VALUE DISTRIBUTION FOR SEQUENCES OF RATIONAL MAPPINGS AND COMPLEX DYNAMICS

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Abstract. We obtain results on the asymptotic equidistribution of the pre-images of linear subspaces for sequences of rational mappings between projective spaces. As an application to complex dynamics, we consider the iterates P_k of a rational mapping P of $\mathbb{P}^n$. We show, assuming a condition on the topological degree λ of P, that there is a probability measure μ on $\mathbb{P}^n$ such that the discrete measures $\lambda^{-k} P_k^* \delta_w$ converge to μ for all $w \in \mathbb{P}^n$ outside a pluripolar set.

1. Introduction

The study of value distribution for sequences of mappings may be regarded as an analogue of Nevanlinna theory; instead of studying the asymptotic behavior of the area of a pre-image of an analytic set in a ball when its radius tends to infinity, one investigates the asymptotics of the pre-images of an analytic set under a sequence of mappings. One of the main reasons for studying this subject is its applications to complex dynamics, although value distribution theory for sequences is also of independent interest.

Many investigations have the Brolin-Lyubich Theorem (see [Br, Ly, FLM]) as their starting point. This theorem can be formulated as follows:

Let $R(z)$ be a rational function of degree $d \geq 2$, and let R_k be its k-th iterate. Then there is an invariant probability measure μ on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ such that for all w outside an exceptional set $\mathcal{E} \subset \mathbb{P}^1$ containing at most 2 points,

$$\frac{1}{d^k}(R_k)^* \delta_w \to \mu.$$

Here $(R_k)^* \delta_w$ is a discrete measure in $\mathbb{P}^1$, counting the number of roots of the equation $R_k(z) = w$ with multiplicities; convergence is in the weak sense.

Date: March 28, 1996.

Shiffman's research was partially supported by National Science Foundation Grant DMS-9500491. The research of both authors at MSRI is partially supported by National Science Foundation Grant DMS-9022140.

Many of the recent papers on holomorphic dynamics in several complex variables (see [BS1, BS2, BS3, BLS1, BLS2, HP, FS1, FS2, FS3, FS4]) have dealt with various extensions of this theorem to n variables in particular cases. This result was also extended by Sodin [So] to the case where $\{R_n\}$ is an arbitrary sequence of rational functions of one variable with rapidly increasing degrees. Sodin proved that if one agrees to omit a larger set $\mathcal{E}$ of exceptional values, then the pre-images of the remaining values $w \in \mathbb{P}^1 \setminus \mathcal{E}$ are equidistributed in a certain sense.

In a recent paper [RS], the phenomenon of equidistribution for sequences of polynomial mappings $\mathbb{C}^n \to \mathbb{P}^m$ was studied. The phenomenon and the corresponding results may be described as follows. We let $\omega = \omega_n$ denote the Kähler form of the Fubini-Study metric on complex projective n-space $\mathbb{P}^n$ (normalized so that $\int_{\mathbb{P}^n} \omega^n = 1$).

Suppose that $\{P_k\}$ is a sequence of polynomial mappings $\mathbb{C}^n \to \mathbb{P}^m$, such that $\sum \frac{1}{\delta_1(P_k)} < \infty$, where $\delta_1(P_k)$ denotes the maximal degree of the components of P_k. Then

- (i) *pre-images of all but an exceptional pluripolar set of complex hyperplanes in $\mathbb{P}^m$ are equidistributed with the pull-backs $P_k^*\omega$.*
- (ii) *if all P_k are non-degenerate, then pre-images of all but a pluripolar set of points in $\mathbb{P}^m$ are equidistributed with $P_k^*\omega^m$ $(m \leq n)$.*

The exceptional sets in (i) and (ii) above are described in terms of a "proximity sequence."

In [RS], only the cases of codimension 1 and m were treated. The present paper deals with all intermediate codimensions $1 \leq \ell \leq \min(n, m)$ and rational mappings $\mathbb{P}^n \to \mathbb{P}^m$ (Theorem 1.1) as well as giving a higher dimensional version (with pluripolar exceptional set) of the Brolin-Lyubich theorem (Theorem 1.3).

In order to state our results on the asymptotic equidistribution of pre-images of linear subspaces of intermediate dimension, we let $\mathbb{G}(\ell, m)$ denote the Grassmannian of projective linear subspaces of codimension ℓ in $\mathbb{P}^m$. Note that $\mathbb{G}(m, m) = \mathbb{P}^m$. If $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is a meromorphic (i.e., rational) map, we let $\delta_\ell(P)$ denote the degree of $P^{-1}(W)$ for generic $W \in \mathbb{G}(\ell, m)$ $(1 \leq \ell \leq \min(n, m))$. (For generic W, $P^{-1}(W)$ has pure codimension ℓ.) One easily sees that $\delta_1(P)$ is the degree of the polynomials in a representation of P (using homogeneous coordinates); if P is holomorphic (i.e., regular), then $\delta_\ell(P) = \delta_1(P)^\ell$. In Section 4, we give an analytic description of the "intermediate degrees" $\delta_\ell(P)$ and we show that $\delta_{k+\ell}(P) \leq \delta_k(P)\delta_\ell(P)$. Intermediate degrees have been considered also in [Fr].

Our first result on the equidistribution of pre-images under an arbitrary sequence of rational maps of projective spaces is as follows:

THEOREM **1.1.** *Let $\{P_k\}$ be a sequence of rational mappings from $\mathbb{P}^n$ to $\mathbb{P}^m$. Let $1 \le \ell \le \min(n,m)$, and let $\{a_k\}$ be a sequence of positive numbers such that*

$$\sum_{k=1}^{\infty} \frac{\delta_{\ell-1}(P_k)}{a_k} < +\infty \ .$$

Then there exists a pluripolar subset $\mathcal{E}$ of $\mathbb{G}(\ell,m)$ such that

$$\frac{1}{a_k}\left(P_k^*[W] - P_k^*\omega^\ell\right) \to 0$$

as $k \to \infty$ for all $W \in \mathbb{G}(\ell,m) \setminus \mathcal{E}$.

Here, convergence means weak convergence in $\mathcal{D}'^{\ell,\ell}(\mathbb{P}^n)$. In Section 6, we give a description of the exceptional set $\mathcal{E}$ in terms of the "proximity sequence." The pull-back $P_k^*\omega^\ell$ is smooth off the indeterminacy locus of P and has locally integrable coefficients. For generic W, $P_k^*[W]$ is given by integration over $P_k^{-1}(W)$. Precise definitions of the pull-back currents $P_k^*[W], P_k^*\omega^\ell \in \mathcal{D}'^{\ell,\ell}(\mathbb{P}^n)$ are given in Section 3.

As an application of Theorem 1.1 to complex dynamics, we have the following result on the equidistribution of iterated pre-images:

COROLLARY **1.2.** *Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational mapping, and let P_k denote the k-th iterate of P. If $a > \delta_{\ell-1}(P)$, then*

$$\frac{1}{a^k}\left(P_k^*[W] - P_k^*\omega^\ell\right) \to 0$$

for all $W \in \mathbb{G}(\ell,m)$ outside a pluripolar set.

To obtain Corollary 1.2 from Theorem 1.1, we use the fact that $\delta_{\ell-1}(P_k) \le \delta_{\ell-1}(P)^k$, which we verify in Section 4 (see Lemma 4.6).

For holomorphic maps $P : \mathbb{P}^n \to \mathbb{P}^n$ of degree $d \ge 2$ it was shown in [HP] and [FS4] that $\frac{1}{d^k}\log|P_k|^2$ converges uniformly to a Green's function G (where P_k is the k-th iterate of P), and hence by Bedford-Taylor [BT],

$$\frac{1}{d^{nk}} P_k^*\omega^n \to (dd^cG)^n = \mu \ ,$$

where μ is a probability measure on $\mathbb{P}^n$ which is invariant in the sense that $P_*\mu = \mu$.

However, Fornaess and Sibony [FS4] showed that for a non-holomorphic map $P : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of degree d, the number of points in the generic fibre of P is strictly less than d^2. (We give a generalization of this fact as Lemma 4.4.) Thus, for such P it follows that

$$\frac{1}{d^{2k}} P_k^*\omega^2 \to 0 \ .$$

So the question of existence of a nontrivial limit measure in the meromorphic case remained open. The following result gives a limit measure and equidistribution for the iterates of meromorphic maps on $\mathbb{P}^n$:

Theorem **1.3.** *Let $P : \mathbb{P}^n - \to \mathbb{P}^n$ be a rational map, let $\lambda = \delta_n(P)$ denote the topological degree of P, and write*

$$\mu_k = \frac{1}{\lambda^k} P_k^* \omega^n \ .$$

If $\lambda > \delta_{n-1}(P)$, then the sequence $\{\mu_k\}$ converges to a probability measure μ on $\mathbb{P}^n$ and

$$\frac{1}{\lambda^k} P_k^*[W] \to \mu$$

for all points $W \in \mathbb{P}^n$ outside a pluripolar set.

