AN INTERPOLATION THEOREM FOR HOLOMORPHIC AUTOMORPHISMS OF $\mathbf{C}^n$

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Abstract

We construct automorphisms of $\mathbf{C}^n$ which map certain discrete sequences one onto another with prescribed finite jet at each point, thus solving a general Mittag-Leffler interpolation problem for automorphisms. Under certain circumstances, this can be done while also approximating a given automorphism on a compact set.

&0. Introduction

Let $\mathbf{C}^n$ be complex Euclidean space of dimension n with coordinates $z = (z_1, \ldots, z_n)$. We shall always assume $n > 1$. We denote by $\mathrm{Aut}\mathbf{C}^n$ the group of all holomorphic automorphisms of $\mathbf{C}^n$, and by $\mathrm{Aut}_1\mathbf{C}^n$ the group of automorphisms $F \in \mathrm{Aut}\mathbf{C}^n$ with complex Jacobian one: $JF \equiv 1$. For some recent developments regarding these groups see the survey [2]. By $\mathbf{B}$ we denote the open unit ball in $\mathbf{C}^n$.

In this paper we construct automorphisms of $\mathbf{C}^n$ which map certain discrete sequences one onto another with prescribed finite jet at each point, thus solving a general Mittag-Leffler interpolation problem for automorphisms.

In approaching such an interpolation problem one must consider the limitations of automorphisms in mapping one sequence to another. Rosay and Rudin [5] showed that, in general, one cannot map an infinite discrete set in $\mathbf{C}^n$ onto another discrete set by an automorphism of $\mathbf{C}^n$. In fact, the infinite discrete sets in $\mathbf{C}^n$ form uncountably many different equivalence classes under this relation [5, sect. 5]. They introduced the following notion:

Definition. *A discrete infinite sequence $\{a_j\} \subset \mathbf{C}^n$ (without repetition) is tame if there exists a holomorphic automorphism $F \in \mathrm{Aut}\mathbf{C}^n$ such that*

$$F(a_j) = je_1 = (j, 0, \ldots, 0), \qquad j = 1, 2, 3, \ldots$$

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The sequence $\{a_j\}$ is very tame if the above holds for some $F \in \mathrm{Aut}_1\mathbf{C}^n$.

Rosay and Rudin [5] showed that points in a tame sequence are permutable by automorphisms of $\mathbf{C}^n$. Hence one can speak about the tameness of an infinite discrete set. On the other hand, they showed that there exist discrete sets $E \subset \mathbf{C}^n$ such that the only nondegenerate holomorphic map $F: \mathbf{C}^n \to \mathbf{C}^n$ satisfying $F(\mathbf{C}^n \backslash E) \subset \mathbf{C}^n \backslash E$ is the identity map [5, theorem 5.1].

In view of this, it is reasonable to consider tame sequences in order to get positive results on jet interpolation. The main result of this paper is the following.

0.1 Theorem. *Assume that $n > 1$, that $\{a_j\}$ and $\{b_j\}$ are tame sequences in $\mathbf{C}^n$ (without repetitions), and that $P_j: \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic polynomial map of degree at most $m_j \geq 1$ such that $P_j(0) = 0$ and $JP_j(0) \neq 0$. Then there exists $F \in \mathrm{Aut}\mathbf{C}^n$ such that for every $j = 1, 2, 3, \ldots$ we have $F(a_j) = b_j$ and*

$$F(z) = b_j + P_j(z - a_j) + O(|z - a_j|^{m_j+1}), \quad z \to a_j. \tag{0.1}$$

If in addition the polynomials P_j satisfy

$$JP_j(z) = 1 + O(|z|^{m_j}), \quad z \to 0, \quad j = 1, 2, 3, \ldots, \tag{0.2}$$

and the sequences $\{a_j\}$ and $\{b_j\}$ are very tame, we may choose $F \in \mathrm{Aut}_1\mathbf{C}^n$.

A version of this theorem in which the degrees of the polynomials P_j are uniformly bounded can be found in [3, theorem 3.1]. In that result, with $a_j = b_j = je_1$ for all j, the automorphism F is a *finite* composition of shears and generalized shears. See the survey [2] for further information on shears and generalized shears.

