO_{2r}, defined over F. We assume that the orthogonal groups are split, and that the unitary groups are quasi-split, and split over a quadratic extension E/F. Let $r_0 = 2r$ if $\mathbf{G} = U_{2r}$ or SO_{2r}, and $r_0 = 2r + 1$ otherwise. Denote by $\mathbf{B} = \mathbf{TU}$ the Borel subgroup of $\mathbf{G}$, where $\mathbf{T}$ contains the maximal split subtorus of diagonal elements, and $\mathbf{U}$ is the subgroup of upper triangular unipotent matrices in $\mathbf{G}$. We use G to denote the F-rational points of $\mathbf{G}$, and use this notational convention for other algebraic groups defined over F.

Denote by $\Phi(\mathbf{G},\mathbf{T})$ the root system of $\mathbf{G}$ with respect to $\mathbf{T}$. We choose the ordering on the roots corresponding to our choice of Borel subgroup. Let $W = W(\mathbf{G},\mathbf{T}_d)$ be the Weyl group of $\mathbf{G}$ with respect to the maximal split subtorus $\mathbf{T}_d$ of $\mathbf{T}$. Thus, $W = N_G(\mathbf{T}_d)/\mathbf{T}$. Then,

$$W \simeq \begin{cases} S_r \ltimes \mathbb{Z}_2^r & \text{if } \mathbf{G} \neq SO_{2r} \\ S_r \ltimes \mathbb{Z}_2^{r-1} & \text{if } \mathbf{G} = SO_{2r}. \end{cases}$$

(See [**Gola**,**Golb**] for a more explicit description of $\mathbf{T}$ and W.) Here we will denote all elements of W as permutations on r_0 letters. Thus, the permutation $(ij) \in S_r$ corresponds to the permutation $(ij)(r_0+1-j\, r_0+1-i)$ in S_{r_0}. Similarly, the sign change c_i which generates that i-th copy of $\mathbb{Z}_2$ corresponds to the permutation $(i\, r_0+1-i)$ in S_{r_0}.

Fix an $\ell < r$ and let $\ell_0 = r_0 - 2\ell$. Let $\mathbf{U}_\ell$ be the subgroup of $\mathbf{U}$ consisting of matrices whose middle $\ell_0 \times \ell_0$ block is the identity matrix. For $1 \le i \le \ell$, let ψ_i be a non-trivial additive character of F if $\mathbf{G}$ is orthogonal, and let ψ_i be the composition of such a character with $Tr_{E/F}$ if $\mathbf{G}$ is unitary. We let $\mathbf{a} = (a_1, a_2, \ldots, a_{\ell_0}) \in F^{\ell_0}$ if $\mathbf{G}$ is orthogonal, and let $\mathbf{a} \in E^{\ell_0}$ if $\mathbf{G}$ is unitary. Then define ψ_{ℓ,a_j} by $\psi_{\ell,a_j}(x) = \psi_\ell(a_j x)$. Now define a character of U_ℓ by

$$\chi\left((u_{ij})\right) = \prod_{i=1}^{\ell-1} \psi_i(u_{i,i+1}) \prod_{j=1}^{\ell_0} \psi_{\ell,a_j}(u_{\ell,\ell+j}).$$

Let

$$\mathbf{M}_\ell = \left\{ \begin{pmatrix} I_\ell & & \\ & g & \\ & & I_\ell \end{pmatrix} \in \mathbf{G} \right\}.$$

Note that $\mathbf{M}_\ell \subset N_{\mathbf{G}}(\mathbf{U}_\ell)$. If $g \in M_\ell$, then define χ^g by $\chi^g(u) = \chi(g^{-1}ug)$. We let

$$M_\chi = \{g \in M_\ell \mid \chi^g = \chi\}.$$

Let $R_\chi = M_\chi U_\ell$. Suppose that ω is an irreducible admissible representation of M_χ. (We will denote this by $\omega \in \mathcal{E}(M_\chi)$.) Let $\omega_\chi = \omega \otimes \chi$ be the associated representation on R_χ.

Definition 1.1. We say that two characters χ_1 and χ_2 of U_ℓ defined as above are *equivalent* if $\chi_1 = \chi_2^g$ for some $g \in N_G(U_\ell)$.

The following result is a consequence of Witt's Theorem.

Lemma 1.2. *Any character χ of U_ℓ which is defined as above, is equivalent to one for which $\mathbf{a} = (\delta, 0, 0 \dots, 1)$, for some δ.*

From now on we assume for convenience that χ is given as in Lemma 1.2.
We let $\ell_1 = \left[\frac{\ell_0}{2}\right] = r - \ell$.

Definition 1.3. Suppose that τ is an admissible representation of G. We say that τ has an ω_χ-*Bessel model* (or *a Bessel model with respect to* ω_χ) if $\mathrm{Hom}_G(\tau, \mathrm{Ind}_{R_\chi}^G(\omega_\chi)) \neq 0$. If χ is a character of U_ℓ, and ℓ_1 is defined as above, then we say that τ has a *rank* ℓ_1 Bessel model.

Remarks.

(1) By Frobenius reciprocity [**BeZb**], we have

$$\mathrm{Hom}_G(\tau, \mathrm{Ind}_{R_\chi}^G(\omega_\chi)) \simeq \mathrm{Hom}_{M_\chi}(\tau_{U_{\ell,\chi}}, \omega_\chi),$$

where $\tau_{U_{\ell,\chi}}$ is the χ-twisted Jacquet module of τ with respect to U_ℓ [**BeZa**]. Thus, the non-vanishing of $\tau_{U_{\ell,\chi}}$, for some ℓ and χ, would imply that τ has a rank ℓ_1 Bessel model with respect to some ω_χ.

(2) A Whittaker model [**Roda**,**Rodb**] is a rank zero Bessel model.

(3) One can make these definitions for any choice of Borel subgroup. We choose the standard one for convenience, but we will need to use others in the sequel.

(4) When $\mathbf{G} = SO_{2r+1}$, Rallis has shown that every irreducible admissible representation of $\mathbf{G}(F)$ has a Bessel model for some choice of χ and ω.

(5) Suppose that τ is irreducible, $\omega \in \mathcal{E}(M_\chi)$, and $\lambda : V_\tau \to V_\omega = V_{\omega_\chi}$ satisfies $\lambda(\tau(x)v) = \delta_{R_\chi}(x)^{1/2}\omega_\chi(x)\lambda(v)$, for all $x \in R_\chi$ and $v \in V_\tau$. (Such a λ is called a Bessel functional.) Let $v \in V_\tau$ and set $B_v(g) = \lambda(\tau(g)v)$. Then the map $v \mapsto B_v$ realizes an intertwining between τ and $\mathrm{Ind}_{R_\chi}^G(\omega_\chi)$. Conversely, if there is an embedding T of τ into $\mathrm{Ind}_{R_\chi}^G(\omega_\chi)$, then setting $\lambda(v) = [T(v)](e)$, we get a map $\lambda : V_\tau \to V_\omega$ with the property specified above. Thus, τ has an ω_χ-Bessel model if and only if a Bessel functional λ exists.

In this paper we shall make use of the following basic uniqueness principle.

Theorem/Conjecture 1.4. *Let $\tau \in \mathcal{E}(G)$. Then for a fixed ω and χ, we have $\dim_{\mathbb{C}} \mathrm{Hom}_G(\tau, \mathrm{Ind}_{R_\chi}^G(\omega_\chi)) \leq 1$. That is, a Bessel model is unique for irreducible representations.* □

Uniqueness for Whittaker models is well-known. For rank one Bessel models, Theorem 1.4 was proved, for both orthogonal and unitary groups, by Novodvorsky [**Nov**]. For Bessel models of arbitrary rank, Theorem 1.4 has been proved when $\mathbf{G}$ is an orthogonal group by S. Rallis ([**Ral**]). Though the argument in the unitary case should be similar, it has not yet been written down in full detail.

In the remainder of this paper we study those Bessel models for which the uniqueness principle above is valid. Thus we assume that Theorem/Conjecture 1.4 is true henceforth. Our results are therefore complete for split orthogonal groups and for rank one Bessel models on quasi-split unitary groups, while they are contingent upon the truth of Theorem/Conjecture 1.4 for higher rank Bessel models in the unitary case.

To conclude this section we introduce the notion of a minimal Bessel model for an admissible representation τ of G. This will be a key notion in what follows.

Definition 1.5. Suppose that τ has an ω_χ-Bessel model which is of rank ℓ_1. We say that this model is *minimal* if τ has no Bessel model of rank $\ell_1 - 1$ with respect to a representation $\omega'_{\chi'}$ obtained as follows: χ' is a character of $U_{\ell+1}$ such that $\chi' = \chi$ on the simple roots of U_ℓ (this implies that $M_{\chi'} \subset M_\chi$), and ω' is a component of $\omega|_{M_{\chi'}}$.

This condition is used in our proof of Proposition 2.4 below; see the discussion following the proof of Lemma 2.10.

If τ has a Bessel model, we denote by $\mathcal{B}(\tau)$ the smallest non-negative integer ℓ_1 such that τ has a Bessel model of rank ℓ_1. For example, τ is generic if and only if $\mathcal{B}(\tau) = 0$. Then any Bessel model for τ of rank $\mathcal{B}(\tau)$ is clearly a minimal Bessel model in the sense of Definition 1.5. In particular, any representation which has a Bessel model has a minimal Bessel model.

§**2 Induction of Bessel models.** In this section we study the behavior of minimal Bessel models under induction and prove an analogue of Rodier's Theorem [**Rodb**] for such models.

Suppose that $\mathbf{P} = \mathbf{MN}$ is an arbitrary parabolic subgroup of $\mathbf{G}$. Then

$$\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_t} \times \mathbf{G}(m) \tag{2.1}$$

if $\mathbf{G}$ is orthogonal, and

$$\mathbf{M} \simeq \mathrm{Res}_F^E(GL_{n_1}) \times \cdots \times \mathrm{Res}_F^E(GL_{n_t}) \times \mathbf{G}(m) \tag{2.2}$$

if $\mathbf{G}$ is unitary, where

$$\mathbf{G}(m) = \begin{cases} SO_{2m+1} & \text{if } \mathbf{G} = SO_{2r+1}; \\ SO_{2m} & \text{if } \mathbf{G} = SO_{2r}; \\ U_{2m+1} & \text{if } \mathbf{G} = U_{2r+1}; \\ U_{2m} & \text{if } \mathbf{G} = U_{2r}, \end{cases}$$

and we take the convention that $SO_1 = \{1\}$. Here $r = n_1 + \cdots + n_t + m$. Let $\pi \in \mathcal{E}(M)$. Then

$$\pi = \sigma_1 \otimes \cdots \otimes \sigma_t \otimes \tau, \tag{2.3}$$

where $\sigma_i \in \mathcal{E}(GL_{n_i}(F))$ or $\mathcal{E}(GL_{n_i}(E))$, accordingly, and $\tau \in \mathcal{E}(G(m))$. Suppose that τ has a Bessel model. We let ℓ_1 be the rank of a minimal Bessel model for τ, $\ell_0 = 2\ell_1 + r_0 - 2r$, and $\ell' = m - \ell_1$. Let $\mathbf{B}' = \mathbf{T}'\mathbf{U}' = \mathbf{B} \cap \mathbf{G}(m)$, and $\mathbf{U}'_{\ell'}$ be the subgroup of $\mathbf{U}'$ consisting of matrices whose middle $\ell_0 \times \ell_0$ block is the identity. Choose a character χ_1 of $U'_{\ell'}$ and $\omega \in \mathcal{E}(M_{\chi_1})$ for which τ has an ω_{χ_1}-Bessel model which is minimal. Suppose that each of the representations σ_i is generic. Let $\ell = r - \ell_1$, and let χ be a character of U_ℓ of the form $\chi = \chi_0 \otimes \chi_1$, where χ_0 is a generic character on each GL block corresponding to a fixed non-trivial additive character ψ of F. (We call this the *ψ–generic* character of the GL component.) Let $\tilde{w}_0$ be the longest element of $W(\mathbf{G}, \mathbf{A}_0)/W(\mathbf{M}, \mathbf{A}_0)$ and fix a representative w_0 for $\tilde{w}_0$. Our first main result is the following.