Remark: We have for any test function φ,

$$(P_* \mu_{k+1}, \varphi) = \frac{1}{\lambda^{k+1}} \int (\varphi \circ P) P_{k+1}^* \omega^n = \frac{1}{\lambda^{k+1}} \int P^*(\varphi P_k^* \omega^n) = \frac{1}{\lambda^k} \int \varphi P_k^* \omega^n = (\mu_k, \varphi) \ .$$

Thus

$$P_* \mu_{k+1} = \mu_k \ . \tag{1}$$

If $\mu(I_P) = 0$, then $P_* \mu$ would be well-defined and hence we could let $k \to \infty$ in (1) to conclude that μ is an invariant measure, i.e., $P_* \mu = \mu$.

Further results will be given in a subsequent paper.

2. Examples and open questions

We begin with several elementary examples of sequences of iterated mappings illustrating our results. These examples are the extensions to $\mathbb{P}^2$ of proper polynomial mappings of $\mathbb{C}^2$. Besides equidistribution, we are going to look at invariance properties of the limit currents and measures.

Example 1. (This example was discussed in [RS].) Let $P_k(z, w)$ be the k-th iterate of the mapping $P = (z^\delta, w^\delta) : \mathbb{C}^2 \to \mathbb{C}^2$. We first restrict our attention to $\mathbb{C}^2$. It is easy to see that the uniform convergence $\frac{1}{\delta^k} \log(1 + |P_k(z, w)|^2) \to G(z, w)$ takes place, where

$$G(z, w) = \sup(\log^+ |z|^2, \log^+ |w|^2)$$

is the plurisubharmonic Green function of the unit polydisk with logarithmic growth at infinity. It follows that

$$dd^c G = \lim_{k \to \infty} \frac{1}{\delta^k} P_k^* \omega.$$

The current $T = dd^c G$ is concentrated on the boundary of the unit polydisk and on the set $\{|z| = |w| > 1\}$. This current is invariant under the mapping P in the sense that

$$P^* T = \delta T \ .$$

(The pull-backs of closed positive $(1,1)$-currents by holomorphic and meromorphic maps are defined in Section 3.) According to the results of [RS], this current is the limit of pre-images of all nonexceptional hyperplanes (lines).

Every hyperplane of the form $z = c$ or $w = c$ has pre-images of similar form. These pre-images tend towards the cylinder $|z| = 1$ or $|w| = 1$, respectively. Thus all such hyperplanes are exceptional. A third family of exceptional hyperplanes consists of those passing through the origin. Pre-images of these hyperplanes tend towards the "cone" $\{|z| = |w|\}$. We now consider the complex projective plane $\mathbb{P}^2 = \mathbb{C}^2 \cup H_\infty$, identifying $(z,w) \in \mathbb{C}^2$ with $(1,z,w) \in \mathbb{P}^2$. We let $\mathbb{P}^{2*} \approx \mathbb{P}^2$ denote the parameter space of hyperplanes in $\mathbb{P}^2$; the point $(\zeta_0, \zeta_1, \zeta_2) \in \mathbb{P}^{2*}$ represents the hyperplane $\{(z,w) \in \mathbb{C}^2 : \zeta_0 + \zeta_1 z + \zeta_2 w = 0\}$. (The point $(1,0,0)$ represents the hyperplane at infinity, H_∞.) Thus we see that the set of exceptional hyperplanes consists of the three pencils in $\mathbb{P}^{2*}$: $\{\zeta_0 = 0\}$, $\{\zeta_1 = 0\}$, $\{\zeta_2 = 0\}$. We note that, besides T, there are at least five more (linearly independent) invariant closed $(1,1)$-currents (on $\mathbb{C}^2$), $T_1 = dd^c \log^+ |z|^2$, $T_2 = dd^c \log^+ |w|^2$, $T_3 = dd^c \log \max(|z|^2, |w|^2)$, $T_4 = [H_1] = dd^c \log |z|^2$, $T_5 = [H_2] = dd^c \log |w|^2$ (where $H_1 = \{z = 0\}, H_2 = \{w = 0\}$) with the same property

$$P^* T_i = \delta T_i.$$

In fact, they are limits of pre-images of the corresponding exceptional hyperplanes. (These currents are invariant on $\mathbb{P}^2$ as well as on $\mathbb{C}^2$, and we have the additional invariant current $T_6 = [H_\infty]$ on $\mathbb{P}^2$.)

Now consider pre-images of points. The measure $\mu = T^2 = (dd^c G)^2$, being the limit of $\dfrac{1}{\delta^{2k}} P_k^* \omega^2$, is concentrated on the distinguished boundary of the unit polydisk; hence pre-images of most points must tend to the torus by Theorem 2 of [RS]. However, all points of the form $(0,c)$ or $(c,0)$ have pre-images of the form $(0,c')$ (respectively $(c',0)$) and are definitely exceptional (as well as the points of H_∞). So in this case the exceptional set is the same union of 3 hyperplanes in $\mathbb{P}^{2*}$.

The measure μ possesses invariance properties, $P^*\mu = \delta^2 \mu$. Note that we also have $\mu = T_1 \wedge T_2$.

The map P extends to a rational mapping: $Q(t,z,w) = (t^\delta, z^\delta, w^\delta) : \mathbb{P}^2 \to \mathbb{P}^2$, which we call the *projectivization* of P. (For this example, Q is holomorphic.) The situation then becomes very symmetric with respect to all variables. To simplify the terminology, we will call the corresponding currents (measures) "projectivizations" of the ones defined on $\mathbb{C}^n$. Instead of, say, $dd^c \log^+ |z|^2 = dd^c \log(|z|^2 \vee 1)$ one has to consider $dd^c \log(|z|^2 \vee |t|^2)$, where $\vee$ stands for maximum, and so on. The same points and hyperplanes are exceptional, and because of the absence of the indeterminacy set, no difficulty occurs.

Example 2. The situation changes if we consider the iterates of the mapping $P = (z^{d_1}, w^{d_2}) : \mathbb{C}^2 \to \mathbb{C}^2$ with $d_1 \neq d_2$. For simplicity, let $P = (z^2, w^3)$. It is easy to see that in this case $\dfrac{1}{3^k}\log(1+|P_k(z,w)|^2) \to G(z,w)$, where now $G(z,w) = \log^+|w|^2$. It follows as before that

$$dd^cG = \lim_{k\to\infty} \frac{1}{3^k} P_k^*\omega;$$

however this time the current $T = dd^cG$ is concentrated on the cylinder $\{|w| = 1\}$. It is invariant for the mapping P:

$$P^*T = 3T\ .$$

Applying the results of [RS], we see that this current is the limit of the pre-images of all nonexceptional hyperplanes.

The set of exceptional hyperplanes in $\mathbb{C}^2$ consists of the two pencils in $\mathbb{P}^{2*}$: $\{\zeta_0 = 0\}$, $\{\zeta_2 = 0\}$. Besides T, there are at least three more invariant currents: $T_1 = dd^c\log^+|z|^2$, $T_2 = dd^c\log|z|^2$, $T_3 = dd^c\log|w|^2$. However this time

$$P^*T_1 = 2T_1\ .$$

(Note that for any smooth form S on $\mathbb{C}^2$, we have $P_*P^*S = 6S$; by smoothing and taking limits, one sees that this identity is also valid if S is a positive $(1,1)$-current on $\mathbb{C}^2$. Hence $P_*T_1 = 3T_1,\ P_*T = 2T$.) The currents T_i are the limits of pre-images of the corresponding families of exceptional hyperplanes. The coefficient 2 for T_1 makes the situation somewhat different as we shall see.

Now consider pre-images of points. The results of [RS] do not provide useful information since now $T^2 = (dd^cG)^2 \equiv 0$. According to Theorem 1.3, this is not a surprise, since $\delta_2(P) = 6$, not 9, and we have a limit

$$\mu = \lim_{k\to\infty} \frac{1}{6^k} P_k^*\omega^2,$$

which is the same measure (concentrated on the distinguished boundary of the unit polydisk) as in Example 1. So pre-images of most points must concentrate there. Again, the exceptional set for points is the same union of 3 hyperplanes in $\mathbb{P}^2$. The measure μ possesses the invariance property $P^*\mu = 6\mu$. Note that we also have $\mu = T \wedge T_1$.

Consider the projectivization of this mapping: $Q(t,z,w) = (t^3, tz^2, w^3) : \mathbb{P}^2 \to \mathbb{P}^2$ and the sequence Q_k of its iterations. Note that although P is holomorphic, Q is only meromorphic and has one indeterminacy point $(0,1,0)$ which is the reason for all "anomalies." Since H_∞ is contracted by Q to the fixed point $(0,0,1)$, the map Q is "generic" in the sense of Fornaess and Sibony [FS3, FS4]. Also, the graph G of the mapping Q is singular. So it is really necessary to resolve the singularities of G to define all our currents correctly (see Section 3). Note that we have $\delta_1(Q_k) = 3^k,\ \delta_2(Q_k) = 6^k < \delta_1^2(Q_k)$. In accordance with the results of [FS3, FS4], there is an invariant $(1,1)$-current $T = dd^c \log(|w|^2 \vee |t|^2)$ which is the projectivization of the above mentioned current on $\mathbb{C}^2$. The projectivization of the other invariant current, T_1, is no longer invariant, since now $Q^* T_1 = 2T_1 + H_\infty$.

