QUASICONFORMALITY AND GEOMETRICAL FINITENESS IN CARNOT–CARATHÉODORY AND NEGATIVELY CURVED SPACES

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ABSTRACT. The paper sketches a recent progress and formulates several open problems in studying equivariant quasiconformal and quasisymmetric homeomorphisms in negatively curved spaces as well as geometry and topology of noncompact geometrically finite negatively curved manifolds and their boundaries at infinity having Carnot–Carathéodory structures. Especially, the most interesting are complex hyperbolic manifolds with Cauchy–Riemannian structure at infinity, which occupy a distinguished niche and whose properties make them surprisingly different from real hyperbolic ones.

1. Introduction

The paper sketches a recent progress and formulates several open problems in studying quasiconformal and quasisymmetric homeomorphisms as well as geometry and topology of noncompact geometrically finite negatively curved manifolds and their boundaries at infinity having Carnot–Carathéodory structures in the sense of M. Gromov [Gr]. Here, complex hyperbolic and Cauchy–Riemannian manifolds occupy a distinguished niche. First, due to their complex analytic nature, a broad spectrum of techniques can contribute to the study, and already obtained results show surprising differences between geometry and topology of noncompact complex and real hyperbolic manifolds, see [BS, BuM, EMM, GoM, KR1-2]. Second, these Kähler manifolds are the most known manifolds with variable pinched negative curvature [BGS, B, Mg1, MaG]. Finally, for complex analytic surfaces, one can apply Seiberg-Witten invariants, decomposition of 4-manifolds along homology 3-spheres, Floer homology and new (homology) cobordism invariants [LB, A11, Sa1-Sa4].

There are three main themes united by using of general Thurston's idea of geometric uniformization of low dimensional manifolds provided some canonical decomposition of them into geometric pieces. First, basic geometric block-manifolds of negative (variable) curvature which possess geometrical finiteness and may however have infinite volume. Second, the interaction between the negatively curved

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geometry of such blocks and the induced Carnot–Carathéodory structure of their boundary manifolds at infinity, especially the interaction between Kähler geometry of geometrically finite complex hyperbolic manifolds and Cauchy–Riemannian structure of their boundaries modeled on the Heisenberg group (which is a particular case of Carnot groups). And finally, deformations of negatively curved manifolds inducing automorphisms of boundaries at infinity which preserves their natural contact structures, together with their metrical (quasiconformal or quasisymmetric) properties.

Our study of geometrical finiteness and homeomorphisms in spaces with negative (variable) curvature exploits a new structural theorem about isometric actions of discrete groups on nilpotent Lie groups (in particular on the Heisenberg group $\mathcal{H}_n$). The problems there have a unique appeal both for the amount of similarity with the model situation of interactions between real hyperbolic geometrically finite manifolds, their boundaries with natural conformal structures and their quasiconformal deformations (see [A1, A11]), and for the interesting ways in which the similarity breaks down.

Besides geometrical finiteness in negatively curved spaces, one of inspiring ideas of our study is a well known theorem of D.Sullivan, which gave rise to many important results in geometry and topology of manifolds and theory of quasiconformal mappings. It states, see [Su1, TV], that homeomorphisms of quasiconformal n-manifolds, $n \neq 4$, can be approximated by quasiconformal ones, where one of important classes is the class of conformal manifolds. Here by the quasiconformality one means the boundedness of distortion with respect to the Euclidean metric. This result rises questions of approximation in the quasiconformal category in another negatively curved geometries and the corresponding sub-Riemannian Carnot–Carathéodory structures which appear at infinity of those geometries as well. We mention here a recent important development by M.Gromov on Carnot–Carathéodory spaces [Gr], where he shown that continues mappings can be approximated by mappings that are Lipschitz with respect to the Carnot–Carathéodory metric. Another important achievement is a recent result by G.Margulis and G.Mostow [MM] that quasiconformal mappings of Carnot–Carathéodory spaces are a.e. differentiable and preserve their contact structures.

Another set of problems is related to the stability theorem proven by D.Sullivan [Su3] for planar Kleinian groups (see also [A11, MaG]). It raises questions on deformations of discrete isometry groups in spaces of negative (variable) curvature, varieties of their representations (Teichmüller spaces) and their boundaries, and quasiconformal/quasisymmetric homeomorphisms induced by isomorphisms of geometrically finite groups (in conformal category, it is known as Tukia isomorphism theorems [Tu]).

2. Complex hyperbolic and Heisenberg manifolds

The natural class of manifolds where the above problems can be considered is the class of geometrically finite locally homogeneous manifolds. That is why we start with studying such manifolds and their boundaries at infinity, especially complex hyperbolic manifolds and Cauchy–Riemannian and Heisenberg manifolds at their infinity.

We recall some facts concerning the link between nilpotent geometry of the Heisenberg group and the Kähler geometry of the complex hyperbolic space (see [Go, GP1, KR1-2]). Let $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ be the unit complex ball equipped with the Bergman metric d which turns the ball into the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ whose sectional curvature is between -1 and -1/4. The automorphism group of $\mathbb{H}^n_{\mathbb{C}} \cong \mathbb{B}^n_{\mathbb{C}}$ is the projective unitary group $PU(n,1)$ whose elements $g \in PU(n,1)$ are biholomorphisms of $\mathbb{B}^n_{\mathbb{C}}$ of the following three types. If g fixes a point in $\mathbb{H}^n_{\mathbb{C}}$, it is called *elliptic*. If g has exactly one fixed point, and it lies in $\partial\mathbb{H}^n_{\mathbb{C}}$, g is called *parabolic*. If g has exactly two fixed points, and they lie in $\partial\mathbb{H}^n_{\mathbb{C}}$, g is called *loxodromic*. These three types exhaust all the possibilities. Since $PU(n,1)$ can be embedded in a linear group (for details, see [AX1]), any finitely generated group $G \subset PU(n,1)$ is residually finite and has a finite index torsion free subgroup.

For any point $\infty \in \partial\mathbb{B}^n_{\mathbb{C}}$, one can identify $\overline{\mathbb{B}^n_{\mathbb{C}}}\backslash\{\infty\}$ with the closure of the Siegel domain $\mathfrak{S}_n$, which is conveniently represented in horospherical coordinates as $\mathbb{C}^{n-1}\times\mathbb{R}\times[0,\infty)$ ([GP1]). Then "the boundary plane" $\mathbb{C}^{n-1}\times\mathbb{R}\times\{0\} = \partial\mathbb{H}^n_{\mathbb{C}}\backslash\{\infty\}$ and the horospheres $H_u = \mathbb{C}^{n-1}\times\mathbb{R}\times\{u\}$, $0 < u < \infty$, centered at ∞ are identified with the Heisenberg group $\mathcal{H}_n = \mathbb{C}^{n-1}\times\mathbb{R}$. It is a 2-step nilpotent group with center $\{0\}\times\mathbb{R} \subset \mathbb{C}^{n-1}\times\mathbb{R}$, and the inverse of (ξ, v) is $(\xi, v)^{-1} = (-\xi, -v)$. The Heisenberg group $\mathcal{H}_n$ isometrically acts on itself and on $\mathbb{H}^n_{\mathbb{C}}$ by left translations:

$$T_{(\xi_0, v_0)} : (\xi, v, u) \longmapsto (\xi_0 + \xi\,,\, v_0 + v + 2\,\mathrm{Im}\langle\langle\xi_0, \xi\rangle\rangle\,,\, u)\,.$$

The unitary group $U(n-1)$ acts on $\mathcal{H}_n$ and $\mathbb{H}^n_{\mathbb{C}}$ by rotations: $A(\xi, v, u) = (A\xi\,,\, v\,,\, u)$ for $A \in U(n-1)$. The semidirect product $\mathcal{H}(n) = \mathcal{H}_n \rtimes U(n-1)$ is naturally embedded in $U(n,1)$ as follows:

$$A \longmapsto \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U(n,1) \quad \text{for} \quad A \in U(n-1)\,,$$

$$(\xi, v) \longmapsto \begin{pmatrix} I_{n-1} & \xi & \xi \\ -\bar{\xi}^t & 1 - \frac{1}{2}(|\xi|^2 - iv) & -\frac{1}{2}(|\xi|^2 - iv) \\ \bar{\xi}^t & \frac{1}{2}(|\xi|^2 - iv) & 1 + \frac{1}{2}(|\xi|^2 - iv) \end{pmatrix} \in U(n,1)$$

where $(\xi, v) \in \mathcal{H}_n = \mathbb{C}^{n-1}\times\mathbb{R}$ and $\bar{\xi}^t$ is the conjugate transpose of ξ.

