HOMOGENEOUS SPECIAL GEOMETRY

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Abstract. Motivated by the physical concept of special geometry two mathematical constructions are studied, which relate real hypersurfaces to tube domains and complex Lagrangean cones respectively. Methods are developed for the classification of homogeneous Riemannian hypersurfaces and for the classification of linear transitive reductive algebraic group actions on pseudo Riemannian hypersurfaces. The theory is applied to the case of cubic hypersurfaces, which is the case most relevant to special geometry, obtaining the solution of the two classification problems and the description of the corresponding homogeneous special Kähler manifolds.

Introduction

The concept of special geometry was developed in the context of string theory and supergravity. It is a leading principle established by Witten et al (s. e.g. [B-W]) that in order to understand and construct Lagrangeans $\mathcal{L}$ for supersymmetric field theories one has to interpret the invariance of the action $\int \mathcal{L} d\mu$ under the gauge super group as *special* geometric property of a Riemannian metric; namely the metric g defined by the coefficients g_{ij} of the kinetic terms of the scalar fields $\phi^1, \ldots, \phi^n$ in

$$\mathcal{L} = \sum_{\mu=1}^{D} \sum_{i,j=1}^{n} g_{ij}(\phi^1, \ldots, \phi^n) \partial_\mu \phi^i \partial^\mu \phi^j + \cdots$$

In a more narrow sense (cf. e.g. [dW-VP0], [dW-VP] and [St]) *special Kähler* geometry is the special geometry associated to $N = 2$ supergravity-Yang-Mills theories in $D = 4$ spacetime dimensions. It has attracted a lot of attention due to its relation to Mirror Symmetry, s. [Y]. E.g. the Weil-Petersson metric of the moduli space of Calabi-Yau 3-folds is special Kähler, s. Example 1 on p. 8. Similarly, *special real* geometry [dW-VP] is defined as the geometry associated to $N = 2$ supergravity-Yang-Mills theories in dimension $D = 5$. Dimensional reduction of supergravity theories [Cr] from dimension $D = 5$ to $D = 4$ induces a correspondence called the r-map [dW-VP] relating special real to special Kähler geometry.

In the first part of the paper, which is a purely mathematical presentation of special geometry, we discuss two natural generalizations of the r-map.

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The basic mathematical object considered is an affine hypersurface $\mathcal{H} \subset \mathbb{R}^n$ defined by a homogeneous polynomial h, which induces on $\mathcal{H}$ a canonical pseudo Riemannian metric, s. 1.1. By the first generalization of the r-map we canonically associate to $\mathcal{H}$ a pseudo Kähler tube domain U, s. 1.2. The second generalization of the r-map associates to $\mathcal{H}$ a complex Lagrangean cone $\mathcal{C} \subset T^*\mathbb{C}^{n+1}$, s. 1.3. If $\mathcal{H}$ is a *cubic* hypersurface, then the pseudo Kähler tube domain U is isomorphic to the projectivized cone $P(\mathcal{C})$ endowed with a canonical pseudo Kähler metric induced by the embedding $\mathcal{C} \subset T^*\mathbb{C}^{n+1}$, s. Theorem 1.12. If $\mathcal{H}$ is moreover a Riemannian cubic hypersurface, then $U \cong P(\mathcal{C})$ is precisely the special Kähler manifold associated to the special real manifold $\mathcal{H}$ via the r-map.

Based on the first construction, we develop a method for the classification of homogeneous Riemannian hypersurfaces of arbitrary degree $d \geq 2$, s. 2.1. It is easily applied to the cases $2 \leq d \leq 3$. As a result, s. Theorem 2.8, the classification of homogeneous cubic Riemannian hypersurfaces and their associated special Kähler (and also quaternionic Kähler) manifolds is obtained, providing a short and conceptual proof for the results of [dW-VP]. Our method uses the basic theory of normal J-algebras (s. Theorem 2.3), avoiding the local cubic tensor calculus of [dW-VP]. A series of homogeneous Riemannian hypersurfaces of arbitrary degree $d \geq 2$ is constructed at the end of 2.1.

Finally, a method for the classification of transitive linear reductive algebraic group actions on pseudo Riemannian hypersurfaces of degree $d \geq 2$ is presented and carried through for $2 \leq d \leq 3$, s. 2.2. We remark that by Corollary 1.3 any transitive linear action of a group $G \subset GL(n, \mathbb{R})$ on a pseudo Riemannian hypersurface $\mathcal{H} \subset \mathbb{R}^n$ induces a transitive affine action of the group $\hat{G} = \mathbb{R}^n \rtimes (\mathbb{R}^+ \times G) \subset Aff(\mathbb{C}^n)$ on the corresponding pseudo Kähler tube domain $U \subset \mathbb{C}^n$ by holomorphic isometries. If $\mathcal{H}$ is cubic, U is special pseudo Kähler by Theorem 1.12 and we obtain a transitive affine action on the special pseudo Kähler manifold U by holomorphic isometries.

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Contents

1. Basic constructions of special geometry

1.1. **The canonical metric of a hypersurface.** Let h be a real or complex polynomial in n variables $x^1, \ldots, x^n$ which is homogeneous of degree d. Consider the level set

$$\mathcal{H}_c(h) = \{X = (x^1, \ldots, x^n) \in \mathbb{K}^n | h(X) = c\}$$

where $c \in \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $c \neq 0$, we can write $\mathcal{H}_c(h) = \mathcal{H}_1(h/c)$.

Definition 1.1. *A* **hypersurface of degree d** *is a smooth, open and connected subset $\mathcal{H} \subset \mathcal{H}_1(h)$. The* **basic polynomial** *h is homogeneous of degree d and not a power of a polynomial of lower degree.*

Remark 1: Similar constructions to the ones which will be discussed in the following can be presented for *projective* hypersurfaces $\mathcal{H} \subset \{\mathbb{K}X \in P(\mathbb{K}^n) | h(X) = 0\}$.

Now we define **the canonical metric** $g = g(h)$ of a hypersurface $\mathcal{H} \subset \mathcal{H}_1(h)$ of degree $d \geq 2$. Let $H \in \vee^d(\mathbb{K}^n)^*$ be the symmetric d-linear form obtained by polarizing h, i.e. $h(X) = H(X, \ldots, X)$. Then we put

$$g_{X_0}(X, X) := -(d-1)H(X_0, \ldots, X_0, X, X)\,,$$

$$X \in T_{X_0}\mathcal{H} = \{X \in \mathbb{K}^n | H(X_0, \ldots X_0, X) = 0\}\,.$$

Formula:

$$g = -\frac{1}{d}\partial^2 h = -\frac{1}{d}\partial^2 \log h \quad \text{on} \quad \mathcal{H}\,. \tag{1}$$

Remark 2: By the preceding formula, we can define the canonical metric without assuming that h is a homogeneous polynomial. However, we will use the homogeneity of h in the next sections, therefore we have assumed it from the very beginning.

Definition 1.2. *A hypersurface $\mathcal{H}$ of degree d is called* **nondegenerate** *if its canonical metric is nondegenerate.*

In the real case ($\mathbb{K} = \mathbb{R}$), the canonical metric g defines on any nondegenerate hypersurface $\mathcal{H}$ a canonical structure of **pseudo Riemannian hypersurface**. In the complex case the canonical metric defines the structure of **complex Riemannian hypersurface** on any nondegenerate hypersurface.

Remark 3: **Special real manifolds** in the sense of de Wit and Van Proeyen [dW-VP] are precisely Riemannian cubic hypersurfaces.

1.2. **The pseudo Kählerian tube domain associated to a pseudo Riemannian hypersurface of degree d.** Let $\mathcal{H} \subset \mathcal{H}_1(h) \subset \mathbb{R}^n$ be a pseudo Riemannian hypersurface of degree d with canonical metric g of signature (k, l). We will construct a totally geodesic isometric embedding $\iota : (\mathcal{H}, g) \hookrightarrow (U, g^c)$ of $\mathcal{H}$ into a pseudo Kähler manifold (U, g^c) of complex dimension n and complex signature $(k+1, l)$. We consider the positive cone $\mathcal{V} := \mathbb{R}^+ \cdot \mathcal{H}$ over $\mathcal{H}$ and the **tube domain** $U = \mathbb{R}^n + i\mathcal{V} \subset \mathbb{C}^n$ with complex coordinates $Z = X + iY$. We define a **canonical** pseudo Kähler **metric** g^c on U by the Kähler potential

$$K(Z) = -\frac{4}{d} \log(h(Y)) \,.$$

Proposition 1.1. *The map $\iota : (\mathcal{H}, g) \hookrightarrow (U, g^c)$ given by $Y \mapsto iY$ is a totally geodesic isometric embedding of the pseudo Riemannian hypersurface $(\mathcal{H}, g)$ of signature (k, l) into the pseudo Kählerian tube domain (U, g^c) of complex signature $(k+1, l)$. The cone $i\mathcal{V} \subset U$ is totally geodesic and isometric to the Riemannian product $(\mathbb{R}, can) \times (\mathcal{H}, g)$ via*

$$\mathbb{R} \times \mathcal{H} \ni (t, Y) \mapsto ie^t Y \in i\mathcal{V} \,.$$

Moreover, (U, g^c) admits the following global isometries preserving the cone $i\mathcal{V}$:

(i) *scaling $Z \mapsto \lambda Z$ by $\lambda \in \mathbb{R}^+$,*
(ii) *translations $Z \mapsto Z + X_0$ by real vectors $X_0 \in \mathbb{R}^n$,*
(iii) *reflection $X + iY \mapsto -X + iY$ and*
(iv) *"inversion" $X + iY \mapsto X + i\frac{Y}{h(Y)^{2/d}}$ with respect to $i\mathcal{H}$.*

Proof: By definition we have

$$g^c_Z = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} K(Z) dz^i \otimes d\bar{z}^j =$$

$$-\frac{1}{d} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial y^i \partial y^j} \log h(Y)(dx^i \otimes dx^j + dy^i \otimes dy^j) \,,$$

where $z^j = x^j + iy^j$ are the standard coordinates on $\mathbb{C}^n \supset \mathbb{R}^n + i\mathcal{V} = U$, and $Z = X + iY = (x^1, \dots, x^n) + i(y^1, \dots, y^n)$. It is clear from this expression, that (ii) and (iii) define isometries of U and from the formula (1) it is obvious that $\iota : (\mathcal{H}, g) \to (i\mathcal{H}, g^c|i\mathcal{H})$ is an isometry. Using the formula

$$\partial^2 \log h = \frac{\partial^2 h}{h} - \frac{(\partial h)^2}{h^2} \tag{2}$$

and the homogeneity of h, it is easy to check that (i) defines an isometric $\mathbb{R}^+$ action on (U, g^c) preserving the cone $i\mathcal{V}$. The induced Killing vector field on the cone is just the radial vector field $R(iY) = iY$. It has unit length and is orthogonal to the hypersurface $i\mathcal{H}$. From this it follows that $i\mathcal{V}$ is a Riemannian product. In particular, the inclusions $i\mathcal{H} \subset i\mathcal{V} \subset U$ are totally geodesic, the cone $i\mathcal{V}$ being totally geodesic as fixed point set of the reflection. Finally, the fact that (iv) is an isometry of U follows from the fact that $iY \mapsto iY/h(Y)^{2/d}$ is an isometry of $i\mathcal{V}$. It corresponds to the isometry $(t, Y) \mapsto (-t, Y)$ of $\mathbb{R} \times \mathcal{H} \cong i\mathcal{V}$. □

Now we show that our construction behaves nicely with respect to group actions.

Proposition 1.2. *Let $G \subset GL(n, \mathbb{R})$ be a group preserving the pseudo Riemannian hypersurface $\mathcal{H} \subset \mathcal{H}_1(h) \subset \mathbb{R}^n$. Then G acts on $(\mathcal{H}, g)$ by isometries with respect to the canonical metric g. Moreover, this action on $i\mathcal{H} \subset U$ is uniquely extended to a (complex) linear action on $\mathbb{C}^n \supset U$ preserving U, namely by the inclusion $G \subset GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$. The extended action on U is isometric (and holomorphic) with respect to the canonical metric g^c on the tube domain U.*

Proof: Assume that $A \in GL(n, \mathbb{R})$ preserves $\mathcal{H} \subset \mathbb{R}^n$. This means that $A^*h = h$ on $\mathcal{H}$ and hence on the open set $\mathcal{V} = \mathbb{R}^+ \cdot \mathcal{H}$, implying $A^*h = h$. In particular, the transformation A preserves the canonical metric g. The action of $A \in GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$ on $\mathbb{C}^n$ is given by $X + iY \mapsto AX + iAY$ and it clearly preserves the tube domain U and its canonical metric g^c. □

From Propositions 1.1 and 1.2 we obtain the following corollary.

Corollary 1.3. *If the hypersurface $\mathcal{H}$ is homogeneous under a subgroup*

$$G \subset GL(n, \mathbb{R}) \subset GL(n, \mathbb{C}),$$

then the associated tube domain (U, g^c) is homogeneous as pseudo Kähler manifold under the subgroup

$$\hat{G} = \mathbb{R}^n \rtimes (\mathbb{R}^+ \times G) \subset Aff(\mathbb{C}^n) = \mathbb{C}^n \rtimes GL(n, \mathbb{C})\,.$$

1.3. **Special pseudo Kähler geometry and the r-map.** First we define special pseudo Kähler geometry in terms of canonical metrics on Lagrangean cones. Second we derive the coordinate expressions (well known in the physical literature) for these metrics in terms of a local holomorphic function homogeneous of degree 2. Then we define a straightforward generalization of the physical r-map. By this we associate a

Lagrangean cone $\mathcal{C}$ to every real hypersurface $\mathcal{H}$ of degree d. If $\mathcal{H}$ is a *cubic* pseudo Riemannian hypersurface, then the projectivized cone $P(\mathcal{C})$ with its special metric, s. Definition 1.5, is isometric to the tube domain (U, g^c) associated to $\mathcal{H}$. This shows, in particular, that (U, g^c) is a special pseudo Kähler manifold.

Consider the following **fundamental algebraic data**:

1) A complex symplectic vector space (V, ω)
2) A compatible real structure, i.e. a $\mathbb{C}$-antilinear involution $\tau : V \to V$ satisfying

$$\omega(\tau X, \tau Y) = \overline{\omega(X, Y)}\,, \quad \forall X, Y \in V\,.$$

Given these data we define a sesquilinear form γ on V by

$$\gamma(X, Y) := i\omega(X, \tau Y)\,.$$

Lemma 1.4. *The form γ is a Hermitian form of signature $(n+1, n+1)$, where $\dim_{\mathbb{C}} V = 2n + 2$.*

Proof: We first check that γ is Hermitian:

$$\overline{\gamma(X, Y)} = -i\,\overline{\omega(X, \tau Y)} = -i\omega(\tau X, Y) = i\omega(Y, \tau X) = \gamma(Y, X)\,.$$

Now we show that the Hermitian form γ has signature $(n+1, n+1)$. The restriction of the symplectic form ω to the fixed point set V^τ of the real structure $\tau : V \to V$ is a real symplectic form on the real vector space V^τ, hence we can choose a basis p_α, q_β, $\alpha, \beta = 1, \ldots, n+1$, of V^τ such that

$$\omega(p_\alpha, q_\beta) = \delta_{\alpha\beta}\,, \quad \omega(p_\alpha, p_\beta) = \omega(q_\alpha, q_\beta) = 0\,.$$

Then p_α, iq_β, $\alpha, \beta = 1, \ldots, n+1$, is a Witt basis for γ, i.e.

$$\gamma(p_\alpha, iq_\beta) = \delta_{\alpha\beta}\,, \quad \gamma(p_\alpha, p_\beta) = \omega(iq_\alpha, iq_\beta) = 0\,.$$

This shows that γ has signature $(n+1, n+1)$. $\square$

Remark that up to isomorphism we can assume that the fundamental data (V, ω, τ) are the following: $V = T^*\mathbb{C}^{n+1} = T^*\mathbb{R}^{n+1} + iT^*\mathbb{R}^{n+1}$, ω the complex bilinear extension of the standard symplectic form on $T^*\mathbb{R}^{n+1}$ and τ complex conjugation with respect to the real form $V^\tau = T^*\mathbb{R}^{n+1}$.