The situation is even more complicated for the point case. The results of [FS3, FS4] do not provide a nontrivial invariant measure since $T^2 = 0$. According to Theorem 1.3, there is a limit measure μ' in this case. In fact, $\mu' = [dd^c \log(|z|^2 \vee |w|^2 \vee |t|^2)]^2$ is the projectivization of μ and is invariant on $\mathbb{P}^2$. We have $\mu' = T \wedge T_1$ as before, although now T_1 is not invariant itself. The measure μ' has the property $Q^*\mu' = 6\mu'$.

Example 3. Finally, consider the sequence of iterations of $P(z,w) = (w^3, z^2)$. This example is not generic in the sense of Fornaess and Sibony [FS3, FS4], since H_∞ is contracted by (the projectivization of) P to the indeterminacy point $(0,1,0)$, so their results cannot be applied. Since $\delta_1(P_k) \neq \delta_1^k(P)$ (which is the case for non-generic maps) we cannot conclude that the sequence of currents $\frac{1}{\delta_1(P_k)} P_k^* \omega$ converges. Instead we have two subsequences

$$\{P_{2k} = (z^{6^k}, w^{6^k})\}\ , \quad \{P_{2k+1} = (w^{3\cdot 6^k}, z^{2\cdot 6^k})\}\ .$$

The first subsequence is the same as in Example 1 with $\delta = 6$, and the corresponding current subsequence thus converges to

$$T_1 = dd^c \log^+(|z|^2 \vee |w|^2).$$

The second subsequence of currents converges to

$$T_2 = dd^c \log^+(|w|^2 \vee |z|^{4/3})\ .$$

These currents are responsible for the distribution of pre-images of hyperplanes. As for the invariance properties, there seems to be no invariant current since we have $P^*T_1 = 3T_2,\ P^*T_2 = 2T_1$.

For the codimension two case, we have $\delta_2(P_k) = \delta_2^k(P) = 6^k$. So there is a limit measure which is the same as in the two previous examples and is invariant.

We state here some open problems. In the Brolin-Lyubich Theorem, the exceptional set consists of two points. In the above examples, the exceptional sets are unions of at most 3 hyperplanes in $\mathbb{P}^2$. This leads to the questions:

Can we further describe the exceptional set in Theorem 1.3? Is the exceptional set in fact algebraic?

Another question is:

Does the measure μ in Theorem 1.3 charge the indeterminacy set of P?

If the answer to this question is "no", then by the remark following Theorem 1.3, $P_*\mu$ would be well-defined and μ would be an invariant measure.

3. Notation and terminology

We let $\mathcal{E}^{p,q}(X)$, $\mathcal{D}^{p,q}(X)$, $\mathcal{D}'^{p,q}(X)$ denote the spaces of (complex-valued) $\mathcal{C}^\infty$ forms, compactly supported $\mathcal{C}^\infty$ forms, and currents, respectively, of bidegree (p,q) on a complex manifold X and we use the standard differentials $d = \partial + \overline{\partial}$, $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \overline{\partial})$. Points in complex projective n-space $\mathbb{P}^n$ are identified with their representations $z = (z_0, z_1, \ldots, z_n)$ in homogeneous coordinates. We shall regard the Grassmannian $\mathbb{G}(\ell, m)$ of projective linear subspaces of codimension ℓ in $\mathbb{P}^m$ as a subvariety of $\mathbb{P}(\bigwedge^{m+1-\ell}\mathbb{C}^{m+1})$.

If $P : \mathbb{P}^n - \to \mathbb{P}^m$ is a non-constant meromorphic map, it is a well-known consequence of Chow's theorem that P must be rational, i.e., P can be written in the form $P = (P_0, \ldots, P_n)$ where $P_j \in \mathbb{C}[z_0, \ldots, z_n]$ and $\deg P_0 = \cdots = \deg P_m = d$. We can assume that the P_j have no common factors; we then say that $\deg P = d$. (This notion of degree should not be confused with the *topological degree* of an equidimensional rational map, which is the number of points in the pre-image of a generic point in the range.) We let $I_P \subset \mathbb{P}^n$ denote the indeterminacy locus of P (the points where P is not holomorphic); I_P is an algebraic subvariety of codimension ≥ 2.

Suppose $f : Y \to X$ is a holomorphic mapping of complex manifolds. If α is a current on X, $f^*\alpha$ is not always defined. However, we shall define $f^*\alpha$ in two special cases: First, we suppose $\alpha = u\gamma$ where $\gamma \in \mathcal{E}^{p,q}(X)$ is a smooth form and u is the difference of plurisubharmonic functions. Assume further that $f(Y)$ is not contained in the $\pm\infty$ locus of u. Then $u \circ f$ is the difference of plurisubharmonic functions on Y and hence is in $\mathcal{L}^1_{\mathrm{loc}}(Y)$. We define $f^*\alpha = (u \circ f) f^*\gamma$, which is clearly independent of the representation $\alpha = u\gamma$. The second case we consider is that of a current of the form $[D] \wedge \gamma$, where $[D]$ is the current of integration over a divisor D on X and γ is a smooth form as before. We assume also that $f(Y) \not\subset \operatorname{Supp} D$ so that f^*D is a divisor on Y. We then define $f^*([D] \wedge \gamma) = [f^*D] \wedge f^*\gamma$. These two definitions are consistent in the following way. Suppose $\alpha = \log|g|^2 \cdot \gamma$ where g is a meromorphic function on X such that neither the zeroes nor the poles of g contain $f(Y)$ and γ is a closed (p,q)-form on X. Then $dd^c\alpha = [D] \wedge \gamma$ where $D = \operatorname{Div}(g)$. Hence

$$f^* dd^c \alpha = [f^*D] \wedge f^*\gamma = dd^c(\log|g \circ f|^2 \cdot f^*\gamma) = dd^c f^*\alpha \,. \tag{2}$$

Let $P : \mathbb{P}^n - \to \mathbb{P}^m$ be a rational map. For a smooth (p,q)-form $\eta \in \mathcal{E}^{p,q}(\mathbb{P}^m)$ we define the pull-back current $P^*\eta \in \mathcal{D}'^{p,q}(\mathbb{P}^m)$ as follows: We let $G_P \subset \mathbb{P}^n \times \mathbb{P}^m$ denote the graph of P (which is an irreducible algebraic subvariety of $\mathbb{P}^n \times \mathbb{P}^m$) and we consider a desingularization $\widetilde{G} \xrightarrow{\rho} G_P$. We have the commutative diagram:

$$\begin{array}{ccc} & \widetilde{G} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}^n & \overset{P}{-\to} & \mathbb{P}^m \end{array} \tag{3}$$

We then define

$$P^*\eta = \pi_{1*}\pi_2^*\eta\ .$$

Note that $P^*\eta$ has coefficients in $\mathcal{L}^1_{\rm loc}$ and has singular support contained in the indeterminacy locus I_P of P. In fact, $(P|_{\mathbb{P}^n\setminus I_P})^*\eta$ is the usual pull-back of the form η.

We consider the current of integration $[W]\in\mathcal{D}'^{\ell,\ell}(\mathbb{P}^m)$ and define the pull-back $\pi_2^*[W]$ to be the current of integration over the algebraic $(n-\ell)$-cycle π_2^*W on $\widetilde{G}$ (using the diagram (3)), whenever $\dim\pi_2^{-1}(W) = n-\ell$. If we represent W as the intersection of hyperplanes $H_1,\dots,H_\ell$ in $\mathbb{P}^m$, then π_2^*W is the intersection of divisors $\pi_2^*H_1\cap\cdots\cap\pi_2^*H_\ell$. For a definition of this intersection, which is a formal sum of the irreducible components of $\pi_2^{-1}(W)$ with positive integer coefficients, see [Ha, Appendix A] or Definitions 2.3 and 2.4.2 (or Example 7.1.10) in [Fu]. This pull-back, or intersection, can also be defined analytically as follows. Let g_j be a local defining function for $\pi_2^*H_j$, for $1\le j\le\ell$, and write $g=(g_1,\dots,g_\ell)$. Then by Griffiths and King [GK, 1.10] (see also [Sh, I.12, Th. 3]), we have the local formula

$$\pi_2^*[W] = dd^c\left(\log|g|^2(dd^c\log|g|^2)^{\ell-1}\right)\ .$$

(Alternately, $\pi_2^*[W] = dd^c\log|g_1|^2\wedge\ldots\wedge dd^c\log|g_\ell|^2$, where the existence of this product of currents is guaranteed by Demailly [De].) One way to verify that these definitions are all equivalent is to first consider generic hyperplanes $H_1,\dots,H_\ell$ so that the divisors $\pi_2^*H_1,\dots,\pi_2^*H_\ell$ are smooth hypersurfaces (of multiplicity 1) in $\widetilde{G}$ intersecting transversely. (This is possible by Lemma 4.1 in the next section.) Then for this case, $\pi_2^*[W]$ is the current of integration over a smooth submanifold. In the general case, the current $\pi_2^*[W]$ is the weak limit of the pull-backs of generic intersections $W^\nu = H_1^\nu\cap\cdots\cap H_\ell^\nu$ converging to W. The existence of the limit follows, for example, from [Fu, Cor. 11.1] for the algebraic definition and from [De] or [FS5, Cor. 3.6] for the analytic definition.