Theorem 2.1. *Let $\mathrm{k} = F$ if $\mathbf{G}$ is orthogonal and E if $\mathbf{G}$ is unitary. Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$, with $\mathbf{M}$ as in (2.1) or (2.2). Let π be as in (2.3) with each σ_i generic. Further suppose that τ has a Bessel model, and that χ_1 is a character of $U_\ell \cap G(m)$ which gives rise to an ω_{χ_1}–Bessel model for τ which is minimal. Let χ be a character of U_ℓ such that $\chi|_{U_\ell \cap GL_{n_i}(\mathrm{k})}$ is ψ–generic for each i and such that $\chi|_{U_\ell \cap G(m)} = \chi_1$. Then $\mathrm{Ind}_P^G(\pi)$ has a unique $\omega_\chi^{w_0}$-Bessel model. Conversely, if any of the σ_i are non-generic, or if τ has no Bessel model, then $\mathrm{Ind}_P^G(\pi)$ has no Bessel model.*

The remainder of this section will be devoted to the proof of Theorem 2.1. The first step is to reduce the theorem to the case of a maximal proper parabolic subgroup. To do this, suppose Theorem 2.1 holds for maximal proper parabolic subgroups and let $\mathbf{P} = \mathbf{MN}$ be an arbitrary parabolic. Then $\mathbf{M}$ is of the form (2.1) or (2.2). Let $\mathbf{P}_1 = \mathbf{M}_1\mathbf{N}_1$ be the standard maximal proper parabolic with $\mathbf{M}_1 = GL_{r-m} \times \mathbf{G}(m)$ or $\mathbf{M}_1 = \mathrm{Res}_F^E(GL_{r-m}) \times \mathbf{G}(m)$ which contains $\mathbf{M}$. Let $\rho = \mathrm{Ind}_{P\cap M_1}^{M_1}(\pi)$. Then $\rho = \rho_1 \otimes \tau$, where ρ_1 is the representation of $GL_{r-m}(\mathrm{k})$ parabolically induced from $\sigma_1 \otimes \cdots \otimes \sigma_t$. Since each σ_i is generic, Rodier's Theorem implies that ρ_1 has a unique generic constituent. Now for each irreducible constituent π_1 of ρ_1, the representation $\pi_1 \otimes \tau$ satisfies the hypothesis of the Theorem. Then, by assumption,

$$\mathrm{Ind}_P^G(\pi) = \mathrm{Ind}_{P_1}^G(\mathrm{Ind}_{P\cap M_1}^{M_1}(\pi) \otimes \mathbf{1}_{N_1})$$

will have a unique Bessel model of the desired type.

Now suppose that $\mathbf{P} = \mathbf{MN}$ is a maximal proper parabolic subgroup of $\mathbf{G}$. Then for some n, $1 \leq n \leq r$, and $m = r - n$ we have $\mathbf{M} \simeq GL_n \times \mathbf{G}(m)$ if $\mathbf{G}$ is orthogonal, and $\mathbf{M} \simeq \mathrm{Res}_{E/F}(GL_n) \times \mathbf{G}(m)$ if $\mathbf{G}$ is unitary. Let $\pi = \sigma \otimes \tau \in \mathcal{E}(M)$, where $\sigma \in \mathcal{E}(GL_n(\mathrm{k}))$ and $\tau \in \mathcal{E}(G(m))$. Suppose that τ has an ω_{χ_1}-Bessel model of rank $\ell_1 \geq 0$, and it is minimal. Assume that χ_1 is of the form given in Lemma 1.2. Let $\ell = r - \ell_1$. Note that $\ell_1 \leq m$ implies $\ell \geq n$. Let $\ell' = \ell - n = m - \ell_1$. Then χ_1 is a character of $U'_{\ell'}$, where $\mathbf{U}'_{\ell'} = \mathbf{U}_{\ell'} \cap \mathbf{G}(m)$. Let χ_0 be the generic character of the upper triangular unipotent subgroup $\mathbf{U}_0$ of GL_n by given by a fixed additive character ψ. Now define the character χ on U_ℓ by $\chi = \chi_0 \otimes \chi_1 \otimes 1_{U'}$, where U' is the complement of $U_0 \times U'_{\ell'}$ in U_ℓ. Note that $M_\chi = M_{\chi_1}$. We will examine the space of ω_χ-Bessel functionals for $\mathrm{Ind}_P^G(\sigma \otimes \tau)$.

In order to carry out our computation, we have to give a description of the $R_\chi - P$ double cosets in G. The given list of such double cosets is exhaustive, but overdefined.

Let $W_{\mathbf{M}} = W(\mathbf{M}, \mathbf{T}_d)$. Then

$$W_{\mathbf{M}} \simeq \begin{cases} S_n \times (S_m \ltimes \mathbb{Z}_2^m) & \text{if } \mathbf{G} \neq SO_{2r} \\ S_n \times (S_m \ltimes \mathbb{Z}_2^{m-1}) & \text{otherwise.} \end{cases}$$

Note that $|W/W_{\mathbf{M}}| = 2^n \begin{pmatrix} r \\ n \end{pmatrix}$. Let w_0 denote the longest element of $W/W_{\mathbf{M}}$. Then

$$w_0 = (1\, r_0)(2\ r_0 - 1) \dots (n\, r_0 + 1 - n),$$

unless $\mathbf{G} = SO_{2r}$ and n is odd, in which case

$$w_0 = (1\, r_0)(2\ r_0 - 1) \dots (n\, r_0 + 1 - n)(r\, r + 1).$$

We now give a list of coset representatives for $W/W_{\mathbf{M}}$. We will say that a permutation $s \in S_{r_0}$ "appears" in w if $w = w's$, for some w' which is disjoint from s. We will also use the convention that if $1 \leq i \leq r$, then $i' = r_0 + 1 - i$.

Lemma 2.2. *Suppose that $w \in W$.*

(a) *If $\mathbf{G} \neq SO_{2r}$, then there is an element w_1 of W so that $w \equiv w_1 \pmod{W_{\mathbf{M}}}$ with w_1 a product of disjoint transpositions in S_{r_0}. More precisely, we may choose w_1 of the form $w'_1 w''_1$, with*

$$w'_1 = \prod_{i=1}^{k} (a_i\ a'_i),$$

for some $\{a_i\} \subset \{1, \dots, n\}$, and

$$w''_1 = \prod_{i=1}^{j} (b_i\, c_i)(c'_i\, b'_i),$$

with $\{b_i\} \subset \{1, \dots, n\}$, and $\{c_i\} \subset \{n+1, n+2, \dots, r_0 - n\}$. Furthermore we may assume that the transpositions appearing in w'_1 and w''_1 are all disjoint.

(b) *If $\mathbf{G} = SO_{2r}$, then $w \equiv w_1 w_2$, where w_1 is of the form given in part (a), and either $w_2 = 1$, $w_2 = (d_0\, d'_0)$, for some $n + 1 \leq d_0 \leq r$, or $w_2 = (i_0\, j'_0\, i'_0\, j_0)$, for some $1 \leq i_0 \leq n < j_0 \leq r$. In each case w_1 and w_2 are disjoint.*

Proof. We first write $w = cs$, with $s \in S_r$ and $c \in \mathbb{Z}_2^r$. Since c acts on the cycles of s independently, we may assume that s is a pair of "companion" cycles,

$(a_1\,a_2\ldots a_t)(a_1'\,a_2'\ldots a_t')$. If $s=1$, or the length of each of the two companion cycles in s is two, then the claim is trivially true, so we assume that the length of each of the cycles is greater than two. Suppose that the claim holds whenever the length of the two cycles in s is less than t. Without loss of generality, we may assume that $a_1 \le n$. If, for some i, we have $a_i, a_{i+1} \le n$, then

$$w \equiv w(a_i\,a_{i+1})(a_i'\,a_{i+1}') = c(a_1\ldots a_{i-1}\,a_i\,a_{i+2}\ldots a_t)(a_1'\ldots a_{i-1}'\,a_i'\,a_{i+2}'\ldots a_t'),$$

and the claim holds by induction. Similarly, we may assume that if $a_i > n$, then $a_{i+1} \le n$. This argument also shows that we may assume that t is even. Now we see that

$$\begin{aligned} w &\equiv cs\cdot(a_1\,a_{t-1}\,a_{t-3}\ldots a_3)(a_1'\,a_{t-1}'\,a_{t-3}'\ldots a_3') \\ &= c(a_1\,a_t)(a_3\,a_2)\ldots(a_{t-1}\,a_{t-2})(a_1'\,a_t')(a_3'\,a_2')\ldots(a_{t-1}'\,a_{t-2}'). \end{aligned}$$

Now write $c=(b_1\,b_1')(b_2\,b_2')\ldots(b_s\,b_s')$, with $b_i \ne b_j$, for $i \ne j$.

If, for a fixed even $i \ge 2$, $\{a_i, a_{i+1}\} \subset \{b_j\}_{j=1}^s$, then the product

$$(a_i\,a_i')(a_{i+1}\,a_{i+1}')(a_{i+1}\,a_i)(a_{i+1}'\,a_i') = (a_i\,a_{i+1}')(a_{i+1}\,a_i')$$

appears in the reduced product for w. The same is true if $\{a_1, a_t\} \subset \{b_j\}_{j=1}^s$, i.e., $(a_1\,a_t')(a_t\,a_1')$ appears in w. If $i \ge 2$ is even and $\{a_i, a_{i+1}\} \cap \{b_j\}_{j=1}^s = \emptyset$, then c commutes with $(a_i\,a_{i+1})(a_i'\,a_{i+1}')$, and so this product of transpositions appears in w. Similarly, if $\{a_1,\, a_t\} \cap \{b_j\}_{j=1}^s = \emptyset$, then $(a_1\,a_t)(a_1'\,a_t')$ appears in w.

Suppose $i \ge 2$ is even and that exactly one element of $\{a_i, a_{i+1}\}$ belongs to $\{b_j\}_{j=1}^s$. Then, if $\mathbf{G} \ne SO_{2r}$, we can replace w by $w(a_i\,a_i')$, and we see that either $(a_{i+1}\,a_i)(a_i'\,a_{i+1}')$ or $(a_{i+1}\,a_i')(a_i\,a_{i+1}')$ appears in $w(a_i\,a_i')$, depending on whether a_{i+1} or a_i is in $\{b_j\}_{j=1}^s$. If $\mathbf{G} = SO_{2r}$ and w either fixes some $d_0 > n$, or interchanges some d_0 and d_0', then we can instead multiply w by $(a_i\,a_i')(d_0\,d_0')$, which shows that one of $(a_{i+1}\,a_i)(a_i'\,a_{i+1}')$ or $(a_{i+1}\,a_i')(a_i\,a_{i+1}')$ appears. We see that the above considerations apply equally well to the pair $\{a_1, a_t\}$. By fixing the element d_0 before starting the above process, we can guarantee that, when we have concluded, $w_2 = 1$ or $w_2 = (d_0\,d_0')$.

Finally suppose that no such d_0 exists. Thus, $w(d) \ne d, d'$ for all $n+1 \le d \le r$. So we may now assume that $d \in \{a_i\}$, for each d, $n+1 \le d \le r$. Suppose that the number of d for which $d = a_i \in \{b_j\}$ with $a_{i+1} \notin \{b_j\}$ is even. (Here we are including $\{a_1,\, a_t\}$ as one possible pair.) Then we see that

$$w \equiv w\prod(d\,d'),$$

where the product is over precisely those $d = a_i$ for which $a_{i+1} \notin \{b_j\}$, is of the form w_1 as claimed. Finally if $\big|\{n+1 \le d \le r | d \in \{b_j\}\}\big|$ is odd, then we fix some such d_0. Without loss of generality, assume that $d_0 = a_t$. Multiplying on the right by the elements $(d\,d')$, for the other such d, we see that we have a factor of $(a_t\,a_t')(a_1\,a_t)(a_1'\,a_t')$ remaining to be dealt with. But this product is indeed $w_2 = (a_1\,a_t'\,a_1'\,a_t)$, as claimed. $\square$

If $\mathbf{G} = SO_{2r}$, and w_2 is of this final form, then there is some flexibility as to the indices appearing in w_2. That is, we may choose, for d_0, any of the $a_i > n$ for which $(a_i\, a_i')$ appears in c, but $(a_{i+1}\, a_{i+1}')$ does not. We will need this below.

Recall that $\ell_0 = r_0 - 2\ell$. Let $s = \ell_0 - 2$. Suppose $\mathbf{x} = (x_1, \ldots, x_s) \in F^s$ if $\mathbf{G}$ is orthogonal and $\mathbf{x} \in E^s$ if $\mathbf{G}$ is unitary. Let

$$n(\mathbf{x}) = \begin{pmatrix} I_\ell & & & & & & & \\ & 1 & x_1 & \ldots & x_s & * & & \\ & & 1 & 0 & \ldots & -\bar{x}_s & & \\ & & & \ddots & 0 & \vdots & & \\ & & & & 1 & -\bar{x}_1 & & \\ & & & & & 1 & & \\ & & & & & & & I_\ell \end{pmatrix},$$

where $\bar{x}$ is the Galois conjugate of x if $\mathbf{G}$ is unitary, and is x if $\mathbf{G}$ is orthogonal.

The next result is a straightforward consequence of Witt's Theorem and the Bruhat decomposition. Here and for the rest of this section, we pass between a Weyl group element and its coset representative without changing the notation.

Proposition 2.3.

(a) *Let $g \in G$. Then for some $w \in W$ and some $\mathbf{x} \in F^s$, we have $R_\chi g P = R_\chi n(\mathbf{x}) w P$. Clearly we can choose w up to $W_{\mathbf{M}}$, i.e., we may assume $w = w_1$ is of the form given in Lemma 2.2.*

(b) *Denote by $||\mathbf{x}||$ the standard length of $\mathbf{x} \in F^s$ or E^s accordingly. If $||\mathbf{x}|| = ||\mathbf{x}_1||$, then $R_\chi n(\mathbf{x}) w P = R_\chi n(\mathbf{x}_1) w P$, for all w.*

If $H \subset G$, we will use $\mathrm{ind}_H^G(\pi)$ to denote the representation of G compactly induced from π [**BeZa**,**Cas**]. Recall that $\mathrm{Ind}_P^G(\pi) = \mathrm{ind}_P^G(\pi)$, by the Iwasawa decomposition. If V is a complex vector space, let $C^\infty(G, V)$ denote the space of locally constant V–valued functions on G, and let $C_c^\infty(G, V)$ denote the subspace of elements of $C^\infty(G, V)$ with compact support. Let $\mathcal{D}(G, V) = C_c^\infty(G, V)^*$ be the space of V–distributions on G.