The action of $\mathcal{H}(n)$ on $\overline{\mathbb{H}^n_{\mathbb{C}}}\backslash\{\infty\}$ also preserves the Cygan metric ρ_c there, which plays the same role as the Euclidean metric does on the upper half-space model of the real hyperbolic space $\mathbb{H}^n$ and is induced by the following norm:

$$||(\xi, v, u)||_c = \left|\,||\xi||^2 + u - iv\right|^{1/2}, \quad (\xi, v, u) \in \mathbb{C}^{n-1}\times\mathbb{R}\times[0,\infty)\,.$$

The relevant geometry on each horosphere $H_u \subset \mathbb{H}^n_{\mathbb{C}}$, $H_u \cong \mathcal{H}_n = \mathbb{C}^{n-1}\times\mathbb{R}$, is the spherical CR-geometry induced by the complex hyperbolic structure. The geodesic perspective from ∞ defines CR-maps between horospheres, which extend to CR-maps between the one-point compactifications $H_u \cup \infty \approx S^{2n-1}$. In the limit, the induced metrics on horospheres fail to converge but the CR-structure remains fixed. In this way, the complex hyperbolic geometry induces CR-geometry on the sphere at infinity $\partial\mathbb{H}^n_{\mathbb{C}} \approx S^{2n-1}$, naturally identified with the one-point compactification of the Heisenberg group $\mathcal{H}_n$.

3. Geometrical finiteness in negative curvature

Our main assumption on a negatively curved n-manifold M is the geometrical finiteness condition on its fundamental group $G \subset \operatorname{Isom} X$ acting by isometries on a simply connected space $X = \widetilde{M}$, which in particular implies (see below) that the discrete group G is finitely generated.

A subgroup $G \subset \operatorname{Isom} X$ is called *discrete* if it is a discrete subset of $\operatorname{Isom} X$. The *limit set* $\Lambda(G) \subset \partial X$ of a discrete group G is the set of accumulation points of (any) orbit $G(y)$, $y \in X$. The complement of $\Lambda(G)$ in ∂X is called the *discontinuity set* $\Omega(G)$. A discrete group G is called *elementary* if its limit set $\Lambda(G)$ consists of at most two points. An infinite discrete group G is called *parabolic* if it has exactly one fixed point $\operatorname{fix}(G)$; then $\Lambda(G) = \operatorname{fix}(G)$, and G consists of either parabolic or elliptic elements. As it was observed by many authors (c.f. [MaG]), parabolicity in the variable curvature case is not as easy a condition to deal with as it is in the constant curvature space. However the results below simplify the situation.

Due to the absence of totally geodesic hypersurfaces in a space X of variable negative curvature, we cannot use the original definition of geometrical finiteness which came from an assumption that the corresponding real hyperbolic manifold $M = \mathbb{H}^n/G$ may be decomposed into a cell by cutting along a finite number of its totally geodesic hypersurfaces, that is the group G should possess a finite-sided fundamental polyhedron, see [Ah]. However, one can use another definition of *geometrically finite* groups $G \subset \operatorname{Isom} X$ as those ones whose limit sets $\Lambda(G) \subset \partial X$ consist of only conical limit points and parabolic (cusp) points p with compact quotients $(\Lambda(G) \backslash \{p\})/G_p$ with respect to parabolic stabilizers $G_p \subset G$ of p, see [BM, Bow]. There are other definitions of geometrical finiteness in terms of ends and the minimal convex retract of the noncompact manifold M, which work well not only in the real hyperbolic spaces $\mathbb{H}^n$ (see [Md, Th, A3, A1]) but also in spaces with variable pinched negative curvature [Bo].

In the case of variable curvature, it is problematic to use geometric methods based on consideration of finite sided fundamental polyhedra. In particular, Dirichlet polyhedra $D_y(G)$ in the complex hyperbolic space $X = \mathbb{H}^n_{\mathbb{C}}$ are fundamental polyhedra for a discrete subgroup $G \subset PU(n,1)$. They are bounded by bisectors in a complicated way (see [Mo2, GP1]), and the bisectors are not totally geodesic hypersurfaces. For discrete parabolic groups $G \subset \operatorname{Isom} X$, one may expect that the Dirichlet polyhedron $D_y(G)$ centered at a point y lying in a G-invariant subspace has finitely many sides. It is true for real hyperbolic spaces [A1] as well as for cyclic and dihedral parabolic groups in complex hyperbolic spaces $X = \mathbb{H}^n_{\mathbb{C}}$. Namely, due to [Ph], Dirichlet polyhedra $D_y(G)$ are always two sided for any cyclic group $G \subset PU(n,1)$ generated by a Heisenberg translation. Due to [GP1], such finiteness also holds for a cyclic ellipto-parabolic group or a dihedral parabolic group $G \subset PU(n,1)$ generated by inversions in asymptotic complex hyperplanes in $\mathbb{H}^n_{\mathbb{C}}$ if the central point y lies in a G-invariant vertical line or $\mathbb{R}$-plane (for any other center y, $D_y(G)$ has infinitely many sides). Our technique easily implies that this finiteness still holds for generic parabolic cyclic groups [AX1]:

Theorem 3.1. *For any discrete group $G \subset PU(n,1)$ generated by a parabolic element, there exists a point $y_0 \in \mathbb{H}^n_{\mathbb{C}}$ such that the Dirichlet polyhedron $D_{y_0}(G)$ centered at y_0 has two sides.*

However, the behavior of Dirichlet polyhedra for parabolic groups $G \subset PU(n,1)$ of rank more than one can be very bad. It is given by our construction [AX1]:

Theorem 3.2. *Let $G \subset PU(2,1)$ be a discrete parabolic group conjugate to the following subgroup Γ of the Heisenberg group $\mathcal{H}_2 = \mathbb{C} \times \mathbb{R}$:*

$$\Gamma = \{(m,n) \in \mathbb{C} \times \mathbb{R} : m, n \in \mathbb{Z}\}\,.$$

Then any Dirichlet polyhedron $D_y(G)$ centered at an arbitrary point $y \in \mathbb{H}^2_{\mathbb{C}}$ has infinitely many sides.

Despite this, we are providing [AX1] a construction of fundamental polyhedra $P(G) \subset \mathbb{H}^n_{\mathbb{C}}$ for arbitrary discrete parabolic groups $G \subset PU(n,1)$, which are bounded by finitely many hypersurfaces (different from Dirichlet bisectors). This construction may be seen as a base for extension of Apanasov's construction [A1] of finite sided pseudo-Dirichlet polyhedra in $\mathbb{H}^n$ to the case of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, which may solve the following problem:

Problem 3.3. *Given geometrically finite group $G \subset \operatorname{Isom} X$ in a negatively curved space X, is there any finitely sided fundamental polyhedron $P(G) \subset X$?*

As another tool for attacking this problem, one can use the following structural theorem for discrete groups acting on nilpotent Lie groups (lying at infinity of X), in particular on the Heisenberg group $\mathcal{H}_n, \mathcal{H}_n \cup \{\infty\} = \partial\mathbb{H}^n_{\mathbb{C}}$ (see [AX1- AX3]):

Theorem 3.4. *Let N be a connected, simply connected nilpotent Lie group, C a compact group of automorphisms of N, and Γ a discrete subgroup of the semidirect product $N \rtimes C$. Then there exist a connected Lie subgroup V of N and a finite index normal subgroup Γ^* of Γ with the following properties:*

1. *There exists $b \in N$ such that $b\Gamma b^{-1}$ preserves V.*
2. *$V/b\Gamma b^{-1}$ is compact.*
3. *$b\Gamma^* b^{-1}$ acts on V by left translations and this action is free.*

Here we remark that

1. Compactness of C is an essential condition because of Margulis [Mg2] construction of nonabelian free discrete subgroups Γ of $R^3 \rtimes GL(3,R)$.
2. This theorem generalizes a result by L.Auslander [Au] who claimed those properties not for whole group Γ but only for some finite index subgroup. We also remark that an extension in [AX1] of Wolf's argument [Wo] to the complex hyperbolic case (based on Margulis Lemma [Mg1, BGS] and geometry of $\mathcal{H}_n$) does not work in general nilpotent groups. A different algebraic proof of Theorem 3.4 can be found in [AX2, AX3].