We now state the Poincaré-Lelong formula for linear subspaces of $\mathbb{P}^m$ and describe its pull-backs by a rational map $P:\mathbb{P}^n\dashrightarrow\mathbb{P}^m$. Let $W\in\mathbb{G}(\ell,m)$ be an $(m-\ell)$-plane in $\mathbb{P}^m$. For each $W\in\mathbb{G}(\ell,m)$, we define the current

$$\Lambda_W = \log\frac{|\zeta|^2|W|^2}{|\zeta\wedge W|^2}\sum_{j=0}^{\ell-1}(d_\zeta d^c_\zeta\log|\zeta\wedge W|^2)^j\wedge\omega_\zeta^{\ell-1-j}\in\mathcal{D}'^{\ell-1,\ell-1}(\mathbb{P}^m_\zeta)\ ,$$

which (by definition) has locally integrable coefficients. We have the *generalized Poincaré-Lelong formula* for W [GK, 1.15] (see also [Sh, II.6, pp. 68-69]),

$$dd^c\Lambda_W = \omega^\ell - [W]\ . \tag{4}$$

Now let $W = H_1\cap\cdots\cap H_\ell$ such that $\dim\pi_2^{-1}(W) = n-\ell$, where we use the notation of (3). Applying the generalized Poincaré-Lelong formula ([GK, 1.15] or [Sh, II.6, pp. 68-69]), to the divisors $\pi_2^*H_1, \ldots, \pi_2^*H_\ell$ of the lifted hyperplane-section bundle $\pi_2^*\mathcal{O}_{\mathbb{P}^m}(1)$ with Chern form $\pi_2^*\omega$, we obtain

$$dd^c\pi_2^*\Lambda_W = \pi_2^*\omega^\ell - \pi_2^*[W]\ , \tag{5}$$

where $\pi_2^*\Lambda_W \in \mathcal{D}'^{\ell-1,\ell-1}(\widetilde{G})$ is given by

$$(\pi_2^*\Lambda_W)(\widetilde{z}) \stackrel{\text{def}}{=} \log\frac{|\pi_2(\widetilde{z})|^2|W|^2}{|\pi_2(\widetilde{z})\wedge W|^2}\sum_{j=0}^{\ell-1}(dd^c\log|\pi_2(\widetilde{z})\wedge W|^2)^j\wedge\pi_2^*\omega^{\ell-1-j} \in \mathcal{D}'^{\ell-1,\ell-1}(\widetilde{G}). \tag{6}$$

In particular, $\pi_2^*\Lambda_W$ has $\mathcal{L}^1_{\text{loc}}$ coefficients and is smooth on $\widetilde{G}\setminus\pi_2^{-1}(W)$. We define the currents

$$\begin{array}{rcl} P^*\Lambda_W & = & \pi_{1*}(\pi_2^*\Lambda_W)\ , \\ P^*[W] & = & \pi_{1*}[\pi_2^{-1}W]\ . \end{array} \tag{7}$$

By applying π_{1*} to (5), we obtain the current identity on $\mathbb{P}^n$,

$$dd^cP^*\Lambda_W = P^*\omega^\ell - P^*[W]. \tag{8}$$

Note that for generic W, $\pi_2^*[W]$ has multiplicity identically 1 and contains no components inside the exceptional locus of π_1, and thus $[P^*W]$ is the current of integration over the closure of $(P|_{\mathbb{P}^n\setminus I_P})^{-1}(W)$.

4. The intermediate degrees of a rational map

In this section, we give some properties of the intermediate degrees $\delta_\ell(P)$ of a rational map P, which we also describe analytically and topologically. We use the following consequence of Bertini's theorem:

Lemma **4.1.** *Let Y be a projective algebraic manifold and let $f : Y \to \mathbb{P}^m$ be a nonconstant holomorphic map. Then for a generic hyperplane $H \subset \mathbb{P}^m$, the divisor f^*H is smooth and has multiplicity* 1.

Proof: Apply Bertini's theorem (see, for example, [GH, p. 137]) to the complete linear system of f^*H. □

For a subvariety $V \subset \mathbb{P}^m$, we write $P^{-1}(V) = \pi_1(\pi_2^{-1}(V))$ (using the notation in (3)). We let $\#(S)$ denote the cardinality of a set S. We begin with a formula for the integral of certain singular forms on $\mathbb{P}^n$:

Lemma **4.2.** *Let $P_j : \mathbb{P}^n - \to \mathbb{P}^{m_j}$, $1 \le j \le n$, be rational maps, and let $I = I_{P_1} \cup \cdots \cup I_{P_n}$.*

i) *For generic hyperplanes $H_1 \subset \mathbb{P}^{m_1}, \ldots, H_n \subset \mathbb{P}^{m_n}$,*

$$\int_{\mathbb{P}^n\setminus I} P_1^*\omega_{m_1}\wedge\cdots\wedge P_n^*\omega_{m_n} = \#\left(\bigcap_{j=1}^n P_j^{-1}(H_j)\setminus I\right)\ ,$$

ii)

$$\int_{\mathbb{P}^n \setminus I} P_1^*\omega_{m_1} \wedge \cdots \wedge P_n^*\omega_{m_n} \leq \prod_{j=1}^{n} \deg P_j \ .$$

To verify Lemma 4.2, we first give a topological description of the integral. Let $X = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}$ and write $\mathcal{P} = (P_1, \ldots, P_n) : \mathbb{P}^n - \rightarrow X$; then $I = I_{\mathcal{P}}$. Consider the commutative diagram

$$\begin{array}{ccc} & \widetilde{G} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}^n & \stackrel{\mathcal{P}}{-\rightarrow} & X \end{array}$$

where $\widetilde{G}$ is a desingularization of the graph of $\mathcal{P}$ and π_1, π_2 are the projections. Let $p_j : X \to \mathbb{P}^{m_j}$ denote the projection to the j-th factor and let $\widetilde{P}_j = p_j \circ \pi_2 : \widetilde{G} \to \mathbb{P}^{m_j}$, for $1 \leq j \leq n$. Let t_m denote the positive generator of $H^2(\mathbb{P}^m, \mathbb{Z})$, and write

$$t_{m_1} \times \cdots \times t_{m_n} = p_1^* t_{m_1} \bullet \cdots \bullet p_n^* t_{m_n} \in H^{2n}(X, \mathbb{Z}) \ ,$$

where $\bullet$ denotes the cup product in the cohomology ring.

Lemma **4.3.** *Using the notation of Lemma 4.2,*

$$\begin{aligned} \int_{\mathbb{P}^n \setminus I} P_1^*\omega_{m_1} \wedge \cdots \wedge P_n^*\omega_{m_n} &= \left(\pi_2^*(t_{m_1} \times \cdots \times t_{m_n}), \widetilde{G}\right) \\ &= (\widetilde{P}_1^* t_{m_1} \bullet \cdots \bullet \widetilde{P}_n^* t_{m_n}, \widetilde{G}) \in \mathbb{Z} \ . \end{aligned}$$

Proof: Since the de Rham class of the Kähler form ω_m on $\mathbb{P}^m$ equals t_m, we have by Section 3,

$$\begin{aligned} \int_{\mathbb{P}^n \setminus I} P_1^*\omega_{m_1} \wedge \cdots \wedge P_n^*\omega_{m_n} &= \int_{\mathbb{P}^n \setminus I_{\mathcal{P}}} \mathcal{P}^*(\omega_{m_1} \times \cdots \times \omega_{m_n}) \\ &= (\pi_{1*}\pi_2^*(\omega_{m_1} \times \cdots \times \omega_{m_n}), 1) \\ &= (\pi_2^*(\omega_{m_1} \times \cdots \times \omega_{m_n}), 1) \\ &= \left(\pi_2^*(t_{m_1} \times \cdots \times t_{m_n}), \widetilde{G}\right) \ . \end{aligned}$$

The second equality follows from our definitions. □

Proof of Lemma 4.2: Let $E \subset \widetilde{G}$ be the exceptional locus of π_1. By Lemma 4.1 applied to $\widetilde{P}_1 : \widetilde{G} \to \mathbb{P}^{m_1}$, there is a hyperplane $H_1 \subset \mathbb{P}^{m_1}$ such that the divisor $\pi_2^*(H_1 \times \mathbb{P}^{m_2} \times \cdots \times \mathbb{P}^{m_n})$ is a smooth hypersurface $Y_1 \subset \widetilde{G}$ of multiplicity 1 with $\dim Y_1 \cap E < n-1$. By Lemma 4.1, we can inductively find hyperplanes $H_2 \subset \mathbb{P}^{m_2}, \ldots, H_n \subset \mathbb{P}^{m_n}$ such that, writing

$$Y_j = \pi_2^{-1}(H_1 \times \cdots \times H_j \times \mathbb{P}^{m_{j+1}} \times \cdots \times \mathbb{P}^{m_n}) \ ,$$