We recall that a complex vector subspace L of a complex symplectic vector space (V, ω) is called **Lagrangean** if it is maximally isotropic.

Definition 1.3. *Given fundamental algebraic data (V, ω, τ) a Lagrangean subspace $L \subset V$ is called* **nondegenerate** *if $\gamma|L$ is nondegenerate. A connected submanifold $\mathcal{C} \subset V$ is called (***nondegenerate***)* **Lagrangean cone** *if $\mathbb{R}^+ \cdot \mathcal{C} = \mathcal{C}$ and if $L = T_v\mathcal{C}$ is a (nondegenerate) Lagrangean subspace of V for all $v \in C$.*

Remark that a Lagrangean subspace $L \subset V$ is nondegenerate if and only if $L \cap \tau L = 0$. From Definition 1.3 it follows that a nondegenerate Lagrangean cone $\mathcal{C}$ is a pseudo Kählerian submanifold of the pseudo Kähler manifold (V, γ).

Definition 1.4. *The induced metric* $g^{\mathcal{C}} = \gamma|\mathcal{C}$ *is called the* **canonical metric** *of the Lagrangean cone* $\mathcal{C} \subset (V, \omega, \tau)$.

Now we define a canonical pseudo Kähler metric $g^{P(\mathcal{C})}$ on the projective image $P(\mathcal{C}) \subset P(V)$ of a nondegenerate Lagrangean cone $\mathcal{C}$. For this we assume that $\gamma(u,u) \neq 0$ for all $u \in \mathcal{C}$. If this additional condition is satisfied we shall say that the cone $\mathcal{C}$ is **properly** nondegenerate. Remark that a Riemannian cone $(\mathcal{C}, g^{\mathcal{C}})$ is automatically properly nondegenerate. Denote by $\pi : V \to P(V)$ the canonical projection. We define

$$g^{P(\mathcal{C})}_{\pi u}(d\pi v, d\pi v) = \frac{\gamma(v,v)}{\gamma(u,u)} - \left|\frac{\gamma(u,v)}{\gamma(u,u)}\right|^2$$

for $u \in \mathcal{C} \subset V$, $v \in T_u\mathcal{C} \subset V$.

Definition 1.5. *The Hermitian metric* $g^{P(\mathcal{C})}$ *is called the* **special metric** *of the projectivized Lagrangean cone* $P(\mathcal{C})$. *A pseudo Kähler manifold is called* **special** *if it is locally isometric to* $(P(\mathcal{C}), g^{P(\mathcal{C})})$ *for some Lagrangean cone* $\mathcal{C} \subset (V, \omega, \tau)$.

Proposition 1.5. *Assume that* $\mathcal{C}$ *is a properly nondegenerate Lagrangean cone. Then the special metric* $g^{P(\mathcal{C})}$ *on* $P(\mathcal{C})$ *is a pseudo Kähler metric with Kähler form*

$$Im(g^{P(\mathcal{C})}) = 2\pi c_1(\mathcal{U}, \gamma),$$

where $c_1(\mathcal{U}, \gamma)$ *is the Chern form of the universal bundle* $p : \mathcal{U} \to P(\mathcal{C})$ *with Hermitian metric induced by* γ. *If the canonical metric* $g^{\mathcal{C}}$ *has Riemannian signature, then the special metric* $g^{P(\mathcal{C})}$ *is a (Riemannian) Kähler metric.*

Proof: Let ζ be a holomorphic section of $\mathcal{U}$. Then

$$\begin{gathered} 2\pi c_1(\mathcal{U}, \gamma) = -i\partial\bar\partial \log \gamma(\zeta,\zeta) = \\ -i\frac{\partial\bar\partial\gamma(\zeta,\zeta)}{\gamma(\zeta,\zeta)} + i\frac{\partial\gamma(\zeta,\zeta)\wedge\bar\partial\gamma(\zeta,\zeta)}{\gamma(\zeta,\zeta)^2} = \\ \frac{Im(\gamma)}{\gamma(\zeta,\zeta)} + i\frac{\gamma(\cdot,\zeta)\wedge\gamma(\zeta,\cdot)}{\gamma(\zeta,\zeta)^2} = Im(g^{P(\mathcal{C})}_{p(\zeta)}) \end{gathered}$$

(here we have used the convention $a \wedge \bar b = (a \otimes \bar b - \bar b \otimes a)/2$). □

Now we describe Lagrangean cones $\mathcal{C}$ and the special metric $g^{P(\mathcal{C})}$ on the projectivization $P(\mathcal{C})$ by a basic function F, thereby relating our presentation to the usual construction of special Kähler geometry in the physical literature. We consider the standard model $V = T^*\mathbb{C}^{n+1}$. Let $F(Z)$, $Z = (z^0, \ldots, z^n) \in \mathbb{C}^{n+1}$, be a locally defined holomorphic function on $\mathbb{C}^{n+1}$ which is homogeneous of degree 2, i.e. $F(\lambda Z) = \lambda^2 F(Z)$, $\lambda \in \mathbb{C} - \{0\}$, where this equation makes sense. We call F a **basic function**. For example, we may take $F = p/q$ to be the quotient of two homogeneous polynomials p, q, $\deg p = \deg q + 2$. We are interested in the image $\mathcal{C}_F \subset T^*\mathbb{C}^{n+1}$ of the differential $dF : Z \mapsto dF|_Z$.

Proposition 1.6. *Let F be a basic function. Then the connected components of $\mathcal{C}_F - \{0\}$ are Lagrangean cones. Conversely, every Lagrangean cone which locally projects isomorphically onto $\mathbb{C}^{n+1}$ is locally of this form for some basic function F.*

Proof: It is a well known fact in symplectic geometry, that a Lagrangean submanifold of $T^*\mathbb{C}^{n+1}$ which locally projects isomorphically onto $\mathbb{C}^{n+1}$ is locally the image $\mathcal{C}_F$ of a differential dF. Now $\mathcal{C}_F$ is a cone if and only if dF is homogeneous of degree one, i.e. if and only if F is homogeneous of degree two. □

Now let $\mathcal{C} = \mathcal{C}_F$ be a Lagrangean cone which is the image of the differential dF of a basic function F. We want to express the metrics $g^{\mathcal{C}}$ and $g^{P(\mathcal{C})}$ in terms of F. Denote by $Z = (z^0, \dots, z^n)$ and $P = (p_0, \dots, p_n)$ be the canonical coordinates for $T^*\mathbb{C}^{n+1}$.

Proposition 1.7. *The canonical metric $g^{\mathcal{C}}$ of the Lagrangean cone $\mathcal{C} = \mathcal{C}_F$ at the point $dF|_Z \in \mathcal{C}_F$ is given by:*

$$g^{\mathcal{C}} = -2\sum_{j,k=0}^{n} Re\left(i\frac{\partial^2 F(Z)}{\partial z^j \partial z^k}\right) dz^j \otimes d\bar{z}^k = 2\sum_{j,k=0}^{n} Im\left(\frac{\partial^2 F(Z)}{\partial z^j \partial z^k}\right) dz^j \otimes d\bar{z}^k\,.$$

Proof: The standard symplectic form ω on $T^*\mathbb{C}^{n+1}$ is

$$\omega = \sum_{j=0}^{n}(dz^j \otimes dp_j - dp_j \otimes dz^j)\,.$$

The corresponding Hermitian metric $\gamma = i\omega(\cdot, \bar{\cdot})$ of signature $(n+1, n+1)$ on $T^*\mathbb{C}^{n+1}$ is given by

$$\gamma = i\sum_{j=0}^{n}(dz^j \otimes d\bar{p}_j - dp_j \otimes d\bar{z}^j)\,.$$

On $\mathcal{C} = \mathcal{C}_F$ we have $p_j = \partial F/\partial z^j$, hence

$$g^{\mathcal{C}} = \gamma|\mathcal{C} = i\sum_{j=0}^{n}(dz^j \otimes d\frac{\partial \bar{F}}{\partial \bar{z}^j} - d\frac{\partial F}{\partial z^j} \otimes d\bar{z}^j)$$

$$\overset{(*)}{=} i\sum_{j,k=0}^{n}\left(\frac{\partial^2 \bar{F}}{\partial \bar{z}^j \partial \bar{z}^k} - \frac{\partial^2 F}{\partial z^j \partial z^k}\right) dz^j \otimes d\bar{z}^k = 2\sum_{j,k=0}^{n} Im\left(\frac{\partial^2 F}{\partial z^j \partial z^k}\right) dz^j \otimes d\bar{z}^k\,,$$

where at $(*)$ we have used that F is holomorphic. □

Since we are assuming that $\mathcal{C} = \mathcal{C}_F$, the cone $\mathcal{C} \subset T^*\mathbb{C}^{n+1}$ is mapped isomorphically onto an open subset of $\mathbb{C}^{n+1}$ under the canonical projection $T^*\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. Therefore, we can use inhomogeneous coordinates of $P(\mathbb{C}^{n+1})$ as coordinates on $P(\mathcal{C})$, e.g. $\zeta^j = z^j/z^0$, $j = 1, \dots, n$. Recall that in order for our definition of the special metric $g^{P(\mathcal{C})}$ to make sense we assume that $\mathcal{C}$ is properly non degenerate, i.e. $\gamma(u,u) \neq 0$ for all $u \in \mathcal{C}$. Without restriction of generality we may assume $\gamma(u,u) > 0$ for all $u \in \mathcal{C}$.

Corollary 1.8. *The special metric $g^{P(\mathcal{C})}$ is given with respect to inhomogeneous coordinates on $P^n_{\mathbb{C}}$ by the pseudo Kähler potential*

$$K^F(\zeta^1,\dots,\zeta^n) = \log\gamma(\zeta,\zeta) = \log\left(2\sum_{j,k=0}^n Im\left(\frac{\partial^2 F(\zeta)}{\partial z^j\partial z^k}\right)\zeta^j\bar\zeta^k\right),$$

where $\zeta = (\zeta^0 = 1, \zeta^1,\dots,\zeta^n)$.

Proof: The first equation follows from Proposition 1.5, since $(\zeta^1,\dots,\zeta^n)\mapsto dF|_\zeta \in \mathcal{C}\subset T^*\mathbb{C}^{n+1}$ is a local section of the universal bundle over the projectivized cone $P(\mathcal{C})$. Remark that $(\zeta^1,\dots,\zeta^n)$ are the inhomogeneous coordinates of the point $\pi dF|_\zeta \in P(\mathcal{C})\subset P(T^*\mathbb{C}^{n+1})$. The second equation follows from Proposition 1.7. Remark that the functions $\frac{\partial^2 F}{\partial z^j\partial z^k}$ are homogeneous of degree zero and hence well defined on $P^n_{\mathbb{C}}$ as they must be. □

By Proposition 1.6 to every local holomorphic function F homogeneous of degree two we can associate a Lagrangean cone $\mathcal{C}_F$ and the special metric $g^F := g^{P(\mathcal{C}_F)}$ on the projectivization $P(\mathcal{C}_F)$. The **r-map** maps by definition any homogeneous cubic polynomial $h(x^1,\dots x^n)$ on $\mathbb{R}^n$ to the (possibly degenerate) special metric g^{F_h} with

$$F_h(Z) = \frac{h(z^1,\dots,z^n)}{z^0}, \quad Z = (z^0,\dots,z^n).$$

We can easily extend the r-map to homogeneous polynomials h of degree d by

$$h\mapsto F_h(Z) = \frac{h(z^1,\dots,z^n)}{(z^0)^{d-2}}.$$

Now we want to compare the special metric g^{F_h} on the projectivized Lagrangean cone $P(\mathcal{C}_{F_h})$ to the canonical metric g^c on the tube domain $U = \mathbb{R}^n + i\mathcal{V}$, $\mathcal{V} = \mathbb{R}^+\cdot\mathcal{H}$, associated to a hypersurface $\mathcal{H}\subset\mathcal{H}_1(h)$. Consider the biholomorphism

$$\varphi : \mathbb{C}^n \supset U \ni Z \mapsto \pi dF_h|_{(1,Z)} \in P(\mathcal{C}_{F_h}) \subset P(T^*\mathbb{C}^{n+1}),$$

where $\pi : T^*\mathbb{C}^{n+1}\to P(T^*\mathbb{C}^{n+1})$ is the canonical projection.

Proposition 1.9. *The metric $g^s = \varphi^* g^{F_h}$ on the tube domain U associated to the homogeneous polynomial h of degree d has the Kähler potential*

$$K^s(Z) = (\varphi^*K^{F_h})(Z) = -\frac{4}{d}\log(2Im\,(-H(Z,\dots,Z)(d-2) + H(Z,\dots,Z,\bar Z)d)),$$

where H is the polarization of h. In particular, g^{F_h} is nondegenerate if and only if $\partial\bar\partial K^s$ is nondegenerate.

Proof: We have to show that

$$(\partial^2 F_h)(\zeta,\bar\zeta) = -H(Z,\dots,Z)(d-2) + H(Z,\dots,Z,\bar Z)d,$$

where $\partial^2 F_h = (\frac{\partial^2 F_h}{\partial z^i \partial \bar{z}^j})_{i,j=0,\dots,n}$ is the complex Hessian form at the point $\zeta = (1, Z)$. We compute:

$$\begin{aligned}
(\partial^2 F_h)(\zeta,\bar\zeta) &= h(Z)(d-2)(d-1) - \partial h \bar{Z}(d-2) \\
&\quad -\partial h Z(d-2) + (\partial^2 h)(Z,\bar{Z}) \\
&= H(Z,\dots,Z)(d-2)(d-1) - H(Z,\dots,Z,\bar{Z})(d-2)d \\
&\quad -H(Z,\dots,Z)(d-2)d + H(Z,\dots,Z,\bar{Z})(d-1)d \\
&= -H(Z,\dots,Z)(d-2) + H(Z,\dots,Z,\bar{Z})d\,. \quad \Box
\end{aligned}$$

Proposition 1.10. *The metrics g^c and g^s on the tube domain associated to a homogeneous cubic polynomial h coincide.*

Proof: The following lemma shows that the Kähler potentials defining g^s and g^c coincide up to an additive constant. $\Box$

Lemma 1.11. *Let $H \in \vee^3(\mathbb{R}^n)^*$ be the polarization of the homogeneous cubic polynomial h and $Z = X + iY \in \mathbb{C}^n$, then*

$$Im\,(-H(Z,Z,Z) + 3H(Z,Z,\bar{Z})) = 4h(Y)\,.$$

The following theorem is a consequence of Propositions 1.1, 1.9 and 1.10.

Theorem 1.12. *The metrics $g^c = g^s = \varphi^* g^{F_h}$ and g^{F_h} associated to a pseudo Riemannian cubic hypersurface $(\mathcal{H}, g) \subset \mathcal{H}_1(h)$ with canonical metric g of signature (k,l) are special pseudo Kählerian of complex signature $(k+1,l)$. In particular, the r-map maps special real manifolds (i.e. Riemannian cubic hypersurfaces) to special Kähler manifolds.*

Example 1: Let X be a Kähler manifold with holonomy algebra $\mathfrak{hol} = \mathfrak{su}(3)$, i.e. a general Calabi-Yau 3-fold. Consider the complex vector space $V = H^3(X,\mathbb{C})$ with standard real structure τ, $V^\tau = H^3(X,\mathbb{R})$. The "intersection" form ω

$$\omega(\xi,\eta) = \int_X \xi \wedge \eta\,, \quad \xi,\eta \in V\,,$$

is a complex skew symmetric bilinear form on V compatible with τ, i.e.

$$\overline{\omega(\xi,\eta)} = \omega(\tau\xi,\tau\eta)\,, \quad \xi,\eta \in V\,.$$

By the Hodge-Riemann bilinear relations for primitive cohomology, s. e.g. [We] Ch. V Sec. 6, the form ω is non degenerate and thus a complex symplectic form. Indeed, the third cohomology is primitive, i.e. $\Omega \wedge \xi = 0 \in H^5(X,\mathbb{C})$ for all $\xi \in H^3(X,\mathbb{C})$, where Ω is the Kähler class of X. This follows from the formula $h_0^{p,q} = h^{p,q} - h^{p-1,q-1}$, which relates the primitive Dolbeault numbers $h_0^{p,q}$ to the usual ones $h^{p,q}$ and from the fact that the only holomorphic form on X (up to scaling) is the volume form, due to our holonomy assumption.