This result allows one to study

Problem 3.5. *Investigate parabolic cusp ends of negatively curved manifolds.*

To solve this problem in the complex hyperbolic case, we use the above Theorem 3.4 to define standard cusp neighborhoods of parabolic cusps. This allows us to prove the following results about cusp ends of complex hyperbolic manifolds (or, equivalently, about the structure of Heisenberg manifolds).

Namely, suppose a point $p \in \partial\mathbb{H}^n_{\mathbb{C}}$ is fixed by some parabolic element of the group G, and G_p is the stabilizer of p in G. Conjugating the group G by an element $h \in PU(n,1)\,, h(p) = \infty$, we have that $G_\infty \subset \mathcal{H}(n)$. In particular, if p is the origin in $\mathcal{H}_n$, we can take such h as the Heisenberg inversion $\mathcal{I}$ in the hyperchain in $\mathcal{H}_n$, which preserves the unit Heisenberg sphere $S_c(0,1)$ and acts in $\mathcal{H}_n$ as follows:

$$\mathcal{I}(\xi, v) = \left(\frac{\xi}{|\xi|^2 - iv}\,,\ \frac{-v}{v^2 + |\xi|^4} \right) \quad \text{where } (\xi, v) \in \mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}\,. \tag{3.6}$$

Then, due to Theorem 3.4, there exists a connected Lie subgroup $\mathcal{H}_\infty \subseteq \mathcal{H}_n$ preserved by G_∞.

Definition. A set $U_{p,r} \subset \overline{\mathbb{H}^n_{\mathbb{C}}} \backslash \{p\}$ is called a *standard cusp neighborhood of radius* $r > 0$ at a parabolic fixed point $p \in \partial\mathbb{H}^n_{\mathbb{C}}$ of a discrete group $G \subset PU(n,1)$ if, for the Heisenberg inversion $\mathcal{I}_p \in PU(n,1)$ with respect to the unit sphere $S_c(p,1) = \{(\xi, v) : \rho_c(p, (\xi, v)) = 1\}$, $\mathcal{I}_p(p) = \infty$, the following conditions hold:

(1) $U_{p,r} = \mathcal{I}_p^{-1}\left(\{x \in \mathbb{H}^n_{\mathbb{C}} \cup \mathcal{H}_n : \rho_c(x, \mathcal{H}_\infty) \geq 1/r\}\right)$;

(2) $U_{p,r}$ is precisely invariant with respect to $G_p \subset G$, that is:

$$\gamma(U_{p,r}) = U_{p,r} \quad \text{for } \ \gamma \in G_p \quad \text{and} \quad g(U_{p,r}) \cap U_{p,r} = \emptyset \quad \text{for } \ g \in G \backslash G_p\,.$$

A parabolic point $p \in \partial\mathbb{H}^n_{\mathbb{C}}$ of $G \subset PU(n,1)$ is a *cusp point* if it has a cusp neighborhood $U_{p,r}$.

We remark that some parabolic points of a discrete group $G \subset PU(n,1)$ may not be cusp points, see [AX1, §5.4]. Applying Theorem 3.4 and [Bo], we have:

Lemma 3.7. *Let $p \in \partial\mathbb{H}^n_{\mathbb{C}}$ be a parabolic fixed point of a discrete group $G \subset PU(n,1)$. Then p is a cusp point if and only if $(\Lambda(G) \backslash \{p\})/G_p$ is compact.*

This fact and [Bo] allows us to use another equivalent definitions of geometrical finiteness. In particular, a group $G \subset PU(n,1)$ is *geometrically finite* if and only if its quotient space $M(G) = [\mathbb{H}^n_{\mathbb{C}} \cup \Omega(G)]/G$ has finitely many ends, and each of them is a cusp end, that is an end whose neighborhood can be taken as $U_{p,r}/G_p \approx (S_{p,r_0}/G_p) \times (0,1]$, where

$$S_{p,r} = \mathcal{I}_p^{-1}\left(\{x \in H^n_{\mathbb{C}} \cup \mathcal{H}_n : \rho_c(x, \mathcal{H}_\infty) = 1/r\}\right)\,.$$

Using the above description of discrete group actions on a nilpotent group (Theorem 3.4), one can study Carnot–Carathéodory manifolds whose fundamental groups act at infinity of a negatively curved space X as discrete parabolic groups. In particular, we establish fiber bundle structures on Heisenberg manifolds which have the form $\mathcal{H}_n/G$ where G is a discrete group freely acting on $\mathcal{H}_n$ by isometries, i.e. a torsion free discrete subgroup of $\mathcal{H}(n) = \mathcal{H}_n \rtimes U(n-1)$, see [AX1]:

Theorem 3.8. *Let $\Gamma \subset \mathcal{H}_n \rtimes U(n-1)$ be a torsion-free discrete group acting on the Heisenberg group $\mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$ with non-compact quotient. Then the quotient $\mathcal{H}_n/\Gamma$ has zero Euler characteristic and is a vector bundle over a compact manifold. Furthermore, this compact manifold is finitely covered by a nil-manifold which is either a torus or the total space of a circle bundle over a torus.*

It gives, in addition to finiteness of generators of geometrically finite groups established in [Bo], the following important finiteness:

Corollary 3.9. *The fundamental groups of Heisenberg manifolds and geometrically finite complex hyperbolic manifolds are finitely presented.*

Due to Theorem 3.4, any Heisenberg manifold $N = \mathcal{H}_n/\Gamma$ is the vector bundle $\mathcal{H}_n/\Gamma \to \mathcal{H}_\Gamma/\Gamma$ where $\mathcal{H}_\Gamma \subset \mathcal{H}_n$ is a minimal Γ-invariant subspace. As simple examples in [AX1] show, such vector bundles are non-trivial in general. However, up to finite coverings, they are trivial [AX1]:

Theorem 3.10. *Let $\Gamma \subset \mathcal{H}_n \rtimes U(n-1)$ be a discrete group and $\mathcal{H}_\Gamma \subset \mathcal{H}_n$ a connected Γ-invariant Lie subgroup on which Γ acts co-compactly. Then there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ such that the vector bundle $\mathcal{H}_n/\Gamma_0 \to \mathcal{H}_\Gamma/\Gamma_0$ is trivial. In particular, any Heisenberg orbifold $\mathcal{H}_n/\Gamma$ is finitely covered by the product of a compact nil-manifold $\mathcal{H}_\Gamma/\Gamma_0$ and an Euclidean space.*

Such finite covering property holds not only for Heisenberg manifolds alone but for geometrically finite complex hyperbolic manifolds, too:

Theorem 3.11. *Let $G \subset PU(n,1)$ be a geometrically finite discrete group. Then G has a subgroup G_0 of finite index such that every parabolic subgroup of G_0 is isomorphic to a discrete subgroup of the Heisenberg group $\mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$. In particular, each parabolic subgroup of G_0 is free Abelian or 2-step nilpotent.*

It seems this property holds for discrete isometry groups in general negatively curved spaces because our proof of Theorem 3.11 in [AX1] is based on the residual finiteness of geometrically finite subgroups $G \subset PU(n,1)$ and the following two lemmas (the second of which generalizes a result for finite volume real hyperbolic manifolds ([AF]).