Y_j is a smooth submanifold of $\widetilde{G}$ of dimension $n-j$, $\dim Y_j \cap E < n-j$, and the divisor $(\widetilde{P}_j|_{Y_{j-1}})^* H_j$ (on Y_{j-1}) has multiplicity 1, or equivalently, $\widetilde{P}_j^{-1}(H_j)$ intersects Y_{j-1} transversely. In particular, $\dim Y_n = 0$ and $Y_n \cap E = \emptyset$. Write $H'_j = p_j^{-1}(H_j)$, for $1 \le j \le n$, so that

$$Y_j = \pi_2^{-1}(H'_1 \cap \cdots \cap H'_j) = \pi_2^{-1}(H'_j) \cap Y_{j-1} \ .$$

A codimension j submanifold S of a complex manifold Y determines the current of integration $[S] \in \mathcal{D}'^{j,j}(Y)$ given by $([S], \varphi) = \int_S \varphi$ for a test form φ. We also let $[S]$ denote the de Rham class in $H^{2j}(Y, \mathbb{R})$ containing the current $[S]$. (If Y is compact of dimension n, then the cohomology class $[S]$ is the Poincaré dual of the $(2n-2j)$-cycle S.) If two submanifolds S_1, S_2 intersect transversely, then $[S_1 \cap S_2] = [S_1] \bullet [S_2]$ in the cohomology ring of Y. In particular, $[H_j] = t_j \in H^2(\mathbb{P}^{m_j}, \mathbb{Z}) \subset H^2(\mathbb{P}^{m_j}, \mathbb{R})$ and

$$[H_1 \times \cdots \times H_n] = [H'_1 \cap \cdots \cap H'_n] = t_{m_1} \times \cdots \times t_{m_n} \ .$$

Furthermore, our construction of the H_j, Y_j implies that

$$[Y_j] = [Y_{j-1}] \bullet \pi_2^*[H'_j] = \pi_2^*[H'_1] \bullet \cdots \bullet \pi_2^*[H'_j] = \pi_2^*[H'_1 \cap \cdots \cap H'_j] \ .$$

Therefore,

$$\pi_2^*(t_{m_1} \times \cdots \times t_{m_n}) = \pi_2^*[H_1 \times \cdots \times H_n] = [Y_n] \tag{9}$$

(where the points of Y_n have multiplicity 1).

Let $A_j : \mathbb{C}^{m_j+1} \to \mathbb{C}$ be a linear map defining the hyperplane H_j, and consider the polynomial

$$Q_j = A_j(P_{j0}, \ldots, P_{jn}) \in \mathbb{C}[z_0, \ldots, z_n] \ ,$$

where $P_j = (P_{j0}, \ldots, P_{jn})$, for $1 \le j \le n$. We then have

$$\{z \in \mathbb{P}^n : Q_j(z) = 0\} = \pi_1\left(\pi_2^{-1}(H'_j)\right) \supset I_{P_j} \ . \tag{10}$$

Since $Y_j \subset \widetilde{G} \setminus E$ and π_1 maps $\widetilde{G} \setminus E$ bijectively to $\mathbb{P}^n \setminus I_{\mathcal{P}}$, we have

$$\begin{aligned}
\pi_1(Y_n) &= \pi_1\left(\pi_2^{-1}(H'_1 \cap \cdots \cap H'_n)\right) \setminus I_{\mathcal{P}} \\
&= \bigcap_{j=1}^n \pi_1\left(\pi_2^{-1}(H'_j)\right) \setminus I_{\mathcal{P}} = \bigcap_{j=1}^n P_j^{-1}(H_j) \setminus I_{\mathcal{P}} \\
&= \{z \in \mathbb{P}^n \setminus I_{\mathcal{P}} : Q_1(z) = \cdots = Q_n(z) = 0\} \ .
\end{aligned}$$

Thus by (9),

$$\left(\pi_2^*(t_{m_1} \times \cdots \times t_{m_n}), \widetilde{G}\right) = \left([Y_n], \widetilde{G}\right) = \#(Y_n) \ .$$

By Bézout's Theorem,

$$\#(Y_n) = \#(\pi_1(Y_n)) = \#\left(\bigcap_{j=1}^n P_j^{-1}(H_j) \setminus I_{\mathcal{P}}\right) \le \prod_{j=1}^n \deg Q_j = \prod_{j=1}^n \deg P_j \ .$$

The conclusion follows from Lemma 4.3. □

Definition: Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map. We define the *intermediate degrees* $\delta_\ell(P)$ of P by the formula

$$\delta_\ell(P) = \int_{\mathbb{P}^n \setminus I_P} P^* \omega_m^\ell \wedge \omega_n^{n-\ell}$$

for $1 \leq \ell \leq \min(n, m)$.

We shall show in Lemma 4.4 below that the intermediate degrees are also given by the geometric definition in the introduction; in particular, if $m = n$, then $\delta_n(P)$ is the *topological degree* of P, which is defined as the cardinality of $P^{-1}(x)$, for a generic point $x \in \mathbb{P}^n$. Clearly, $\delta_\ell(P) > 0$ if and only if rank $P \geq \ell$. It is easy to verify that $\delta_1(P) = \deg P$, and if P is holomorphic (this can happen only if $m \geq \operatorname{rank} P = n$), then $\delta_\ell(P) = (\deg P)^\ell$, for $\ell \leq n$.

It follows from Lemma 4.2 applied to the maps $P_1 = \cdots = P_\ell = P$, $P_{\ell+1} = \cdots = P_n = \mathrm{Id}_{\mathbb{P}^n}$ that in general,

$$\delta_\ell(P) \leq (\deg P)^\ell , \quad \text{for } \ 1 \leq \ell \leq n . \tag{11}$$

We shall give a more general inequality in Lemma 4.7 below.

LEMMA **4.4.** *Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map. Then*

$$\delta_\ell(P) = \deg P^{-1}(W) \leq (\deg P)^\ell$$

for generic $W \in \mathbb{G}(\ell, m)$, with equality if and only if $\operatorname{codim} I_P > \ell$. *In particular, $\delta_n(P) \leq (\deg P)^n$, with equality if and only if P is holomorphic.*

Proof: We shall apply Lemma 4.2 with $P_1 = \cdots = P_\ell = P$, $P_{\ell+1} = \cdots = P_n = \mathrm{Id}_{\mathbb{P}^n}$. By part (ii) of the lemma, $\delta_\ell(P) \leq (\deg P)^\ell$. For generic $W = H_1 \cap \cdots \cap H_\ell$, $\pi_2^{-1}(W)$ is of codimension ℓ and has no components contained in the exceptional locus of π_1, and thus $P^{-1}(W)$ has pure dimension $n - \ell$ and $\dim P^{-1}(W) \cap I_P < n - \ell$. Hence for generic hyperplanes $H_1, \ldots , H_n$, we have

$$\bigcap_{j=1}^n P_j^{-1}(H_j) \setminus I_P = P^{-1}(W) \cap H_{\ell+1} \cap \cdots \cap H_n ,$$

where $W = H_1 \cap \cdots \cap H_\ell$, and thus by part (i),

$$\delta_\ell(P) = \# \left(P^{-1}(W) \cap H_{\ell+1} \cap \cdots \cap H_n \right) = \deg P^{-1}(W) .$$

Furthermore, using the notation in the proof of Lemma 4.2,

$$\{z \in \mathbb{P}^n : Q_1(z) = \cdots = Q_\ell(z) = 0\} = P^{-1}(W) \cup I_P .$$

Since $\deg Q_1 = \cdots = \deg Q_\ell = \deg P$, it follows from Bézout's theorem (see for example [Fu, Example 8.4.6]) that $\delta_\ell(P) < (\deg P)^\ell$ if $\dim I_P \geq n - \ell$. If $\dim I_P <$

$n-\ell$, then $P^{-1}(W) \supset I_P$ (by dimension considerations) and Bézout's theorem gives equality. □

In fact, if $n=2$ in Lemma 4.4, then $\delta_2(P) = (\deg P)^2 - q$, where q is the number of points of I_P counting multiplicity. This is illustrated by the following example.

Example: Let $P : \mathbb{P}^2 - \to \mathbb{P}^2$ be given by

$$P(z_0, z_1, z_2) = (z_1 z_2, z_0 z_2, z_0 z_1) = \left(\frac{1}{z_0}, \frac{1}{z_1}, \frac{1}{z_2}\right) .$$

Then $\delta_1(P) = 2,\ \delta_2(P) = 1$. Note that in this example I_P consists of the three points $(1,0,0), (0,1,0), (0,0,1)$.

Lemma **4.5.** *Suppose that $P : \mathbb{P}^n - \to \mathbb{P}^m$ is a rational map and $L : \mathbb{C}^{m+1} \to \mathbb{C}^{M+1}$ is a linear map such that* Image$P \not\subset \mathbb{P}(L^{-1}(0))$. *Let $P_L = \widehat{L} \circ P : \mathbb{P}^n - \to \mathbb{P}^M$, where $\widehat{L} : \mathbb{P}^m - \to \mathbb{P}^M$ is the map induced from L. Then*

$$\int_{\mathbb{P}^n \setminus I} (P_L^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} \le \delta_\ell(P)$$

for $1 \le k \le \ell \le n$, where $I = I_P \cup I_{P_L}$.