Summing up, to every general Calabi-Yau 3-fold we have associated the fundamental algebraic data (V, ω, τ), cf. definition on p. 4.

Again by the Hodge-Riemann bilinear relations, the complementary subspaces $W = H^{3,0}(X) + H^{1,2}(X)$ and $\overline{W} := \tau W$ are Lagrangean (with respect to ω), the Hermitian form $\gamma = i\omega(\cdot, \tau\cdot)$ is positively defined on $W \times \overline{W}$ and $H^{3,0}(X)$ and $H^{1,2}(X)$ are γ-orthogonal.

Now we consider the moduli space of X, i.e deformations of its complex structure. It is known (s. [Ti], [To], cf. [Kn]) that there exists a local universal deformation $\delta : \mathcal{X} \to S$, $X_s := \delta^{-1}(s)$, $X_{s_0} = X$, of X. The base (S, s_0) is the Kuranishi moduli space, which can be identified with a connected open subset of $H^{2,1}(X)$.

Since X is Kähler the Dolbeault numbers $h^{p,q}(X_s)$ are independent of the deformation parameter s, by well known results of Kodaira and Spencer's deformation theory (combine [We] Ch. V Cor. 6.6 and [M-K] Thm. 4.6). After identification of the fibres of $\delta : \mathcal{X} \to S$ by a diffeomorphism $\mathcal{X} \cong S \times X$ we can define the period map

$$Per : S \ni s \mapsto H^{3,0}(X_s) \in P(H^3(X, \mathbb{C})) = P(V)\,.$$

The differential $dPer_s$ maps $T_sS \cong H^{2,1}(X)$ isomorphically onto $d\pi_p(H^{3,0}(X_s) + H^{2,1}(X_s)) = d\pi_p H^{2,1}(X_s)$, where $\pi : V \to P(V)$ is the canonical projection and $0 \neq p \in Per(s) = H^{3,0}(X_s)$. Remark that the subspace $W_s = H^{3,0}(X_s) + H^{2,1}(X_s) \subset V$ is still Lagrangean, because a local deformation of a Kähler manifold is again Kähler, s. [M-K] Thm. 4.6. This shows that the lines $Per(s) \subset V$, $s \in S$, form a Lagrangean cone $\mathcal{C} = \cup_s Per(s) \subset V$.

The special metric $g^{P(\mathcal{C})}$ on $P(\mathcal{C}) = Per(S)$ is negatively defined and hence $-g^{P(\mathcal{C})}$ is a genuine Kähler metric. In fact, γ is negatively defined on $H^{2,1}(X_s) = Per(s)^{\perp} \cap T_p\mathcal{C}$, $p \in Per(s) - \{0\}$; here $\perp$ denotes the γ-orthogonal complement. The (negatively defined) special "Kähler" metric $g^{P(\mathcal{C})}$ is known as Weil-Petersson metric.

Remark 4: The Weil-Petersson metric on the Kuranishi moduli space for general Calabi-Yau m-folds of arbitrary dimension m can be defined similarly. The reasons why we have concentrated on the case $m = 3$ are the following: first $Per(S)$ is always a totally isotropic cone with respect to the intersection form, but in general not maximally isotropic in $H^m(X, \mathbb{C})$, and second the intersection form is symmetric if m is even.

2. Homogeneous case

2.1. **Classification of homogeneous Riemannian hypersurfaces and corresponding Siegel domains.** Let $\mathcal{H} \subset \mathcal{H}_1(h) \subset \mathbb{R}^n$ be a pseudo Riemannian hypersurface of degree $d \geq 2$ with basic polynomial h and canonical metric g. We define the real algebraic linear group

$$Aut(h) = \{\varphi \in GL(n, \mathbb{R}) | \varphi^* h = h\}\,,$$

which acts naturally on $\mathcal{H}_1(h)$.

Definition 2.1. *A pseudo Riemannian hypersurface $\mathcal{H} \subset \mathcal{H}_1(h)$ of degree $d \geq 2$ is said to be* **homogeneous** *if the connected component $Aut_0(h)$ acts transitively on $\mathcal{H}$.*

From Definition 2.1 and the definition of the canonical metric g (p. 1) it follows that a homogeneous pseudo Riemannian hypersurface $(\mathcal{H}, g)$ of degree $d \geq 2$ admits a transitive group of isometries, namely $Aut_0(h) \subset Isom(\mathcal{H}, g)$. By Corollary 1.3 to any homogeneous (pseudo) Riemannian hypersurface of degree d we have canonically associated a homogeneous (pseudo) Kähler manifold, which is special (pseudo) Kähler if $d = 3$. This motivates the interest in the classification problem for homogeneous such hypersurfaces, which we study in this section.

The first step is to reduce the classification of homogeneous pseudo Riemannian hypersurfaces $(\mathcal{H}, g)$ of degree $d \geq 2$ to the case of hypersurfaces admitting a transitive triangular subgroup $\mathcal{L} \subset Aut_0(h)$. By a decomposition theorem for real algebraic groups due to Vinberg [V] we prove now that this reduction is possible if the canonical metric g is Riemannian.

Theorem 2.1. *Let $(\mathcal{H}, g)$ be a homogeneous Riemannian hypersurface of degree $d \geq$ 2. Then the homogeneous pseudo Kählerian tube domain $U = \mathbb{R}^n + i\mathcal{V}$ associated to $\mathcal{H} \subset \mathcal{H}_1(h) \subset \mathbb{R}^n$ (cf. Corollary 1.3) is a* **Siegel domain of type I**, *i.e. $\mathcal{V}$ is convex and does not contain any line. Moreover, $\mathcal{V}$ is a connected component of $\mathbb{R}^n - \mathcal{H}_0(h)$. Finally, $Aut_0(h)$ admits the polar decomposition*

$$Aut_0(h) = \mathcal{K} \cdot \mathcal{L}\,,$$

where $\mathcal{K}$ is the stabilizer of a point $v \in \mathcal{H}$ and is a maximal compact connected subgroup and $\mathcal{L}$ is a maximal triangular subgroup acting simply transitively on $\mathcal{H}$. In particular, $\mathcal{H}$ is contractible.

Proof: The stated properties of the cone $\mathcal{V}$ follow, by well known arguments of Koszul [K], from the existence of the $(\mathbb{R}^+ \times Aut_0(h))$-invariant closed 1-form $\alpha = -d \log h$ on the homogeneous cone $\mathcal{V} = \mathbb{R}^+\mathcal{H} \subset \mathbb{R}^\backslash$ with positively defined Euclidean covariant derivative $\partial\alpha > 0$. Recall that the Hessian bilinear form $\partial\alpha = -\partial^2 \log h$ defines up to a positive factor the canonical $(\mathbb{R}^+ \times Aut_0(h))$-invariant Riemannian metric $g^{\mathcal{V}} \hat{=} g^c|i\mathcal{V}$ of the cone $(\mathcal{V}, g^{\mathcal{V}}) \cong (i\mathcal{V}, g^c|i\mathcal{V}) \subset (U, g^c)$, cf. Propositions 1.1 and 1.2.

The polar decomposition follows from Vinberg's theorem [V] and the convexity of the cone. In fact, since $Aut(h)$ is defined by a polynomial equation, it is a real algebraic linear group. Hence, by [V] we have $Aut_0(h) = \mathcal{K} \cdot \mathcal{L}$, where $\mathcal{K}$ is maximal compact and connected and $\mathcal{L}$ is maximal triangular.

From the convexity of the cone $\mathcal{V}$ it follows that $\mathcal{K}$ fixes a point $v \in \mathcal{H}$ (centre of gravity of a compact orbit), hence $\mathcal{K}$ is contained in the isotropy group $\mathcal{K}^1$ of $Aut_0(h)$ at v. Since $Aut_0(h)$ acts by isometries of the Riemannian metric g, the isotropy group $\mathcal{K}^1$ is compact and hence $\mathcal{K} = \mathcal{K}^1$. □

The second step is to reformulate our classification problem in terms of metric Lie algebras.

Definition 2.2. *A* **metric Lie group** $(\mathcal{L}, g)$ *is a Lie group* $\mathcal{L}$ *together with a left-invariant (pseudo) Riemannian metric* g. *Its* **metric Lie algebra** $(\mathfrak{l}, \langle\cdot,\cdot\rangle)$ *consists of the Lie algebra* $\mathfrak{l} = Lie\,\mathcal{L}$ *together with the scalar product* $\langle\cdot,\cdot\rangle = g_e$, $e \in \mathcal{L}$ *the identity. A* **(pseudo) Kähler Lie group** $(\mathcal{L}, g, \tilde{J})$ *is a metric Lie group* $(\mathcal{L}, g)$ *together with a parallel, left-invariant and orthogonal complex structure* $\tilde{J}$. *Its* **(pseudo) Kähler Lie algebra** $(\mathfrak{l}, \langle\cdot,\cdot\rangle, J)$ *consists of its metric Lie algebra* $(\mathfrak{l}, \langle\cdot,\cdot\rangle)$ *together with the orthogonal complex structure* $J := \tilde{J}_e$ *on* $\mathfrak{l}$. *We say that* $(\mathfrak{l}, \langle\cdot,\cdot\rangle)$ *(resp.* $(\mathfrak{l}, \langle\cdot,\cdot\rangle, J)$*) is metric (resp. pseudo Kähler) Lie algebra* **for** *the pseudo Riemannian (resp. pseudo Kähler) manifold* M *if it is the metric (resp. pseudo Kähler) Lie algebra of a metric (resp. pseudo Kähler) Lie group which as pseudo Riemannian (resp. as pseudo Kähler) manifold is isomorphic to* M. *A Kähler Lie algebra* $(\mathfrak{l}, \langle\cdot,\cdot\rangle, J)$ *is called a* **normal J-algebra** *if* $\mathfrak{l}$ *is splittable solvable and if its Kähler form* $\langle\cdot, J\cdot\rangle$ *is the differential of a 1-form* ω *on* $\mathfrak{l}$, *i.e. if* $\omega([X,Y]) = -\langle X, JY\rangle$ *for all* $X, Y \in \mathfrak{l}$.

It is known [PS] that any normal J-algebra is Kähler Lie algebra for a bounded homogeneous domain, the Kähler metric not necessarily being the Bergmann metric. **Example 2:** The basic examples of normal J-algebras are the following: A **key algebra** $\mathfrak{f} = span\{G, H\}$ **with root** $\mu > 0$ is defined in terms of the orthonormal basis $G = JH, H$ by the formula $[H, G] = \mu G$. Given $\mathfrak{f}$ and a Euclidean vector space $\mathfrak{x}$ with orthogonal complex structure, $\mathfrak{e} = \mathfrak{f} + \mathfrak{x}$ carries a canonical Euclidean scalar product $\langle\cdot,\cdot\rangle$ and complex structure J. The structure of **elementary Kählerian Lie algebra** with key subalgebra $\mathfrak{f}$ is defined on $\mathfrak{e}$ by the formulas

$$ad_H|\mathfrak{x} = \frac{\mu}{2} Id, \quad ad_G|\mathfrak{x} = 0 \quad \text{and} \quad [X, Y] = \mu\langle JX, Y\rangle G \quad \text{for} \quad X, Y \in \mathfrak{x}\,.$$

The metric Lie algebra $\mathfrak{e}$ is determined up to isomorphism (i.e. orthogonal isomorphism of Lie algebras) by $n = \dim_{\mathbb{C}} \mathfrak{x}$ and μ. If we wish to specify these parameters, we shall write $\mathfrak{e} = \mathfrak{e}(n+1, \mu)$. The elementary Kählerian Lie algebra $\mathfrak{e}(n+1,\mu)$ is Kähler Lie algebra for the complex hyperbolic space $H^{n+1}_{\mathbb{C}}$ with suitably normalized holomorphic sectional curvature. Remark that as Lie algebra $\mathfrak{e}(n+1, \mu)$ is isomorphic to the Iwasawa Lie algebra of the semisimple Lie algebra $\mathfrak{su}(1, n+1)$.

Let $(\mathcal{H}, g)$ be a homogeneous Riemannian hypersurface of degree $d \geq 2$ with basic polynomial h and $\mathcal{B}_0 \subset Aut_0(h)$ a simply transitive triangular subgroup, which exists by Theorem 2.1. By Corollary 1.3 the Lie group $\mathcal{B} := \mathbb{R}^+ \times \mathcal{B}_0 \subset GL(n, \mathbb{R})$ (resp. $\mathcal{U}_0 := \mathbb{R}^n \rtimes \mathcal{B} \subset Aff(\mathbb{C}^n)$) acts simply transitively and isometrically (resp. and isometrically and biholomorphically) on the cone $(\mathcal{V}, g^{\mathcal{V}}) \cong (i\mathcal{V}, g^c|i\mathcal{V})$ (resp. on the Siegel domain $(U = \mathbb{R}^n + i\mathcal{V}, g^c)$). The orbit map

$$\varphi : \mathcal{U}_0 \ni u \mapsto u(ip) \in U\,, \quad p \in \mathcal{H}\,,$$

induces diffeomorphisms $\mathcal{B}_0 \cong \mathcal{B}_0(ip) = i\mathcal{H} \cong \mathcal{H}$, $\mathcal{B} \cong \mathcal{B}(ip) = i\mathcal{V} \cong \mathcal{V}$ and $\mathcal{U}_0 \cong \mathcal{U}_0(ip) = U$ defining the structure of metric Lie group on $\mathcal{B}_0$, $\mathcal{B}$ and $\mathcal{U}_0$ by the condition that φ be an isometry. Moreover, the pull back of the complex structure of the tube domain via φ is a left-invariant, parallel orthogonal complex structure on $\mathcal{U}_0$. Hence we have on $\mathcal{U}_0$ the structure of Kähler Lie group. Its Kähler Lie algebra $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ contains the (metric) Lie algebras $\mathfrak{b}_0 = Lie\,\mathcal{B}_0$ and $\mathfrak{b} = Lie\,\mathcal{B}$ as metric subalgebras.

Proposition 2.2. *The Kähler Lie algebra $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ associated to a homogeneous Riemannian hypersurface $\mathcal{H} \subset \mathcal{H}_1(h)$ of degree $d \geq 2$ is a normal J-algebra and admits the orthogonal (semidirect) decompositions:*

$$\mathfrak{u}_0 = \mathfrak{b} \ltimes J\mathfrak{b}\,, \quad \mathfrak{b} = \mathbb{R}B_0 \oplus \mathfrak{b}_0\,,$$

where $J\mathfrak{b}$ is an Abelian ideal and the vector B_0 is in the centre of $\mathfrak{b}$ and satisfies the equations

$$ad_{B_0} JX = ad_X JB_0 = JX \quad \textit{for all} \quad X \in \mathfrak{b}\,.$$

The polynomial $h_0 = h \circ d\varphi_e|J\mathfrak{b}$ is homogeneous of degree d and invariant under the adjoint action of $\mathcal{B}_0$ on the ideal $J\mathfrak{b} \subset \mathfrak{u}_0$. Moreover, the differential $d\varphi_e$ of the orbit map φ maps the $\mathcal{B}_0$-orbit $\mathcal{H}^0 = \mathcal{B}_0(JB_0) \subset \mathcal{H}_1(h_0) \subset J\mathfrak{b}$ of the vector JB_0 diffeomorphically onto the hypersurface $\mathcal{H} \subset \mathcal{H}_1(h) \subset \mathbb{R}^n \subset T_{ip}U$, $ip = \varphi(e)$. It is an isometry with respect to the canonical metrics on $\mathcal{H}^0$ and $\mathcal{H}$.

Proof: The decompositions of metric Lie algebras are induced by the decompositions $\mathcal{U}_0 = \mathcal{B} \ltimes \mathbb{R}^n$ and $\mathcal{B} = \mathbb{R}^+ \times \mathcal{B}_0$ of metric Lie groups.