Lemma 3.12. *Let $G \subset \mathcal{H}_n \rtimes U(n-1)$ be a discrete group and $\mathcal{H}_G \subset \mathcal{H}_n$ a minimal G-invariant connected Lie subgroup (given by Theorem 4.1). Then G acts on $\mathcal{H}_G$ by translations if G is either Abelian or 2-step nilpotent.*

Lemma 3.13. *Let $G \subset \mathcal{H}_n \rtimes U(n-1)$ be a torsion free discrete group, F a finite group and $\phi : G \longrightarrow F$ an epimorphism. Then the rotational part of $\ker(\phi)$ has strictly smaller order than that of G if one of the following happens:*

1. *G contains a finite index Abelian subgroup and F is not Abelian;*
2. *G contains a finite index 2-step nilpotent subgroup and F is not a 2-step nilpotent group.*

We conclude this section by pointing out that the problem of geometrical finiteness is very different in complex dimension two. Namely, it is a well known fact that any finitely generated discrete subgroup of $PU(1,1)$ is geometrically finite. This and Goldman's local rigidity theorem for uniform lattices $G \subset U(1,1) \subset PU(2,1)$ (see [GM]) suggest the following intrigue question:

Problem 3.14. *Are all finitely generated discrete groups $G \subset PU(2,1)$ with non-empty discontinuity set $\Omega(G) \subset \partial\mathbb{H}^2_{\mathbb{C}}$ geometrically finite?*

To solve this problem, one can try to decompose a given finitely generated discrete group $G \subset PU(2,1)$ with non-empty discontinuity set $\Omega(G) \subset \partial\mathbb{H}^2_{\mathbb{C}}$ into free amalgamated products of elementary groups. More arguments for an affirmative solution of this problem are given by the following two facts. First, due to

Chen-Greenberg [CG], all pure loxodromic subgroups $G \subset PU(2,1)$ are discrete. The second fact is due to the trace classification of projective transformations [Go, VI.2]. Namely, in contrast to Kleinian groups on the plane, the subset of groups in a deformation space of a pure loxodromic group $G \subset PU(2,1)$ which have accidental parabolic elements may have real codimension 1.

4. The boundary at infinity of negatively curved manifolds

Another set of problems is related to geometry and topology of Carnot–Carathéodory manifolds which are the boundaries at infinity of negatively curved non-compact manifolds.

In a sharp contrast to the real hyperbolic case, for a compact complex manifold $M(G) = (\mathbb{H}^n_{\mathbb{C}} \cup \Omega(G))/G$, an application of Kohn-Rossi analytic extension theorem shows that the boundary of this manifold $M(G)$ is connected, and the limit set $\Lambda(G)$ is in some sense small (see [EMM] and, for quaternionic and Cayley hyperbolic manifolds, [C1]). Moreover, according to a recent result of D.Burns (based on an uniformization theorem [BuM] for isolated ends of complex analytic spaces), the same claim about connectedness of the boundary $\partial M(G)$ still holds if only a boundary component is compact.

However, if $\partial M(G)$ has no compact components, and there is no finiteness condition on the holonomy group of $M(G)$, our algorithmical construction in [AX1] shows that the situation is completely different:

Theorem 4.1. *For any integers k, k_0, $k \geq k_0 \geq 0$, and $n \geq 2$, there exists a complex hyperbolic n-manifold $M = \mathbb{H}^n_{\mathbb{C}}/G$, $G \subset PU(n,1)$, whose boundary at infinity splits up into k connected $(n-1)$-manifolds, $\partial_\infty M = N_1 \cup \cdots \cup N_k$. Moreover, for each boundary component N_j, $j \leq k_0$, the inclusion $i_j : N_j \subset M(G)$ induces a homotopy equivalence of N_j to $M(G)$.*

The construction in the proof of this theorem firstly provides discrete groups $G \subset PU(n,1)$ with two connected components of the discontinuity set $\Omega(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$. Here we essentially use properties of Heisenberg inversions (3.6) in hyperchains in $\mathbb{H}^n_{\mathbb{C}}$. Then we apply an idea due to A.Tetenov [Te, KAG] in order to construct groups G with any given number k of topological G-invariant balls $\Omega_i \subset \Omega(G)$ with common boundary $\partial\Omega_i = \Lambda(G)$. The groups we construct in the proof are all however infinitely generated. The finitely generated case has a close relation to Problem 3.14:

Problem 4.2. *How many boundary components are there in a complex manifold $M(G)$ with finitely generated fundamental group $G \subset PU(n,1)$?*

Toward this problem, we can show that the situation described in Theorem 4.1 is impossible if the complex hyperbolic manifold M is geometrically finite. Namely, if the manifold $M(G)$ has non-compact boundary $\partial M = \Omega(G)/G$ with a component $N_0 \subset \partial M$ homotopy equivalent to $M(G)$, then there exists a compact homology cobordism $M_c \subset M(G)$ homotopy equivalent to $M(G)$, and $M(G)$ can be easily reconstructed from M_c by gluing up a finite number of standard open "Heisenberg collars", see [AX1]:

Theorem 4.3. *Let $G \subset PU(n,1)$ be a geometrically finite non-elementary torsion free discrete group whose Kleinian manifold $M(G)$ has non-compact boundary $\partial M = \Omega(G)/G$ with a component $N_0 \subset \partial M$ homotopy equivalent to $M(G)$. Then there exists a compact homology cobordism $M_c \subset M(G)$ such that $M(G)$ can be reconstructed from M_c by gluing up a finite number of open collars $M_i \times [0,\infty)$ where each M_i is finitely covered by the product $E_k \times B^{2n-1-k}$ of a closed $(2n-1-k)$-ball and a closed k-manifold E_k which is either flat or a nil-manifold (with 2-step nilpotent fundamental group).*

We refer the reader to [AX1] for more precise formulation and proof of this cobordism theorem.

This result allows one to study complex surfaces and Cauchy–Riemannian 3-manifolds at their infinity by using decomposition of such 4-manifolds along homology 3-spheres and applying the gauge theory together with homology (cobordism) invariants. We mention that, due to Milnor [Ml], all Seifert homology 3-spheres can be seen as the boundaries at infinity of (geometrically finite) complex hyperbolic 2-orbifolds. Along this line, one can also investigate the following question (which is also related to Problem 5.4):

Problem 4.4. *Are there Cauchy–Riemannian structures on homology 3-spheres of plumbing type or on real hyperbolic homology 3-spheres?*

One can be especially interested in this question for homology spheres obtained by splicing of two Seifert homology spheres along their singular fibers, see [Sa4]. Another interesting fact (due to Livingston-Myers construction [My]) is that any homology 3-sphere is homology cobordant to a hyperbolic one. On the other hand, as it was shown by C.T.C.Wall [Wa], the assignment of the appropriate geometry (when available) gives a detailed insight into the intrinsic structure of a complex surface. We mention here Yau's uniformization theorem [Y] which implies that every smooth complex projective 2-surface M with positive canonical bundle and satisfying the topological condition that $\chi(M) = 3\,\mathrm{Signature}(M)$, is a complex hyperbolic manifold. The necessity of homology sphere decomposition in dimension four is due to M.Freedman and L.Taylor result ([FT]):

Let M be a simply connected 4-manifold with intersection form q_M which decomposes as a direct sum $q_M = q_{M_1} \oplus q_{M_2}$, where M_1, M_2 are smooth manifolds. Then the manifold M can be represented as a connected sum $M = M_1 \#_\Sigma M_2$ along a homology sphere Σ.

We refer to [Sa1-Sa4] and [Mat] for recent advances in this direction, in particular, for results on Floer homology of homology 3-spheres and a new Saveliev's (presumably, homology cobordism) invariant based on Floer homology.

5. Homeomorphisms induced by group isomorphisms

The main problem we are concerned in this section is about geometric realizations of isomorphisms of geometrically finite discrete groups $G, H \subset \mathrm{Isom}\, X$ of isometries of a negatively curved space X:

Problem 5.1. *Given an isomorphism $\varphi : G \to H$ of geometrically finite discrete groups $G, H \subset \mathrm{Isom}\, X$, find subsets $X_G, X_H \subset \overline{X}$ invariant for the action of groups*

G and H, respectively, and an equivariant homeomorphism $f_\varphi : X_G \to X_H$ which induces the isomorphism φ. Determine metric properties of f_φ, in particular, whether it is either quasisymmetric or quasiconformal with respect to the given negatively curved metric d in X (or the induced sub-Riemannian structure on the Carnot–Carathéodory space at infinity $Y = \overline{X} \backslash \{\infty\}$).