Under certain circumstances we can obtain the above result while at the same time approximating a given automorphism on a polynomially convex set.

0.2 Theorem. *In addition to the hypotheses of theorem 0.1, assume that $\Phi \in \mathrm{Aut}\mathbf{C}^n$ is an automorphism and $K \subset \mathbf{C}^n \backslash \{a_j\}_{j=1}^{\infty}$ is a compact, polynomially convex set such that $\Phi(K) \subset \mathbf{C}^n \backslash \{b_j\}_{j=1}^{\infty}$. Then for each $\epsilon > 0$ there exists an $F \in \mathrm{Aut}\mathbf{C}^n$ satisfying theorem 0.1 and*

$$|F(z) - \Phi(z)| < \epsilon, \qquad z \in K. \tag{0.3}$$

If the volume preserving assumptions in theorem 0.1 hold and if $\Phi \in \mathrm{Aut}_1\mathbf{C}^n$, we may choose $F \in \mathrm{Aut}_1\mathbf{C}^n$.

The paper is organized as follows. In section 1 we recall a result concerning holomorphic automorphisms of $\mathbf{C}^n$ which have prescribed jets of finite order at a finite set of points, and which are close to the identity on a given polynomially convex set, as well as a result about convergence of compositions of automorphisms. Both of these results are proved in [3].

In section 2 we provide the proof of theorem 0.1. The outline of the proof is as follows. Since $\{a_j\}$ and $\{b_j\}$ are tame, we may first assume that they both lie in the z_1-axis. Next, if $\{c_j\}$ is a tame set in the z_1-axis, we can find $\Psi \in \mathrm{Aut}_1\mathbf{C}^n$ such that at each point a_j, Ψ agrees with the translation $z \mapsto c_j + (z - a_j)$ to order m_j. In fact, we can find such a Ψ which is a composition of three shears of the form $z \mapsto z + f(\lambda(z))v$, where f is entire on $\mathbf{C}$, $v \in \mathbf{C}^n\backslash\{0\}$, and λ is a $\mathbf{C}$-linear form with $\lambda(v) = 0$.

Hence it suffices to solve the following simpler problem: Given polynomials P_j as above, construct a tame sequence $\{c_j\}$ contained in the z_1-axis and $F \in \mathrm{Aut}\mathbf{C}^n$ such that F fixes each c_j and has prescribed jet P_j at c_j (with appropriate modifications in the volume-preserving case).

To construct F and $\{c_j\}$, we use an inductive procedure. We take $c_1 = 3e_1$ and use the results of section 1 to find an automorphism F_1 with prescribed jet P_1 at c_1 and which is near the identity on $K_1 = 2\overline{\mathbf{B}}$, where $\mathbf{B}$ is the unit ball in $\mathbf{C}^n$. We then choose a point c_2 on the z_1-axis so that c_2 and $F_1(c_2)$ lie outside the convex hull K_2 of $2\overline{\mathbf{B}} \cup K_1 \cup F_1(2\overline{\mathbf{B}}) \cup \{c_1\}$. Again the results of section 1 allow us to find an automorphism Ψ_2 which is near the identity on K_2 and which agrees with the identity to order m_1 at c_1, which maps $F_1(c_2)$ to c_2, and which has an appropriate jet at $F_1(c_2)$. Then $F_2 = \Psi_2 \circ F_1$. We then repeat this procedure, choosing c_{k+1} so that c_{k+1} and $F_k(c_{k+1})$ lie outside the convex hull K_{k+1} of $(k+1)\overline{\mathbf{B}} \cup K_k \cup F_k((k+1)\overline{\mathbf{B}}) \cup \{c_k\}$, then choosing Ψ_k to be near the identity on K_{k+1}, equal to the identity to order m_j at c_j for $j \leq k$, mapping $F(c_{k+1})$ to c_{k+1} and having appropriate jet at $F_k(c_{k+1})$.

In this way we define a sequence of automorphisms $F_1, F_2, \ldots$ with $F_{k+1} = \Psi_{k+1} \circ F_k$. Then the second result of section 1 implies that this sequence converges to an automorphism of $\mathbf{C}^n$ with the desired properties.