Let V_σ be the space of σ, V_τ be the space of τ, and V_ω the space of ω (and hence the space of ω_χ). We let $V_\pi = V_\sigma \otimes V_\tau$. Denote by V the vector space $\tilde{V}_\omega \otimes V_\pi$, where $\tilde{V}_\omega$ is the space of the smooth contragredient $\tilde{\omega}$ of ω.

We wish to analyze the space $\mathrm{Hom}_G(\mathrm{Ind}_P^G(\pi), \mathrm{Ind}_{R_\chi}^G(\omega_\chi))$. Dualizing, and using Theorem 2.4.2 of [**Cas**], this is isomorphic to the space

$$\mathrm{Hom}_G(\mathrm{ind}_{R_\chi}^G(\tilde{\omega}_\chi), \mathrm{Ind}_P^G(\tilde{\sigma} \otimes \tilde{\tau})).$$

This space, in turn, is isomorphic to the space of intertwining forms on

$$\mathrm{ind}_{R_\chi}^G(\tilde{\omega}_\chi) \bigotimes \mathrm{Ind}_P^G(\pi)$$

[**Har**, Lemma 4]. Now by Bruhat's thesis (see [**Rodb**, Theorem 4]) this is isomorphic to the space of V–distributions T on G satisfying

$$\varepsilon(r) * T * \varepsilon(p^{-1}) = \delta_P^{1/2}(p) T \circ [\tilde{\omega}_\chi(r) \otimes \pi(p)], \tag{2.4}$$

for all $r \in R_\chi$ and $p \in P$.

The analysis of this space of distributions will make use of the following proposition. Its proof requires a combinatorial argument, and will be given in several steps later in this section.

Proposition 2.4. *If there is a non-zero V-distribution T satisfying* (2.4) *for all $r \in R_\chi$ and $p \in P$ which is supported on $R_\chi n(\mathbf{x})wP$, then $R_\chi n(\mathbf{x})wP = R_\chi w_0 P$ and σ is generic.*

Lemma 2.5. *Suppose that T satisfies* (2.4). *Then T is completely determined by its restriction to $R_\chi w_0 P$.*

Proof. First note that a straightforward matrix computation shows that $n(\mathbf{x})w_0 = w_0 n(\mathbf{x})$, for any $\mathbf{x}$. Thus

$$R_\chi w_0 P = \bigcup_{\mathbf{x}} R_\chi n(\mathbf{x}) w_0 P = P w_0 P,$$

is open. Therefore $C = G \setminus R_\chi w_0 P$ is closed. Therefore, we have the exact sequence [**BeZa**, §1.7]

$$0 \longrightarrow C_c^\infty(R_\chi w_0 P) \longrightarrow C_c^\infty(G) \longrightarrow C_c^\infty(C) \longrightarrow 0.$$

Then, by tensoring with V, the above exact sequence yields the exact sequence

$$0 \longrightarrow C_c^\infty(R_\chi w_0 P, V) \longrightarrow C_c^\infty(G, V) \longrightarrow C_c^\infty(C, V) \longrightarrow 0.$$

Dualizing, we get the exact sequence

$$0 \longrightarrow \mathcal{D}(C, V) \longrightarrow \mathcal{D}(G, V) \longrightarrow \mathcal{D}(R_\chi w_0 P, V) \longrightarrow 0.$$

Let $\mathcal{D}_{R_\chi, P}$ be the subspace of distributions satisfying (2.4). Then Proposition 2.4 implies that if $T \in \mathcal{D}(G, V)_{R_\chi, P}$ and $f \in C_c^\infty(C, V)$, then $T(f) = 0$. Thus, the above sequence tells us that $\mathcal{D}(G, V)_{R_\chi, P} \simeq \mathcal{D}(R_\chi w_0 P, V)_{R_\chi, P}$, which completes the proof of the Lemma. $\square$

Let $R_\chi^{w_0} = w_0^{-1} R_\chi w_0$, and denote by $\omega_\chi^{w_0}$ the representation of $R_\chi^{w_0}$ defined by $\omega_\chi^{w_0}(r) = \omega_\chi(w_0 r w_0^{-1})$. Recall that $\mathbf{P} = \mathbf{MN}$ is the Levi decomposition of $\mathbf{P}$.

Lemma 2.6. *There exists an isomorphism between the vector space $\mathcal{D}(R_\chi w_0 P, V)_{R_\chi, P}$ and the vector space of distributions in $\mathcal{D}(U_\ell) \otimes \mathcal{D}(P, V)$ of the form*

$$\chi(u)\, du \otimes \delta_P^{-1/2}(m)\, dQ(m)\, dn,$$

where $Q \in \mathcal{D}(M, V)$ satisfies

$$\varepsilon(r) * Q * \varepsilon(m^{-1}) = Q \circ [\tilde{\omega}_\chi^{w_0}(r) \otimes \pi(m)], \tag{2.5}$$

for all $r \in R_\chi^{w_0} \cap M$, $m \in M$.

Proof. Define a projection

$$\mathcal{P} : C_c^\infty(U_\ell) \otimes C_c^\infty(P, V) \to C_c^\infty(U_\ell w_0 P, V)$$

by specifying that for all $f_1 \in C_c^\infty(U_\ell)$ and $f_2 \in C_c^\infty(P, V)$, one has

$$\mathcal{P}(f_1 \otimes f_2)(uw_0p) = \int\limits_{U_\ell \cap w_0 P w_0^{-1}} f_1(uu_1)\, f_2(w_0^{-1}u_1^{-1}w_0p)\, du_1.$$

Then it follows from [**Sil**, Lemma 1.2.1] that $\mathcal{P}$ is onto. Let $T \in \mathcal{D}(R_\chi w_0 P, V)_{R_\chi, P}$. For f_1, f_2 as above, define $T' \in \mathcal{D}(U_\ell) \otimes \mathcal{D}(P, V)$ by

$$T'(f_1 \otimes f_2) = T(\mathcal{P}(f_1 \otimes f_2)).$$

Then one sees easily that (2.4) implies the equality

$$\varepsilon(u) * T' * \varepsilon(p^{-1}) = \tilde{\omega}_\chi(u) T \circ [\pi(p)] \tag{2.6}$$

for all $u \in U_\ell$, $p \in P$ (where π acts on the second factor of V). As in [**Sil**, Section 1.8], this implies that T' is in fact a pure tensor of the form

$$\chi(u)\, du \otimes \delta_P(m)^{-1/2}\, dQ(m)\, dn. \tag{2.7}$$

where $Q \in \mathcal{D}(M, V)$. (Here we are using that $\pi(mn) = \pi(m)$.) It is a formal consequence of the definitions that (2.6) implies that

$$Q * \varepsilon(m^{-1}) = Q \circ [\pi(m)]$$

for all $m \in M$. We claim that, more strongly, equation (2.5) holds. To see this, write

$$dQ(p) = \delta_P(m)^{-1/2}\, dQ(m)\, dn.$$

Let $f_1 \in C_c^\infty(U_\ell)$, $f_2 \in C_c^\infty(P, V)$ and $r \in R_\chi \cap w_0 M w_0^{-1}$. Then by (2.4) we have

$$\begin{aligned}
\int\limits_{U_\ell} f_1(u)\chi(u)\, du \int\limits_{P} \tilde{\omega}_\chi(r) f_2(p)\, dQ(p) &= \int\limits_{U_\ell \times P} f_1(u)\, \tilde{\omega}_\chi(r) f_2(p)\, dT'(u, p) \\
&= T\left(\tilde{\omega}_\chi(r)\mathcal{P}(f_1 \otimes f_2)\right) \\
&= \int\limits_{U_\ell w_0 P} \mathcal{P}(f_1 \otimes f_2)(ruw_0p)\, dT(uw_0p).
\end{aligned}$$

But $ruw_0p = (rur^{-1})w_0(w_0^{-1}rw_0p)$, so this expression is equal to

$$\begin{aligned}
&\int\limits_{U_\ell \times P} f_1(rur^{-1}) f_2(w_0^{-1}rw_0p)\, dT'(u, p) \\
&\qquad = \int\limits_{U_\ell} f_1(rur^{-1})\chi(u)\, du \int\limits_{P} f_2(w_0^{-1}rw_0p)\, dQ(p) \\
&\qquad = \int\limits_{U_\ell} f_1(u)\chi(u)\, du \int\limits_{P} f_2(p)\, d(\varepsilon(w_0^{-1}rw_0) * Q)(p),
\end{aligned}$$

where in this last equality the defining properties of $R_\chi = M_\chi U_\ell$ have been used to simplify the U_ℓ integral. Since this holds for all $f_1 \in C_c^\infty(U_\ell)$ one concludes that

$$\varepsilon(w_0^{-1} r w_0) * Q = Q \circ \tilde{\omega}_\chi(r)$$

for all $r \in R_\chi \cap w_0 M w_0^{-1}$, as desired.

Conversely, given a distribution Q satisfying equation (2.5), one reverses the above steps to arrive at a distribution $T' \in \mathcal{D}(U_\ell) \otimes \mathcal{D}(P, V)$ satisfying (2.6). Since the map $\mathcal{P}$ is onto, one may define a distribution $T \in \mathcal{D}(U_\ell w_0 P, V)$ by the formula

$$T(\mathcal{P}(f_1 \otimes f_2)) = T'(f_1 \otimes f_2)$$

provided one shows that if $\mathcal{P}(\sum_i f_{1,i} \otimes f_{2,i}) = 0$, then $T'(\sum_i f_{1,i} \otimes f_{2,i}) = 0$. This follows as in [**HeR**, Theorem 15.24]. Since $M_\chi \subseteq w_0^{-1} M w_0$ and $R_\chi = U_\ell M_\chi$, it follows from (2.5) and (2.7) that the T so-obtained satisfies (2.4).

The maps $T \mapsto Q$, $Q \mapsto T$ described above are clearly inverses. This completes the proof of the Lemma. □

We now complete the proof of Theorem 2.1, modulo the proof of Proposition 2.4. Let Q be as in the proof of Lemma 2.6. Then by Bruhat's thesis once again, Q corresponds to an element of

$$\mathrm{Hom}_M(\mathrm{ind}_{R_\chi^{w_0} \cap M}^M(\tilde{\omega}_\chi^{w_0}), \tilde{\sigma} \otimes \tilde{\tau}),$$

which, by duality gives an element of $\mathrm{Hom}_M(\pi, \mathrm{Ind}_{R_\chi^{w_0} \cap M}^M(\omega_\chi^{w_0}))$. Since $M = GL_n(\mathsf{k}) \times G(m)$, where G_1 is either $GL_n(F)$ or $GL_n(E)$, depending on whether $\mathbf{G}$ is orthogonal or unitary, we see that this last space is exactly the space of Whittaker models for σ tensored with the space of $\omega_\chi^{w_0}$–Bessel models for τ. □

Proof of Proposition 2.4. The remainder of the section will consist of a proof of Proposition 2.4. This is carried out in several steps. We begin by showing that, on many double cosets, the compatibility condition $\pi(p) = \omega_\chi(wpw^{-1})$ can not be satisfied for some $p \in P$ with $r = wpw^{-1} \in R_\chi$. By [**Sil**, Theorem 1.9.5], this is sufficient to imply the Proposition.

Let $\Sigma_{\mathbf{P}}^+$ denote the set of positive roots in $\mathbf{N}$. Let Δ denote the simple roots of $\mathbf{T}$ in $\mathbf{G}$ which give rise to our choice of Borel subgroup. If $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, then we let X_α be the corresponding element of a Chevalley basis for the Lie algebra of $\mathbf{U}$ or $\bar{\mathbf{U}}$. Let α_i denote the root $e_i - e_{i+1}$, and $\beta = e_\ell + e_{\ell+1}$. Let $X = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta\}$. Then X is the set of roots where the character χ is non-trivial. For $\alpha \in X$ we have $\chi(I + tX_\alpha) = \psi_\alpha(t)$. Also, note that $X \cap \Sigma_{\mathbf{P}}^+ = \{\alpha_n\}$. We list the elements of $\Sigma_{\mathbf{P}}^+$, for future reference. If $\mathbf{G} = SO_{2r+1}$, then

$$\begin{aligned}\Sigma_{\mathbf{P}}^+ = &\{e_i \pm e_j \,|\, 1 \le i \le n < j \le r\} \cup \\ &\{e_i + e_j \,|\, 1 \le i < j \le n\} \cup \{e_i | 1 \le i \le n\}.\end{aligned}$$

If $\mathbf{G} = U_{2r}$, then

$$\begin{aligned}\Sigma_{\mathbf{P}}^+ = &\{e_i \pm e_j \,|\, 1 \le i \le n < j \le r\} \cup \\ &\{e_i + e_j \,|\, 1 \le i < j \le n\} \cup \{2e_i | 1 \le i \le n\}.\end{aligned}$$

If $\mathbf{G} = U_{2r+1}$, then

$$\begin{aligned}\Sigma_{\mathbf{P}}^{+} = &\{e_i \pm e_j \mid 1 \leq i \leq n < j \leq r\} \cup \\ &\{e_i + e_j \mid 1 \leq i < j \leq n\} \cup \{e_i, 2e_i | 1 \leq i \leq n\}.\end{aligned}$$

Finally, if $\mathbf{G} = SO_{2r}$, then

$$\begin{aligned}\Sigma_{\mathbf{P}}^{+} = &\{e_i \pm e_j \mid 1 \leq i \leq n < j \leq r\} \cup \\ &\{e_i + e_j \mid 1 \leq i < j \leq n\}.\end{aligned}$$

We list the various $I + X_\alpha$, which generate the root subgroups $\mathbf{U}_\alpha$ of $\mathbf{U}$. Let E_{ij} denote the elementary matrix whose only non-zero entry is a 1 in the ij-th entry. We recall the convention that $i' = r_0 + 1 - i$. Suppose $E = F(\gamma)$, where $\bar{\gamma} = -\gamma$, and $a \mapsto \bar{a}$ is the Galois automorphism of E/F. If $\alpha = e_i - e_j$, then $I + X_\alpha = I + E_{ij} - E_{j'i'}$ if $\mathbf{G}$ is orthogonal, and $I + X_\alpha = I + \gamma E_{ij} - \gamma E_{j'i'}$ if $\mathbf{G}$ is unitary. If $\alpha = e_i + e_j$, then $I + X_\alpha = I + E_{ij'} - E_{ji'}$ if $\mathbf{G}$ is orthogonal, and $I + X_\alpha = I + \gamma E_{ij'} - \gamma E_{ji'}$ if $\mathbf{G}$ is unitary. If $\alpha = e_i$, then $I + X_\alpha = I + E_{i,r+1} - E_{r+1,i'}$ if $\mathbf{G} = SO_{2r+1}$, and $I + X_\alpha = I + \gamma E_{i,r+1} - \gamma E_{r+1,i'}$ if $\mathbf{G} = U_{2r+1}$. Finally, if $\mathbf{G}$ is unitary and $\alpha = 2e_i$, then $I + X_\alpha = I + \gamma E_{ii'}$.