Such type problems were studied by several authors. In the case of lattices G and H in rank 1 symmetric spaces X, G.Mostow [Mo1] proved in his celebrated rigidity theorem that such isomorphisms $\varphi : G \to H$ can be extended to inner isomorphisms of X, provided that there is no analytic homomorphism of X onto $PSL(2, \mathbb{R})$. For that proof, it was essential to prove that φ can be induced by a quasiconformal homeomorphism of the sphere at infinity $\overline{X}$ which is a one point compactification of a Carnot group N. Quasiconformal mappings between general Carnot groups have been studied by P.Pansu [P]. For the case of complex hyperbolic spaces $X = \mathbb{H}^n_{\mathbb{C}}$ and the Heisenberg groups at their infinity, foundations of the theory of quasiconformal mappings has been made by A.Koranyi and M.Reimann [KR1, KR2].

The essential component of such rigidity results is the fact that quasiconformal mappings of Carnot–Carathéodory spaces are almost everywhere differentiable and preserve the contact structures of these spaces (horizontal vector fields). This fact is due to Pansu [P] in the case of graded nilpotent groups. In the case of general Carnot–Carathéodory spaces determined by "horizontal" subbundles of their tangent bundles, this result has been recently proven by G.Margulis and G.Mostow [MM]. The second important ingredient for such rigidity is the ACL-property on lines for quasiconformal mappings, with respect to the Carnot–Carathéodory metric at infinity $\overline{X} \backslash \{\infty\}$, see [Mo3, KR2, Vo1-2, VG].

If the groups $G, H \subset \operatorname{Isom} X$ are neither lattices nor trivial, the only results on geometric realization of their isomorphisms are known for real hyperbolic spaces X (of constant negative curvature). In dimension $\dim X \geq 3$ they are due to P.Tukia [Tu] and use a natural "type-preserving" condition on the isomorphism φ, that parabolic elements are carried out only to parabolic ones.

In the case of variable negative curvature, it is problematic to use geometric properties of convex hulls which provide a powerful tool for studying metric properties of G-equivariant mappings in the real hyperbolic space. However, as a first step in solving the above Problem 5.1, we can analyze the metric of X and the induced metrics on horospheres in X. Such analysis, a generalization of W.Floyd construction of the group completion [Fl], and the described above analysis of geometrical finiteness in X are crucial components of our approach to constructing canonical homeomorphisms of subsets in the sphere at infinity $\overline{X}$, which induce type-preserving isomorphisms $\varphi : G \to H$ of geometrically finite groups G and H (not necessarily lattices). In particular, as a first result in this direction, we have the following isomorphism theorem [A12]:

Theorem 5.2. *Let $\varphi : G \to H$ be a type preserving isomorphism of two non-elementary geometrically finite discrete subgroups $G, H \subset PU(n, 1)$. Then there exists a unique equivariant homeomorphism $f_\varphi : \Lambda(G) \to \Lambda(H)$ of their limit sets that induces the isomorphism ϕ.*

Upon existence of such equivariant homeomorphism f_φ, the above problem would be reduced to the question whether f_φ is quasisymmetric with respect to

the Carnot–Carathéodory metric. In the case of an affirmative answer, it may be possible to find a global G-equivariant homeomorphism (of the sphere at infinity ∂X or even the whole space $\overline{X}$ inducing the isomorphism φ. However, in contrast to the real hyperbolic case, here we have an interesting fenomenon related to possible noncompactness of the boundary $\partial M(G) = \Omega(G)/G$. Namely, even for the simplest case of cyclic groups $G \cong H \subset PU(n,1)$, the homeomorphic CR-manifolds $\partial M(G) = \mathcal{H}^n/G$ and $\partial M(H) = \mathcal{H}^n/H$ may be not quasiconformally equivalent, see [Mn]. So it may be possible to (affirmatively) answer the following problem:

Problem 5.3. *When are there quasisymmetric homeomorphisms in a Carnot–Carathéodory space $\partial X\backslash\{\infty\}$ compatible with the action of discrete geometrically finite groups $G, H \subset \operatorname{Isom} X$ but quasisymmetrically non-extendable to the whole space?*

We note that, besides the metrical (quasisymmetric) part of this problem, some topological obstructions for extensions of equivariant homeomorphisms f_φ of the limit sets, $f_\varphi : \Lambda(G) \to \Lambda(H)$, may exist. It follows from the following example.

Example 5.4. *Let $G \subset PU(1,1) \subset PU(2,1)$ and $H \subset PO(2,1) \subset PU(2,1)$ be two geometrically finite (loxodromic) groups isomorphic to the fundamental group $\pi_1(S_g)$ of a compact oriented surface S_g of genus $g > 1$. Then the equivariant homeomorphism $f_\varphi : \Lambda(G) \to \Lambda(H)$ cannot be homeomorphically extended to the whole sphere $\partial\mathbb{H}^2_{\mathbb{C}} \approx S^3$.*

The obstruction in this example is topological and is due to the fact that the quotient manifolds $\mathbb{H}^2_{\mathbb{C}}/G$ and $\mathbb{H}^2_{\mathbb{C}}/H$ are not homeomorphic. Namely, these complex surfaces are disk bundles over the surface S_g and have different Toledo invariants: $\tau(\mathbb{H}^2_{\mathbb{C}}/G) = 2g-2$ and $\tau(\mathbb{H}^2_{\mathbb{C}}/H) = 0$, see [To].

We remark that some of Carnot–Carathéodory spaces are very rigid. In fact, due to P.Pansu [Pa], any quasiconformal map on the boundary ∂X of a quaternionic or octanionic hyperbolic space X (which are symmetric spaces of rank 1) is necessarily an extension of an isometry in X. This shows that any non-trivial homeomorphism $f_\varphi : \Lambda(G) \to \Lambda(H)$ (non-isometry) is non-extendable to a quasiconformal map of some open subset of ∂X.

In order to attack the above problem, in particular to construct such non-trivial geometric isomorphisms φ in both rigid (like quaternionic and octanionic spaces) and more flexible (like $\mathbb{H}^n_{\mathbb{C}}$) spaces, one probably may to generalize our block-building method [A1, A5] (whose usefulness in conformal category has been also demonstrated in [A6, A11]). Another aspect of such constructions should be

Problem 5.5. *Develop geometric combination theorems generalizing well known Maskit combination theorems for Kleinian groups, see* [Mas, A1].

So far, there are the only simplest versions (free products) of such combination theorems, see [FG]. The general case of free amalgamated product is still unknown even in the complex hyperbolic case, where one can however use a week Maskit combination, see [Be].

As the first steps in investigating Problem 5.1, it appears to be promising to study G-equivariant homeomorphisms in Carnot groups $Y = \partial X\backslash\{\infty\}$, with Carnot–Carathéodory metric ρ, from the following two classes of embeddings $f : A \hookrightarrow Y$, $A \subset Y$.

The first one consists of well known quasisymmetric embeddings f, see [TV].

The second class consists of embeddings $f: A \hookrightarrow Y$, $A \subset Y$, which generalize the so-called quasi-Möbius embeddings introduced in conformal category by V.Aseev [As, AsT] and J.Väisälä [V]. They have bounded distortion of the cross-ratio (or, equivalently, of the complex cross-ratio in the sense of Koranyi-Reimann [CR4]),

$$CR(q) = \rho(x_1, x_2)\rho(x_3, x_4)\rho(x_1, x_3)^{-1}\rho(x_2, x_4)^{-1}\,, \tag{5.6}$$

of quadruples $q = \{x_1, x_2, x_3, x_4\} \in (Y)^4$ in the following sense.

Let $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ be a given homeomorphism, $\omega(0) = 0$. Then, for any quadruple $q \in A^4$ and its f-image $f(q) = \{f(x_1, \dots, f(x_4\} \in Y^4$, we require that

$$CR(f(q)) \le \omega(CR(q))\,, \tag{5.7}$$

and call such embeddings f *quasi-CR embeddings*. By continuity, the cross ration can trivially be extended to the one-point compactification $Y \cup \{\infty\}$, so we can consider embeddings which are not necessarily fixing ∞.