In section 3 we prove theorem 0.2. We use the result of theorem 0.1, but must take extra care to insure that we can in addition approximate the given map. The main addition here is lemma 3.1: given a compact, polynomially convex set $K \subset \mathbf{C}^n$, an automorphism H of $\mathbf{C}^n$, and a discrete sequence $\{a_j\}$ contained in the z_1-axis outside K, there is an $R > 0$ and an automorphism $\Psi \in \mathrm{Aut}\mathbf{C}^n$ that approximates H on K and satisfies $\Psi(a_j) = (R + j)e_1$ for each j.

&1. Interpolation on a finite set and convergence of automorphisms

The results of this section are taken directly from [3]. We do not reproduce the proofs here. The first result concerns jet interpolation by automorphisms on a finite set.

1.1 Proposition. *Let $n > 1$. Assume that*

(a) *$K \subset \mathbf{C}^n$ is a compact, polynomially convex set,*

(b) *$\{a_j\}_{j=1}^k \subset K$ is a finite subset of K,*

(c) *p and q are arbitrary points in $\mathbf{C}^n\backslash K$ (not necessarily distinct),*

(d) *N is a nonnegative integer, and*

(e) *$P\colon \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic polynomial map of degree at most $m \geq 1$, satisfying $P(0) = 0$ and $JP(0) \neq 0$.*

Then for each $\epsilon > 0$ there exists an automorphism $F \in \mathrm{Aut}\mathbf{C}^n$ satisfying

(i) *$F(z) = q + P(z - p) + O(|z - p|^{m+1})$ as $z \to p$,* *(ii)* $F(z) = z + O(|z - a_j|^N)$ *as* $z \to a_j$ *for each* $j = 1, 2, \ldots, k$, *and*

(iii) $|F(z) - z| + |F^{-1}(z) - z| < \epsilon$ *for each* $z \in K$.

If in addition the polynomial map P *satisfies*

(e') $P(0) = 0$ *and* $JP(z) = 1 + O(|z|^m)$ *as* $z \to 0$,

then there exists a polynomial automorphism F *with* $JF = 1$ *satisfying (i)–(iii).*

The next result concerns convergence of a sequence of automorphisms. It gives sufficient conditions for such a sequence to converge to a biholomorphic map in some domain and describes the domain of convergence and the image of the limit map.

1.2 Proposition. *Let* D *be a connected open set in* $\mathbf{C}^n$ *which is exhausted by compact sets* $K_0 \subset K_1 \subset K_2 \subset \cdots \subset \cup_{j=0}^{\infty} K_j = D$ *such that* $K_{j-1} \subset \mathrm{Int} K_j$ *for each* $j \in \mathbf{N}$. *Choose numbers* ϵ_j *(*$j = 1, 2, 3, \ldots$*) such that*

$$0 < \epsilon_j < \mathrm{dist}(K_{j-1}, \mathbf{C}^n \backslash K_j) \quad (j \in \mathbf{N}), \qquad \sum_{j=1}^{\infty} \epsilon_j < \infty. \tag{1.1}$$

Suppose that for each $j = 1, 2, 3, \ldots$, Ψ_j *is a holomorphic automorphism of* $\mathbf{C}^n$ *satisfying*

$$|\Psi_j(z) - z| < \epsilon_j, \quad z \in K_j. \tag{1.2}$$

Set $\Phi_m = \Psi_m \circ \Psi_{m-1} \circ \cdots \circ \Psi_1$. *Then there is an open set* $\Omega \subset \mathbf{C}^n$ *such that* $\lim_{m \to \infty} \Phi_m = \Phi$ *exists on* Ω *(uniformly on compacts), and* Φ *is a biholomorphic map of* Ω *onto* D. *In fact,* $\Omega = \cup_{m=1}^{\infty} \Phi_m^{-1}(K_m)$.

Finally, we include some elementary observations about power series. If $F\colon U \subset \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic map, $a \in U$, and $m \geq 1$ an integer, we write

$$F(z) = F(a) + F_{m,a}(z - a) + O(|z - a|^{m+1}), \quad z \to a.$$

Thus $F_{m,a}$ is just the Taylor polynomial of F of order m at a without the constant term.

The following lemma is evident by composing the power series.