Suppose that, for some $\alpha \in \Sigma_{\mathbf{P}}^{+}$, we have $\alpha' = w\alpha \in X$. Choose some $t \in F^\times$ for which $\psi_{\alpha'}(t) \neq 1$. Now set $p = I + tX_\alpha$, which is in P. Then $r = wpw^{-1} = I + tX_{\alpha'} \in U_\ell \subset R_\chi$. Note that $\pi(p) = 1 \neq \omega_\chi(r) = \psi_\alpha(t)$. Thus, if w has the above property, $R_\chi w P$ can support no distribution of the desired type.

Lemma 2.7. *Let $\mathbf{G} = SO_{2r}$. Suppose that, as in Lemma 2.2, $w \in W$ is equivalent* $\operatorname{mod} W_{\mathbf{M}}$ *to $w_1 w_2$, with $w_2 = (i_0\, j_0'\, i_0'\, j_0)$, for some $1 \leq i_0 \leq n < j_0 \leq r$. Then $w\Sigma_{\mathbf{P}}^{+} \cap X \neq \emptyset$.*

Proof. From the proof of Lemma 2.2, we may assume that for each $n+1 \leq k \leq r$, we have $w(k) = i_k$ or i_k', for some $1 \leq i_k \leq n$. First suppose that $(i_{n+1}\, n+1)(i_{n+1}'\, (n+1)')$ appears in w. Consider first the case that for all k, $n+1 \leq k \leq \ell$, we have a permutation $(i_k\, k)(i_k'\, k')$ appearing in w. Since $\ell + 1 = w(i_{\ell+1})$ or $\ell + 1 = w(i_{\ell+1}')$, we have $w(e_{i_\ell} + e_{i_{\ell+1}}) = e_\ell \pm e_{\ell+1}$, which will be in X. So now we may suppose that either $j_0 \leq \ell$, or $(i_k\, k')(i_k'\, k)$ appears in w, for some k with $n+1 \leq k \leq \ell$. Since w changes an even number of signs, we see that in the former case there must be some k with $n+1 \leq k \leq r$, so that $(i_k\, k')(i_k'\, k)$ appears in w. Now we can multiply on the right by $(j_0\, j_0')(k\, k')$, to see that, in fact, we may assume that $(i_{k_0}\, k_0')(i_{k_0}'\, k_0)$ is appearing, for some k_0, with $n+1 \leq k_0 \leq \ell$. Choosing the minimal such k_0, we know that $k_0 = w(i_{k_0}')$, while $k_0 - 1 = w(i_{k_0-1})$. Thus, $\alpha_{k_0-1} = w(e_{i_{k_0-1}} + e_{i_{k_0}}) \in w\Sigma_{\mathbf{P}}^{+} \cap X$, and the Lemma holds.

Thus, we may assume that either $(i_{n+1}\, (n+1)')(i_{n+1}'\, n+1)$ appears in w, or that $(i_{n+1}\, (n+1)'\, i_{n+1}'\, n+1)$ does. In the former case, we may multiply on the right by $(n+1\, (n+1)')(j_0\, j_0')$, to get an equivalent w for which the latter is true, i.e, we may assume that $i_0 = i_{n+1}$. First suppose $w(n) = n$. Then $w(e_n + e_{i_0}) = \alpha_n \in w\Sigma_{\mathbf{P}}^{+} \cap X$ and we are done. Suppose instead that $w(n) = n'$, i.e., that $(n\, n')$ appears in w. Let i be the smallest positive integer so that $w(n-i) \neq n - i'$. (By our assumption on the form of w, such an i exists.) Then $n - i = i_k$, for some $k \geq n+1$, and $n - i = w(k)$ or $w(k')$. Therefore, α_{n-i} is equal to either $w(e_{n-i+1} - e_k)$ or to

$w(e_{n-1+1}+e_k)$. In either case, $\alpha_{n-i}\in w\Sigma_{\mathbf{P}}^+\cap X$. Finally, we may suppose that either $n=i_0$ or that one of $(n\,k)(n'\,k')$ or $(n\,k')(n'\,k)$ appears in w, for some k, $n+1\le k\le r$. If $(n\,k)(n'\,k')$ appears in w, then we may multiply on the right by

$$((n+1)\,k)((n+1)'\,k')(i_0\,n)(i_0'\,n'),$$

to replace w by an equivalent element with $i_0=n$. Similarly, if $(n\,k')(n'\,k)$ appears in w, then we may multiply on the right by

$$(n+1\,k)((n+1)'\,k')(i_0\,n)(i_0'\,n')(k\,k')(n+1\,(n+1)'),$$

to see that we may assume that $i_0=n$. We are thus reduced to the case where $(n\,(n+1)'\,n'\,n+1)$ appears in w. In this case, $w(e_n+e_{n+1})=\alpha_n\in w\Sigma_{\mathbf{P}}^+\cap X$. Thus, in all cases, the Lemma holds. $\square$

Remark. For future use we make note of the following fact. If w is as in Lemma 2.7, and if $w^{-1}\alpha_\ell\in\Sigma_{\mathbf{P}}^+$, then the proof of Lemma 2.7 shows that either $w^{-1}\alpha_\ell=e_i+e_j$, for some $i,j\le n$, or that $w\Sigma_{\mathbf{P}}^+\cap X\ne\{\alpha_\ell\}$.

We now describe those w which have the property that $w\Sigma_{\mathbf{P}}^+\cap X=\emptyset$. By Lemmas 2.2 and 2.7, we may assume that w is a product of disjoint transpositions.

Lemma 2.8. *Suppose that $w\in W$ is a representative for a class in $W/W_{\mathbf{M}}$, and w is in the form specified by Lemma 2.2. Further suppose that, for all $\alpha\in\Sigma_{\mathbf{P}}^+$, we have $w\alpha\notin X$. Then the following hold:*

(a) *For all k with $n+1\le k\le\ell$, we have $w(k)>n$;*
(b) *For all i with $1\le i\le n$, we have $w(i)\ne i$.*

Proof. (a) First suppose that $w(\ell)\le n$. If $w(\ell+1)\le n$, then $w(\beta)\in\Sigma_{\mathbf{P}}^+$, contradicting our choice of w. If $w(\ell+1)=\ell+1$, then again $w(\beta)\in\Sigma_{\mathbf{P}}^+$. Finally, if $w(\ell+1)\ge n'$, then $w(\alpha_\ell)\in\Sigma_{\mathbf{P}}^+$. So we must have $w(\ell)>n$.

Now suppose that for some k, $n+1\le k\le\ell-1$, we have $w(k)\le n$. If $w(k+1)=k+1$, or $w(k+1)\ge n'$, then $w(\alpha_k)\in\Sigma_{\mathbf{P}}^+$, which is a contradiction. Therefore, $w(k+1)\le n$. However, this implies, by induction, that $w(\ell)\le n$, which we have already seen is impossible. Therefore, $w(k)>n$.

(b) Suppose that $w(i)=i$ for some i, $1\le i\le n-1$. If $w(i+1)\ne i+1$, then $w(i+1)>n$, and so $w(\alpha_i)\in\Sigma_{\mathbf{P}}^+$. Since this contradicts our choice of w, we have $w(i+1)=i+1$. We may thus suppose that w fixes n. Now by part (a), we have $w(n+1)\ge n+1$, and therefore, $w\alpha_n\in\Sigma_{\mathbf{P}}^+$. This again is a contradiction, so w cannot fix n. Therefore, w fixes none of the integers $1,2,\ldots,n$. $\square$

Lemma 2.9. *Suppose that w is as in Lemma 2.8 and assume that $w(n)\ne n'$. Then for $n+1\le k\le\ell$, we have $w(k)=k$.*

Proof. By Lemma 2.8(a) it is enough to show that it is impossible that $w(k)\ge n'$ for any such k. Suppose to the contrary that there is some k, with $n+1\le k\le\ell$, for which $w(k)\ge n'$. Then there some $i\le n$, for which $k=w(i')$. If $w(k-1)=k-1$, then $\alpha_{k-1}=w(e_i+e_{k-1})\in w\Sigma_{\mathbf{P}}^+\cap X$. Since this contradicts our choice of w, we must have $w(k-1)\ge n'$. Therefore, by (downwards) induction, $w(n+1)\ge n'$. Set $w(n+1)=i'$. Since $w(n)\ne n$, and, by assumption, $w(n)\ne n'$, either $w(n)=k$ or $w(n)=k'$ for some k, with $n+1\le k\le n'-1$. Therefore, $\alpha_n=w(e_i+e_k)$ or $\alpha_n=w(e_i-e_k)$. Either one of these possibilities contradicts our assumption on w. Thus $w(n+1)<n'$, which then implies the result of the Lemma. $\square$

Lemma 2.10. *Suppose that w is as in Lemma 2.8. Suppose that there is some i, $2 \le i \le n$, for which $w(i) = i'$. Then $w(i-1) = (i-1)'$.*

Proof. Suppose $w(i-1) \ne (i-1)'$. By Lemma 2.8(b), we can choose k, with $n+1 \le k \le r$ so that $w(i-1) = k$ or $w(i-1) = k'$. Now $\alpha_{i-1} = w(e_i - e_k)$ or $\alpha_{i-1} = w(e_i + e_k)$. Since this contradicts our choice of w we conclude $w(i-1) = (i-1)'$. $\square$

Thus, if w is chosen as in Lemma 2.2 with $w\Sigma_{\mathbf{P}}^+ \cap X = \emptyset$ and $w(n) = n'$, then $w = w_0$. If $w\Sigma_{\mathbf{P}}^+ \cap X = \emptyset$, and $w(n) \ne n'$, then by Lemma 2.9, $w(\ell) = \ell$. If $w(\ell+1) \ne \ell+1$, then either α_ℓ or β would be of the form $w\alpha$ for some $\alpha \in \Sigma_{\mathbf{P}}^+$. Consequently, $w(\ell+1) = \ell+1$, and therefore $w(\alpha) = \alpha$ for $\alpha \in \{\alpha_{n+1} \dots, \alpha_\ell, \beta\}$. Thus, for some $i_0 < n$, and some $\{a_i\} \subset \{\ell+2, \dots, (\ell+2)'\}$,

$$w = (1\, r_0)(2\, r_0 - 1) \dots (i_0\, i_0')(i_0+1\, a_{i_0+1})((i_0+1)'\, a_{i_0+1}') \dots (n\, a_n)(n'\, a_n'). \tag{2.8}$$

Let $a = a_n$ if $a_n \le r$, and $a = a_n'$ otherwise. Then $w\alpha_n = \pm e_a - e_{n+1}$. Let $X_1 = \{\alpha_{n+1} \dots, \alpha_\ell, \beta, w\alpha_n\}$. Note that

$$X_1 = w(\{\alpha_n, \alpha_{n+1} \dots, \alpha_\ell, \beta\}),$$

and is thus a linearly independent subset of the root system $\Phi(\mathbf{G}(m), \mathbf{T}')$, where we recall that $\mathbf{T}' = \mathbf{T} \cap \mathbf{G}(m)$. We extend $X_1 \setminus \{\beta\}$ to a set of simple roots for $\mathbf{G}(m)$. Set $\mathbf{B}' = \mathbf{T}_1\mathbf{U}''$ to be the corresponding Borel subgroup of $\mathbf{G}(m)$, and suppose that $U_{\ell'+1}''$ is the subgroup of $\mathbf{U}''$ which is conjugate to $\mathbf{U}_{\ell'+1}'$ and generated by the elements of X_1. (Recall that $U_{\ell'}'$ is the subgroup supporting the character χ_1 which gives rise to the model for τ.) Now let χ' be the character of $U_{\ell'+1}''$ so that $\chi'(I + tX_\alpha) = \psi_\alpha(t)$, for $\alpha \in X_1 \setminus \{\alpha_\ell\}$, and $\chi'(I + tX_{\alpha_\ell}) = \psi_{\alpha_\ell}(\delta t)$. Let $M_{\chi'}$ be the corresponding normalizer in $M_{\ell'+1}$. Note that $M_{\chi'} \subset M_\chi$. Suppose that $m' \in M_{\chi'}$. If the distribution T satisfies (2.4), then $\varepsilon(m') * T = T \circ \omega(m')$. So for some component ω' of $\omega|_{M_{\chi'}}$, we have

$$\varepsilon(r) * T * \varepsilon(h) = T \circ [\tilde{\omega}_{\chi'}'(r) \otimes \tau(h)],$$

for all $h \in \mathbf{G}(m)$, and $r \in R_{\chi'} = M_{\chi'}U_{\ell'+1}''$. If T is non-zero, this now implies that τ has a Bessel model with respect to U'', χ' and ω'. However, since $U_{\ell+1}''$ is isomorphic to $U_{\ell+1}$, this is a rank $\ell_1 - 1$ Bessel model for τ. This contradicts the minimality of the ω_{χ_1}-Bessel model for τ. Hence, no such T exists.