In particular, for homeomorphism $\omega(t) = M \cdot \eta_\alpha(t)$ with given constants $M > 0$ and $\alpha \ge 1$ where

$$\eta_\alpha(t) = \begin{cases} t^\alpha, & \text{for } t \ge 1 \\ t^{1/\alpha}, & \text{for } 0 \le t < 1\,, \end{cases}$$

we call such embeddings f satisfying (5.7) as (M, α)-CR-embeddings.

In particular, this approach can be used to study the geometric realization Problem 5.1 in the case of free geometrically finite groups $G, H \subset \operatorname{Isom} X$ whose limit sets are discontinua. Here we have the following

Conjecture 5.8. *An embedding $f: A \hookrightarrow \overline{Y}$, $A \subset \overline{Y}$, in a Carnot–Carathéodory space Y is quasi-CR if and only if f preserves the notion of bounded μ-density of subsets $\Sigma \subset A$.*

Here a set Σ is called *μ-dense* for some $\mu > 1$ if any two points $a, b \in \Sigma$ can be connected by a *μ-chain* $\{x_i\}$ of points $x_i \in \Sigma$ where $\lim_{i\to-\infty} x_i = a$, $\lim_{i\to+\infty} x_i = b$ and $\ln(CR(a, x_i, x_{i+1}, b)) \le \ln \mu$. Naturally, the μ-density property is a characteristic property of the limit discontinua for geometrically finite groups. We also suspect that that property distinguishes parabolic subgroups of the groups G and H which cannot be quasiconformally conjugate (we expect that they should have non-maximal rank). On the other hand, we expect that an embedding $f: A \hookrightarrow \overline{Y}$ of a μ-dense set A in a Carnot–Carathéodory space Y is quasi-CR if and only if it is an (M, α)-CR embedding with some finite M and α. One can expect an affirmative solution of the above problems in the case of geometrically finite groups whose limit discontinua consist only of conical points (for the Heisenberg group Y, compare [Mn]). For study of these problems, one can use methods developed by Koranyi and Reimann [KR1-4], Vodop'yanov [Vo1,VG] as well as a generalization of the technique known as Sullivan's microscope (see the next section).

6. Deformations of discrete groups and Sullivan's stability

Here we concern with problems related to deformation spaces of a geometrically finite discrete group $G \subset PU(n, 1)$ acting by isometries in the complex hyperbolic

space $\mathbb{H}^n_{\mathbb{C}}$ and to the stability theorem of D.Sullivan [Su3] (which has been originally proved for planar Kleinian groups, see also [A11, MaG]). We restrict our attention to the complex hyperbolic case because other rank 1 spaces with negative variable curvature such as quaternionic and octanionic hyperbolic spaces are more rigid. In particular, due to Corlette's rigidity theorem for harmonic maps [C2], finite volume manifolds locally modeled on these spaces are super-rigid, completely analogous to Margulis super-rigidity in higher rank, see [Mg1]. Moreover, due to Gromov and Schoen [GS], all such finite volume manifolds are arithmetic.

Let $G \subset PU(n,1)$ be a geometrically finite discrete group. Then we have a fundamental problem concerning the deformation space of the inclusion (or of the representation of the group G obtained by restricting a natural inclusion $PU(n,1) \to PU(n+1,1)$ to G). The corresponding problem for deformation spaces of a hyperbolically rigid real hyperbolic lattice $G \subset O(n,1)$, $n \geq 3$, has its roots in the first construction by the author [A2] of non- trivial curves in the Teichmüller space $T(G)$ of conjugacy classes of representations of G in $O(n+1,1)$. Then the situation was greatly clarified by Thurston's "Mickey Mouse" example [Th] showing that such deformations of a hyperbolic surface are in fact bendings along geodesics. Since that time, several authors studied such deformation spaces, see for example [A4, A6, A8, A9-A11, JM, Ko].

In the case of deformations of a complex hyperbolic group $G \subset PU(n,1)$, we would like to assume that G is not a uniform lattice, that is the quotient $\mathbb{H}^n_{\mathbb{C}}/G$ is noncompact. Otherwise, due to a fundamental result on local rigidity of deformations [GoM], the set of "Fuchsian" representations of G, $\{\mathrm{Ad}\, h : G \to hGh^{-1},\ h \in PU(n+1,1)\}$, is a connected component $\mathcal{R}_0$ of the variety of representations $\mathrm{Hom}(G, PU(n+1,1)$.

On the other hand, we may deform an isomorphic uniform lattice $G' \subset PO(2,1)$, as well as another isomorphic convex co-compact groups $G' \subset PU(2,1)$, $G' \cong G$, if such groups have non-elementary subgroups preserving totally geodesic real planes [AG1]. Such quasiconformal deformations bend the corresponding complex surfaces along any simple closed geodesic in its totally real geodesic 2-dim subsurface. For another deformations of groups $G' \subset PO(2,1)$, see also [GP2]. We can also deform the above groups $G' \subset PO(2,1)$ in $PO(3,1) \subset PU(3,1)$, so it makes sense to consider such deformations in higher dimensions, as it has been done by the author in the real hyperbolic case, compare [A9-A11].

The main problem we deal with is as follows.

Conjecture 6.1. *Let $G \subset PU(n,1)$ be a geometrically finite group which is either convex cocompact or having parabolic subgroups of rank at least 3. Then every its representation $\rho : G \to PU(n+k,1)$, $k \geq 0$, that close enough to a natural inclusion, is in fact discrete and faithful and, furthermore, is a quasiconformal conjugation. That means that there is an equivariant quasiconformal self-homeomorphism f of the extended Heisenberg group $\overline{\mathcal{H}^{n+k}}$ such that $\rho G = f_* G = fGf^{-1}$.*

This conjecture generalizes a remarkable structural stability theorem of D. Sullivan [Su3] for Kleinian subgroups G in $PSL(2,\mathbb{C})$, which shows that an algebraic structural stability for holomorphic perturbations implies a hyperbolicity property for the action of G on its limit set. In conformal category, we refer to [A11] for our proof of its high dimensional analog. The complex case provides a possibility to

generalize a crucial Sullivan's argument, the so-called λ-Lemma proved by Mañe, Sad and Sullivan [MSS]. Here topological stability is an intermediate open problem. We note also that the negative curvature property is crucial for algebraic stability of convex cocompact subgroups, due to G.Martin [MaG]:

Theorem 6.2. *Let $\{G_t : t \in \mathbb{R}\}$ be a continuous deformation of a torsion free convex cocompact group $G \subset \mathrm{Isom}\, X$ in a space X of pinched negative curvature, with convex cocompact G_t. Then $\{G_t : t \in \mathbb{R}\}$, and its closure in the topology of algebraic convergence in $\mathrm{Isom}\, X$, consists entirely of isomorphic groups.*

For geometrically finite groups without parabolics, it is possible to use a generalization of the so-called Sullivan's microscope (see [Su2, A11]) to construct probable quasiconformal conjugation of the group actions on their limit sets. It is possible because one can find expanding Heisenberg coverings of the limit set of such a group, which defines a good Cauchy–Riemannian dynamics there. As another result important for a generalization of Sullivan's arguments, we mention an uniformization theorem for Cauchy–Riemannian structures which is due to Falbel and Gusevskii [FG].

Finally, we would like to mention another problems which are linked with Conjecture 6.1 and, in dimension $n = 2$, with Problem 3.14 about geometrical finiteness. These problems are related to the boundaries of the deformation spaces $\mathcal{R}(G,k) = \mathrm{Hom}(G, PU(n+k,1)$ and $\mathcal{T}(G,k) = \mathcal{R}(G,k)/PU(n+k,1)$, $k \geq 0$, to the number of their connected components, discreteness and geometrical finiteness of boundary representations. In particular:

Problem 6.3. *Is any representation ρ obtained as the limit of geometrically finite representations $\rho_i \in \mathcal{R}(G,k)$ discrete and faithful, that is $\rho(G) \subset PU(n+k)$ is a discrete group isomorphic to a given geometrically finite group $G \subset PU(n,1)$?*

We note that, due to G.Martin [MaG], the subset of discrete faithful representations is closed in $\mathcal{R}(G,k)$.