Proof: We can assume without loss of generality that $M=m$. We first consider the case where L is nonsingular and thus $\widehat{L}$ is biholomorphic. Therefore

$$\int_{\mathbb{P}^n \setminus I} (P_L^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} = \int_{\mathbb{P}^n \setminus I} (P^* \omega')^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell}$$

where $\omega' = \widehat{L}^*\omega$. Since ω' and ω are in the same de Rham class, it follows from Section 3 (or by the proof of Lemma 4.3) that

$$\int_{\mathbb{P}^n \setminus I} (P^* \omega')^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} = \int_{\mathbb{P}^n \setminus I} (P^*\omega)^{\ell} \wedge \omega^{n-\ell} = \delta_\ell(P) .$$

We now suppose that L is singular. Choose a sequence $\{L_\nu\}$ of nonsingular linear operators on $\mathbb{C}^{m+1}$ such that $L_\nu \to L$. We can write

$$(P_{L_\nu}^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} = f_\nu \omega^n , \quad (P_L^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} = f \omega^n$$

where f_ν, f are non-negative $\mathcal{C}^\infty$ functions on $\mathbb{P}^n \setminus I$. Then $f_\nu \to f$ pointwise on $\mathbb{P}^n \setminus I$, and hence by Fatou's Lemma,

$$\int_{\mathbb{P}^n \setminus I} (P_L^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} = \int_{\mathbb{P}^n \setminus I} f\omega^n \le \liminf_{\nu\to\infty} \int_{\mathbb{P}^n \setminus I} f_\nu \omega^n = \delta_\ell(P) .$$

□

Lemma **4.6.** *Let $P : \mathbb{P}^n - \to \mathbb{P}^m$, $Q : \mathbb{P}^m - \to \mathbb{P}^r$ be rational maps. Then*

$$\delta_\ell(Q \circ P) \le \delta_\ell(P)\delta_\ell(Q) .$$

Proof: Let $\eta = Q^*\omega^\ell \in \mathcal{D}'^{\ell,\ell}(\mathbb{P}^n)$. We smooth η by an approximate identity $\{\psi_\varepsilon\}_{\varepsilon>0}$ with respect to a Haar measure h on $GL(n+1,\mathbb{C})$ to obtain

$$\eta_\varepsilon \stackrel{\text{def}}{=} \int_{GL(n+1,\mathbb{C})} (g^*\eta)\psi_\varepsilon(g)dh(g) \in \mathcal{E}^{\ell,\ell}(\mathbb{P}^n) \ .$$

Then $\eta_\varepsilon \to \eta$ pointwise as $\varepsilon \to 0$, $\eta_\varepsilon \geq 0$, and we have the identity in de Rham cohomology,

$$[\eta_\varepsilon] = [\eta] = \delta_\ell(Q)[\omega^\ell] \in H^{2\ell}(\mathbb{P}^n,\mathbb{Z}) \ .$$

Using the commutative diagram (3), we then have

$$\int_{\mathbb{P}^n} P^*\eta_\varepsilon \wedge \omega^{n-\ell} = \int_{\widetilde{G}} \pi_2^*\eta_\varepsilon \wedge \pi_1^*\omega^{n-\ell} = \int_{\widetilde{G}} \delta_\ell(Q)\pi_2^*\omega^\ell \wedge \pi_1^*\omega^{n-\ell} = \delta_\ell(Q)\delta_\ell(P) \ .$$

Therefore, by Fatou's lemma,

$$\delta_\ell(Q\circ P) = \int_{\mathbb{P}^n\setminus I_{Q\circ P}} P^*\eta \wedge \omega^{n-\ell} \leq \liminf_{\varepsilon\to 0} \int_{\mathbb{P}^n} P^*\eta_\varepsilon \wedge \omega^{n-\ell} = \delta_\ell(Q)\delta_\ell(P) \ .$$

□

LEMMA **4.7.** *Let $P : \mathbb{P}^n - \to \mathbb{P}^m$ be a rational map. Then*

$$\delta_{k+\ell}(P) \leq \delta_k(P)\delta_\ell(P) \ .$$

Proof: Let $\eta = P^*\omega^\ell \in \mathcal{D}'^{\ell,\ell}(\mathbb{P}^m)$ and consider the smooth forms η_ε as in the above proof. As before,

$$[\eta_\varepsilon] = \delta_\ell(P)[\omega^\ell] \in H^{2\ell}(\mathbb{P}^n,\mathbb{Z}) \ ,$$

and

$$\begin{aligned}\int_{\mathbb{P}^n} P^*\omega^k \wedge \eta_\varepsilon \wedge \omega^{n-k-\ell} &= \int_{\widetilde{G}} \pi_2^*\omega^k \wedge \pi_1^*(\eta_\varepsilon \wedge \omega^{n-k-\ell}) \\ &= \delta_\ell(P)\int_{\widetilde{G}} \pi_2^*\omega^k \wedge \pi_1^*\omega^{n-k} = \delta_\ell(P)\delta_k(P) \ .\end{aligned}$$

The conclusion follows as above by letting $\varepsilon \to 0$ and applying Fatou's lemma. □

5. The proximity function

Let $P : \mathbb{P}^n - \to \mathbb{P}^m$ be a rational map, and let $1 \leq \ell \leq \min(n,m)$. Recall that if $\operatorname{codim} P^{-1}(W) = \ell$, then equations (6), (7) define a current $P^*\Lambda_W \in \mathcal{D}'^{\ell-1,\ell-1}(\mathbb{P}^n)$ with locally integrable coefficients. Hence we can define the *proximity function* m_P^ℓ : $\mathbb{G}(\ell,m) \to [0,+\infty]$ by

$$m_P^\ell(W) = \begin{cases} (P^*\Lambda_W, \omega^{n-\ell+1}) = \int_{\mathbb{P}^n\setminus(I_P\cup I_{P,W})} P^*\Lambda_W \wedge \omega^{n-\ell+1} & \text{if } \operatorname{codim} P^{-1}(W) = \ell \\ +\infty & \text{if } \operatorname{codim} P^{-1}(W) < \ell \end{cases}$$

for $W \in \mathbb{G}(\ell, m)$, where $I_{P,W}$ is the indeterminacy locus of the map $z \mapsto P(z) \wedge W \in \mathbb{P}(\bigwedge^{m+2-\ell} \mathbb{C}^{m+1})$. We give $\mathbb{G}(\ell, m)$ the Kähler metric ω induced from the natural embedding $\mathbb{G}(\ell, m) \subset \mathbb{P}(\bigwedge^{m+1-\ell} \mathbb{C}^{m+1})$.

The following key estimate is used in our proof of Theorem 1.1.

Lemma **5.1.** *If $P : \mathbb{P}^n - \to \mathbb{P}^m$ is a rational map, then $m_P^\ell \in \mathcal{L}^1(\mathbb{G}(\ell, m))$ and*

$$dd^c m_P^\ell \leq \ell \delta_{\ell-1}(P)\omega\ ,$$

for $1 \leq \ell \leq \min(n, m)$.

Proof: (The techniques used for this proof are somewhat similar to those of [Sk1, Sk2].) Assume first that $\ell \geq 2$. (The estimate of Lemma 5.1 is straightforward for the hyperplane case $\ell = 1$; we give the argument for this case at the end of this proof. See also [RS] for a complete treatment of pre-images of hyperplanes.) Write $E = \bigwedge^{m+1-\ell} \mathbb{C}^{m+1}$, $\widetilde{E} = \bigwedge^{m+2-\ell} \mathbb{C}^{m+1}$ and let $\lambda_0 : E \to \mathbb{C}$, $\lambda_1 : \widetilde{E} \to \mathbb{C}$ be linear functions of unit norm. For $\zeta \in \mathbb{C}^{m+1}$, $W \in E$, we define the *augmented exterior product*

$$\zeta \tilde{\wedge} W = (\zeta_0 W_0, \zeta \wedge W) \in \mathbb{C} \oplus \widetilde{E}$$

where $W_0 = \lambda_0(W)$. Let

$$X = \mathbb{P}^n_z \times \mathbb{G}(\ell, m)_W \times \mathbb{P}^m_\zeta \times \mathbb{P}(\widetilde{E})_\theta \times \mathbb{P}(\mathbb{C} \oplus \widetilde{E})_\eta\ .$$

(The subscripts z, W, ζ, θ, η serve to identify the variables used in this discussion.)