Note that this argument shows that $\mathrm{Ind}_P^G(\pi)$ cannot have any Bessel model of rank less than $\mathcal{B}(\tau)$ supported on $R_\chi wP$.

Finally, suppose that $w = w_0$. Let $u \in \bar{U} \cap GL_n(F)$. Set $r = w_0^{-1}uw_0$. Then $r \in U_\ell$, and $\chi(r) = \chi_0^{w_0}(u)$. Since

$$\varepsilon(r) * T = T \circ [\chi_0^{w_0}(u)] = T * \varepsilon(u) = T \circ [\sigma(u)],$$

we see that σ must be generic if T is non-zero [**Rodb**]. This completes the proof of Proposition 2.4 for the cosets $R_\chi wP$, with $w \in W/W_{\mathbf{M}}$.

We now examine the double cosets represented by $n(\mathbf{x})w$, where $\mathbf{x} = (x_1, \ldots, x_s)$ is a vector. Recall that

$$n(\mathbf{x}) = \begin{pmatrix} I_\ell & & & & & & \\ & 1 & x_1 & \ldots & x_s & * & \\ & 0 & 1 & 0 & \ldots & -\bar{x}_s & \\ & & & \ddots & 0 & & \\ & 0 & 0 & \ldots & 1 & -\bar{x}_1 & \\ & 0 & 0 & 0 & \ldots & 1 & \\ & & & & & & I_\ell \end{pmatrix},$$

where $\bar{x}$ is the Galois conjugate of x if $\mathbf{G}$ is unitary, and $\bar{x} = x$ if $\mathbf{G}$ is orthogonal.

We assume that w is of the form given in Lemma 2.2. First note that if $\alpha \in X \setminus \{\alpha_\ell\}$, then $n(\mathbf{x})(I + tX_\alpha)n(\mathbf{x})^{-1} = I + tX_\alpha$. Suppose that $w\Sigma_{\mathbf{P}}^+ \cap (X \setminus \{\alpha_\ell\}) \neq \emptyset$. Choose $\alpha' \in \Sigma_{\mathbf{P}}^+$ with $w\alpha' = \alpha \in X \setminus \{\alpha_\ell\}$, and t for which $\psi_\alpha(t) \neq 1$. Setting $p = I + tX_{\alpha'}$, we have $n(\mathbf{x})wpw^{-1}n(\mathbf{x})^{-1} = I + tX_\alpha \in R_\chi$. Furthermore $\omega_\chi(x) = \psi_\alpha(t) \neq 1$, while $\pi(p) = 1$. Thus, $R_\chi n(\mathbf{x})wP$ supports no distributions satisfying (2.4).

Now suppose that $w\Sigma_{\mathbf{P}}^+ \cap X = \{\alpha_\ell\}$. First suppose that $w^{-1}\alpha_\ell = e_i + e_j$, with $i, j \leq n$. Without loss of generality, assume that $w(i) = \ell$, and $w(j) = (\ell+1)'$. Suppose that $\ell + 2 \leq k \leq r$. If $w(k) = i_k \leq n$ then $w^{-1}(e_\ell + e_k) = e_i + e_{i_k} \in \Sigma_{\mathbf{P}}^+$. If instead $w(k) = i_k'$ for some $i_k \leq n$, then $w^{-1}(e_\ell - e_k) = e_i + e_{i_k}$. Finally, if $w(k) = k$, then $w^{-1}(e_\ell \pm e_k) = e_i \pm e_k \in \Sigma_{\mathbf{P}}^+$. Choose $s_0 \leq s$ for which $x_{s_0} \neq 0$. Let $y = x_{s_0}$. Choose k_0 with the property that either $w^{-1}(e_\ell + e_{k_0})$ or $w^{-1}(e_\ell - e_{k_0})$ is an element of $\Sigma_{\mathbf{P}}^+$. Denote the root $e_\ell \pm e_{k_0}$ as α_0, with $\pm$ chosen so that $w^{-1}\alpha_0 \in \Sigma_{\mathbf{P}}^+$. We may also assume that X_{α_0} has -1 as its $(r + \ell_1, \ell + s_0)$ entry (see Lemma 2.3). Now note that

$$n(\mathbf{x})(I + tX_{\alpha_0})n(\mathbf{x})^{-1} = (I + tX_{\alpha_0})(I + ytX_\beta).$$

Thus, if $\psi_{\alpha_0}(yt) \neq 0$, and $p = I + tX_{w^{-1}\alpha_0} \in N$, then $\pi(p) = 1$, while

$$\omega_\chi(n(\mathbf{x})wpw^{-1}n(\mathbf{x})^{-1}) = \psi_{\alpha_0}(yt) \neq 1.$$

Consequently, $R_\chi n(\mathbf{x})wP$ cannot support a V-distribution of the desired form.

We are left with the cases $w\Sigma_{\mathbf{P}}^+ \cap X = \{\alpha_\ell\}$, but $w^{-1}\alpha_\ell \neq e_i + e_j$ for all $i, j \leq n$, or $w\Sigma_{\mathbf{P}}^+ \cap X = \emptyset$. For the second of these two cases, the form of w is given by (2.8). In order to complete the proof we will determine the form of w in the first case. To do so we need a few lemmas.

Lemma 2.11. *Suppose that $w\Sigma_{\mathbf{P}}^+ \cap X = \{\alpha_\ell\}$, but $w^{-1}\alpha_\ell \neq e_i + e_j$, for all $i, j \leq n$. Then $w(\ell) = \ell$.*

Proof. If $w^{-1}(\ell) = j'$ for some $j \leq n$, then $w^{-1}\alpha_\ell \notin \Sigma_{\mathbf{P}}^+$, which is a contradiction. Suppose $w^{-1}(\ell) = j \leq n$. If $w(\ell+1) = \ell+1$, then $w^{-1}(\beta) = e_j + e_{\ell+1} \in \Sigma_{\mathbf{P}}^+$, contradicting our choice of w. If $w(\ell+1) = i \leq n$, then $w^{-1}\alpha_\ell \notin \Sigma_{\mathbf{P}}^+$, which also contradicts our choice of w. Finally, if $w^{-1}(\ell+1) = i'$ for some $i \leq n$, then $w^{-1}\alpha_\ell = e_j + e_i$, which is again a contradiction. Thus, $w(\ell) = \ell$. $\square$

Lemma 2.12. *If w is as in Lemma 2.11, then for $n+1 \leq k \leq \ell-1$, we have $w^{-1}(k) > n$.*

Proof. Suppose that $w^{-1}(k) = j \leq n$. If $w(k+1) = k+1$, then $w^{-1}\alpha_k = e_j - e_{k+1} \in \Sigma^+_{\mathbf{P}}$. If $w(k+1) = i'$ for some $i \leq n$, then $w^{-1}\alpha_k = e_j + e_i$. Either case contradicts our hypotheses. Therefore $w^{-1}(k+1) \leq n$. Now by induction, $w^{-1}(\ell-1) \leq n$. On the other hand, by Lemma 2.11, $w(\ell) = \ell$. Therefore $w^{-1}\alpha_{\ell-1} \in \Sigma^+_{\mathbf{P}}$, contradicting our choice of w. Consequently, $w^{-1}(k) > n$. $\square$

Lemma 2.13. *Suppose that w is as in Lemma 2.11. Then $w(n) \neq n$.*

Proof. Suppose that $w(n) = n$. If $w(n+1) = n+1$, then w fixes α_n, which is in the intersection of X and $\Sigma^+_{\mathbf{P}}$. If $w^{-1}(n+1) = j'$ for some $j \leq n$, then $w^{-1}\alpha_n = e_n + e_j$. Both of these possibilities contradict our choice of w. By Lemma 2.12, $w^{-1}(n+1) > n$, and so these are the only two choices for $w(n+1)$. Since each leads to a contradiction, $w(n) \neq n$. $\square$

Lemma 2.14. *Suppose that w is as in Lemma 2.11.*

(a) *For all $i \leq n$ we have $w(i) \neq i$.*
(b) *If $w(i_0) = i_0'$, for some $i_0 \leq n$ then $w(i) = i'$ for all $i \leq i_0$.*

Proof. (a) Suppose that $w(i) = i$ for some $i \leq n$. Choose the maximal such i. By Lemma 2.13, $i < n$. Suppose that $w(i+1) = (i+1)'$. Then $w^{-1}\alpha_i = e_i + e_{i+1} \in \Sigma^+_{\mathbf{P}}$. Thus in this case we have a contradiction. If $w(i+1) = k$ or $w(i+1) = k'$ for some $n+1 \leq k \leq r$, then $w^{-1}\alpha_i = e_i \pm e_k \in \Sigma^+_{\mathbf{P}}$. This is also a contradiction, and hence no $i \leq n$ can be fixed by w.

(b) Suppose that $w(i) = i'$, for some $i \leq n$. If $w(i-1) = k$ or k' for some $n+1 \leq k \leq r$, then $w^{-1}\alpha_{i-1} = e_i \pm e_k \in \Sigma^+_{\mathbf{P}}$. But by part (a), $w(i-1) \neq i-1$, so the only remaining possibility is $w(i-1) = (i-1)'$. This gives the claim by induction. $\square$

Corollary 2.15. *If w is as in Lemma 2.11, then $w(n) = k_0$ or $w(n) = k_0'$ for some $n+1 \leq k_0 \leq r$.* $\square$

Lemma 2.16. *Suppose that w is as in Lemma 2.11. Then $w(k) = k$ for all k with $n+1 \leq k \leq \ell-1$.*

Proof. Suppose that $w^{-1}(n+1) = j'$ for some $j \leq n$. Then, by Corollary 2.15, $w^{-1}\alpha_n = e_j \pm e_{k_0} \in \Sigma^+_{\mathbf{P}}$, contradicting our choice of w. Thus, by Lemma 2.12, $w(n+1) = n+1$.

Now suppose $w^{-1}(k) = j_k'$ for some k with $n+2 \leq k \leq \ell-1$, and some $j_k \leq n$. If $w(k-1) = k-1$, then $w^{-1}\alpha_{k-1} \in \Sigma^+_{\mathbf{P}}$, which is a contradiction. Therefore, by Lemma 2.12, $w^{-1}(k-1) = j_{k-1}'$, for some $j_{k-1} \leq n$. By induction, this gives $w(n+1) \neq n+1$, while we have just shown that $w(n+1) = n+1$. Therefore, $w(k) = k$. $\square$

Lemma 2.17. *Suppose that w is as in Lemma 2.11.*

(a) *Suppose that $\mathbf{G} \neq SO_{2r}$. Then, for some n_1, with $0 \leq n_1 < n$, and some*

$$\{k_j | 1 \leq j \leq n - n_1\} \subset \{\ell+1, \ell+2, \ldots, (\ell+1)'\},$$

we have

$$w = (1\, r_0)(2\, r_0 - 1)\dots(n_1\, n_1')(n_1+1\, k_1)(k_1'\,(n_1+1)')\dots \\ (n\, k_{n_2})(k_{n_2}'\, n').$$

Here $n = n_1 + n_2$. *Furthermore,* $k_j = (\ell+1)'$ *for some* j.

(b) *If* $\mathbf{G} = SO_{2r}$, *and we write* $w = w_1 w_2$ *as in Lemma 2.2, then* $w_2 = 1$ *or* $w_2 = (d\, d')$, *for some* $\ell + 2 \le d \le r$. *Furthermore* w_1 *is of the form*

$$w_1 = (1\, r_0)(2\, r_0 - 1)\dots(n_1\, n_1')(n_1+1\, k_1)(k_1'\,(n_1+1)')\dots \\ (n\, k_{n_2})(k_{n_2}'\, n'),$$

with $n = n_1 + n_2$, *and the integers* k_j *are as in part (a). Moreover,* $k_j = (\ell+1)'$ *for some* j.

Proof. First note that if $\mathbf{G} = SO_{2r}$, and $w = w_1 w_2$, then Lemma 2.16 and the remark following Lemma 2.7 imply that w_2 is not of the form $(i\, j'\, i'\, j)$, for some $1 \le i \le n < j \le r$. Moreover, since $w^{-1}\alpha_\ell \in \Sigma_{\mathbf{P}}^+$, Lemmas 2.16 and 2.11 imply that if $w_2 = (d\, d')$, then $\ell + 2 \le d \le r$. If $\mathbf{G} \ne SO_{2r}$, let $w_2 = 1$.