Another question is about the boundary of the Teichmüller space $\mathcal{T}(G,k)$ of a convex cocompact group $G \subset PU(n,1)$, $n \geq 2, k \geq 0$.

Problem 6.4. *Are there convex cocompact faithful representations $\rho_t \in \mathcal{R}(G,k)$, $t \in \mathbb{R}$, of a convex cocompact group $G \subset PU(n,1)$ which converge to a boundary discrete representation ρ whose image $\rho(G) \subset PU(n+k)$ has accidental parabolic elements?*

We remark that I.Belegradek [Be] constructed a discrete faithful representation ρ of the fundamental group G of a compact hyperbolic surface into $PU(2,1)$ such that $\rho(G)$ has parabolic elements. This construction uses a Maskit combination theorem, and it is unclear whether $\rho(G)$ lies in the boundary $\partial\mathcal{R}_0(G)$ of a component of the variety of convex co-compact representations $G \to PU(2,1)$. However the Teichmüller space $\mathcal{T}(G) = \mathcal{T}(G,0)$ does have "cusps" due to a recent result by Apanasov and Gusevskii [AG2]:

Theorem 6.5. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a uniform lattice isomorphic to the fundamental group of a closed surface S_g of genus $p \geq 2$. Then, for any simple closed geodesic $\alpha \subset S_p = H^2_{\mathbb{R}}/G$, there is a continuous deformation $\rho_t = f_t^*$ induced by G-equivariant quasiconformal homeomorphisms $f_t : \mathbb{H}^2_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}$ whose*

limit representation ρ_∞ corresponds to a boundary cusp point of the Teichmüller space $\mathcal{T}(G)$. In other words, the boundary group $\rho_\infty(G)$ has an accidental parabolic element $\rho_\infty(g_\alpha)$ where $g_\alpha \in G$ represents the simple closed geodesic $\alpha \subset S_p$.

We note that, due to our construction of such continuous quasiconformal deformations in [AG2], such deformations are independent if the corresponding simple closed geodesics $\alpha_i \subset S_p$ are disjoint. It implies the existence of a boundary group in $\partial\mathcal{T}(G)$ with "maximal" number of non-conjugate accidental parabolic subgroups:

Corollary 6.6. *Let $G \subset PO(2,1) \subset PU(2,1)$ be as in Theorem 6.5. Then there is a continuous deformation $R : \mathbb{R}^{2p-2} \to \mathcal{T}(G)$ whose boundary group $G_\infty = R(\infty)(G)$ has $2p-2$ non-conjugate accidental parabolic subgroups.*

Finally, we mention another aspect of the intrigue Problem 3.14:

Problem 6.7. *Construct a geometrically infinite (finitely generated) group $G \subset PU(n,1)$, $n \geq 2$, whose limit set is the whole sphere at infinity, $\Lambda(G) = \partial\mathbb{H}^n_{\mathbb{C}} = \overline{\mathcal{H}^n}$, and which is the limit of convex cocompact groups $G_i \subset PU(n,1)$ from the Teichmüller space $\mathcal{T}(G)$ of a convex cocompact group $G \subset PU(n,1)$. Is that possible for a Schottky group G?*

REFERENCES

[Ah] Lars V.Ahlfors, Fundamental polyhedra and limit point sets of Kleinian groups. - Proc. Nat. Acad. Sci. USA, **55**(1966), 251-254.

[As] V.V.Aseev, Quasi-symmetric embeddings and mappings with bounded distortion of moduli. - Sib. Matem. Zh./VINITI, Dep 7190-84, 1984 (Russian).

[AsT] V.V. Aseev and D.A. Trotsenko, Quasi-symmetric embeddings, quadruples of points, and moduli distortion. - Sibirsk. Matem. Zh. **28:4** (1987), 32-38 (Russian); Engl. Transl.: Siberian Math. J. **28** (1987).

[Au] L. Auslander, Bieberbach's theorem on space groups and discrete uniform subgroups of Lie groups, II. - Amer. J. Math., **83** (1961), 276-280.

[A1] Boris Apanasov, Discrete groups in Space and Uniformization Problems. - Math. and Appl. **40**, Kluwer Academic Publishers, Dordrecht, 1991.

[A2] _____, Nontriviality of Teichmüller space for Kleinian group in space. - Riemann Surfaces and Related Topics: Proc. 1978 Stony Brook Conference (I. Kra and B. Maskit, Eds), Ann. of Math. Studies **97**, Princeton Univ. Press, 1981, 21-31.

[A3] _____, Geometrically finite hyperbolic structures on manifolds. - Ann. of Glob. Analysis and Geom. **1:3** (1983), 1-22.

[A4] _____, Thurston's bends and geometric deformations of conformal structures. - Complex Analysis and Applications'85, Publ. Bulgarian Acad. Sci., Sofia, 1986, 14-28.

[A5] _____, Quasisymmetric embeddings of a closed ball inextensible in neighborhoods of any boundary point. - Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), 243-255.

[A6] _____, Nonstandard uniformized conformal structures on hyperbolic manifolds. - Invent. Math. **105** (1991), 137-152.

[A7] _____, Kobayashi conformal metric on manifolds, Chern-Simons and η-invariants. - Intern. J. Math. **2** (1991), 361-382.

[A8] _____, Deformations of conformal structures on hyperbolic manifolds. - J. Diff. Geom. **35** (1992), 1-20.

[A9] _____, Quasisymmetric knots and Teichmüller spaces.- Russian Acad. Sci. Dokl. Math. **343** (1995).

[A10] _____, Varieties of discrete representations of hyperbolic 3-lattices.- M.I.Kargopolov Memorial Volume (Yu.L.Ershov a.o., eds), W. de Gruyter-Verlag, Berlin, 1996, 7-20.

[A11] _____, Conformal geometry of discrete groups and manifolds. - W. de Gruyter, Berlin - New York, to appear.

[A12] _____, Canonical homeomorphisms in Heisenberg groups induced by isomorphisms of discrete subgroups of $PU(n,1)$. - Russian Acad. Sci. Dokl. Math., to appear.

[AG1] Boris Apanasov and Nikolay Gusevskii, Bending deformations of complex hyperbolic surfaces. - In preparation.

[AG2] _____, The boundary of Teichmüller space of complex hyperbolic surfaces. - In preparation.

[AX1] B.Apanasov and X.Xie, Geometrically finite complex hyperbolic manifolds. - Preprint, Univ. of Oklahoma, 1995, 51 pp.

[AX2] _____, Manifolds of negative curvature and nilpotent groups. - Preprint, Univ. of Oklahoma, 1995.

[AX3] _____, Discrete isometry groups of nilpotent Lie groups. - Preprint, Univ. of Oklahoma, 1995.

[AF] C.S. Aravinda and F.T. Farrell, Rank 1 aspherical manifolds which do not support any nonpositively curved metric. - Comm. Anal. Geom. **2** (1994), 65-78.

[BGS] W. Ballman, M. Gromov and V. Schroeder, Manifolds of nonpositive curvature. - Birkhäuser, 1985.

[BM] A. Beardon and B. Maskit, Limit points of Kleinian groups and finite sided fundamental polyhedra. - Acta Math. **132** (1974), 1-12.

[Be] I. Belegradek, Discrete surface groups actions with accidental parabolics on complex hyperbolic plane. - Preprint, 1995.

[Bo] B. Bowditch, Geometrical finiteness with variable negative curvature. - Duke J. Math. **77** (1995), 229-274.

[BuM] D. Burns and R. Mazzeo, On geometry of cusps for $SU(n,1)$. - Preprint, Univ. of Michigan, 1994.

[BS] D. Burns and S. Shnider, Spherical hypersurfaces in complex manifolds. - Inv. Math., **33**(1976), 223-246.

[C1] K. Corlette, Hausdorff dimensions of limit sets. - Invent. Math. **102** (1990), 521-542.

[C2] _____, Archimedian superrigidity and hyperbolic geometry. - Ann. of Math. **135** (1992), 165-182.