1.3 Lemma. *If* F *and* G *are nondegenerate holomorphic maps on certain open subsets of* $\mathbf{C}^n$, *with values in* $\mathbf{C}^n$, *then for each integer* $m \geq 1$ *we have*

$$(G \circ F)_{m,a}(z) = G_{m,F(a)} \circ F_{m,a}(z) + O(|z|^{m+1}), \quad z \to 0.$$

at each point a *where the composition* $G \circ F$ *is defined.*

This generalizes to composition of several maps. We also have

$$(F^{-1})_{m,F(a)} \circ F_{m,a}(z) = z + O(|z|^{m+1}), \quad z \to 0.$$

&2. Proof of the interpolation theorem

In this section we prove theorem 0.1. The main construction is contained in the following lemma.

2.1 Lemma. *Given P_j and m_j as in theorem 0.1, there exists a discrete sequence $\{c_j\}$ contained in the z_1-axis and $F \in \mathrm{Aut}\mathbf{C}^n$ such that for all j we have*

$$F(z) = c_j + P_j(z - c_j) + O(|z - c_j|^{m_j+1}), \quad z \to c_j. \tag{2.1}$$

If in addition each polynomial P_j satisfies (0.2), we may choose $F \in \mathrm{Aut}_1\mathbf{C}^n$.

Proof. For $j \geq 1$, let $\epsilon_j = 2^{-j}$. We will construct the sequence $\{c_j\}$ inductively and construct F as the limit of a composition of automorphisms, each chosen inductively.

Let $K_0 = \emptyset$ and $K_1 = 2\overline{\mathbf{B}}$. Let $c_1 = 3e_1$. By proposition 1.1 with $K = K_1$, there exists $\Psi_1 \in \mathrm{Aut}\mathbf{C}^n$ such that $\Psi_1(z) = c_1 + P_1(z - c_1) + O(|z - c_1|^{m_1+1})$, $z \to c_1$ and $|\Psi_1(z) - z| + |\Psi_1^{-1}(z) - z| < \epsilon_1$ for $z \in K_1$. If also P_1 satisfies (0.2), then we may choose $\Psi_1 \in \mathrm{Aut}_1\mathbf{C}^n$. Let $F_0(z) = z$ and $F_1 = \Psi_1$.

We will inductively choose automorphisms Ψ_j and let $F_k = \Psi_k \circ \cdots \circ \Psi_1$. For the induction, suppose we have the following.

(1) Compact, convex sets $K_0 \subset K_1 \subset \cdots \subset K_k$ with $j\mathbf{B} \cup F_{j-1}(j\mathbf{B}) \subset K_j$ and

$$\mathrm{dist}(K_{j-1} \cup F_{j-1}(j\mathbf{B}), \mathbf{C}^n \backslash K_j) > \epsilon_j$$

for each $1 \leq j \leq k$.

(2) Points $c_j \in K_{j+1} \backslash K_j$ for $1 \leq j \leq k-1$ and $c_k \in \mathbf{C}^n \backslash K_k$ such that each c_j is contained in the z_1-axis.

(3) $\Psi_j \in \mathrm{Aut}\mathbf{C}^n$ for $1 \leq j \leq k$ with

$$|\Psi_j(z) - z| + |\Psi_j^{-1}(z) - z| < \epsilon_j$$

for $z \in K_j$ (and $\Psi_j \in \mathrm{Aut}_1\mathbf{C}^n$ if P_j satisfies (0.2)).

(4) $F_k = \Psi_k \circ \Psi_{k-1} \circ \cdots \circ \Psi_1$ satisfying (2.1) for $1 \leq j \leq k$.

Given this, let K_{k+1} be a compact, convex set in $\mathbf{C}^n$ with

$$(k+1)\mathbf{B} \cup K_k \cup F_k((k+1)\mathbf{B}) \cup \{c_k\} \subset K_{k+1}$$

and

$$\mathrm{dist}(K_k \cup F_k((k+1)\mathbf{B}), \mathbf{C}^n \backslash K_{k+1}) > \epsilon_{k+1}.$$