We further write $\lambda_1(\theta) = \theta_1$ for $\theta \in \widetilde{E}$. If $\eta = (c, \theta) \in \mathbb{C} \oplus \widetilde{E}$, we write $\eta_0 = c$, $\eta_1 = \theta_1 = \lambda_1(\theta)$. By making a linear change of coordinates in $\mathbb{P}^m_\zeta$ we can assume without loss of generality that $\mathrm{Image}(P) \not\subset \{\zeta \in \mathbb{P}^m : \zeta_0 = 0\}$. We consider the current

$$\Omega = \log \frac{|\zeta|^2 |W|^2 |\theta_1|^2 |\eta_0|^2}{|\zeta_0|^2 |W_0|^2 |\theta|^2 |\eta_1|^2} \omega_z^{n-\ell+1} \wedge \sum_{j=0}^{\ell-1} \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j} \in \mathcal{D}'^{n-1,n-1}(X)\ .$$

Let $Q : \mathbb{P}^n \times \mathbb{G}(\ell, m) - \to X$ be the meromorphic (rational) map given by

$$Q(z, W) = (z, W, P(z), P(z) \wedge W, P(z) \tilde{\wedge} W)\ . \tag{12}$$

(The redundancy in the map Q is needed to facilitate our proof.) We can assume that λ_1 was chosen so that $\lambda_1(P(z) \wedge W) \not\equiv 0$. We shall show (see Lemma 5.2) that

$$m_P^\ell = \pi_{2*} Q^* \Omega\ ,$$

where $\pi_2 : \mathbb{P}^n \times \mathbb{G}(\ell, m) \to \mathbb{G}(\ell, m)$ is the projection. Of course, since Q is not regular, we must define the pull-back $Q^*\Omega$, which we do as follows. Let

$$Y \xrightarrow{\rho} \mathrm{Image}(Q) \subset X$$

be a desingularization of the image of Q. (Note that $\mathrm{Image}(Q)$ is an algebraic subvariety of X and can be identified with the graph of Q.) Let $\pi_1 : X \to \mathbb{P}^n \times \mathbb{G}(\ell, m)$ be the projection, and write $\rho_1 = \pi_1 \circ \rho, \rho_2 = \pi_2 \circ \rho_1$ so that we have the commutative diagram:

$$\begin{array}{ccccc} & & \mathbb{P}^n \times \mathbb{G}(\ell, m) & \xleftarrow{\rho_1} & Y \\ & & Q \downarrow & \rho \swarrow \quad \downarrow \rho_1 & \searrow \rho_2 \\ & & X \xrightarrow{\pi_1} & \mathbb{P}^n \times \mathbb{G}(\ell, m) & \xrightarrow{\pi_2} \mathbb{G}(\ell, m) \end{array}$$

We then define $Q^*\Omega = \rho_{1*}\rho^*\Omega$, where $\rho^*\Omega$ is given as in Section 3. Let $I_Q \subset \mathbb{P}^n \times \mathbb{G}(\ell, m)$ denote the indeterminacy locus of Q; I_Q is an algebraic subvariety of codimension ≥ 2. Write $U = \mathbb{P}^n \times \mathbb{G}(\ell, m) \setminus I_Q$, and let $Q_0 = Q|_U : U \to X$. Since ρ_1 maps $\rho_1^{-1}(U)$ biholomorphically onto U, we have $Q_0 = \rho \circ \rho_1^{-1}|_U$. Hence

$$(Q^*\Omega)|_U = (\rho_1^{-1}|_U)^* \rho^*\Omega = Q_0^*\Omega \ ,$$

so our definition of Q^* agrees with the usual one at regular points. Since $Q^*\Omega$ has coefficients in $\mathcal{L}^1_{\mathrm{loc}}$, it is the extension to $\mathbb{P}^n \times \mathbb{G}(\ell, m)$ of $Q_0^*\Omega$ with zero mass on I_Q.

We first note the following:

LEMMA **5.2.** *$m_P^\ell = \rho_{2*}\rho^*\Omega = \pi_{2*}Q^*\Omega \in \mathcal{L}^1(\mathbb{G}(\ell, m))$.*

Proof: By the definition of $Q^*\Omega$ we have

$$\rho_{2*}\rho^*\Omega = \pi_{2*}\rho_{1*}\rho^*\Omega = \pi_{2*}Q^*\Omega \ . \tag{13}$$

Since $Q^*\Omega$ has coefficients in $\mathcal{L}^1_{\mathrm{loc}}$ and its degree, $2n$, is the fibre dimension of π_2, $\pi_{2*}Q^*\Omega \in \mathcal{L}^1(\mathbb{G}(\ell, m))$ and

$$(\pi_{2*}Q^*\Omega)(W) = \int_{\mathbb{P}^n_z \times \{W\} \setminus I_Q} Q^*\Omega < +\infty \ , \quad \text{for a.a. } W \in \mathbb{G}(\ell, m) \ . \tag{14}$$

Since $\eta_1 \circ Q = \theta_1 \circ Q$ and $\eta_0 \circ Q = P_0(z)W_0 = (\zeta_0 W_0) \circ Q$, we have

$$Q^*\Omega|_U = Q_0^*\Omega = \log \frac{|P(z)|^2 |W|^2}{|P(z) \wedge W|^2} \omega_z^{n-\ell+1} \wedge \sum_{j=1}^{\ell-1} (dd^c \log |P(z) \wedge W|^2)^j \wedge (P^*\omega_\zeta)^{\ell-1-j} \ .$$

Thus for a.a. $W \in \mathbb{G}(\ell, m)$, $I_Q \not\supset \mathbb{P}^n \times \{W\}$ and

$$m_P^\ell(W) = \int_{\mathbb{P}^n \setminus (I_P \cup I_{P,W})} P^*\Lambda_W \wedge \omega^{n-\ell+1} = \int_{\mathbb{P}^n_z \times \{W\} \setminus I_Q} Q^*\Omega \leq +\infty \ . \tag{15}$$

The desired identity follows from (13), (14), and(15). □

Remark: We could use (6), (7) to define $m_P^\ell(W)$ for all $W \in \mathbb{P}(\bigwedge^{m+1-\ell} \mathbb{C}^{m+1})$; then Lemma 5.1 remains valid on $\mathbb{P}(\bigwedge^{m+1-\ell} \mathbb{C}^{m+1})$.

We are now ready to compute $dd^c m_P^\ell$. We let H_ζ^0, H_W^0, H_θ^1, H_η^0, H_η^1 denote the hyperplanes in X given by the divisors of $\zeta_0, W_0, \theta_1, \eta_0, \eta_1$ respectively, and we let

$$D = H_\zeta^0 + H_W^0 - H_\theta^1 + H_\eta^1 - H_\eta^0 = \mathrm{Div}\left(\frac{\zeta_0 W_0 \eta_1}{\theta_1 \eta_0}\right) \ .$$

We have

$$
\begin{aligned}
dd^c\Omega &= (\omega_\zeta + \omega_W - \omega_\theta - D) \wedge \omega_z^{n-\ell+1} \wedge \sum_{j=0}^{\ell-1} \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j} \\
&= \left[\omega_\zeta^\ell - \omega_\theta^\ell + (\omega_W - D) \wedge \sum_{j=0}^{\ell-1} \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j} \right] \wedge \omega_z^{n-\ell+1} \, .
\end{aligned}
\tag{16}
$$

LEMMA **5.3.** *The divisor $\rho^* D$ is effective (i.e., is locally the divisor of a holomorphic function).*

Proof: (Our choices of ζ_0 and θ_1 guarantee that $\rho(Y) \not\subset \operatorname{Supp} D$ so that $\rho^* D$ is defined.) Let $y_0 \in Y$ be arbitrary, and let

$$X' = \{(z, W, \zeta, \theta, \eta) \in X : z_i \neq 0, \lambda(W) \neq 0, \zeta_j \neq 0, \lambda'(\theta) \neq 0\}$$

where $0 \leq i \leq n$, $0 \leq j \leq m$ and $\lambda : E \to \mathbb{C}, \lambda' : \widetilde{E} \to \mathbb{C}$ are linear functions chosen so that $\rho(y_0) \in X'$. Let

$$g = \frac{\widetilde{\zeta}_0 \widetilde{W}_0}{\widetilde{\theta}_1} \frac{\eta_1}{\eta_0} \in \operatorname{Mer}(X')$$

where $\widetilde{\zeta}_0 = \zeta_0/\zeta_j$, $\widetilde{W}_0 = W_0/\lambda(W) = (\lambda_0/\lambda)(W)$, $\widetilde{\theta}_1 = \theta_1/\lambda'(\theta) = (\lambda_1/\lambda')(\theta)$. Then $\operatorname{Div}(g) = D|_{X'}$. We must show that $g \circ \rho$ is holomorphic at y_0. Now

$$
\begin{aligned}
g \circ Q &= \frac{P_0(z)}{P_j(z)} \frac{W_0}{\lambda(W)} \frac{\lambda'(P(z) \wedge W)}{\lambda_1(P(z) \wedge W)} \frac{\lambda_1(P(z) \wedge W)}{P_0(z) W_0} \\
&= \frac{\lambda'(P(z) \wedge W)}{P_j(z) \lambda(W)} = \frac{\lambda'(P(\widetilde{z}) \wedge \widetilde{W})}{P_j(\widetilde{z})}
\end{aligned}
$$

where $\widetilde{z} = z_i^{-1} z \in \mathbb{C}^{n+1}$, $\widetilde{W} = \lambda(W)^{-1} W \in E$. Write, for $y \in Y$,

$$\rho_1(y) = (z(y), W(y)) \, ,$$

$\widetilde{z}(y) = z_i^{-1}(y) z(y), \widetilde{W}(y) = \lambda(W(y))^{-1} W(y)$. Since ρ is holomorphic and $\rho(y_0) \notin \operatorname{Div}(\zeta_j)$, there is a neighborhood Y_0 of y_0 so that $P_\mu(\widetilde{z}(y)) = \varphi(y) f_\mu(y)$ for $y \in Y_0$ and $0 \leq \mu \leq m$, where $\varphi, f_0, \ldots, f_m \in \mathcal{O}(Y_0), \varphi \not\equiv 0$, and $f_j(y_0) \neq 0$. Thus

$$g \circ \rho = g \circ Q \circ \rho_1 = \frac{\lambda'(\varphi F \wedge \widetilde{W})}{\varphi f_j} = \frac{\lambda'(F \wedge \widetilde{W})}{f_j}$$

on Y_0, where $F = (f_0, \ldots f_m)$. Since $f_j(y_0) \neq 0$, $g \circ \rho$ is holomorphic at y_0. □