By Lemma 2.14(a), $w(i) \ne i$ for all $i \le n$. By Lemma 2.11, Corollary 2.15, and Lemma 2.16, $w(n) = k$ or k', for some $\ell + 1 \le k \le r$. Let n_1 be the largest nonnegative integer for which $n_1 < n$ and $w(n_1) = n_1'$. If $n_1 > 0$, then by Lemma 2.14(b) $w = (1\, r_0)(2\, r_0 - 1)\dots(n_1\, n_1')w_2 w'$, where $w'(i) = i$ for all $i \le n_1$, and w_2 and w' are disjoint. Now $w'(i) \ne i$ and $w'(i) \ne i'$ for $n_1 + 1 \le i \le n$, and therefore $n + 1 \le w'(i) \le n' - 1$. However, by Lemma 2.16, $\ell + 1 \le w'(i) \le (\ell+1)'$. Thus,

$$w' = (n_1+1\, k_1)(k_1'\, n_1' - 1)\dots(n\, k_{n_2})(k_{n_2}'\, n'),$$

as claimed. Finally, Lemma 2.11 implies $w(\ell+1) \ne \ell+1$, and so we must have $(w')^{-1}(\ell+1) = w^{-1}(\ell+1) = j'$, for some $j \le n$. $\square$

We now finish the proof of Proposition 2.4. If $w = w_0$, then $n(\mathbf{x})w_0 = w_0 n(\mathbf{x})$, and since $n(\mathbf{x}) \in P$, we have $R_\chi w_0 P = R_\chi n(\mathbf{x}) w_0 P$. If $w\Sigma_{\mathbf{P}}^+ \cap X = \{\alpha_\ell\}$, or $w\Sigma_{\mathbf{P}}^+ \cap X = \emptyset$, then Lemma 2.16 and equation (2.8) show that $w(e_n + e_\ell) = e_\ell \pm e_k$, for some k, with $\ell + 2 \le k \le r$. Let $\alpha = w(e_n + e_\ell)$, and denote $I + tX_{e_n + e_\ell}$ by p. As before, choose $\mathbf{x}_0$ so that $R_\chi n(\mathbf{x}_0) wP = R_\chi n(\mathbf{x}) wP$, and such that $\mathbf{x}_0$ has a non-zero entry y with $\omega_\chi(n(\mathbf{x}) wpw^{-1} n(\mathbf{x})^{-1}) = \psi_\alpha(yt)$. Note that $\pi(p) = 1$. Choosing t for which $\psi_\alpha(yt) \ne 1$, we see that $R_\chi n(\mathbf{x}) wP$ cannot support a V-distribution of the desired form. $\square$

From the argument above, it is apparent that $\mathrm{Ind}_P^G(\pi)$ cannot have a Bessel model of rank less than $\mathcal{B}(\tau)$. Hence we obtain the following Corollary.

Corollary 2.18. *Let the notation be as in Theorem 2.1. Suppose that the* ω_χ*-Bessel model for* τ *is of rank* $\mathcal{B}(\tau)$. *Then the* $\omega_\chi^{w_0}$*-Bessel model for* $\mathrm{Ind}_P^G(\pi)$ *is also minimal, and of rank* $\mathcal{B}(\tau)$.

The proof of Theorem 2.1 also gives the following result.

Corollary 2.19. *For any σ, τ, ℓ, χ, and ω, the support of the twisted Jacquet functor $\pi_{U_\ell,\chi}$ is a finite number of double cosets.* $\square$

§3 Holomorphicity and local coefficients.

In this section we prove the holomorphicity of the Bessel functional and the existence of a local coefficient. To do so, we first adapt the argument used by Banks [**Ban**] to prove the holomorphicity of Whittaker functions for metaplectic covers of GL_n. Banks's result is an extension of Bernstein's Theorem, which establishes the meromorphicity under uniqueness and regularity hypotheses. We show that the desired regularity holds in the case of Bessel functionals. We then use an argument similar to Harish-Chandra's and to Shahidi's in the generic case to establish the existence of the local coefficient under certain conditions (Theorem 3.8). Corollary 3.9 shows that the local coefficient factors in a manner analogous to the generic case.

Let $\mathbf{G}$ be as in Section 1. We use the conventions found in [**Cas**, §1,**Shaa**] for subsets of simple roots, Weyl groups, and arbitrary parabolic subgroups. Suppose that Δ is the collection of simple roots corresponding to our choice of Borel subgroup. Let $\theta \subset \Delta$ be a collection of simple roots and set $\mathbf{P} = \mathbf{P}_\theta$. Then $\mathbf{P}$ has Levi decomposition $\mathbf{P} = \mathbf{M}_\theta \mathbf{N}_\theta$, with

$$\mathbf{M} = \mathbf{M}_\theta \simeq GL_{n_1} \times \cdots \times GL_{n_k} \times \mathbf{G}(m),$$

for some n_i, m such that $r = n_1 + \cdots + n_k + m$. We abbreviate this by writing $\mathbf{M} \simeq \mathbf{G}_1 \times \mathbf{G}(m)$. We also write $\mathbf{N} = \mathbf{N}_\theta$.

Let $\mathbf{A} = \mathbf{A}_\theta$ be the split component of $\mathbf{M}$. Denote by $\mathfrak{a}^*_{\mathbb{C}} = (\mathfrak{a}_\theta)^*_{\mathbb{C}}$ the complexified dual of the real Lie algebra of $\mathbf{A}$, q_F the residual characteristic of F, and denote by H_P the Harish-Chandra homomorphism [**Har**,**Shaa**]. Suppose that $\sigma \in \mathcal{E}(G_1)$ and $\tau \in \mathcal{E}(G(m))$, and let $\pi = \sigma \otimes \tau$. For $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, let $I(\nu, \pi, \theta)$ denote the induced representation

$$\mathrm{Ind}_P^G(\pi \otimes q_F^{<\nu, H_P()>})$$

and let $V(\nu, \pi, \theta)$ denote the space of associated functions. We also use Π_ν to denote the representation $I(\nu, \pi, \theta)$.

Assume that σ is generic and that τ has an $\omega_{\chi'}$–Bessel model which is minimal and of rank ℓ_0. Let χ be the character of U_ℓ whose restriction to $U_\ell \cap G(m)$ is χ' and whose restriction to $G_1(F) \cap U_\ell$ is a ψ–generic character χ_1. We will construct a non-zero functional $\Lambda_\chi(\nu, \pi, \theta)$ on $X_\nu = I(\nu, \pi, \theta) \otimes \tilde{V}_\omega$ so that, for a certain character δ of M_χ,

$$\Lambda_\chi(\nu, \pi, \theta)(\Pi_\nu(mu)(f_\nu \otimes \tilde{v})) = \delta(m)\chi(u)^{-1}\Lambda_\chi(\nu, \pi, \theta)(f_\nu \otimes \tilde{\omega}(m^{-1})\tilde{v}),$$

for all choices of $f_\nu \otimes \tilde{v} \in X_\nu$ and $mu \in R_\chi$. Then we will show in Theorem 3.6 that the function $\nu \mapsto \Lambda(\nu, \pi, \theta)(x_\nu)$ is holomorphic, for a holomorphic section $\nu \mapsto x_\nu$.

Let $K = G(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers in F. Then K is a good maximal compact subgroup of G [**Cas**]. Let K_m be the corresponding m-th principal congruence subgroup. Then each K_m is normal in K. Let Γ_m be a complete set of coset representatives for $P \cap K \backslash K / K_m$. Note that Γ_m is of finite cardinality. Let

$$Y = \left\{ f \in C^\infty(K, V_\pi) \,\middle|\, f(pk) = (\pi)\,(p) f(k),\, \forall p \in P \cap K,\, k \in K \right\}.$$

Then $F \mapsto F|_K$ is a K-isomorphism from $V(\nu, \pi, \theta)$ to Y, by the Iwasawa decomposition of G. We will define a certain functional on Y, and use this realization to define an associated functional on X_ν. Let

$$Y_m = \{f \in Y | f(kk_1) = f(k), \forall k \in K, k_1 \in K_m\}.$$

Thus, Y_m is the set of K_m–fixed vectors of Y under the action of K. Furthermore, the Iwasawa decomposition allows us to realize $I(\nu, \pi, \theta)$ on Y for each ν. Denote by $V_{\pi,m}$ the subspace of V_π consisting of $P \cap K_m$–fixed vectors. Since π is admissible, $V_{\pi,m}$ is finite dimensional.

The next three results are standard. We include the proof of the first two for completeness. The third is a straightforward consequence of the Iwasawa decomposition.

Lemma 3.1. *Y_m has a basis $\{f_j\}$ which satisfies the following properties:*

(1) *If $\gamma \in \Gamma_m$, then the non-zero vectors among $\{f_j(\gamma)\}$ are a basis for $V_{\pi,m}$.*
(2) *If f_j is fixed, then $f_j(\gamma) \neq 0$ for some $\gamma \in \Gamma_m$.*

Proof. Suppose that $f \in Y_m$. Then $f(pkk_1) = \pi(p)f(k)$, for all $p \in P \cap K$, $k \in K$, and $k_1 \in K_m$. Thus, f is completely determined by its values on Γ_m. Fix $\gamma \in \Gamma_m$, and let $p \in P \cap K_m$. Since $\gamma^{-1}K_m\gamma = K_m$, we have $\gamma^{-1}p\gamma \in K_m$. Therefore,

$$f(\gamma) = f(\gamma\gamma^{-1}p\gamma) = f(p\gamma) = \pi(p)f(\gamma).$$

This says that $f(\gamma)$ is an element of $V_{\pi,m}$. Fix a basis $\{v_{m,i}\}$ of $V_{\pi,m}$. Let $f_{\gamma,i} : K \longrightarrow V_{\pi,m}$ be given by

$$f_{\gamma,i}(k) = \begin{cases} \pi(p)v_{m,i} & \text{if } k = p\gamma k_1, \text{ for some } p \in P \cap K, k_1 \in K_m, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is immediate that $f_{\gamma,i}$ is a well-defined element of Y_m. We claim that $\{f_{\gamma,i}\}$ is a basis for Y_m.

Suppose $f \in Y_m$. If $\gamma' \in \Gamma_m$, $p \in P \cap K$, and $k \in K_m$, then

$$f(p\gamma' k) = \pi(p)f(\gamma').$$

Since $f(\gamma') \in V_{\pi,m}$,

$$f(\gamma') = \sum_i c_{\gamma',i}v_{m,i}.$$

This implies that

$$f(p\gamma' k) = \sum_i c_{\gamma',i}\pi(p)v_{m,i} = \sum_i c_{\gamma',i}f_{\gamma',i}(p\gamma' k).$$

Now, taking the collection $\{c_{\gamma,i}\}$ for all $\gamma \in \Gamma_m$, and noting that $f_{\gamma,i}(p\gamma' k) = 0$ for $\gamma \neq \gamma'$,

$$f(p\gamma' k) = \sum_{\gamma,i} c_{\gamma,i}f_{\gamma,i}(p\gamma' k),$$

which says that $\{f_{\gamma,i}\}$ spans Y_m.

On the other hand, suppose that $\sum_{\gamma,i} c_{\gamma,i} f_{\gamma,i} = 0$. Then, for any $\gamma' \in \Gamma_m$, we have

$$\sum_{\gamma,i} c_{\gamma,i} f_{\gamma,i}(\gamma') = 0,$$

which implies that

$$\sum_i c_{\gamma',i} f_{\gamma',i}(\gamma') = \sum_i c_{\gamma',i} v_{m,i} = 0.$$

But, since the $v_{m,i}$ are linearly independent, $c_{\gamma',i} = 0$, for each γ' and i. Thus, $\{f_{\gamma,i}\}$ are also linearly independent. The collection $f_{\gamma,i}$ clearly has properties (1) and (2). □

Denote by X the space $Y \otimes \tilde{V}_\omega$. For each $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ let $X_\nu = V(\nu, \pi, \theta) \otimes \tilde{V}_\omega$. For $f \in Y$ denote by f_ν the unique element of $V(\nu, \pi, \theta)$ satisfying $f_\nu|_K = f$. Then $\{f_\nu \otimes \tilde{v} | f \in Y, \tilde{v} \in \tilde{V}_\omega\}$ spans X_ν. Recall that Π_ν can be realized on Y via $\Pi_\nu(g)f = [\Pi_\nu(g)f_\nu]\,|_K$. This gives the context in which we discuss the holomorphicity of the map $\nu \mapsto \Pi_\nu(g)f_\nu$ for a fixed choice of g and f.