The definition of the Kähler form Ω of the tube domain U by the Kähler potential $K(Z) = -\frac{4}{d}\log h(Y)$ on p. 2 shows that Ω is exact in the complex of $\mathcal{U}_0$-invariant differential forms on U. In fact, $\omega = \overline{\partial}\log h(Y)$ is an $\mathcal{U}_0$-invariant 1-form on the tube domain U and $d\omega = \partial\overline{\partial}\log h(Y)$ is proportional to Ω. Since, by Theorem 2.1, $\mathfrak{u}_0$ is splittable solvable, this shows that $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ is a normal J-algebra.

The last statements follow from the fact that the representations $\mathcal{B}_0 \to GL(J\mathfrak{b})$ and $\mathcal{B}_0 \hookrightarrow GL(\mathbb{R}^n)$ are isomorphic via

$$d\varphi_e|J\mathfrak{b} : \mathfrak{b} + J\mathfrak{b} = \mathfrak{u}_0 \supset J\mathfrak{b} \xrightarrow{\sim} \mathbb{R}^n \subset T_{ip}U = \mathbb{R}^n + i\mathbb{R}^n\,. \quad \square$$

Definition 2.3. *Let $\mathfrak{x}$, $\mathfrak{y}$ and $\mathfrak{z}$ be pseudo Euclidean vector spaces. A bilinear map $\psi : \mathfrak{x} \times \mathfrak{y} \to \mathfrak{z}$ is said to be* **isometric**, *if*

$$\langle\psi(X,Y), \psi(X,Y)\rangle = \langle X,X\rangle\langle Y,Y\rangle$$

for all $X \in \mathfrak{x}$ and $Y \in \mathfrak{y}$. The number $k = \dim\mathfrak{x}$ is the **order** *of the isometric map. An isometric map $\psi : \mathfrak{x} \times \mathfrak{y} \to \mathfrak{z}$ is called special if $\dim\mathfrak{y} = \dim\mathfrak{z}$.*

The **transpose** *$\psi^t : \mathfrak{x} \times \mathfrak{z} \to \mathfrak{y}$ of an isometric map $\psi : \mathfrak{x} \times \mathfrak{y} \to \mathfrak{z}$ is defined by*

$$\langle\psi^t(X,Z), Y\rangle := \langle Z, \psi(X,Y)\rangle\,, \quad X \in \mathfrak{x}\,, \quad Y \in \mathfrak{y} \quad \textit{and} \quad Z \in \mathfrak{z}\,.$$

Remark that the transpose ψ^t of an isometric map ψ is isometric if and only if ψ (and hence ψ^t) is a special isometric map.

The following important structure result can be extracted from [PS], cf. [G-PS-V].

Theorem 2.3. *(S.G. Gindikin, I.I. Pyateckiĭ-Shapiro, E.B. Vinberg) Any normal J-algebra* $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ *has an orthogonal semidirect decomposition*

$$\mathfrak{u}_0 = \mathfrak{e}_1 + \mathfrak{e}_2 + \cdots + \mathfrak{e}_l \tag{3}$$

into elementary Kählerian subalgebras $\mathfrak{e}_j = \mathfrak{f}_j + \mathfrak{x}_j$ *with root* μ_j, $j = 1, 2, \ldots, l$. *More precisely,* $[\mathfrak{f}_i, \mathfrak{e}_j] = 0$ *and* $[\mathfrak{x}_i, \mathfrak{e}_j] \subset \mathfrak{x}_i$ *if* $i < j$.

The normal J-algebra $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ *is Kähler Lie algebra for a Siegel domain of type I if and only if we have the following orthogonal decompositions*

$$\mathfrak{u}_0 = \mathfrak{b} \mathbin{+\!\!\!\!\supset} J\mathfrak{b}\,, \quad \mathfrak{b} = \mathbb{R}B_0 \oplus \mathfrak{b}_0\,, \tag{4}$$

where $J\mathfrak{b}$ *is an Abelian ideal and* $B_0 = \sum_{i=1}^{l} \frac{1}{\mu_i} H_i$ *is in the centre of* $\mathfrak{b}$. *Under this assumption, we have orthogonal decompositions*

$$\mathfrak{x}_i = \sum_{j=i+1}^{l} \mathfrak{x}_{ij}\,, \quad \mathfrak{x}_{ij} = \mathfrak{x}_{ij}^- + \mathfrak{x}_{ij}^+\,, \quad \mathfrak{x}_{ij}^+ = J\mathfrak{x}_{ij}^-\,, \quad i = 1, \ldots, l-1\,, \quad \mathfrak{x}_l = 0\,,$$

such that

$$\mathfrak{b} = \mathfrak{a} + \sum_{j>i} \mathfrak{x}_{ij}^-\,, \quad \text{where} \quad \mathfrak{a} = span\{H_i | i = 1, \ldots, l\}$$

and we have the commutator relations

$$[\mathfrak{f}_k, \mathfrak{x}_{ij}] = 0 \quad \text{if} \quad i < j,\ i < k \quad \text{and} \quad j \neq k\,;$$

$$\begin{aligned} ad_{H_j}|\mathfrak{x}_{ij}^{\pm} &= \pm\frac{\mu_j}{2} Id\,, \\ ad_{G_j}|\mathfrak{x}_{ij}^{+} &= 0\,, \\ ad_{G_j}|\mathfrak{x}_{ij}^{-} &= -\mu_j J \quad \text{if} \quad i < j\,; \end{aligned}$$

$$[\mathfrak{x}_{st}, \mathfrak{x}_{ij}] = 0 \quad \text{if} \quad s < t,\ i < j,\ i < s \quad \text{and} \quad s \neq j \neq t\,;$$

$$\begin{aligned} [\mathfrak{x}_{jk}^{\pm}, \mathfrak{x}_{ij}^{-}] &\subset \mathfrak{x}_{ik}^{\pm}\,, \\ [\mathfrak{x}_{jk}, \mathfrak{x}_{ij}^{+}] &= 0 \quad \text{if} \quad i < j < k\,; \end{aligned}$$

$$\begin{aligned} [\mathfrak{x}_{kj}^{\pm}, \mathfrak{x}_{ij}^{\mp}] &\subset \mathfrak{x}_{ik}^{+}\,, \\ [\mathfrak{x}_{kj}^{\pm}, \mathfrak{x}_{ij}^{\pm}] &= 0 \quad \text{if} \quad i < k < j\,. \end{aligned}$$

The Lie bracket $[\cdot,\cdot] : \mathfrak{x}_{jk}^{\pm} \times \mathfrak{x}_{ij}^{-} \to \mathfrak{x}_{ik}^{\pm}$ *for* $i < j < k$ *is given by an isometric map* $\psi_{ijk} : \mathfrak{x}_{jk}^{-} \times \mathfrak{x}_{ij}^{-} \to \mathfrak{x}_{ik}^{-}$ *as follows:*

$$\begin{aligned} [X, Y] &= \frac{1}{\sqrt{2}} \psi_{ijk}(X, Y)\,, \\ [JX, Y] &= J[X, Y]\,, \quad X \in \mathfrak{x}_{jk}^{-}\,,\ Y \in \mathfrak{x}_{ij}^{-}\,. \end{aligned}$$

The Lie bracket $[\cdot,\cdot] : \mathfrak{x}^{\pm}_{kj} \times \mathfrak{x}^{\mp}_{ij} \to \mathfrak{x}^{+}_{ik}$ *for* $i < k < j$ *is given by*

$$\langle [X,Y], Z\rangle = -\frac{1}{\sqrt{2}}\langle JY, \psi_{ikj}(X, JZ)\rangle\,,$$

$$[X,Y] = [JX, JY]\,,\quad X \in \mathfrak{x}^{-}_{kj}\,,\; Y \in \mathfrak{x}^{+}_{ij}\,,\; Z \in \mathfrak{x}^{+}_{ik}\,.$$

The number l of elementary Kählerian subalgebras in the decomposition (3) is called the **rank** of the normal J-algebra $\mathfrak{u}_0$. The normal J-algebras for Siegel domains of type I will be called normal J-algebras **of type I**.

Lemma 2.4. *Let* $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ *be a normal J-algebra for a Siegel domain of type I. Then the orbit* $\mathcal{H} = \mathcal{B}_0(JB_0)$ *of the vector* $JB_0 \in J\mathfrak{b}$ *under the adjoint action of* $\mathcal{B}_0 \subset \mathcal{U}_0$ *on the ideal* $J\mathfrak{b} \subset \mathfrak{u}_0 = \mathfrak{b} + J\mathfrak{b}$ *is a smooth hypersurface.*

Proof: It is sufficient to prove that the map

$$\mathfrak{b}_0 \ni X \mapsto ad_X JB_0 \in J\mathfrak{b}$$

has maximal rank. This follows from the equation $ad_X JB_0 = JX$ $(X \in \mathfrak{b})$, which characterizes the vector $B_0 \in \mathfrak{b}$. $\square$

By Proposition 2.2 and Lemma 2.4 the classification of homogeneous Riemannian hypersurfaces of degree $d \geq 2$ reduces to the following problem, which can be studied using Theorem 2.3.

Problem 1: Classify all normal J-algebras $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ for Siegel domains of type I (i.e. which admit the decompositions (4)) such that the hypersurface $\mathcal{H} = \mathcal{B}_0(JB_0)$ is contained in the level set $\mathcal{H}_1(h) \subset J\mathfrak{b}$ of a homogeneous polynomial h of degree $d \geq 2$ on $J\mathfrak{b}$ and the canonical metric of $\mathcal{H}$ is Riemannian.

Remark that if $d = 3$, then to any solution $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ to Problem 1 we can associate a homogeneous *special* Kähler manifold, namely the Kählerian Siegel domain of type I associated to the homogeneous Riemannian *cubic* hypersurface $\mathcal{H}$.

For this reason we will give the complete solution to Problem 1 for $2 \leq d \leq 3$. Examples of homogeneous Riemannian hypersurfaces of arbitrary degree $d \geq 2$ and the corresponding homogeneous Kählerian Siegel domains of type I will be presented at the end of this section. An interesting class of homogeneous pseudo Riemannian cubic hypersurfaces and the corresponding homogeneous special pseudo Kähler and pseudo quaternionic Kähler manifolds will be discussed in 2.3.

Before studying the general case, we consider the normal J-algebras of the form $\mathfrak{u}_0 = \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \cdots \oplus \mathfrak{f}_l$, i.e. all the elementary Kählerian subalgebras are key algebras. We have the orthogonal decomposition $\mathfrak{u}_0 = \mathfrak{b} + J\mathfrak{b}$ with

$$\mathfrak{b} = \mathfrak{a} = span\{H_i | i = 1, \ldots, l\} = \mathbb{R}B_0 \overset{\perp}{\oplus} \mathfrak{b}_0\,,\quad B_0 = \sum_{i=1}^{l} \frac{1}{\mu_i} H_i\,.$$

Lemma 2.5. *If $\mathfrak{u}_0 = \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \cdots \oplus \mathfrak{f}_l$ is an orthogonal direct sum of key algebras, then a $\mathcal{B}_0$-invariant homogeneous function f is defined near the point $JB_0 \in J\mathfrak{b}$ by*

$$f(\eta) = \prod_{j=1}^{l} a_j^{\prod_{k\neq j}\mu_k^2}, \quad \eta = \sum_{j=1}^{l} a_j G_j \in J\mathfrak{b} = span\{G_j | j = 1, \dots, l\}\,.$$

Any such (local) function is proportional to a real power of f.

Proof: To prove that f is $\mathcal{B}_0$-invariant it is sufficient to check that f is annihilated by the adjoint action of $\mu_1 H_1 - \mu_2 H_2 \in \mathfrak{b}_0$. The last statement follows from the fact that a homogeneous $\mathcal{B}_0$-invariant function defined on a neighborhood of JB_0 in $J\mathfrak{b}$ is uniquely determined by its degree $\in \mathbb{R}$ and its constant value on the hypersurface $\mathcal{H} = \mathcal{B}_0(JB_0)$. □

Proposition 2.6. *Under the assumption of Lemma 2.5 there exists a non constant $\mathcal{B}_0$-invariant homogeneous polynomial h on $J\mathfrak{b}$ if and only if*

$$\left(\frac{\mu_i}{\mu_j}\right)^2 \in \mathbb{Q} \quad \textit{for all} \quad i, j = 1, \dots, l\,.$$

Under this condition we can, up to homothety of metric Lie algebras, assume that $\mu_j^2 = p_j/q_j$ is a reduced fraction for all j and $p_1 = q_1 = 1$. Then the unique (up to scaling) non constant $\mathcal{B}_0$-invariant homogeneous polynomial of lowest degree d is

$$h(\eta) = \prod_{j=1}^{l} a_j^{(q_j \prod_{k\neq j} p_k)/N}, \quad N = \gcd\{q_j \prod_{k\neq j} p_k | j = 1, \dots, l\}\,,$$

*where "*gcd*" stands for "greatest common divisor". The degree d of h is*

$$d = \frac{1}{N}\sum_{j=1}^{l}(q_j \prod_{k\neq j} p_k)\,.$$

Proof: A real power of f is a non constant polynomial if and only if there exists a $\lambda \in \mathbb{R}^*$ such that

$$\prod_{k\neq j}\mu_k^2 \in \lambda\mathbb{Q} \quad \text{for all} \quad j = 1, \dots, l$$

or equivalently such that for all $i, j = 1, \dots, l$ we have

$$\frac{\mu_i^2}{\mu_j^2} = \frac{\prod_{k\neq j}\mu_k^2}{\prod_{k\neq i}\mu_k^2} \in \mathbb{Q}\,.$$

By scaling the scalar product of the normal J-algebra $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ we can assume that $\mu_1 = 1$ and hence $\mu_2^2, \dots \mu_l^2 \in \mathbb{Q}$. Now the remaining statements are immediate. □

Lemma 2.7. *The solutions $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ to Problem 1 (p. 14) which admit a direct orthogonal decomposition $\mathfrak{u}_0 = \mathfrak{f}_1 + \mathfrak{f}_2 + \cdots + \mathfrak{f}_l$ into key algebras and for which $2 \le d \le 3$ are up to scaling listed in the following table, where $h(\eta)$, $\eta = \sum_{j=1}^{l} a_j G_j$, is a basic polynomial for the (flat) homogeneous Riemannian hypersurface $\mathcal{H} = \mathcal{B}_0(JB_0) \subset \mathcal{H}_1(h) \subset J\mathfrak{b}$ of degree d.*

d	l	$(\mu_i^2,\quad i = 1, \ldots, l)$	$h(\eta)$
2	2	(1,1)	a_1a_2
3	2	(1,2)	$a_1^2a_2$
3	3	(1,1,1)	$a_1a_2a_3$

Proof: It is straightforward, using Proposition 2.6, to determine the solutions to Problem1 for which $\mathfrak{u}_0 = \mathfrak{f}_1 + \mathfrak{f}_2 + \cdots + \mathfrak{f}_l$ and $2 \le d \le 3$. If we normalize $\mu_1 = 1$, then $d = 2$ implies $l = 2$, $\mu_2 = 1$ and $h(\eta) = a_1a_2$ and $d = 3$ implies either $l = 3$, $\mu_2 = \mu_3 = 1$ and $h(\eta) = a_1a_2a_3$ or $l = 2$ and $\mu_2 \in \{\sqrt{2}, 1/\sqrt{2}\}$. In the last two cases the corresponding polynomials are $h(\eta) = a_1^2a_2$ and $a_1a_2^2$ respectively. Up to scaling the scalar product of $\mathfrak{u}_0$, it is sufficient to consider the case $\mu_2 = \sqrt{2}$ and $h(\eta) = a_1^2a_2$. □

Remark 5: The Riemannian hypersurfaces defined by the two polynomials $a_1^2a_2$ and $a_1a_2^2$ are isometric via the *linear* transformation $a_1G_1 + a_2G_2 \mapsto a_2G_1 + a_1G_2$. In particular, the Kählerian tube domains associated to these hypersurfaces are isomorphic.