Since K_{k+1} is compact, we can choose c_{k+1} in the z_1-axis so that $c_{k+1}, F_k(c_{k+1}) \in \mathbf{C}^n \backslash K_{k+1}$. Let $N = \max\{m_1, \ldots, m_k\} + 1$. Let $d_{k+1} = F_k(c_{k+1})$, and let $Q_{k+1} = P_{k+1} \circ (F_k^{-1})_{m_{k+1}, d_k}$. Using proposition 1.1, choose $\Psi_{k+1} \in \mathrm{Aut}\mathbf{C}^n$ (with $\Psi_{k+1} \in \mathrm{Aut}_1\mathbf{C}^n$ if P_{k+1} satisfies (0.2) and $F_k \in \mathrm{Aut}_1\mathbf{C}^n$)) so that

(i) $\Psi_{k+1}(z) = c_{k+1} + Q_{k+1}(z - d_{k+1}) + O(|z - d_{k+1}|^{m_{k+1}+1})$ as $z \to d_{k+1}$,

(ii) $\Psi_{k+1}(z) = z + O(|z - c_j|^N)$ as $z \to c_j$, $1 \leq j \leq k$,

(iii) $|\Psi_{k+1}(z) - z| + |\Psi_{k+1}^{-1}(z) - z| < \epsilon_{k+1}$ for each $z \in K_{k+1}$.

Taking $F_{k+1} = \Psi_{k+1} \circ F_k$, we obtain the induction hypotheses at stage $k+1$.

By proposition 1.2, the sequence $\{F_j\}$ converges uniformly on compact subsets of $\Omega = \cup_{j=1}^{\infty} F_j^{-1}(K_j)$ to a biholomorphic map F from Ω onto $D = \cup_{j=1}^{\infty} K_j$. Since $j\mathbf{B} \subset K_j$, we see that $D = \mathbf{C}^n$. Moreover, $F_j^{-1}(K_j) = F_{j-1}^{-1}\Psi_j^{-1}(K_j)$, and by Rouché's theorem [1, p.110] and induction hypotheses (1) and (3), we see that $F_{j-1}(j\mathbf{B}) \subset \Psi_j^{-1}(K_j)$. Hence $j\mathbf{B} \subset F_j^{-1}(K_j)$, so also $\Omega = \mathbf{C}^n$.

Hence $F \in \mathrm{Aut}\mathbf{C}^n$ satisfies (2.1) for all j since each F_k satisfies (2.1) for $1 \le j \le k$. Finally, if each P_j satisfies (0.2), then each $\Psi_j \in \mathrm{Aut}_1\mathbf{C}^n$, so $F \in \mathrm{Aut}_1\mathbf{C}^n$. ♠

Remark. Given $R > 0$, $1 > \epsilon > 0$, we can replace K_1 by $(R+1)\overline{\mathbf{B}}$, c_1 by $(R+2)e_1$, and ϵ_j by $\epsilon(\epsilon_j/2)$ to construct a sequence $\{c_j\}$ and F satisfying the conclusions of the theorem and also so that $|F(z) - z| + |F^{-1}(z) - z| < \epsilon$ on $R\mathbf{B}$.

We next use a classical 1-variable interpolation result together with theorem 0.1 to find an automorphism fixing each je_1 and having prescribed jet there. Our technique of using shears to map a given discrete set in the z_1-axis to another was also used in the proof of proposition 3.1 in [5]. However, for the current application, we need the map to be tangent to a translation to high order at each point in the discrete set.

2.2 Corollary. *Let P_j and m_j be as in theorem 0.1. For each $R \ge 0$ and $\epsilon > 0$ there exists $F \in \mathrm{Aut}\mathbf{C}^n$ such that*

$$\begin{aligned} &F(z) = je_1 + P_j(z - je_1) + O(|z - je_1|^{m_j+1}), \quad z \to je_1,\ j > R, \\ &|F(z) - z| + |F^{-1}(z) - z| < \epsilon, \quad |z| < R. \end{aligned}$$

If each P_j satisfies (0.2), then we can choose $F \in \mathrm{Aut}_1\mathbf{C}^n$.

Proof. We shall first prove the corollary without the last condition on F (i.e., taking $R = 0$). By lemma 2.1, there exists $G \in \mathrm{Aut}\mathbf{C}^n$ and a sequence $\{c_j\}$ contained in the z_1-axis so that $G(z) = c_j + P_j(z - c_j) + O(|z - c_j|^{m_j+1})$ as $z \to c_j$. Also, $G \in \mathrm{Aut}_1\mathbf{C}^n$ if each P_j satisfies (0.2).