Lemma 3.2. *Fix $g \in G$, $f \in Y$ and $\tilde{v} \in \tilde{V}_\omega$. Then the function $\nu \mapsto \Pi_\nu(g)f \otimes \tilde{v}$ is a regular function from $\mathfrak{a}^*_{\mathbb{C}}$ to X.*

Proof. Choose m_0 so that $f \in Y_{m_0}$, and choose $m > m_0$ satisfying $g^{-1}K_m g \subset K_{m_0}$. Then $f \in Y_m$ and, for all $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ and $k \in K_m$,

$$\Pi_\nu(k)(\Pi_\nu(g)f_\nu)(x) = f_\nu(xgg^{-1}kg) = f_\nu(xg) = \Pi_\nu(g)f_\nu(x),$$

which says that $\Pi_\nu(g)f \in Y_m$ for all ν. Now, by Lemma 3.1,

$$\Pi_\nu(g)f = \sum_{\gamma,i} c_{\gamma,i}(\nu) f_{\gamma,i}$$

for a unique choice of $c_{\gamma,i}(\nu) \in \mathbb{C}$. It suffices to show that $c_{\gamma,i} : \mathfrak{a}^*_{\mathbb{C}} \longrightarrow \mathbb{C}$ is holomorphic. Fix $\gamma' \in \Gamma_m$. Then $\gamma' g = p\gamma'' k$, for some $p \in P$, $\gamma'' \in \Gamma_m$, and $k \in K_m$. Then

$$\begin{aligned}
\Pi_\nu(g)f(\gamma') &= q_F^{<\nu, H_P(p)>} \delta_P^{1/2}(p)\pi(p)f(\gamma'' k) \\
&= q_F^{<\nu, H_P(p)>} \delta_P^{1/2}(p)\pi(p) \sum_{\gamma,i} c_{\gamma,i}(\nu) f_{\gamma,i}(\gamma'') \\
&= q_F^{<\nu, H_P(p)>} \delta_P^{1/2}(p)\pi(p) \sum_i c_{\gamma'',i}(\nu) v_i.
\end{aligned}$$

Set $c'_{\gamma'',i}(\nu) = q_F^{<-\nu, H_p(p)>} \delta_P^{-1/2}(p) c_{\gamma'',i}(\nu)$. Then

$$\pi(p)f(\gamma'') = \sum_i c_{\gamma'',i}(\nu) v_i,$$

for all ν. Since the left hand side in the equation above is independent of ν and the v_i are linearly independent, $c'_{\gamma'',i}(\nu)$ is constant for each i. This implies that $c_{\gamma'',i}(\nu)$ is holomorphic. $\square$

From now on we need to distinguish between a Weyl group element $\tilde{w} \in W(\mathbf{G}, \mathbf{A})$, for some torus $\mathbf{A}$, and a representative $w \in N_G(A)$ for $\tilde{w}$. Let $\tilde{w}_\theta = \tilde{w}_{l,\Delta}\tilde{w}_{l,\theta}$, where $\tilde{w}_{l,\Delta}$ is the longest element of the Weyl group $W(\mathbf{G}, \mathbf{T})$, and $\tilde{w}_{l,\theta}$ is the longest element of $W(\mathbf{G}, \mathbf{A}_\theta)$. Fix a representative w_θ for $\tilde{w}_\theta$ with $w_\theta \in K$. Note that $\tilde{w}_\theta(\theta) \subset \Delta$. Now let $\mathbf{M}' = \mathbf{M}_{\tilde{w}_\theta(\theta)} = w_\theta \mathbf{M}_\theta w_\theta^{-1}$. Then $\mathbf{M}'$ is a standard Levi subgroup of $\mathbf{G}$. Let $\mathbf{N}'$ be the standard unipotent subgroup of $\mathbf{U}$ so that $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$ is a standard parabolic subgroup of $\mathbf{G}$. Since $\mathbf{M}' \simeq \mathbf{M}$, we have $\mathbf{U}_\ell \supset \mathbf{N}'$.

Lemma 3.3. *For each $m > 0$, we have $w_\theta^{-1}N' \cap Pw_\theta^{-1}K_m$ is compact.*

For $m \in M_\chi$, let $\delta(m) = (\delta_P^{-1/2}\delta_{P'})(m)$. Let $X_{R_\chi,\omega,\nu,\theta}$ be the subspace spanned by functions of the form

$$\Pi_\nu(mu)f \otimes \tilde{v} - \delta(m)\chi(u)f \otimes \tilde{\omega}(m^{-1})\tilde{v},$$

for $m \in M_\chi$, $u \in U_\ell$, $f \in Y$, and $\tilde{v} \in V_{\tilde{\omega}}$. Then a non-zero functional Λ on X is a $(\delta\omega_\chi)$-Bessel functional for Π_ν if and only if $\Lambda|_{X_{R_\chi,\omega,\nu,\theta}} \equiv 0$. By Theorem 2.1 the space of such functionals is one-dimensional. Thus $X/X_{R_\chi,\omega,\nu,\theta}$ is one-dimensional.

The construction of this functional will be obtained by taking a direct limit of functionals given by integrating over compact subsets of N'. We show that such a limit exists and is not identically zero. Moreover, we show that there is a function in X which is a complement to $X_{R_\chi,\omega,\nu,\theta}$ for all ν. This will give the regularity condition necessary to apply Bernstein's Theorem and to obtain the holomorphicity of the functional.

Now let us fix a Whittaker functional for σ and a Bessel functional for ω. (Actually, for notational convenience, we twist ω by $\delta_{R\chi}^{-1/2}$.) That is, suppose that $\lambda_\chi : V_\pi \otimes \tilde{V}_\omega \longrightarrow \mathbb{C}$ satisfies

$$\lambda_\chi((\sigma(u_1) \otimes \tau(mu_2))(v_1 \otimes v_2) \otimes \tilde{v}) = \chi_1(u_1)\chi'(u_2)\lambda_\chi(v_1 \otimes v_2 \otimes \tilde{\omega}(m^{-1})\tilde{v})$$

for all $u_1 \in U_\ell \cap G_1$, $u_2 \in U_\ell \cap G(m)$, and $m \in M_\chi$. Let Ω be a compact subgroup of N'. Define a functional on X by

$$\lambda^\Omega_{\pi,\nu,\theta}(f \otimes \tilde{v}) = \int_\Omega \lambda_\chi(\Pi_\nu(w_\theta^{-1}u)f_\nu(e) \otimes \tilde{v})\chi(u)^{-1}\,du. \tag{3.1}$$

This functional depends on the choice of the representative w_θ for $\tilde{w}_\theta$.

Since N' is exhausted by compact subgroups, the compact subgroups of N' form a directed set. The following Lemma was suggested to the authors by Prof. Steve Rallis.

Lemma 3.4. *For every $f \otimes \tilde{v} \in X$, the limit $\lim\limits_{\Omega} \lambda^\Omega_{\pi,\nu,\theta}(f \otimes \tilde{v})$ exists, where the limit is the direct limit taken over all compact subgroups of N'.*

Proof. Fix $f \otimes \tilde{v} \in X$. Let Ω_r be the subset of N' of elements with all entries of absolute value at most q_F^r. It suffices to show that there is an r sufficiently large such that if $\Omega \supset \Omega_r$ then $\lambda^{\Omega}_{\pi,\nu,\theta}(f \otimes \tilde{v}) = \lambda^{\Omega_r}_{\pi,\nu,\theta}(f \otimes \tilde{v})$. This follows since, for r sufficiently large, if $\gamma \in N' \setminus \Omega_r$ then

$$\int\limits_{\Omega_r \gamma} \lambda_\chi(\Pi_\nu(w_\theta^{-1}u)f_\nu(e) \otimes \tilde{v})\chi(u)^{-1}\,du = 0.$$

Indeed, there is a subgroup $K \subset \Omega_r$ such that $f_\nu(w_\theta^{-1}ku) = f_\nu(w_\theta^{-1}u)$ for $k \in K$, $u \in \Omega_r\gamma$, but such that χ is not identically 1 on K. For one may choose K such that the relevant minors of $w_\theta^{-1}u$ and $w_\theta^{-1}ku$ are highly congruent, and use the Iwasawa decomposition. $\square$

Define a functional on X by

$$\Lambda_\chi(\nu,\pi,\theta)(f \otimes \tilde{v}) = \lim_{\Omega} \lambda^{\Omega}_{\pi,\nu,\theta}(f \otimes \tilde{v}). \tag{3.2}$$

Again, this functional depends on the choice of w_θ.

Proposition 3.5. *Let $\Lambda_\chi(\nu,\pi,\theta)$ be defined as in (3.2), and extend Λ_χ to X_ν by the section $f \otimes \tilde{v} \mapsto f_\nu \otimes \tilde{v}$. Then $\Lambda_\chi(\nu,\pi,\theta)$ defines a non-zero $\delta\omega_\chi$–Bessel functional for Π_ν.* $\square$

Proof. Suppose that $u_1 \in U_\ell$. Since $U_\ell \subset P'$, we can write $u_1 = m_1 n_1$, with $m_1 \in M' \cap U_\ell$, and $n_1 \in N'$. Suppose first that $u_1 = n_1 \in N'$. Since N' is exhausted by compact subgroups, we can choose Ω_0 compact with $n_1 \in \Omega_0$. If $\Omega_0 \subset \Omega$, then

$$\begin{aligned}
\lambda^{\Omega}_{\pi,\nu,\theta}(\Pi_\nu(n_1)f \otimes \tilde{v}) &= \int\limits_{\Omega} \lambda_\chi(f_\nu(w_\theta^{-1}un_1) \otimes \tilde{v})\chi^{-1}(u)\,du \\
&= \int\limits_{\Omega} \lambda_\chi(f_\nu(w_\theta^{-1}u) \otimes \tilde{v})\chi^{-1}(un_1^{-1})\,du \\
&= \chi(n_1)\lambda^{\Omega}_{\pi,\nu,\theta}(f \otimes \tilde{v}).
\end{aligned}$$

Therefore,

$$\Lambda_\chi(\nu,\pi,\theta)(\Pi_\nu(n_1)f \otimes \tilde{v}) = \chi(n_1)\Lambda_\chi(\nu,\pi,\theta)(f \otimes \tilde{v}).$$

If $u = m_1 \in U_\ell \cap M'$, then since $\chi|_{G_1 \cap U_\ell}$ is ψ–generic, $\chi^{w_\theta}(m_1) = \chi(m_1)$. Thus,

$$\begin{aligned}
\lambda^{\Omega}_{\pi,\nu,\theta}(\Pi_\nu(m_1)f \otimes \tilde{v}) &= \int\limits_{\Omega} \lambda_\chi(f_\nu(w_\theta^{-1}um_1) \otimes \tilde{v})\chi^{-1}(u)\,du \\
&= \int\limits_{\Omega} \lambda_\chi(f_\nu(w_\theta^{-1}m_1w_\theta w_\theta^{-1}m_1^{-1}um_1) \otimes \tilde{v})\chi^{-1}(u)\,du \\
&= \int\limits_{\Omega} \lambda_\chi(\pi(w_\theta^{-1}m_1w_\theta)f_\nu(w_\theta^{-1}m_1^{-1}um_1) \otimes \tilde{v})\chi^{-1}(u)\,du \\
&= \chi(m_1) \int\limits_{m_1^{-1}\Omega m_1} \lambda_\chi(f_\nu(w_\theta^{-1}u) \otimes \tilde{v})\chi^{-1}(u)\,du \\
&= \chi(m_1)\lambda^{m_1^{-1}\Omega m_1}_{\pi,\nu,\theta}(f \otimes \tilde{v}).
\end{aligned}$$

Therefore,

$$\Lambda_\chi(\nu,\pi,\theta)(\Pi_\nu(m_1)f\otimes\tilde v)=\chi(m_1)\Lambda_\chi(\nu,\pi,\theta)(f\otimes\tilde v).$$

Similarly, if $m\in M_\chi\subset M'$, then

$$\lambda^{\Omega}_{\pi,\nu,\theta}(\Pi_\nu(m)f\otimes\tilde v)=\int\limits_{\Omega}\lambda_\chi(f_\nu(w_\theta^{-1}um)\otimes\tilde v)\chi^{-1}(u)\,du$$

$$=\int\limits_{\Omega}\lambda_\chi(\pi(w_\theta^{-1}mw_\theta)\delta_P^{1/2}(w_\theta^{-1}mw_\theta)f_\nu(w_\theta^{-1}m^{-1}um)\otimes\tilde v)\chi^{-1}(u)\,du$$

$$=\delta_P^{1/2}(w_\theta^{-1}mw_\theta)\int\limits_{m^{-1}\Omega m}\lambda_\chi(f_\nu(w_\theta^{-1}m^{-1}um)\otimes\tilde\omega(m^{-1})\tilde v)\chi^{-1}(u)\,du$$

$$=\delta_P^{-1/2}\delta_{P'}(m)\int\limits_{m^{-1}\Omega m}\lambda_\chi(f_\nu(w_\theta^{-1}u)\otimes\tilde\omega(m^{-1})\tilde v)\chi^{-1}(mum^{-1})\,du$$

$$=\delta(m)\lambda^{m^{-1}\Omega m}_{\pi,\nu,\theta}(f\otimes\tilde\omega(m^{-1})\tilde v).$$

Taking the limit on Ω on the right and left sides of the above equation completes the proof that $\Lambda_\chi(\nu,\pi,\theta)$ is a Bessel functional for Π_ν with respect to the representation $\delta(m)\omega_\chi$.