Now we consider the general case of a normal J-algebra for a Siegel domain of type I. We use the notations and decompositions introduced above, cf. Theorem 2.3; in particular,

$$\mathfrak{u}_0 = \mathfrak{e}_1 + \mathfrak{e}_2 + \cdots + \mathfrak{e}_l\,, \quad \mathfrak{e}_j = \mathfrak{e}(n_j + 1, \mu_j)\,, \quad j = 1, \ldots, l\,, \quad n_l = 0\,.$$

Consider the decomposition $J\mathfrak{b} = A_{1,0} + A_{0,1}$, where

$$A_{1,0} := J\mathfrak{a} = span\{G_j | j = 1, \ldots, l\}\,, \quad A_{0,1} := \sum_{j>i} \mathfrak{x}_{ij}^{+}\,.$$

It defines an $\mathfrak{a}$-invariant decomposition for the homogeneous polynomials of degree d on $J\mathfrak{b}$:

$$\vee^d (J\mathfrak{b})^* = \sum_{p+q=d} A^{p,q} := \pi\left((A^{1,0})^{\otimes p} \otimes (A^{0,1})^{\otimes q}\right)\,,$$

where $A^{1,0} \cong A_{1,0}^*$ and $A^{0,1} \cong A_{0,1}^*$ are the subspaces of $(J\mathfrak{b})^*$ which annihilate $A_{0,1}$ and $A_{1,0}$ respectively and $\pi : \otimes^d (J\mathfrak{b})^* \to \vee^d (J\mathfrak{b})^*$ denotes the natural projection from the tensor product to the symmetric tensor product. We can decompose any homogeneous polynomial $h \in \vee^d (J\mathfrak{b})^*$ into its pure components:

$$h = \sum_{p+q=d} h^{p,q}\,, \quad h^{p,q} \in A^{p,q}\,.$$

Put $\mathfrak{a}_0 = \mathfrak{b}_0 \cap \mathfrak{a} = \{A \in \mathfrak{a} | \langle A, B_0 \rangle = 0\}$. Remark that if h is a $\mathfrak{b}_0$-invariant homogeneous polynomial, then h and its pure components $h^{p,q}$ are $\mathfrak{a}_0$-invariant. Next we will apply these general considerations to the case of degree $2 \le d \le 3$.

To simplify the terminology we will use the following definition.

Definition 2.4. *We say that a (pseudo) Kähler Lie algebra $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ for a homogeneous (pseudo) Kähler manifold U, s. Definition 2.2, is* **of degree d** *if, up to scaling the metric, U is isometric to the (pseudo) Kählerian tube domain associated to a homogeneous (pseudo) Riemannian hypersurface of degree d.*

Theorem 2.8. *The quadratic normal J-algebras are (up to scaling) precisely the normal J-algebras of type I (s. Theorem 2.3) of the form*

$$\mathfrak{u}_0 = \mathfrak{e}_1 + \mathfrak{f}_2\,, \quad \mu_1 = \mu_2 = 1\,.$$

The $\mathfrak{b}_0$-invariant homogeneous quadratic polynomial on $J\mathfrak{b} = J\mathfrak{a} + \mathfrak{x}_{12}^+$,

$$J\mathfrak{a} = span\{G_1, G_2\},$$

is given by

$$h(\eta) = a_1 a_2 - \frac{1}{2}\langle X, X\rangle\,, \quad \eta = a_1 G_1 + a_2 G_2 + X\,, \quad X \in \mathfrak{x}_{12}^+\,.$$

All cubic normal J-algebras have rank $l = 2$ or 3. The cubic normal J-algebras of rank 2 are (up to scaling) precisely the normal J-algebras of type I of the form

$$\mathfrak{u}_0 = \mathfrak{e}_1 + \mathfrak{f}_2\,, \quad \mu_1 = 1\,,\ \mu_2 = \frac{1}{\sqrt{2}}\,,$$

$$h(\eta) = a_1 a_2^2 - \frac{1}{\sqrt{2}} a_2 \langle X, X\rangle\,.$$

The cubic normal J-algebras of rank 3 are (up to scaling) precisely the normal J-algebras of type I of the form

$$\mathfrak{u}_0 = \mathfrak{u}_0(\psi) = \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{f}_3\,, \quad \mu_1 = \mu_2 = \mu_3 = 1\,,$$

determined by an isometric map $\psi := \psi_{123} : \mathfrak{x}_{23}^- \times \mathfrak{x}_{12}^- \mapsto \mathfrak{x}_{13}^-$, which has to be special or of order zero, s. Definition 2.3. The $\mathfrak{b}_0$-invariant homogeneous cubic polynomial on $J\mathfrak{b} = J\mathfrak{a} + \mathfrak{x}_{23}^+ + \mathfrak{x}_{13}^+ + \mathfrak{x}_{12}^+$ is

$$h(\eta) = a_1 a_2 a_3 - \frac{1}{2}\sum_{\alpha=1}^{3} a_\alpha \langle X_{\beta\gamma}, X_{\beta\gamma}\rangle + \frac{1}{\sqrt{2}}\langle \psi(JX_{23}, JX_{12}), JX_{13}\rangle\,,$$

where $\eta = \sum_{\alpha=1}^{3}(a_\alpha G_\alpha + X_{\beta\gamma})$, $X_{\beta\gamma} \in \mathfrak{x}_{\beta\gamma}$, $\beta < \gamma$ and $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$.

Corollary 2.9. *The quadratic normal J-algebras are the normal J-algebras for the Hermitian symmetric spaces $SO_0(2, 2+p)/(SO(2) \times SO(2+p))$. The cubic normal J-algebras are precisely the normal J-algebras for the special Kähler submanifolds of the Alekseevsky spaces, s. 2.3.*

Proof(of the theorem)**:** Let $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ be a quadratic normal J-algebra and h the basic polynomial of the quadratic Riemannian hypersurface $\mathcal{H} = \mathcal{B}_0(JB_0) \subset J\mathfrak{b}$. We have the decomposition

$$h = h^{2,0} + h^{1,1} + h^{0,2}$$

into $\mathfrak{a}_0$-invariant homogeneous quadratic polynomials. The polynomial $h^{2,0}$ is nonzero because otherwise the canonical metric of $\mathcal{H}$ defined by h would be zero on the subspace $J\mathfrak{a}_0 \subset J\mathfrak{b}_0 = T_{JB_0}\mathcal{H}$, which is impossible for a Riemannian metric. Hence $h^{2,0}$ defines a $\mathfrak{a}_0$-invariant nonzero homogeneous quadratic polynomial on $J\mathfrak{a}$. The canonical metric defined by $h^{2,0}$ on a hypersurface in $J\mathfrak{a}$ through the point $JB_0 \in J\mathfrak{a}$ is Riemannian. By Lemma 2.7 this can only occur if $l = 2$, i.e. $\mathfrak{u}_0 = \mathfrak{e}_1 + \mathfrak{f}_2$, $\mu_1 = \mu_2 = 1$ (up to scaling) and $h^{2,0}(\eta) = a_1a_2$ for $\eta = a_1G_1 + a_2G_2$.

In the cubic case we have

$$h = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3}$$

and $h^{3,0}$ is a nonzero $\mathfrak{a}_0$-invariant homogeneous cubic polynomial. As above, we conclude in this case that $l = 2$ or $l = 3$. If $l = 2$ we have (up to scaling) $\mu_1 = 1$ and either $\mu_2 = \sqrt{2}$ and $h^{3,0}(\eta) = a_1^2a_2$ or $\mu_2 = \frac{1}{\sqrt{2}}$ and $h^{3,0}(\eta) = a_1a_2^2$ for $\eta = a_1G_1 + a_2G_2$. If $l = 3$ we have (up to scaling) $\mu_1 = \mu_2 = \mu_3 = 1$ and $h^{3,0}(\eta) = a_1a_2a_3$ for $\eta = a_1G_1 + a_2G_2 + a_3G_3$.

Using the decomposition of $\vee^3(J\mathfrak{b})^*$ introduced above we show next that in the first case the polynomial $h^{3,0}(\eta) = a_1^2a_2$ cannot be extended to a $\mathfrak{b}_0$-invariant homogeneous cubic polynomial on $J\mathfrak{b}$ unless $\mathfrak{u}_0 = \mathfrak{e}_1 \oplus \mathfrak{f}_2 = \mathfrak{f}_1 \oplus \mathfrak{f}_2$ is a direct sum of key algebras.

Assume that $h = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3}$ is such an extension, denote by $\pi_{p,q} : \vee^3(J\mathfrak{b})^* \to A^{p,q}$, $p+q = 3$, the natural projection and by $Y \mapsto ad_Y^*$ the representation of $\mathfrak{b}_0$ on $\vee^3(J\mathfrak{b})^*$ induced by the adjoint representation of $\mathfrak{b}_0$ on the ideal $J\mathfrak{b} \subset \mathfrak{u}_0$. Let $Y \in \mathfrak{x}_{12}^- \subset \mathfrak{b}_0$. Then the equation $ad_Y^*h = 0$ implies the equations

$$\begin{aligned} \pi_{2,1}ad_Y^*h &= ad_Y^*h^{3,0} + \pi_{2,1}ad_Y^*h^{1,2} = 0\,, \\ \pi_{0,3}ad_Y^*h &= \pi_{0,3}ad_Y^*h^{1,2} = 0 \end{aligned}$$

for the polynomial

$$h^{1,2}(\eta) = c_1a_1q_1(X,X) + c_2a_2q_2(X,X)\,, \quad \eta = a_1G_1 + a_2G_2 + X\,,$$

where q_1 and q_2 are quadratic forms on $\mathfrak{x}_{12}^+ \ni X$ and c_1 and c_2 are real constants. Since $ad_Y^*h^{3,0}(\eta) = -2a_1a_2\langle JY, X\rangle$, the first equation is satisfied for all $Y \in \mathfrak{x}_{12}^- \neq 0$ if and only if $h^{1,2}(\eta) = -\frac{1}{\sqrt{2}}a_1\langle X, X\rangle$. However, this implies

$$\pi_{0,3}ad_Y^*h^{1,2}(\eta) = -\frac{1}{\sqrt{2}}\langle JY, X\rangle\langle X, X\rangle\,,$$

so the second equation is not satisfied. This shows that the case $\mu_2 = \sqrt{2}$ is impossible if $\mathfrak{x}_{12}^- \neq 0$.

To conclude the proof in the case $l=2$ we have to check that if $\mu_2 = 1$ (resp. if $\mu_2 = \frac{1}{\sqrt{2}}$) $h^{2,0}(\eta) = a_1 a_2$ (resp. $h^{3,0}(\eta) = a_1 a_2^2$) is extended by h given in the theorem to a homogeneous $\mathfrak{b}_0$-invariant quadratic (resp. cubic) polynomial on $J\mathfrak{b}$, that the canonical metric of $\mathcal{H} = \mathcal{B}_0(JB_0) \subset \mathcal{H}_1(h) \subset J\mathfrak{b}$ is Riemannian and, more precisely, that that $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ is Kähler Lie algebra for the Kählerian Siegel domain associated to $\mathcal{H}$. This is a special case of Proposition 2.10.

Now we show in the case $l=3$ that if $h = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3}$ is a $\mathfrak{b}_0$-invariant polynomial on $J\mathfrak{b}$ and $h^{3,0}(\eta) = a_1 a_2 a_3$, then h must be the polynomial given above.

It is easy to check that the only $\mathfrak{a}_0$-invariant element of $A^{2,1}$ is zero, the $\mathfrak{a}_0$-invariant elements of $A^{1,2}$ are spanned by the three polynomials

$$a_\alpha \langle X_{\beta\gamma}, X_{\beta\gamma}\rangle\,,$$

and finally, the $\mathfrak{a}_0$-invariant elements f of $A^{0,3}$ are given by trilinear functions

$$t : \mathfrak{x}_{23} \times \mathfrak{x}_{13} \times \mathfrak{x}_{12} \to \mathbb{R}\,, \quad \text{i.e.} \quad f(X_{23} + X_{13} + X_{12}) = t(X_{23}, X_{13}, X_{12})\,,$$

where

$$X_{\beta\gamma} \in \mathfrak{x}_{\beta\gamma}\,, \quad \{\alpha,\beta,\gamma\} = \{1,2,3\}\,,\ \beta < \gamma\,.$$

This implies

$$h^{2,1} = 0\,, \quad h^{1,2}(\eta) = \sum_{\alpha=1}^{3} c_\alpha a_\alpha \langle X_{\beta\gamma}, X_{\beta\gamma}\rangle\,, \quad h^{0,3}(\eta) = t(X_{23}, X_{13}, X_{12})\,,$$

where c_α are real constants, t is a trilinear function as above and

$$\eta = \sum_{\alpha=1}^{3} (a_\alpha G_\alpha + X_{\beta\gamma})\,, \quad X_{\beta\gamma} \in \mathfrak{x}_{\beta\gamma}\,, \quad \{\alpha,\beta,\gamma\} = \{1,2,3\}\,,\ \beta<\gamma\,.$$

Now we let $X \in \sum_{i<j} \mathfrak{x}_{ij}^-$ and consider the equations

$$\begin{aligned}
0 = \pi_{2,1} ad_X^* h &= ad_X^* h^{3,0} + \pi_{2,1} ad_X^* h^{1,2}\\
0 = \pi_{1,2} ad_X^* h &= \pi_{1,2} ad_X^* h^{1,2} + \pi_{1,2} ad_X^* h^{0,3}\\
0 = \pi_{0,3} ad_X^* h &= \pi_{0,3} ad_X^* h^{1,2} + \pi_{0,3} ad_X^* h^{0,3}\,.
\end{aligned}$$

The first equation is satisfied if and only if $c_1 = c_2 = c_3 = -\frac{1}{2}$. Then the second equation is equivalent to

$$t(X_{23}, X_{13}, X_{12}) = \frac{1}{\sqrt{2}} \langle \psi(JX_{23}, JX_{12}), JX_{13}\rangle\,.$$

Now the third equation is satisfied only if $\mathfrak{x}_{23}^- = 0$ or if ψ is a special isometric map. In fact, if e.g. $X \in \mathfrak{x}_{23}^-$ then

$$[X, \sum_{\alpha=1}^{3} (a_\alpha G_\alpha + X_{\beta\gamma})] = a_3 JX + \langle JX, X_{23}\rangle G_2 + \frac{1}{\sqrt{2}} J\psi^t(X, JX_{13})$$

and the third equation reads

$$\begin{aligned} 0 &= \frac{1}{2}\langle JX, X_{23}\rangle\langle X_{13}, X_{13}\rangle + \frac{1}{2}\langle \psi(JX_{23}, \psi^t(X, JX_{13}), JX_{13}\rangle \\ &= \frac{1}{2}\langle JX, X_{23}\rangle\langle X_{13}, X_{13}\rangle + \frac{1}{2}\langle \psi^t(X, JX_{13}), \psi^t(JX_{23}, JX_{13})\rangle\,. \end{aligned}$$

If $\mathfrak{r}_{23}^- \neq 0$ we can choose $X = JX_{23} \neq 0$ and the last equation shows that $\psi^t(X,\cdot) : \mathfrak{r}_{13}^- \to \mathfrak{r}_{12}^-$ is injective, hence ψ is a special isometric map.

It only remains to check that the scalar product of the normal J-algebra $\mathfrak{u}_0$ is (up to scaling) induced by the canonical metric of the tube domain associated to the hypersurface $\mathcal{H} = \mathcal{B}_0(JB_0) \subset \mathcal{H}_1(h) \subset J\mathfrak{b}$. This is a straightforward computation, cf. Proposition 2.10. Remark that the pure component $h^{0,3}$ involving the special isometric map ψ plays no role in this calculation, since its Hessian vanishes at the point JB_0. □

Now we give examples of normal J-algebras of arbitrary degree $d = 2, 3, 4, \ldots$. For every degree $d \geq 2$ we construct a series $(\mathfrak{u}_0(p, d-1), \langle\cdot,\cdot\rangle, J)$, $p \in \mathbb{N}_0$, of normal J-algebras of degree d and rank 2. The subalgebra $\mathfrak{b}_0$ for these series is up to scaling isomorphic to the Iwasawa algebra of $\mathfrak{so}(1, p+1)$ with scalar product induced by the Riemannian metric of hyperbolic $(p+1)$-space $H^{p+1}_{\mathbb{R}} = SO_0(1, p+1)/SO(p+1)$.