To obtain the corollary we need only find $\Psi \in \mathrm{Aut}_1\mathbf{C}^n$ which maps each je_1 to c_j with $\Psi_{m_j, je_1}(z) = z - je_1$, since then lemma 1.3 implies that $\Psi^{-1} \circ G \circ \Psi$ has the desired properties.

To do this, let $\xi_j \in \mathbf{C}$ such that $c_j = \xi_j e_1$. By a standard 1-variable interpolation result [4, cor. 1.5.4], there exists an entire function f_1 of one variable with $f_1(\zeta) = j + O(|\zeta - j|^{m_j+1})$ as $\zeta \to j$ for all $j \ge 1$. Let $\Psi_1(z) = z + f_1(z_1)e_2$, where $e_2 = (0, 1, 0, \ldots, 0)$. Then $\Psi_1 \in \mathrm{Aut}_1\mathbf{C}^n$ and $\Psi_1(z) = j(e_1 + e_2) + (z - je_1) + O(|z - je_1|^{m_j+1})$ as $z \to je_1$. Thus Ψ_1 maps je_1 to $j(e_1 + e_2)$ and agrees with a translation to order $m_j + 1$ at je_1.

Likewise, choosing f_2 entire with $f_2(\zeta) = \xi_j - j + O(|\zeta - j|^{m_j+1})$ as $\zeta \to j$, and taking $\Psi_2(z) = z + f_2(z_2)e_1$, we see that Ψ_2 maps $j(e_1 + e_2)$ to $\xi_j e_1 + je_2$ and agrees with a translation to order $m_j + 1$ at $j(e_1 + e_2)$. Similarly, we can find $\Psi_3(z) = z_1 + f_3(z_1)e_2$ so that Ψ_3 maps $\xi_j e_1 + je_2$ to $\xi_j e_1 = c_j$ and agrees with a translation to order $m_j + 1$ at $\xi_j e_1 + je_2$.

Let $\Psi = \Psi_3 \circ \Psi_2 \circ \Psi_1$. Then $\Psi(z) = c_j + (z - je_1) + O(|z - je_1|^{m_j+1})$ as $z \to je_1$, and by lemma 1.3, we see that $\Psi^{-1}(z) = je_1 + (z - c_j) + O(|z - c_j|^{m_j+1})$ as $z \to c_j$. Also, $\Psi \in \mathrm{Aut}_1\mathbf{C}^n$ since each Ψ_l has Jacobian one.

Hence taking $F = \Psi^{-1} \circ G \circ \Psi$, and applying lemma 1.3, we obtain the desired automorphism.

Our proof shows that the sequence $\{je_1\}$ could clearly be replaced by any discrete sequence $\{d_j\}$ without repetition in the z_1-axis. Moreover, to get the last condition on F, suppose that $\{d_j\}$ and $\{c_j\}$ lie outside the ball $R\overline{\mathbf{B}}$ (in the z_1 axis). A simple argument using Runge's theorem and the Weierstrass theorem shows that we can choose each f_l as above and so that $|f_l|$ is small on a neighborhood of the closed disk of radius R in $\mathbf{C}$. Hence by the remark after the proof of lemma 2.1, we see that we may choose F satisfying the conclusions of corollary 2.2 with d_j in place of je_1 and so that $|F(z) - z| + |F^{-1}(z) - z| < \epsilon$ on $R\mathbf{B}$. ♠

Proof of theorem 0.1. Since $\{a_j\}$ and $\{b_j\}$ are tame, there exist $H_1, H_2 \in \mathrm{Aut}\mathbf{C}^n$ such that $H_1(a_j) = H_2(b_j) = je_1$, and if the sequences are very tame, then $H_1, H_2 \in \mathrm{Aut}_1\mathbf{C}^n$.