It remains to show that $\Lambda_\chi(\nu,\pi,\theta)$ is not identically zero. Let $\bar{\mathbf{P}}'$ be the parabolic opposite to $\mathbf{P}'$. Then $\bar{\mathbf{P}}'=w_\theta\mathbf{P}w_\theta^{-1}$. By Lemma 3.4, $\bar P'K_m$ is compact, and if $pw_\theta^{-1}k\in Pw_\theta^{-1}K_m\cap N'$, then in fact $p\in P\cap K_m$. Choose a $v\in V_\pi$ and $\tilde v\in\tilde V_\omega$ such that $\lambda_\chi(v\otimes\tilde v)\neq 0$. Choose $m>>0$ such that $v\in V_{\pi,m}$ and such that $\chi|N'\cap\bar P'K_m\equiv 1$. Consider the function in Y defined by

$$f_0(k)=\begin{cases}\pi(p)v & \text{if } k=pw_\theta^{-1}k_1, p\in P\cap K, k_1\in K_m\\ 0 & \text{otherwise.}\end{cases}\tag{3.3}$$

Then

$$\Lambda_\chi(\nu,\pi,\theta)(f_0\otimes\tilde v)=\int\limits_{N'\cap\bar P'K_m}\lambda_\chi(f_\nu(w_\theta^{-1}u)\otimes\tilde v)\,du$$

$$=\lambda_\chi(v\otimes\tilde v)\,|N'\cap\bar P'K_m|\neq 0.$$

Thus, $\Lambda_\chi(\nu,\pi,\theta)$ is non-zero, and f_0 is a complement to $X_{R_\chi,\omega,\nu,\theta}$ for all ν. □

Suppose $r=mu\in R_\chi$, $f\in Y$, and $\tilde v\in\tilde V_\omega$. Define an X–valued function on $\mathfrak{a}^*_{\mathbb{C}}$ by

$$x_{r,f,\tilde v,\theta}(\nu)=\Pi_\nu(r)(f)\otimes\tilde v-\delta(m)\chi(u)(f\otimes\tilde\omega(m^{-1})\tilde v).$$

Theorem 3.6. *The function $\nu\mapsto\Lambda_\chi(\nu,\pi,\theta)(x)$ is holomorphic for each $x\in X$.*

Proof. We will apply Banks's extension of Bernstein's Theorem. Let

$$\mathcal{R}=\{(r,f\otimes\tilde v)|r\in R_\chi,f\in Y,\tilde v\in V_{\tilde\omega}\}\cup\{*\}.$$

For $\alpha = (r, f \otimes \tilde{v}) \in \mathcal{R}$, we let $x_\alpha(\nu) = x_{r,f,\tilde{v},\theta}(\nu)$ in X and let $c_\alpha(\nu) = 0$. Fix m, $\tilde{v}$, v, and f_0 as in (3.3). For $\alpha = *$, we set $x_*(\nu) = f_0 \otimes \tilde{v}$ and $c_*(\nu) = |N' \cap \bar{P}' K_m \,|\, \lambda_\chi(v \otimes \tilde{v})$. Now for every $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, we consider the systems of equations in $X \times \mathbb{C}$ given by:

$$\Xi(\nu) = \{(x_\alpha(\nu), c_\alpha(\nu)) |\, \alpha \in \mathcal{R}\}\,.$$

By Lemma 3.2, the function $\nu \mapsto x_\alpha(\nu)$ is holomorphic for each α of the form $(r, f \otimes \tilde{v})$. For $\alpha = *$, the function $x_\alpha(\nu) = f_0 \otimes \tilde{v}$ is constant on $\mathfrak{a}^*_{\mathbb{C}}$. Note that each c_α is constant, hence holomorphic as well.

Now, for each ν the functional $\Lambda_\chi(\nu, \pi, \theta)$ is a solution to the system $\Xi(\nu)$. Moreover, such a solution is unique by the results of Section 2. Thus, Banks's extension of Bernstein's theorem [**Ban**] implies that $\nu \mapsto \Lambda(\nu, \pi, \theta)(f \otimes \tilde{v})$ is holomorphic for all choices of f and $\tilde{v}$. $\square$

We turn to the question of local coefficients. Let $\tilde{w} \in W$, and fix a representative w for $\tilde{w}$ with $w \in K$. We recall that the intertwining operator

$$A(\nu, \pi, w) : V(\nu, \pi, \theta) \to V(\tilde{w}(\nu), \tilde{w}\pi, \tilde{w}(\theta))$$

is defined for $\nu >> 0$ by

$$A(\nu, \pi, w) f(g) = \int_{N_{\tilde{w}}} f(w^{-1} n g)\, dn, \tag{3.4}$$

where $\mathbf{N}_{\tilde{w}} = \mathbf{U} \cap w \bar{\mathbf{N}} w^{-1}$, and $\bar{\mathbf{N}}$ is the unipotent radical opposite to $\mathbf{N}$. Then $A(\nu, \pi, w)$ is defined on all of $\mathfrak{a}^*_{\mathbb{C}}$ by analytic continuation. Note that the intertwining operator depends on the choice of w representing $\tilde{w}$.

We also recall the Langlands decomposition of the intertwining operator, described in Lemma 2.1.2 of [**Shaa**]. For the convenience of the reader, let us restate this here. For two associate subsets θ and θ' of Δ, we let

$$W(\theta, \theta') = \{\tilde{w} \in W \,|\, \tilde{w}\theta = \theta'\}\,.$$

Lemma 3.7 (Langlands (see [Shaa, Lemma 2.1.2])). *Suppose that $\theta, \theta' \subset \Delta$ are associate. Let $\tilde{w} \in W(\theta, \theta')$. Then there exists a family $\theta_1, \theta_2, \ldots, \theta_n \subset \Delta$ so that*

1. *$\theta_1 = \theta$ and $\theta_n = \theta'$;*
2. *For each $1 \leq i \leq n$ there is a root $\alpha_i \in \Delta \setminus \theta_i$ so that θ_{i+1} is the conjugate of θ_i in $\Delta_i = \theta_i \cup \{\alpha_i\}$;*
3. *For each $1 \leq i \leq n-1$, we let $\tilde{w}_i = \tilde{w}_{\ell,\Delta_i} \tilde{w}_{\ell,\theta_i}$ in $W(\theta_i, \theta_{i+1})$. Then $\tilde{w} = \tilde{w}_{n-1} \ldots \tilde{w}_1$;*
4. *Set $\tilde{w}'_1 = \tilde{w}$, and $\tilde{w}'_{i+1} = \tilde{w}'_i \tilde{w}_i^{-1}$ for $1 \leq i \leq n-1$. Then $\tilde{w}'_n = 1$ and*

$$\mathfrak{n}_{\tilde{w}'_i} = \mathfrak{n}_{\tilde{w}_i} \oplus \mathrm{Ad}(w_i^{-1}) \mathfrak{n}_{\tilde{w}'_{i+1}}\,.$$

Here $\mathfrak{n}$ is the Lie algebra of $\mathbf{N}$.

Let $\theta_* \subset \theta$ and let ρ be an irreducible supercuspidal representation of M_{θ_*}. If ρ is generic, then Rodier's Theorem implies that there is a unique constituent π of $\mathrm{Ind}_{P_{\theta_*}}^{M}(\rho)$ which is generic with compatible character. For this constituent, Shahidi proved that there is a complex number $C_\chi(\nu, \pi, \theta, w)$ which satisfies

$$\Lambda_\chi(\nu, \pi, \theta) = C_\chi(\nu, \pi, \theta, w)\Lambda_\chi(\tilde{w}\nu, \tilde{w}\pi, \tilde{w}\theta)A(\nu, \pi, w),$$

where Λ_χ is the Whittaker functional. Moreover, the function $\nu \mapsto C_\chi(\nu, \pi, \theta, w)$ is a meromorphic function on $(\mathfrak{a}_\theta)^*_{\mathbb{C}}$. The value of the local coefficient depends on the choice of representative w for $\tilde{w}$.

Now suppose that ρ is any irreducible supercuspidal which has a minimal Bessel model of a particular type. Then we prove a similar result for the constituent π of $\mathrm{Ind}_{P_{\theta_*}}^{M}(\rho)$ which has a Bessel model of compatible type; such a constituent exists and is unique by Theorem 2.1.

Theorem 3.8. *Let θ and θ' be associate subsets of Δ. Let $\theta_* \subset \theta$ and let ρ be an irreducible supercuspidal representation of M_{θ_*}. Suppose that ρ has an ω_χ–Bessel model which is minimal. Let π be the constituent of $\mathrm{Ind}_{P_{\theta_*}}^{M}(\rho)$ such that π has an $\omega_\chi^{w_0}$-Bessel model, as in Theorem 2.1. For each $\tilde{w} \in W(\theta, \theta')$ fix a representative w for $\tilde{w}$. Then there is a complex number $C_\chi(\nu, \pi, \theta, w)$ so that*

$$\Lambda_\chi(\nu, \pi, \theta) = C_\chi(\nu, \pi, \theta, w)\Lambda_\chi(\tilde{w}\nu, \tilde{w}\pi, \tilde{w}\theta)A(\nu, \pi, w). \tag{3.5}$$

Moreover, the function $\nu \mapsto C_\chi(\nu, \pi, \theta, w)$ is meromorphic on $\mathfrak{a}^*_{\mathbb{C}}$, and depends only on the class of π and the choice of w.

Proof. We first show how to define $C_\chi(\tilde{\nu}, \pi, \theta_*, w)$ for $\tilde{\nu} \in (\mathfrak{a}_{\theta_*})^*_{\mathbb{C}}$. By [**Sil**, Theorem 5.4.3.7] the representation $I(\tilde{\nu}, \rho, \theta_*)$ is irreducible unless the Plancherel measure $\mu(\tilde{\nu}, \rho) = 0$ and $(\tilde{\nu}, \rho)$ is fixed by a nontrivial element of the Weyl group W_{θ_*} (i.e., is singular). Thus, on an open dense subset of $(\mathfrak{a}_{\theta_*})^*_{\mathbb{C}}$ the representation $I(\tilde{\nu}, \rho, \theta_*)$ is irreducible, and so $\Lambda_\chi(\tilde{w}\tilde{\nu}, \tilde{w}\rho, \tilde{w}\theta_*)A(\tilde{\nu}, \rho, w)$ defines a non-zero Bessel functional on $V(\tilde{\nu}, \rho, \theta_*) \otimes \tilde{V}_\omega$. By the uniqueness of such a functional (Theorem/Conjecture 1.4), we get the existence of $C(\tilde{\nu}, \rho, \theta_*, w)$ satisfying

$$\Lambda_\chi(\tilde{\nu}, \rho, \theta_*) = C_\chi(\tilde{\nu}, \rho, \theta_*, w)\Lambda_\chi(\tilde{w}\tilde{\nu}, \tilde{w}\rho, \tilde{w}\theta_*)A(\tilde{\nu}, \rho, w)$$

on the open dense subset. Moreover, it is holomorphic there since both

$$\Lambda_\chi(\tilde{w}\tilde{\nu}, \tilde{w}\rho, \tilde{w}\theta_*)$$

and $A(\tilde{\nu}, \rho, w)$ are holomorphic there. Thus, $C(\tilde{\nu}, \rho, \theta_*, w)$ extends to a meromorphic function on $(\mathfrak{a}_{\theta_*})^*_{\mathbb{C}}$. Now, write $\tilde{w} = \tilde{w}_{n-1} \ldots \tilde{w}_1$ as in Lemma 3.7. Since $C_\chi(\tilde{\nu}, \rho, \theta_*, w)$ is now defined, it admits a factorization compatible with the decomposition of the intertwining operators given in Lemma 3.7. (See corollary 3.9.) This implies that on an open dense subset of $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$, the local coefficient $C_\chi(\nu, \rho, \theta_*, w)$ may be defined by the equation $C_\chi(\nu, \rho, \theta_*, w) = C_\chi(\tilde{\nu}, \rho, \theta_*, w)$, where $\tilde{\nu}$ is the restriction of ν to $(\mathfrak{a}_{\theta_*})_{\mathbb{C}}$. Suppose that, for some ν in this open dense subset, $\Lambda_\chi(\tilde{w}\nu, \tilde{w}\pi, \tilde{w}\theta)A(\nu, \pi, w)$ was the zero functional. Then, by inducing in stages and using the discussion preceding this Theorem, we would conclude that $\Lambda_\chi(\tilde{w}\tilde{\nu}, \tilde{w}\rho, \tilde{w}\theta_*)A(\tilde{\nu}, \rho, w)$ is also zero. However, since $C_\chi(\tilde{\nu}, \rho, \theta_*, w)$ is defined there, this would be a contradiction. Thus we may define $C(\nu, \pi, \theta, w)$ by the relation (3.5) on this open dense subset, and we have $C(\nu, \pi, \theta, w) = C(\tilde{\nu}, \rho, \theta_*, w)$. Since $A(\nu, \pi, w)$ has a meromorphic continuation to $(\mathfrak{a}_\theta)^*_\mathbb{C}$, and $\Lambda_\chi(\tilde{w}\nu, \tilde{w}\pi, \tilde{w}\theta)$ is holomorphic on $(\mathfrak{a}_\theta)^*_\mathbb{C}$, the function $\nu \mapsto C_\chi(\nu, \pi, \theta, w)$ must have a meromorphic continuation. $\square$

Corollary 3.9. *Let the notation be as in Lemma 3.7 and Theorem 3.8. Let $\pi_1 = \pi$, and $\nu_1 = \nu$. For each i, $2 \leq i \leq n-1$, set $\pi_i = \tilde{w}_i\pi_{i-1}, \nu_i = \tilde{w}\nu_{i-1}$. Then the local coefficient factors as*

$$C_\chi(\nu, \pi, \theta, w) = \prod_{i=1}^{n-1} C_\chi(\pi_i, \theta_i, w_i).$$

Proof. Let $f_1 = f \in V(\nu, \pi, \theta)$ and for $2 \leq i \leq n-1$, let $f_i = A(\nu_{i-1}, \pi_{i-1}, w_{i-1})f_{i-1}$. Then

$$\begin{aligned}\Lambda_\chi(\nu_i, \pi_i, \theta_i)f_i =& C_\chi(\nu_i, \pi_i, \theta_i, w_i)\Lambda_\chi(\nu_{i+1}, \pi_{i+1}, \theta_{i+1}) \\ &\cdot A(\nu_i, \pi_i, w_i)f_i,\end{aligned}$$

for each $1 \leq i \leq n-1$. The corollary now follows immediately from Lemma 3.7 and iteration of the above equality. $\square$

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