[CG] S. Chen and L. Greenberg, Hyperbolic spaces. - Contributions to Analysis, Academic Press, New York, 1974, 49-87.

[DS] S.K. Donaldson and D. Sullivan, Quasiconformal 4-manifolds. - Acta Math. **163** (1989), 181-252.

[EHS] P.Eberlein, U.Hamenstädt and V.Schroeder, Manifolds of nonpositive curvature. - Proc. Symp. Pure Math. **54:3**, 1993, 179-227.

[EMM] C. Epstein, R. Melrose and G. Mendoza, Resolvent of the Laplacian on strictly pseudoconvex domains. - Acta Math. **167** (1991), 1-106.

[FG] E. Falbel and N. Gusevskii, Spherical CR-manifolds of dimension 3. - Bol. Soc. Bras. Mat. **25** (1994), 31-56.

[FT] M.H. Freedman and L. Taylor, Λ-splitting 4-manifolds. - Topology **16** (1977), 181-184.

[GM] F.W. Gehring and G.J. Martin, Discrete quasiconformal groups. - Proc. London Math. Soc. **55** (1987), 331-358.

[Go] W. Goldman, Complex hyperbolic geometry. - Oxford Univ. Press, to appear.

[GoM] W. Goldman and J. Millson, Local rigidity of discrete groups acting on complex hyperbolic space. - Invent. Math., **88** (1987), 495-520.

[GP1] W. Goldman and J. Parker, Dirichlet polyhedron for dihedral groups acting on complex hyperbolic space. - J. of Geom. Analysis **2:6** (1992), 517-554.

[GP2] _____, Complex hyperbolic ideal triangle groups.- J. reine angew. Math. **425** (1992), 71-86.

[Gr] M. Gromov, Carnot–Carathéodory spaces seen from within.- Preprint IHES, Bures-sur-Yvette, 1994.

[JM] D. Johnson and J. Millson, Deformation spaces associated to compact hyperbolic manifolds. - Discrete Groups in Geometry and Analysis,Boston, 1987, 48-106.

[KR1] A. Koranyi and M. Reimann, Quasiconformal mappings on the Heisenberg group. - Invent. Math. **80** (1985), 309-338.

[KR2] _____, Foundations for the theory of quasiconformal mappings on the Heisenberg group. - Adv. Math. **111** (1995), 1-87.

[KR3] _____, Contact transformations as limits of symplectomorphisms. - C. R. Acad. Sci. Paris **318** (1994), 1119-1124.

[KR4] _____, The complex cross ratio on the Heisenberg group.- L'Ens. Math. **33** (1987), 291-300.

[Ko] C. Kourouniotis, Deformations of hyperbolic structures.- Math. Proc. Cambr. Phil. Soc. **98** (1985), 247-261.

[LB] Claude LeBrun, Einstein metrics and Mostow rigidity. - Preprint, Stony Brook, 1994.

[MSS] R.Mañe, P.Sad and D.Sullivan, On the dynamics of rational maps.- Ann. Sci. École Norm. Sup. **16** (1983), 193-217.

[Md] A. Marden, The geometry of finitely generated Kleinian groups. - Ann. of Math. **99** (1974), 383-462.

[Mg1] G.A. Margulis, Discrete groups of motions of manifolds of nonpositive curvature. - Amer. Math. Soc. Transl. **109** (1977), 33-45.

[Mg2] _____, Free properly discontinuous groups of affine transformations. - Dokl. Acad. Sci. USSR **272** (1983), 937-940.

[MM] G.A. Margulis and G.D. Mostow, The differential of a quasiconformal mapping of a Carnot-Carathéodory space. - Geom. Funct. Anal. **5** (1995), 402-433.

[MaG] G. Martin, On discrete isometry groups of negative curvature.- Pacif. J. Math. **160** (1993), 109-127.

[Mas] B.Maskit, Kleinian groups. - Springer-Verlag, 1987.

[Mat] R. Matveyev, A decomposition of smooth simply-connected h-cobordant 4-manifolds. - Preprint, Michigan State Univ. at E.Lansing, 1995.

[Ml] J. Milnor, On the 3-dimensional Brieskorn manifolds $M(p,q,r)$.- Knots, groups and 3-manifolds, Ann. of Math. Studies **84**, Princeton Univ. Press, 1975, 175–225.

[Mn] R. Miner, Quasiconformal equivalence of spherical CR manifolds.- Ann. Acad. Sci. Fenn. Ser. A I Math. **19** (1994), 83-93.

[Mo1] George D. Mostow, Strong rigidity of locally symmetric spaces.- Princeton Univ. Press, 1973.

[Mo2] _____, On a remarkable class of polyhedra in complex hyperbolic space. - Pacific J. Math., **86** (1980), 171-276.

[Mo3] _____, A remark on quasiconformal mappings on Carnot groups.- Mich. Math. J. **41** (1994), 31-37.

[My] R. Myers, Homology cobordisms, link concordances, and hyperbolic 3-manifolds.- Trans. Amer. Math. Soc. **278** (1983), 271-288.

[Pa] P. Pansu, Métriques de Carnot–Carathéodory et quasiisométries des espaces symmétries de rang un.- Ann. Math. **129** (1989), 1-60.

[Ph] M.B. Phillips, Dirichlet polyhedra for cyclic groups in complex hyperbolic space. - Proc. Amer. Math. Soc. **115** (1992), 221-228.

[Sa1] Nikolai Saveliev, Floer homology and 3-manifold invariants.- Ph.D. Thesis, Univ. of Oklahoma, 1995.

[Sa2] _____, Floer homology and invariants of homology cobordism.- Preprint, Univ. of Oklahoma, 1995.

[Sa3] _____, On homology cobordisms of plumbed homology 3-spheres.- Preprint, Univ. of Oklahoma, 1995.

[Sa4] _____, Notes on homology cobordisms of plumbed homology 3-spheres.- Preprint, Univ. of Michigan, 1995.

[Su1] D. Sullivan, Hyperbolic geometry and homeomorphisms.- Geometric topology, Acad. Press, 1979.

[Su2] _____, Seminar on conformal and hyperbolic geometry.- Preprint IHES, 1982.

[Su3] _____, Quasiconformal homeomorphisms and dynamics, II: Structural stability implies hyperbolicity for Kleinian groups. - Acta Math. **155** (1985), 243–260.

[Te] A.V. Tetenov, The discontinuity set for a Kleinian group and topology of its Kleinian manifold. - Intern. J. Math. **4** (1993), 167-177.

[Th] W.P. Thurston, The geometry and topology of three-manifolds. - Lect. Notes, Princeton Univ., 1979.

[To] D. Toledo, Representations of surface groups on complex hyperbolic space. - J. Diff. Geom. **29** (1989), 125-133.

[Tu] P. Tukia, On isomorphisms of geometrically finite Kleinian groups. - Publ. Math. IHES **61** (1985), 171-214.

[TV] P. Tukia and J. Väisälä, Quasisymmetric embeddings of metric spaces.- Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), 97-114.

[V] J. Väisälä, Quasimöbius maps.- J. Analyse Math. **50** (1984/85), 218-234.

[Vo1] S.K. Vodop'yanov, Quasiconformal mappings on Carnot groups.- Russian Dokl. Math., to appear.

[Vo2] _____, Monotone functions on Carnot groups.- Siberian Math. J. **37:5** (1996), to appear.

[VG] S.K. Vodop'yanov and A.V.Greshnov, Analytic properties of quasiconformal mappings on Carnot groups. - Siberian Math. J. **36:6** (1995), 1317-1327.

[VU] S.K. Vodop'yanov and A.D. Ukhlov, Weakly contact transformations and a change-of-variable formula on nilpotent groups. - Russian Acad. Sci. Dokl. Math. **341:4** (1995), 439-441.

[Wa] C.T.C. Wall, Geometric structures on compact complex analytic surfaces. - Topology, **25** (1986), 119-153.

[Wo] J. Wolf, Spaces of constant curvature.- Publ. or Perish, Berkeley, 1977.

[Y] S.T.Yau, Calabi's conjecture and some new results in algebraic geometry.- Proc. Nat. Acad. Sci **74** (1977), 1798-1799.

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