From now on we write $\mathfrak{u}_0$ instead of $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$. The scalar product and complex structure are understood and always denoted by $\langle\cdot,\cdot\rangle$ and J respectively.

For every $s = 1, 2, \ldots$ consider the Kählerian Lie algebra

$$\mathfrak{u}_0 = \mathfrak{u}_0(p, s) = \mathfrak{e}(p+1, 1) \oplus\!\!\!\!+ \,\mathfrak{e}(1, \frac{1}{\sqrt{s}}) = (\mathfrak{f}_1 + \mathfrak{r}_1) + \mathfrak{f}_2\,,$$

where the semidirect orthogonal sum of the key algebra $\mathfrak{f}_2$ with root $\mu = \frac{1}{\sqrt{s}}$ and the ideal $\mathfrak{f}_1 + \mathfrak{r}_1$ is defined by the condition that $ad_{\mathfrak{f}_2}|\mathfrak{r}_1$ has weight decomposition $\mathfrak{r}_1 = \mathfrak{r}_{12}^- + \mathfrak{r}_{12}^+$, $p = \dim \mathfrak{r}_{12}^-$.

Proposition 2.10. *$\mathfrak{u}_0 = \mathfrak{u}_0(p, s) = \mathfrak{b} + J\mathfrak{b}$, $\mathfrak{b} = \mathbb{R}B_0 \oplus \mathfrak{b}_0$, s. Theorem 2.3, is a normal J-algebra of degree $d = s+1$ with $\mathfrak{b}_0$-invariant polynomial*

$$h(\eta) = a_1(\mu a_2)^s - \frac{1}{2}(\mu a_2)^{s-1}\langle X, X\rangle\,, \quad \mu = \frac{1}{\sqrt{s}}\,,$$

where $\eta = a_1G_1 + a_2G_2 + X$, $X \in \mathfrak{r}_{12}^+$. Its subalgebra $\mathfrak{b}_0$ is isomorphic to the Iwasawa algebra of $\mathfrak{so}(1, p+1)$. In particular, the Iwasawa subgroup of $SO_0(1, p+1)$ acts simply transitively on a Riemannian hypersurface $\mathcal{H}$ of degree d and of constant negative curvature. The Kählerian Siegel domain U associated to the Riemannian hypersurface $\mathcal{H} = \mathcal{B}_0(JB_0) \subset J\mathfrak{b}$ of degree d is symmetric only if $\mathcal{H}$ is quadratic and in this case U is isometric to the Hermitian symmetric space $SO_0(2, p+2)/(SO(2) \times SO(p+2))$. If $\mathcal{H}$ is cubic (cf. Theorem 2.8), then U is special Kähler; in fact , it is the special Kähler submanifold of the Alekseevvsky space $\mathcal{T}(p)$, s. 2.3.

Proof: First we observe that $\mathfrak{u}_0 = \mathfrak{b} + J\mathfrak{b}$, where $\mathfrak{b} = span\{H_1, H_2\} + \mathfrak{x}_{12}^-$ has centre $B_0 = H_1 + \frac{1}{\mu}H_2$ if $p \neq 0$. Remark that the metric Lie algebra $\mathfrak{b}_0 \subset \mathfrak{b} = \mathbb{R}B_0 \overset{\perp}{\oplus} \mathfrak{b}_0$ is isomorphic to the Iwasawa algebra of $\mathfrak{so}(1, p+1)$ with scalar product induced (up to scaling) by the hyperbolic metric of $H_{\mathbb{R}}^{p+1} = SO_0(1, p+1)/SO(p+1)$.

Consider now the faithful linear representation $\rho = ad| : \mathfrak{b} \to \mathfrak{gl}(J\mathfrak{b})$. We have $\rho(B_0) = Id$ and $\rho(B)JB_0 = JB$ for all $B \in \mathfrak{b}$. The corresponding representation $R : \mathcal{B} \to GL(J\mathfrak{b})$ of the simply connected Lie group $\mathcal{B} = \mathbb{R}^+ \times \mathcal{B}_0$ with Lie algebra $\mathfrak{b} = \mathbb{R}B_0 + \mathfrak{b}_0$ has the open orbit $R(\mathcal{B})JB_0$ and $\mathcal{H} = \mathcal{B}_0(JB_0) = R(\mathcal{B}_0)JB_0$ is a codimension one orbit of the simply connected Lie group $\mathcal{B}_0$ with Lie algebra $\mathfrak{b}_0$.

We will prove that $\mathcal{H}$ is defined by a homogeneous polynomial h of degree d and that the simply connected metric Lie group $\mathcal{U}_0$ with metric Lie algebra $\mathfrak{u}_0$ is up to scaling isomorphic to the Kählerian Siegel domain U associated to the hypersurface $\mathcal{H}$.

For $\eta = a_1G_1 + a_2G_2 + X \in J\mathfrak{b}$, $X \in \mathfrak{x}_{12}^+$, we define

$$h(\eta) = a_1(\mu a_2)^s - \frac{1}{2}(\mu a_2)^{s-1}\langle X, X\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product of the metric Lie algebra $\mathfrak{u}_0$. It is an easy computation to check that h is invariant under the linear action of $\mathfrak{b}_0$ on $\vee^d(J\mathfrak{b})^*$ induced by $\rho|\mathfrak{b}_0 : \mathfrak{b}_0 \to \mathfrak{gl}(J\mathfrak{b})$. Hence $\mathcal{H} = R(\mathcal{B}_0)JB_0$ is contained in $\mathcal{H}_1(h)$. Let us now consider the orbit map

$$\varphi : \mathcal{B}_0 \to \mathcal{H}\,, \quad b \mapsto R(b)JB_0\,.$$

We check that the Euclidean scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{b}_0 = T_e\mathcal{B}_0$ is

$$\langle\cdot,\cdot\rangle \quad = \quad \frac{1}{d}(\varphi^*g)_e\,, \tag{5}$$

where $g = -\frac{1}{d}\partial^2 h$ is the canonical metric of the hypersurface $\mathcal{H}$ defined by the basic polynomial h. We compute

$$\begin{aligned}
(\varphi^*g)_e(H_1 - \mu H_2, H_1 - \mu H_2) &= g_{JB_0}(G_1 - \mu G_2, G_1 - \mu G_2)\\
&= g(G_1, G_1) - 2\mu g(G_1, G_2) + \mu^2 g(G_2, G_2)\\
&= 0 + \frac{1}{d}(2\mu s(\mu\frac{1}{\mu})^{s-1}\mu - \mu^2 s(s-1)(\mu\frac{1}{\mu})^{s-2}\mu^2)\\
&= \frac{1}{d}(2\mu^2 s - \mu^4 s(s-1)) = \frac{1}{d}(2 - \frac{s-1}{s})\\
&= \frac{1}{d}(1+\mu^2) = \frac{1}{d}\langle H_1 - \mu H_2, H_1 - \mu H_2\rangle \quad \text{and}\\
(\varphi^*g)_e(X_-, X_-) &= g_{JB_0}(JX_-, JX_-)\\
&= \frac{1}{d}(\mu\frac{1}{\mu})^{s-1}\langle JX_-, JX_-\rangle = \frac{1}{d}\langle X_-, X_-\rangle\,,
\end{aligned}$$

where $X_- \in \mathfrak{r}_{12}^-$. This proves equation (5).

Now we extend the linear representation $\rho : \mathfrak{b} \to \mathfrak{gl}(J\mathfrak{b}) \subset \mathfrak{gl}((J\mathfrak{b})\otimes\mathbb{C})$ to an affine representation $\rho : \mathfrak{u}_0 \to \mathfrak{aff}((J\mathfrak{b})\otimes\mathbb{C})$ of the real Lie algebra $\mathfrak{u}_0$ on the complex vector space $(J\mathfrak{b})\otimes\mathbb{C} = J\mathfrak{b} + iJ\mathfrak{b}$. For $B \in \mathfrak{b}$ we define

$$\rho(JB) = JB \in J\mathfrak{b} \subset (J\mathfrak{b})\otimes\mathbb{C} \subset (J\mathfrak{b})\otimes\mathbb{C} + \mathfrak{gl}((J\mathfrak{b})\otimes\mathbb{C}) = \mathfrak{aff}((J\mathfrak{b})\otimes\mathbb{C}) .$$

The corresponding extension of $R : \mathcal{B} \to GL(J\mathfrak{b}) \subset GL((J\mathfrak{b})\otimes\mathbb{C})$ will be also denoted by the same letter:

$$R : \mathcal{U}_0 \to Aff((J\mathfrak{b})\otimes\mathbb{C}) .$$

The tube domain $U = J\mathfrak{b} + i\mathcal{V}$, $\mathcal{V} = \mathbb{R}^+\mathcal{H} \subset J\mathfrak{b}$, is precisely the orbit of the point $iJB_0 \in i\mathcal{V} \subset U$ under the affine representation R of the group $\mathcal{U}_0$ on $(J\mathfrak{b})\otimes\mathbb{C}$, that is: $U = R(\mathcal{U}_0)(iJB_0)$.

Now we prove that the scalar product $\langle\cdot,\cdot\rangle$ of the metric Lie algebra $\mathfrak{u}_0 = T_e\mathcal{U}_0$ is

$$\langle\cdot,\cdot\rangle \quad = \quad \frac{1}{d}(\phi^* g^c)_e ,$$

where g^c is the canonical metric of the tube domain U and

$$\phi : \mathcal{U}_0 \to U , \quad u \mapsto R(u)(iJB_0)$$

is the orbit map.

First we remark that that

$$(d\phi)_e : \mathfrak{u}_0 = \mathfrak{b} + J\mathfrak{b} \to T_{iJB_0}U = J\mathfrak{b} + iJ\mathfrak{b}$$

maps the orthogonal subspaces $\mathfrak{b}$ and $J\mathfrak{b}$ of $\mathfrak{u}_0$ onto the orthogonal subspaces $iJ\mathfrak{b}$ and $J\mathfrak{b}$ of $T_{iJB_0}U$ respectively:

$$(d\phi)_e|\mathfrak{b} : \mathfrak{b} \to iJ\mathfrak{b} , \quad B \mapsto iJB$$

$$(d\phi)_e|J\mathfrak{b} = Id : J\mathfrak{b} \to J\mathfrak{b} .$$

Moreover, J and multiplication by $-i$ are orthogonal endomorphisms of $\mathfrak{u}_0$ and $T_{iJB_0}U$ respectively which make the following diagram commutative:

$$\begin{array}{ccc} \mathfrak{u}_0 & \xrightarrow{(d\phi)_e} & T_{iJB_0}U \\ \downarrow J & & \downarrow \quad -i \\ \mathfrak{u}_0 & \xrightarrow{(d\phi)_e} & T_{iJB_0}U \end{array}$$

Now since J interchanges the subspaces $\mathfrak{b}$ and $J\mathfrak{b}$ of $\mathfrak{u}_0$ and $-i$ interchanges the subspaces $iJ\mathfrak{b}$ and $J\mathfrak{b}$ of $T_{iJB_0}U$, it is sufficient to check that the map $(d\phi)_e|\mathfrak{b} : \mathfrak{b} \to iJ\mathfrak{b} \subset T_{iJB_0}U$ becomes a linear isometry after scaling it by $\sqrt{d}$. We have checked this already on $\mathfrak{b}_0 \subset \mathfrak{b} = \mathbb{R}B_0 + \mathfrak{b}_0$, s. (5). The vector $(d\phi)_e B_0 = iJB_0$ is precisely the radial vector at the point $iJB_0 \in i\mathcal{V}$, so it has unit length with respect to the canonical metric g^c, thanks to Proposition 1.1. On the other hand,

$\langle B_0, B_0\rangle = 1 + \frac{1}{\mu^2} = 1 + s = d$. This proves that $(\phi^* g^c)_e = \frac{1}{d}\langle\cdot,\cdot\rangle$. All remaining statements are easily checked. $\Box$

2.2. **Classification of transitive reductive group actions on pseudo Riemannian hypersurfaces.** Now we explain how it is possible to use results from invariant theory for reductive algebraic groups to classify pseudo Riemannian hypersurfaces of degree d admitting a transitive reductive algebraic group of linear transformations. We will give a complete classification for (quadratic and) cubic hypersurfaces, which is the case relevant to homogeneous special geometry.

Let G be a real algebraic reductive group and V an (algebraic) G-module. Assume that the connected component G_0 acts transitively on the hypersurface $\mathcal{H} \subset \mathcal{H}_1(h) \subset V$, where h is the basic polynomial, s. Def. 1.1. Then G preserves h, i.e. h is a G-invariant. In fact, any G-invariant is of the form ch^k, $c \in \mathbb{C}$, $k \in \mathbb{N}_0$.

Consider now the one dimensional extension $G^1 = \mathbb{R}^* \times G$ and on V the canonical structure of G^1-module, where $\mathbb{R}^*$ acts by the standard scalar multiplication on V. The G^1-module V has an open orbit, it is a **prehomogeneous vector space** (P.V.) in the terminology of [S-K].

In [S-K] T. Kimura and M. Sato have classified all *irreducible* P.V.s for *complex* algebraic reductive groups. If we consider the $(G^1)^{\mathbb{C}}$-module $V^{\mathbb{C}}$ and assume that it is irreducible, then it must appear in the classification [S-K]. The $(G^1)^{\mathbb{C}}$-module $V^{\mathbb{C}}$ is irreducible if and only if the $G^{\mathbb{C}}$-module $V^{\mathbb{C}}$ is irreducible. We will assume first that this condition is satisfied and treat the reducible case later. Without restriction of generality we also assume that $G^{\mathbb{C}}$ acts almost faithfully, i.e. with only discrete kernel. Then $V^{\mathbb{C}}$ is an irreducible almost faithful P.V. of $(G^1)^{\mathbb{C}}$ and by a theorem of Cartan (Thm. 1 of [S-K]) G must be semisimple. Remark that since h is a G-invariant, it must be a **relative** G^1- and $(G^1)^{\mathbb{C}}$-**invariant**, i.e. h is preserved up to scaling. A P.V. was called **regular** in [S-K] if there exists a relative invariant with not identically vanishing Hessian determinant. Remark that a one dimensional P.V. is always regular.

Lemma 2.11. *If $\mathcal{H} = G_0 v \subset \mathcal{H}_1(h) \subset V$ is a pseudo Riemanian hypersurface in the G-module $V \ni v$, then $V^{\mathbb{C}}$ is a regular P.V. of $(G^1)^{\mathbb{C}}$.*

Proof: If $\partial^2 h|T_v\mathcal{H}$ is nondegenerate, then $\partial^2 h|_v$ is nondegenerate, since h is a homogeneous polynomial of degree $d \geq 2$. $\Box$

An analogeous discussion applies to complex Riemannian hypersurfaces $\mathcal{H} \subset V \cong \mathbb{C}^n$ admitting a transitive irreducible linear action of a complex algebraic reductive group G. We can easily deduce the classification of such group actions from the classification of P.V.s in [S-K]. We will consider two representations $R : G \to GL(V)$ and $R' : G' \to GL(V')$ as **equivalent** if there is an isomorphism $GL(V) \xrightarrow{\sim} GL(V')$ mapping $R(G)$ onto $R(G')$.