Hence it suffices to construct $G \in \mathrm{Aut}\mathbf{C}^n$ (or $G \in \mathrm{Aut}_1\mathbf{C}^n$) such that $G(je_1) = je_1$ and $(H_2^{-1}GH_1)_{m_j,a_j} = P_j$. By lemma 1.3, this latter condition can be satisfied by making

$$G_{m_j,je_1}(z) = (H_2)_{m_j,b_j} \circ P_j \circ (H_1^{-1})_{m_j,je_1}(z) + O(|z|^{m_j+1}),\ z \to 0$$

for each j. By corollary 2.2, we can find $G \in \mathrm{Aut}\mathbf{C}^n$ satisfying this condition and $G(je_1) = je_1$. Moreover, if each P_j satisfies (0.2) and $H_1, H_2 \in \mathrm{Aut}_1\mathbf{C}^n$, then we can choose $G \in \mathrm{Aut}_1\mathbf{C}^n$.

Finally, taking $F = H_2^{-1} \circ G \circ H_1$, we obtain the desired automorphism. ♠

&3. Interpolation with approximation

We begin with the following lemma.

3.1 Lemma. *Let $H \in \mathrm{Aut}\mathbf{C}^n$, let K be a compact, polynomially convex set in $\mathbf{C}^n$, and let $\{a_j\}_{j=1}^{\infty}$ be a discrete sequence disjoint from K and contained in the z_1-axis. Let $R > 0$ with $H(K) \subset R\mathbf{B}$ and let $\epsilon > 0$. Then there exists $\Psi \in \mathrm{Aut}\mathbf{C}^n$ such that $|\Psi(z) - H(z)| < \epsilon$ on K, $|\Psi^{-1}(z) - H^{-1}(z)| < \epsilon$ on $H(K)$, and $\Psi(a_j) = (R + j)e_1$ for all j. If $H \in \mathrm{Aut}_1\mathbf{C}^n$, then we can choose $\Psi \in \mathrm{Aut}_1\mathbf{C}^n$.*

Proof. Let $r > 0$ with $K \subset r\mathbf{B}$. Let $K_0 \subset r\mathbf{B}$ be compact and polynomially convex with $K \subset \mathrm{Int}K_0$, $K_0 \cap \{a_j\} = \emptyset$, and $H(K_0) \subset R\mathbf{B}$. Let $a_{j_1}, \dots, a_{j_m}$ be the points in $\{a_j\} \cap r\overline{\mathbf{B}}$, and let $\delta > 0$. Using the fact that the union of K_0 with finitely many points is again polynomially convex, we can apply proposition 1.1 m times to find $\Psi_1 \in \mathrm{Aut}_1\mathbf{C}^n$ such that $|\Psi_1(z) - z| + |\Psi_1^{-1}(z) - z| < \delta$ for $z \in K_0$ and $\Psi_1(a_{j_k}) = H^{-1}((R + j_k)e_1)$ for $k = 1, \dots, m$.

Let $\pi_2(z) = z_2$. For fixed $v \in \mathbf{C}^n\backslash\{0\}$ and $j \in \mathbf{N}$ we consider the 1-variable function $g_j(\zeta) = \pi_2 H\Psi_1(a_j + \zeta v)$. Since the kernel of $D_{a_j}(\pi_2 \circ H \circ \Psi_1)$ is an $(n-1)$-dimensional subspace for each j, we may choose v arbitrarily near e_2 such that g_j is nonconstant for each j and hence so that the image of g_j omits at most one point in $\mathbf{C}$. In particular, we may choose such a v so that there exists a $\mathbf{C}$-linear form λ with $\lambda(v) = 0$, $\lambda(e_1) = 1$, and $\lambda(a_j) \notin \lambda(r\overline{\mathbf{B}})$ if $j \notin \{j_1, \ldots, j_m\}$.

Choose f entire on $\mathbf{C}$ such that $|f| < \delta/2$ on $\lambda(r\mathbf{B})$, $f(\lambda(a_{j_k})) = 0$ for each $k = 1, \ldots, m$, and $|\pi_2 H\Psi_1(a_j + f(\lambda(a_j))v)| = R + j$ for each $j \notin \{j_1, \ldots, j_m\}$. Let $\Psi_2(z) = z + f(\lambda(z))v$. Then $\Psi_2 \in \mathrm{Aut}_1\mathbf{C}^n$, $|\Psi_2(z) - z| + |\Psi_2^{-1}(z) - z| < \delta$ on $r\mathbf{B}$, $H\Psi_1\Psi_2(a_{j_k}) = (R + j_k)e_1$ for $k = 1, \ldots, m$, and $|\pi_2 H\Psi_1\Psi_2(a_j)| = R + j$ for $j \notin \{j_1, \ldots, j_m\}$.