Theorem 2.12. *The following list gives, up to equivalence, all irreducible G-modules V of a connected complex algebraic reductive group G which induce a transitive action on a complex Riemannian cubic hypersurface $\mathcal{H} \subset \mathcal{H}_1(h) \subset V$. The quadruples (V^n, G, h, K) below contain the G-module V of (complex) dimension n, the basic cubic polynomial h (unique up to multiplicative constant) and the isotropy group K of a point $v \in \mathcal{H}$ as abstract group.*
1) $(V^9 = U \otimes \mathbb{C}^3,\ H \times SL(3,\mathbb{C}),\ \det,\ H)$, *where* $U = \mathbb{C}^3$ *is the standard 3-dimensional module of* $H = SL(3,\mathbb{C})$, $H = SO(3,\mathbb{C})$ *or* $H = \{e\}$ *and* $\mathbb{C}^3$ *is the standard* $SL(3,\mathbb{C})$-*module.*
2) $(V^6 = \vee^2\mathbb{C}^3,\ SL(3,\mathbb{C}),\ \det,\ SO(3,\mathbb{C}))$,
3) $(V^{15} = \wedge^2\mathbb{C}^6,\ SL(6,\mathbb{C}),\ Pff,\ Sp(3,\mathbb{C}))$, *where* Pff *is the Pfaffian of a skew symmetric* 6×6-*matrix:*

$$Pff(A) = \sum_{\sigma \in S_6} sgn(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} a_{\sigma(5)\sigma(6)}\,, \quad A = (a_{ij})\,,\ i,j = 1,\dots,6\,.$$

(Here we use the notation $Sp(n,\mathbb{C})$ *for the symplectic group of* $\mathbb{C}^{2n}$ *as in* [S-K]*.)*
4) $(V^{27} = Herm_3(\mathbb{O}) \otimes \mathbb{C} = Herm_3(\mathcal{C}) \otimes \mathbb{C},\ E_6,\ \det,\ F_4)$, *where* $Herm_3(\mathbb{O})$ *(resp.* $Herm_3(\mathcal{C})$*) denotes the Hermitian* 3×3 *matrices over the octonions (resp. over the real split Cayley algebra* $\mathcal{C}$*), which is an irreducible module of the real form* $E_6^{(-26)}$ *(resp.* $E_6^{(6)}$*) of* E_6*.*

Now we classify all real G-modules V of real algebraic reductive groups G such that the connected component G_0 acts transitively on a cubic pseudo Riemannian hypersurface $\mathcal{H} \subset \mathcal{H}_1(h) \subset V$ under the condition that the $G^{\mathbb{C}}$-module $V^{\mathbb{C}}$ is irreducible. If the last condition is satisfied, we shall say that the module (V,G) is **totally irreducible**. It is clear that under these conditions the complexified module $(V^{\mathbb{C}}, G^{\mathbb{C}})$ must be one of the list in Theorem 2.12. In other words, we must study the real forms of the complex modules classified in Theorem 2.12.

Let now V be a complex G-module of a connected complex algebraic reductive group G.

Definition 2.5. *A* **real structure** *for the module* (V,G) *is given by*

1) *a (real algebraic) antiholomorphic automorphism* $\tau : G \to G$ *of the complex algebraic group* G *and*
2) *a complex antilinear involution* $\tau : V \to V$ *of the complex vector space* V *(denoted by the same letter) such that*

$$\tau(av) = \tau(a)\tau(v)\,, \quad a \in G\,, \quad v \in V\,.$$

The pair (V^τ, G^τ) *is called a* **real form** *for* (V,G), *where the superscript* τ *stands for fixed point set of* τ.

Now we assume that $\mathcal{H} = Gv \subset \mathcal{H}_1(h)$ is a complex Riemannian hypersurface for some $v \in V$. Remark that then Gv is a complex Riemannian hypersurface for all $v \in V - S$, where the singular set S is Zariski closed. Given a real structure τ for (V, G) we may assume that h is real, i.e. $\overline{h(v)} = h(\tau v)$ for all $v \in V$. In fact, multiplying h by a suitable complex constant we may assume that $h^\tau := Re\, h|_{V^\tau} \neq 0$. Then the complex extension $h^\tau_{\mathbb{C}}$ is a G-invariant polynomial on V of the same degree as h and hence $h^\tau_{\mathbb{C}} = ch$ for some $c \in \mathbb{C}^*$. Finally, we can assume that $\mathcal{H} = Gv$ for some $v \in V^\tau$. Indeed, the singular set S is a Zariski closed proper subset of V and hence $V^\tau \not\subset S$. Under these assumptions we have the following lemma.

Lemma 2.13. *The real structure τ preserves the isotropy group K of G at v and the real form K^τ of K is precisely the isotropy group of v in G^τ. Moreover, the orbit $(G^\tau)_0 v \cong (G^\tau)_0/(K^\tau \cap (G^\tau)_0)$ of the connected component $(G^\tau)_0$ of G^τ coincides with the connected component $(\mathcal{H}^\tau)_0$ of v in the fixed point set $\mathcal{H}^\tau$, $\mathcal{H} = Gv$. Finally, $(\mathcal{H}^\tau)_0$ is pseudo Riemannian if and only if $\mathcal{H}$ is complex Riemannian.*

Proof: It is clear that $K^\tau = K \cap G^\tau$ is the isotropy group of v in G^τ; in particular $G^\tau v \cong G^\tau/K^\tau$ and $(G^\tau)_0 v \cong (G^\tau)_0/(K^\tau \cap (G^\tau)_0)$. From $v \in V^\tau$ it follows that $\tau K = K$, $\tau(Gv) = Gv$ and $G^\tau v \subset (Gv)^\tau$, so K^τ is a real form of K and

$$\begin{aligned} \dim_{\mathbb{C}} Gv &\geq \dim_{\mathbb{R}} (Gv)^\tau \geq \dim_{\mathbb{R}} G^\tau v \\ &= \dim_{\mathbb{R}} G^\tau/K^\tau = \dim_{\mathbb{C}} G/K = \dim_{\mathbb{C}} Gv \,. \end{aligned}$$

This implies $\dim(Gv)^\tau = \dim G^\tau v$ and $(G^\tau)_0 v = ((Gv)^\tau)_0$. The last statement of the lemma is trivial since we are assuming that the basic polynomial h is real. □

Theorem 2.14. *The totally irreducible modules of real algebraic reductive groups inducing a transitive action of the group's identity component on a cubic pseudo Riemannian hypersurface are (up to equivalence) the real forms (V^τ, G^τ) of the complex modules (V, G) classified in Theorem 2.12. We list the triples (V^τ, G^τ, K^τ), where K^τ is the isotropy group of $v \in V^\tau$; $\mathcal{H} = Gv \subset V$ the underlying complex Riemannian hypersurface. The corresponding homogeneous pseudo Riemannian cubic hypersurfaces are all locally symmetric. (The real cubic hypersurfaces with positively or negatively defined canonical metric correspond to the triples (V^τ, G^τ, K^τ) with compact isotropy group K^τ.)*

1)$(U^0 \otimes \mathbb{R}^3,\ H^0 \times SL(3, \mathbb{R}),\ H^0)$, where $U^0 = \mathbb{R}^3$ is the standard 3-dimensional module of $H^0 = SL(3, \mathbb{R})$, $SO(1, 2)$, $SO(3)$ or $\{e\}$.

2)$(\vee^2 \mathbb{R}^3,\ SL(3, \mathbb{R}),\ SO(3))$,

3) $(\wedge^2 \mathbb{R}^6,\ SL(6, \mathbb{R}),\ Sp(3, \mathbb{R}))$, $((\wedge^2_{\mathbb{C}} \mathbb{H}^3)^\tau,\ SL(3, \mathbb{H}),\ Sp(3))$, where the real structure τ on $\wedge^2_{\mathbb{C}} \mathbb{H}^3 \subset \mathbb{H}^3 \otimes_{\mathbb{C}} \mathbb{H}^3$ is the square of the quaternionic structure on the complex vector space $\mathbb{H}^3$.

4) $(Herm_3(\mathbb{O}),\ E_6^{(-26)},\ F_4^{(-52)})$, $(Herm_3(\mathcal{C}),\ E_6^{(6)},\ F_4^{(4)})$, for the notation s. Theorem 2.12, 4).

Proof: The fact that the above list gives all real forms for the complex modules of Theorem 2.12 can be checked using Tits' tables [Tt]. □

Now we study *reducible* G-modules of connected complex algebraic reductive groups G inducing a transitive G-action on a nondegenerate hypersurface. We use the following theorem as lemma.

Lemma 2.15. [S-K] *Let (V, G) be a prehomogeous vector space of a connected complex algebraic reductive group G. Then the following conditions are equivalent.*

(i) *(V, G) is regular.*
(ii) *The generic isotropy group G_v, $v \in V$, is reductive.*
(iii) *The singular set $S = V - Gv$ is a hypersurface.*

Theorem 2.16. *Let (V, G) be a faithful module of a simply connected and complex algebraic reductive group G acting transitively on a nondegenerate cubic hypersurface $\mathcal{H} \subset V$ with basic polynomial h. Consider its canonical extension (V, G^1) to a regular prehomogeneous vector space V of the group $G^1 = \mathbb{C}^* \times G$. Then (V, G^1) is the sum of at most 3 irreducible regular P.V.s. Moreover, only the three following possibilities can occur.*

(1) *The module (V, G^1) and the basic polynomial h are irreducible and G is semisimple.*
(2) *The G^1-module V is the direct sum $V = V_1 \oplus V_2$ of two irreducible P.V.s and $\dim V_1 = 1$. The polynomial h is the product $h = lq$ of a linear function l on V_1 and a quadratic G-invariant q on V_2. Finally, $G \cong \mathbb{C}^* \times G'$, where G' is semisimple or trivial.*
(3) *The G^1-module V is the direct sum of three one-dimensional P.V.s and $G \cong \mathbb{C}^* \times \mathbb{C}^*$.*

Proof: First we remark that the G^1-module V is the direct sum $V = \oplus_{j=1}^r V_j$ of irreducible submodules, because G^1 is a reductive algebraic group. Moreover, the irreducible summands V_j are again prehomogeneous vector spaces.

From the assumption that G acts transitively on a *nondegenerate* hypersurface it follows that (V, G^1) is a *regular* P.V. and by Lemma 2.15 the generic isotropy group G_v is reductive. This implies that the generic isotropy group for the irreducible summands V_j is also reductive and hence the V_j are regular P.V.s. In particular, each of them admits a non-constant relative invariant h_j.

We have already remarked that (V, G^1) being irreducible and faithful implies the semisimplicity of G. Now we prove that if (V, G^1) is irreducible, then the basic polynomial h is also irreducible. In fact, assume that $h = fg$ is the product of two non-constant polynomials. Then f and g are relative G^1-invariants and one of them must be linear. However, a module V with a linear relative invariant l splits as $V = V_1 \oplus V_2$, where $\dim V_1 = 1$ and $V_2 = ker\, l$.

Any simply connected complex algebraic reductive group G is the direct product $G = Z \times G'$ of its centre Z, which is an algebraic torus, and a semisimple group G'.

If the G^1-module $V = \oplus_{j=1}^r V_j$ is the sum of r irreducible submodules V_j, then it has r algebraically independent relative G^1-invariants $h_1, \ldots, h_r$, which are necessarily G'-invariant. Remark that at a generic point $v \in V$ the differentials $dh_1, \ldots, dh_r$ are linearly independent. Since G acts transitively on a hypersurface in V, the group G' has an orbit of codimension $\leq \dim Z + 1$. This implies $r \leq \dim Z + 1$. On the other hand, since $\mathbb{C}^* \times Z$ acts faithfully on V, we have that $1 + \dim Z \leq r$, hence

$$r \quad = \quad \dim Z + 1\,.$$

Now we show that $r \leq 3$ and hence $\dim Z = r - 1 \leq 2$. A homogeneous polynomial $f = f_1 \cdots f_r$ on V which is a product of homogeneous polynomials f_j of degree d_j on V_j will be called a **monomial of degree** $(d_1, \ldots, d_r)$, with respect to the G-invariant decomposition $V = \oplus_{j=1}^r V_j$. Any homogeneous polynomial f on V of degree d can be decomposed into monomials f_δ of degree $\delta = (d_1, \ldots, d_r) \in \mathbb{N}_0^r$, $|\delta| = d_1 + \cdots + d_r = d$:

$$f \quad = \quad \sum_{|\delta| = d} f_\delta\,.$$

If f is G-invariant, then the f_δ are G-invariant. It follows that the basic cubic polynomial h is a monomial $h = h_1 \cdots h_r$ and the degrees $d_j \geq 0$ of the homogeneous polynomials h_j must add up to three: $d_1 + \cdots + d_r = 3$. Now we use the fact that h has nondegenerate Hessian form $\partial^2 h$ to conclude that $d_j \geq 1$ for all $j = 1, \ldots, r$. This implies $r \leq 3$, as claimed above.

If $r = 1$, V and hence h is irreducible. If $r = 2$, $h = h_1 h_2$, where we can assume that $h_1 = l$ is a linear polynomial on V_1 and $h_2 = q$ is a quadratic polynomial on V_2. From the irreducibility of V_2 (and nondegeneracy of $\partial^2 h$) it follows that either q is irreducible (and a nondegenerate quadratic form) or it is the square of a linear polynomial and $\dim V_2 = 1$. The second case implies $G' = \{e\}$ by the faithfullness of the G-action. Finally if $r = 3$, $h = h_1 h_2 h_3$ is the product of three relative G^1-invariant polynomials h_j on V_j. In particular, $\dim V_j = 1$ and $G' = \{e\}$. $\square$.

Theorem 2.16 reduces the classification of homogeneous pseudo Riemannian cubic hypersurfaces of reductive real algebraic groups G to three cases. The first case is that of totally irreducible G-modules and was treated in Theorem 2.14. The third case corresponds to the real forms of the flat complex Riemannian cubic hypersurface in $\mathbb{C}^3$ with basic polynomial $h(z^1, z^2, z^3) = z^1 z^2 z^3$ and transitive action of the algebraic torus $\mathbb{C}^* \times \mathbb{C}^*$. The second case reduces to the classification of homogeneous pseudo Riemannian quadratic hypersurfaces, which we give now.

Every homogeneous quadratic polynomial q on $\mathbb{K}^n$ such that $\partial^2 q$ is nondegenerate is of the form

$$q(X) = \langle X, X \rangle\,, \quad X \in \mathbb{K}^n\,,$$

for some nondegenerate symmetric $\mathbb{K}$-bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{K}^n$. If $\mathbb{K} = \mathbb{C}$, we call $\mathcal{H} = \mathcal{H}_1(q)$ a **complex sphere**. If $\mathbb{K} = \mathbb{R}$, the connected components $\mathcal{H}$ of $\mathcal{H}_1(q)$

are called **pseudo spheres**. The canonical metric g of $\mathcal{H}$ at $X_0 \in \mathcal{H}$ is precisely

$$g_{X_0}(X, X) = -\langle X, X \rangle\,, \quad X \in T_{X_0}\mathcal{H} = \{X \in \mathbb{K}^n | \langle X_0, X \rangle = 0\}\,.$$

In the real case this shows that $(\mathcal{H}, g)$ has signature $(l, k-1)$ if $\langle \cdot, \cdot \rangle$ has signature (k, l). In particular, the canonical metric of the hyperboloid $\mathcal{H}_1(q) \subset \mathbb{R}^{1,n-1}$ is positively defined and that of the sphere $\mathcal{H}_1(q) \subset \mathbb{R}^{n,0}$ is negatively defined; where $q(X) = x_1^2 + \cdots x_k^2 - x_{k+1}^2 - \cdots - x_n^2$ for $\mathbb{R}^{k,l}$, $k + l = n$.

By Witt's Theorem, $SO_0(k, l)$ acts transitively on the (connected) pseudo spheres $\mathcal{H} \subset \mathbb{R}^{k,l}$. The identity component of the isotropy group of $SO_0(k, l)$ at $(1, 0, \dots, 0)$ is $SO_0(k-1, l)$. Moreover, the canonical metric defines on $\mathcal{H}$ the structure of pseudo Riemannian symmetric space. The symmetry $\sigma_{X_0} : \mathcal{H} \to \mathcal{H}$ at $X_0 \in \mathcal{H}$ is given by:

$$\sigma_{X_0}(X) = -\left(X - \frac{\langle X, X_0 \rangle}{\langle X_0, X_0 \rangle} X_0\right) + \frac{\langle X, X_0 \rangle}{\langle X_0, X_0 \rangle} X_0\,.$$

We sum up our discussion.