Using a composition of two shears as in the proof of corollary 2.2, we can find $\Psi_3 \in \mathrm{Aut}_1\mathbf{C}^n$ such that $|\Psi_3(z) - z| + |\Psi_3^{-1}(z) - z| < \delta$ on $R\mathbf{B}$, $\Psi_3((R + j_k)e_1) = (R + j_k)e_1$ for $k = 1, \ldots, m$, and $\Psi_3 H\Psi_1\Psi_2(a_j) = (R + j)e_1$ for each $j \notin \{j_1, \ldots, j_m\}$.

Let $\Psi = \Psi_3 \circ H \circ \Psi_1 \circ \Psi_2$. Then $\Psi(a_j) = (R + j)e_1$ for all j, and for δ sufficiently small, we have $|\Psi(z) - H(z)| < \epsilon$ on K and $|\Psi^{-1}(z) - H^{-1}(z)| < \epsilon$ on $H(K)$, so the lemma follows. Since each $\Psi_l \in \mathrm{Aut}_1\mathbf{C}^n$, we see that $\Psi \in \mathrm{Aut}_1\mathbf{C}^n$ if $H \in \mathrm{Aut}_1\mathbf{C}^n$. ♠

Proof of theorem 0.2. Since the sequences $\{a_j\}$ and $\{b_j\}$ are tame, there exist automorphisms H_1, H_2 of $\mathbf{C}^n$ such that $H_1(a_j) = je_1$ and $H_2(b_j) = je_1$ for each j. Replacing Φ by $H_2 \circ \Phi \circ H_1^{-1}$, K by $H_1(K)$, and adjusting the jets P_j as in the proof of theorem 0.1, we reduce the problem to the case when $a_j = b_j = je_1$ for all j.

Choose a larger polynomially convex set $L \subset \mathbf{C}^n \backslash \{je_1 : j \in \mathbf{N}\}$ such that $K \subset \mathrm{Int}\, L$. Fix $\eta > 0$, and choose an integer $r > 0$ such that $L \cup \Phi(L) \subset r\mathbf{B}$. By lemma 3.1 there exist automorphisms $\Psi, \Theta \in \mathrm{Aut}\mathbf{C}^n$ such that

$$
\begin{aligned}
&\Psi(je_1) = \Theta(je_1) = (r + j)e_1, \quad j = 1, 2, 3, \ldots, \\
&|\Psi(z) - \Phi(z)| < \eta, \quad z \in L, \\
&|\Theta(z) - z| + |\Theta^{-1}(z) - z| < \eta, \quad z \in \Phi(L).
\end{aligned}
$$

By corollary 2.2, there exists $G \in \mathrm{Aut}\mathbf{C}^n$ such that $G((r + j)e_1) = (r + j)e_1$ for $j \in \mathbf{N}$, $|G(z) - z| < \eta$ for $|z| \le r$, and such that for each $j \in \mathbf{N}$ the jet $Q_j = G_{m_j,(r+j)e_1}$ satisfies

$$
P_j(z) = (\Theta^{-1})_{m_j,(r+j)e_1} \circ Q_j \circ \Psi_{m_j,je_1}(z) + O(|z|^{m_j+1}), \quad z \to 0.
$$

Let $F = \Theta^{-1} \circ G \circ \Psi$. Then $F(je_1) = je_1$ for each $j \ge 1$, F satisfies (0.3) provided that $\eta > 0$ is chosen sufficiently small (depending on ϵ and $\mathrm{dist}(K, \mathbf{C}^n \backslash L)$), and lemma 1.3 shows that F satisfies (0.1), with $a_j = b_j = je_1$.

Finally, if the sequences a_j and b_j are very tame, the P_j's satisfy (0.2), and $\Phi \in \mathrm{Aut}_1\mathbf{C}^n$, then the automorphisms Ψ, Θ, and G can be chosen with Jacobian one. ♠

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