Proposition 2.17. *Any pseudo Riemannian quadratic hypersurface (always with the canonical metric) is a pseudo sphere $\mathcal{H} \subset \mathbb{R}^{k,l}$ and, in particular, a pseudo Riemannian symmetric space. Its full connected group of linear automorphisms is $SO_0(k, l)$, which acts irreducibly on $\mathbb{R}^{k,l}$.*

To round up our discussion, we give the classification of transitive linear algebraic actions of reductive groups on complex and on pseudo Riemannian quadratic hypersurfaces. As before, the complex case can be easily extracted from [S-K]:

Theorem 2.18. *Any G-module V of a complex algebraic reductive group which induces a transitive action on a nondegenerate quadratic hypersurface is (up to equivalence) one in the following list. All of them are irreducible and hence their basic quadratic polynomial q is the (up to scaling) unique G-invariant quadratic form on V. The corresponding quadratic hypersurfaces are complex spheres. The last entry K in the triples (V^n, G, K) is, as before, the generic isotropy group as abstract group and $n = \dim V$.*
1) $(V^4 = U \times \mathbb{C}^2,\ H \times SL(2, \mathbb{C}),\ H)$, $U = \mathbb{C}^2$, $H = SL(2, \mathbb{C})$ or $= \{e\}$,
2) $(V^4 = \vee^2\mathbb{C}^2,\ SL(2, \mathbb{C}),\ SO(2, \mathbb{C}))$,
3) $(V^{4n} = \mathbb{C}^{2n} \otimes \mathbb{C}^2,\ Sp(n, \mathbb{C}) \times SL(2, \mathbb{C}),\ Sp(n-1, \mathbb{C}) \times Sp(1, \mathbb{C}))$, $n \geq 2$. In this case the basic polynomial q can be described as follows. Consider $\mathbb{C}^{2n} \otimes \mathbb{C}^2$ as vector space of complex $(2n) \times 2$-matrices $A = (a_{ij})$, then

$$q(A) = Pff(A^t J A)\,, \quad J = \begin{pmatrix} 0 & \mathbf{1_n} \\ -\mathbf{1_n} & 0 \end{pmatrix}\,,$$

is the Pfaffian of the skew symmetric matrix $A^t J A$.
4) $(V^n = \mathbb{C}^n,\ SO(n, \mathbb{C}),\ SO(n-1, \mathbb{C}))$,
5) $(V^8 = $ spinor module$,\ Spin(7, \mathbb{C}),\ G_2)$,

6) (V^{16} *= spinor module,* $Spin(9,\mathbb{C})$, $Spin(7,\mathbb{C})$),
7) (V^7 *with highest weight* Λ_2, G_2, $SL(3,\mathbb{C})$).

Theorem 2.19. *The real algebraic G-modules V of reductive groups G with transitive action of G_0 on a quadratic pseudo Riemannian hypersurface $\mathcal{H} \subset V$ are obtained from the real forms of the complex modules in Theorem 2.18 and are listed below. The hypersurface $\mathcal{H} \subset \mathcal{H}_1(q) \subset (V, q)$ is a pseudo sphere in the pseudo Euclidean vector space (V, q), where q is a real basic G-invariant quadratic polynomial of the irreducible G-module V.*
1) $(\mathbb{R}^2 \times \mathbb{R}^2,\ H^0 \times SL(2,\mathbb{R}))$, *where* $H^0 = SL(2,\mathbb{R})$ *or* $= \{e\}$, $(\mathbb{R}^4,\ SO(4))$, *cf. 4),*
2) $(\vee^2\mathbb{R}^2,\ SL(2,\mathbb{R}))$,
3) $(\mathbb{R}^{2n} \otimes \mathbb{R}^2,\ Sp(n,\mathbb{R}) \times SL(2,\mathbb{R}))$, $(\mathbb{H}^n,\ Sp(n-l,l) \times Sp(1))$, $0 \le l \le n \ge 2$,
4) $(\mathbb{R}^{k,l},\ SO(k,l))$, $k + l = n$,
5) spinor module of $Spin(7,0)$ *and* $Spin(3,4)$ *and semi spinor modules of* $Spin(0,7)$ *and* $Spin(4,3)$,
6) spinor module of $Spin(8,1)$, $Spin(4,5)$ *and* $Spin(0,9)$ *and semi spinor modules of* $Spin(1,8)$, $Spin(5,4)$ *and* $Spin(9,0)$,
7) $(Im\,\mathbb{O},\ G_2^{(-14)} = Aut\,\mathbb{O})$, $(Im\,\mathcal{C},\ G_2^{(2)} = Aut\,\mathcal{C})$, *where* $Im\,\mathbb{O}$ *(resp.* $Im\,\mathcal{C}$*) denotes the imaginary part and* $Aut\,\mathbb{O}$ *(resp.* $Aut\,\mathcal{C}$*) the full irreducible automorphism group of the octonions (resp. of the real split Cayley algebra).*

Proof: The real forms are obtained again using tables, s. [Tt] and [L-M]. □

2.3. **Appendix: Special Kähler submanifolds of Alekseevsky's quaternionic Kähler manifolds and their pseudo Riemannian analogues.** First of all we explain how one can obtain pseudo Kählerian versions of the cubic (and also of the quadratic) normal J-algebras classified in Theorem 2.8. Consider e.g. the cubic normal J-algebras $(\mathfrak{u}_0(p), \langle\cdot,\cdot\rangle, J)$ of rank 2. If $p \neq 0$, we can define a new pseudo Kähler Lie algebra $(\mathfrak{u}_0(p)', \langle\cdot,\cdot\rangle', J)$ changing the scalar product on the subspace $\mathfrak{x}_1 \subset \mathfrak{u}_0(p)$ only by a sign. The formulas defining the Lie bracket on $\mathfrak{u}_0(p)'$ are the same as for $\mathfrak{u}_0(p)$ only the scalar product occuring in these formulas is substituted by $\langle\cdot,\cdot\rangle'$.

Recall (Theorem 2.3) that to every isometric map $\psi : \mathfrak{x}_{23}^- \times \mathfrak{x}_{12}^- \to \mathfrak{x}_{13}^-$ of Euclidean vector spaces $\mathfrak{x}_{23}^-$, $\mathfrak{x}_{12}^-$ and $\mathfrak{x}_{13}^-$ we can associate a normal J-algebra $\mathfrak{u}_0(\psi)$ of type I. If moreover $\mathfrak{x}_{23}^- = 0$ or if $\dim \mathfrak{x}_{12}^- = \dim \mathfrak{x}_{13}^-$, then $\mathfrak{u}_0(\psi)$ is a cubic normal J-algebra, s. Theorem 2.8. Similarly, to every isometric map $\psi : \mathfrak{x}_{23}^- \times \mathfrak{x}_{12}^- \to \mathfrak{x}_{13}^-$ of *pseudo* Euclidean vector spaces $\mathfrak{x}_{23}^-$, $\mathfrak{x}_{12}^-$ and $\mathfrak{x}_{13}^-$ we can associate a *pseudo* Kählerian Lie algebra $\mathfrak{u}_0(\psi)$ and we have the following proposition.

Proposition 2.20. *The pseudo Kähler Lie algebra $(\mathfrak{u}_0(p)', \langle\cdot,\cdot\rangle', J)$ is cubic. Let $\psi : \mathfrak{x}_{23}^- \times \mathfrak{x}_{12}^- \to \mathfrak{x}_{13}^-$ be an isometric map of pseudo Euclidean vector spaces $\mathfrak{x}_{23}^-$, $\mathfrak{x}_{12}^-$ and $\mathfrak{x}_{13}^-$ and assume that ψ is special or of order 0, s. Definition 2.3. Then the pseudo Kählerian Lie algebra $(\mathfrak{u}_0(\psi), \langle\cdot,\cdot\rangle, J)$ associated to ψ is cubic. In particular,*

$(\mathfrak{u}_0(p)', \langle\cdot,\cdot\rangle', J)$ and $(\mathfrak{u}_0(\psi), \langle\cdot,\cdot\rangle, J)$ are pseudo Kähler Lie algebras for special pseudo Kähler tube domains.

Proof: The proof is analogeous to the proof of Theorem 2.8. □

Next we recall some basic facts about Alekssevsky's quaternionic Kähler manifolds, s. [A], [C] and [A-C] for details.

Definition 2.6. *An* **Alekseevsky space** *is a quaternionic Kähler manifold M admitting a simply transitive (non-Abelian) splittable solvable group $\mathcal{L}$ of isometries.*

We can present M as metric Lie group $(\mathcal{L}, g)$ and consider its metric Lie algebra $(\mathfrak{l}, \langle\cdot,\cdot\rangle)$. The quaternionic Kähler structure of M induces a quaternionic structure $\mathfrak{q} = span\{J_1, J_2, J_3\}$ on the Euclidean vector space $(\mathfrak{l}, \langle\cdot,\cdot\rangle)$. The triple $(\mathfrak{l}, \langle\cdot,\cdot\rangle, \mathfrak{q})$ associated to the Alekseevsky space M is called its **Alekseevskian Lie algebra**. According to [A], [dW-VP] and [C] there are (up to symmetric spaces) 3 series of Alekseevsky spaces: $\mathcal{T}$, $\mathcal{W}$ and $\mathcal{V}$-spaces. Their Alekseevskian Lie algebras $(\mathfrak{l}, \langle\cdot,\cdot\rangle, \mathfrak{q})$ are constructed as quaternionic Kähler extensions of normal J-algebras $\mathfrak{u}$.

More precisely, given a normal J-algebra of the form $\mathfrak{u} = \mathfrak{f}_0 \overset{\perp}{\oplus} \mathfrak{u}_0$ which admits a so called **Q-representation** $T : \mathfrak{u} \to End(\tilde{\mathfrak{u}})$ we can canonically define the structure $(\mathfrak{l}, \langle\cdot,\cdot\rangle, \mathfrak{q})$ of Alekseevskian Lie algebra on the vector space $\mathfrak{l} = \mathfrak{u} + \tilde{\mathfrak{u}}$ such that $(\mathfrak{u}, \langle\cdot,\cdot\rangle|\mathfrak{u}, J = J_1|\mathfrak{u})$ is a Kählerian subalgebra, $\tilde{\mathfrak{u}} = J_2\mathfrak{u}$, $[\tilde{\mathfrak{u}}, \tilde{\mathfrak{u}}] \subset \mathfrak{u}$ and $[\mathfrak{u}, \tilde{\mathfrak{u}}] \subset \tilde{\mathfrak{u}}$ is given by T. We remark that the definition of Q-representation and this construction can be naturally generalized to pseudo Kählerian Lie algebras; the homogeneous spaces associated to $(\mathfrak{l}, \langle\cdot,\cdot\rangle, \mathfrak{q})$ being *pseudo* quaternionic Kähler manifolds.

The 3 series of $\mathcal{T}$, $\mathcal{W}$ and $\mathcal{V}$-spaces are defined by 3 series of normal J-algebras $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ and corresponding Q-representations. By direct comparison of these series of normal J-algebras $\mathfrak{u}_0$ with the cubic normal J-algebras (s. Theorem 2.8) one can easily check the following facts.

The Alekseevsky spaces $\mathcal{V}(\psi)$ are defined by the Q-representation of the normal J-algebras $\mathfrak{u} = \mathfrak{f}_0 \oplus \mathfrak{u}_0(\psi)$ for which $\mathfrak{u}_0(\psi)$ is a cubic normal J-algebra of rank 3 defined by a non zero special isometric map ψ.

For the spaces $\mathcal{T}(p)$, $p = 0, 1, 2, \ldots,$

$$\mathfrak{u}_0(p) = \mathfrak{e}(p+1, 1) \oplus \mathfrak{e}(1, \frac{1}{\sqrt{2}})$$

are the cubic normal J-algebras of rank 2, cf. Proposition 2.10.

Finally, for the spaces $\mathcal{W}(p,q) \cong \mathcal{W}(q,p)$, $p, q = 0, 1, 2, \ldots,$

$$\mathfrak{u}_0(p,q) = \mathfrak{e}(p+q+1, 1) \oplus \mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$$

are the cubic normal J-algebras of rank 3 associated to isometric maps of order zero.

The preceding facts motivate the following definition.

Definition 2.7. *Let $M = (\mathcal{L}, g)$ be an Alekseevsky space with Alekseevskian Lie algebra $(\mathfrak{l}, \langle\cdot,\cdot\rangle, \mathfrak{q})$, $\mathfrak{l} = \mathfrak{u} + \tilde{\mathfrak{u}}$, $\mathfrak{u} = \mathfrak{f}_0 \oplus \mathfrak{u}_0$ the decompositions introduced above and $(\mathcal{U}_0, g, \tilde{J})$ the Kähler Lie group associated to the normal J-algebra $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J = J_1|\mathfrak{u}_0)$. It is naturally identified with a (totally geodesic) Kähler submanifold of M, which is called the* **special Kähler submanifold** *of M. The normal J-algebra $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J = J_1|\mathfrak{u}_0)$ is called the* **special Kähler subalgebra** *of the Alekseevskian Lie algebra $(\mathfrak{l}, \langle\cdot,\cdot\rangle, \mathfrak{q})$.*

We remark that to the cubic pseudo Kählerian Lie algebras of Proposition 2.20 one can associate pseudo Riemannian analogues of the Alekseevsky spaces.

Proposition 2.21. *The cubic pseudo Kählerian Lie algebras $(\mathfrak{u}_0(p)', \langle\cdot,\cdot\rangle', J)$ and $\mathfrak{u}_0(\psi)$ associated to a an isometric map ψ of pseudo Euclidean vector spaces, which is special or of zero order, admit a Q-representation. In particular, to any such map ψ we can associate a homogeneous pseudo quaternionic Kähler manifold.*

Proof: The proof does not depend on the signature of the scalar products, s. [A], [C]. $\square$

Remark 6: Being a totally geodesic Kählerian submanifold of a symmetric space, the special Kähler submanifold of a symmetric Alekseevsky spaces is Hermitian symmetric. It can be described as follows. Let $M = G/K$ be a symmetric Alekseevsky space, G its maximal connected isometry group and $K = Sp(1) \cdot H$ its isotropy subgroup; G and H are semisimple. We denote by $(\mathfrak{l}, \langle\cdot,\cdot\rangle, \mathfrak{q})$ the Alekseevskian Lie algebra corresponding to M and by $(\mathfrak{u}_0, \langle\cdot,\cdot\rangle, J)$ its special Kähler subalgebra. Then $\mathfrak{l}$ is isomorphic to the Iwasawa Lie algebra of G and $\mathfrak{u}_0$ is isomorphic to the Iwasawa Lie algebra of a non compact semisimple Lie group $H^0 \subset G$. The special Kähler submanifold $U \subset M$ is the orbit of the point $K = eK \in M = G/K$ under the group H^0 and is a Hermitian symmetric space of non compact type.

Consider the twistor space $Z = G/(U(1) \cdot H)$ of M. It carries a natural structure of homogeneous complex contact manifold, s. [W]. The contact hyperplane $D_S \subset T_S Z$, $S \in Z$, carries an H-invariant complex symplectic structure ω. The representation of H on D_S is irreducible, preserves this complex symplectic structure and the orbit of the highest root vector is the base of a Lagrangean cone $\hat{\mathcal{C}}$. Its projectivization $P(\hat{\mathcal{C}})$ is isomorphic to the compact dual U^* of the Hermitian symmetric space U.

We can consider the action of the complex semisimple Lie group $H^{\mathbb{C}}$ on the complex symplectic vector space D_S and on the compact Hermitian symmetric space $U^* \cong P(\hat{\mathcal{C}})$. Now we remark that the Lie group $H^0 = Isom(U)$ is a (non compact) real form of $H^{\mathbb{C}}$, hence we can also consider the action of H^0 on D_S and on $U^* \cong P(\hat{\mathcal{C}})$. In this way we can realize U as open orbit $U = P(\mathcal{C}) \subset P(\hat{\mathcal{C}})$ of H^0 on the compact Hermitian symmetric space $U^* \cong P(\hat{\mathcal{C}})$.

One can check that D_S carries a H^0-invariant real structure τ, such that the canonical special Kähler metric of the projectivized cone $P(\mathcal{C})$ defined by the data (D_S, ω, τ)

coincides with the (nonpositively curved) Hermitian symmetric metric given by the inclusion $U \subset M$.

It is known[1] that the compact projectivized cone $P(\hat{\mathcal{C}}) \cong U^*$ is precisely the projectivization of the cone $\hat{\mathcal{C}} \subset T_S Z^*$ of tangent directions to minimal rational curves through a point $S \in Z^*$ in the twistor space Z^* of the Wolf space $M^* = G^*/K$. $M^* = G^*/K$ is the compact symmetric quaternionic Kähler manifold which is dual to the symmetric Alekseevsky space $M = G/K$.

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