We now complete the proof of Lemma 5.1: By Lemma 5.3 and (16), we have

$$\rho^* dd^c \Omega \leq \rho^* \left[\left(\omega_\zeta^\ell + \omega_W \wedge \sum_0^{\ell-1} \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j} \right) \wedge \omega_z^{n-\ell+1} \right] \, .$$

By Section 3, $\rho^* dd^c\Omega = dd^c\rho^*\Omega$. Hence by Lemma 5.2,

$$\begin{aligned} dd^c m_P^\ell &= dd^c(\rho_{2*}\rho^*\Omega) = \rho_{2*}\rho^* dd^c\Omega \\ &\le \rho_{2*}\rho^* \left[(\omega_\zeta^\ell + \omega_W \wedge \textstyle\sum_0^{\ell-1} \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j}) \wedge \omega_z^{n-\ell+1}\right] . \end{aligned}$$

Since $\rho_{2*}\rho^* = \pi_{2*}Q^*$ and

$$Q_0^*(\omega_\zeta^\ell \wedge \omega_z^{n-\ell+1}) = (dd^c \log |P(z)|^2)^\ell \wedge \omega_z^{n-\ell+1} = 0 ,$$

we obtain

$$\begin{aligned} dd^c m_P^\ell &\le \sum_{j=0}^{\ell-1} \pi_{2*}Q^* \left(\omega_W \wedge \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j} \wedge \omega_z^{n-\ell+1}\right) \\ &= \sum_{j=0}^{\ell-1} \left(\int_{\mathbb{P}_z^n \times \{W\}} (dd^c \log |P(z) \wedge W|^2)^j \wedge (P^*\omega_\zeta)^{\ell-1-j} \wedge \omega_z^{n-\ell+1}\right) \omega_W . \end{aligned}$$

By Lemma 4.5 with $L : \mathbb{C}^{m+1} \to \widetilde{E}$ given by $L(\zeta) = \zeta \wedge W$,

$$\int_{\mathbb{P}_z^n \times \{W\}} (dd^c \log |P(z) \wedge W|^2)^j \wedge (P^*\omega_\zeta)^{\ell-1-j} \wedge \omega_z^{n-\ell+1} \le \delta_{\ell-1}(P) ,$$

and the desired inequality follows.

We now modify (and simplify) the above argument for the case $\ell = 1$: Identify

$$G(1, m) = \mathbb{P}(\textstyle\bigwedge^m \mathbb{C}^{m+1}) = \mathbb{P}(\mathbb{C}^{m+1*}) = \mathbb{P}^{m*} .$$

Let $X = \mathbb{P}_z^n \times \mathbb{P}_W^{m*} \times \mathbb{P}_\zeta^m$ and consider the current

$$\Omega = \log \frac{|\zeta|^2 |W|^2}{|(W, \zeta)|^2} \omega_z^n \in \mathcal{D}'^{n,n}(X) .$$

Then

$$dd^c\Omega = (\omega_\zeta + \omega_W - D) \wedge \omega_z^n ,$$

where $D = \mathrm{Div}(W, \zeta)$. Then $m_P^1 = \pi_{2*}Q^*\Omega$, where

$$Q = (z, W, P(z)) : \mathbb{P}_z^n \times \mathbb{P}_W^{m*} \to X .$$

We conclude as before that $dd^c m_P^1 = \pi_{2*}Q^* dd^c\Omega \le \omega_W$. □

6. Description of the exceptional set

In this section we prove Theorem 1.1, giving a description of the exceptional set $\mathcal{E}$ in terms of the proximity function as follows. Let $\mathcal{P} = \{P_k\}$ be a sequence of rational mappings from $\mathbb{P}^n$ to $\mathbb{P}^m$ as in Theorem 1.1 and fix a sequence $\mathcal{A} = \{a_k\}$ of positive numbers. We define the *exceptional set*

$$\mathcal{E}_\ell(\mathcal{P}, \mathcal{A}) = \left\{ W \in \mathbb{G}(\ell, m) : \limsup_{k\to\infty} \frac{m_k(W)}{a_k} > 0 \right\} , \tag{17}$$

where $m_k = m^\ell_{P_k}$. (Thus by definition, $\mathcal{E}_\ell(\mathcal{P},\mathcal{A})$ contains those planes W such that infinitely many of the pre-images $P_k^{-1}(W)$ have codimension less than ℓ.) The following two propositions yield Theorem 1.1 with $\mathcal{E} = \mathcal{E}_\ell(\mathcal{P},\mathcal{A})$.

Proposition **6.1.** *Let $\mathcal{P} = \{P_k\}$ be a sequence of rational mappings from $\mathbb{P}^n$ to $\mathbb{P}^m$ and let $\mathcal{A} = \{a_k\}$ be a sequence of positive numbers. Let $1 \le \ell \le \min(n,m)$. Then for all $W \in \mathbb{G}(\ell,m) \setminus \mathcal{E}_\ell(\mathcal{P},\mathcal{A})$,*

$$\frac{1}{a_k}\left(P_k^*[W] - P_k^*\omega^\ell\right) \to 0$$

as $k \to \infty$.

Proposition **6.2.** *Let $\mathcal{P} = \{P_k\}$, $\mathcal{A} = \{a_k\}$ be as in Proposition 6.1. If*

$$\sum_{k=1}^\infty \frac{\delta_{\ell-1}(P_k)}{a_k} < +\infty\ ,$$

then $\mathcal{E}_\ell(\mathcal{P},\mathcal{A})$ is pluripolar in $\mathbb{G}(\ell,m)$.

Before proving Propositions 6.1 and 6.2, we note a corollary to Theorem 1.1 on the equidistribution of pre-images for subsequences of a given sequence of rational mappings. This corollary uses the following terminology: For a current $T \in \mathcal{D}'^{p,p}(\mathbb{P}^n)$ of order 0, we let $\|T\|$ denote the total variation measure of $\mathbb{P}^n$, which is the regular measure on $\mathbb{P}^n$ given by

$$\|T\|(U) = \sup\left\{|(T,\varphi)| : \varphi \in \mathcal{D}^{n-p,n-p}(U), \|\varphi\| \le 1\right\}$$

for U open in $\mathbb{P}^n$. Here, $\|\varphi\|$ denotes the comass norm of a compactly supported form φ (see Federer [Fe, 1.8.1, 4.1.7]). The quantity $\|T\|(\mathbb{P}^n)$ is called the mass of T. If T is positive, then it follows from Wirtinger's inequality (see for example, [Fe, 1.8.2]) that $\|T\|(\mathbb{P}^n) = (T,\omega^{n-p})$. In particular, $\|\omega^p\|(\mathbb{P}^n) = 1$.

Corollary **6.3.** *Let $\mathcal{P} = \{P_k\}$ be as in Theorem 1.1, let $1 \le \ell \le \min(n,m)$, and suppose that*

$$\frac{\delta_{\ell-1}(P_k)}{\delta_\ell(P_k)} \to 0\ .$$

Let $\mathcal{M} \subset \mathcal{D}'^{\ell,\ell}(\mathbb{P}^n)$ be the set of weak cluster points of $\{\frac{1}{\delta_\ell(P_k)} P_k^*\omega^\ell\}$. Then $\mathcal{M} \ne \emptyset$, and for every current $\eta \in \mathcal{M}$,

i) *η is a positive current*
ii) *η has mass* 1
iii) *there is a subsequence $\{P'_k\}$ of $\{P_k\}$ and a pluripolar set $\mathcal{E}$ such that*

$$\frac{1}{\delta_\ell(P'_k)} {P'}^*_k[W] \to \eta$$

for all $W \in \mathbb{G}(\ell,m) \setminus \mathcal{E}$.

Proof (assuming Theorem 1.1): Since

$$\|\frac{1}{\delta_\ell(P_k)}P_k^*\omega^\ell\|(\mathbb{P}^n) = \frac{1}{\delta_\ell(P_k)}\int P_k^*\omega^\ell\wedge\omega^{n-\ell} = 1,$$

it follows that $\mathcal{M}\neq\emptyset$. Let

$$\eta = \lim_{k\to\infty}\frac{1}{\delta_\ell(P_k')}P_k'^*\omega^\ell\in\mathcal{M}$$

for some subsequence $\{P_k'\}$ of $\mathcal{P}$. Then (i) is obvious, and (ii) follows from

$$\|\eta\|(\mathbb{P}^n) = (\eta,\omega^{\ell-1}) = \lim_{k\to\infty}\frac{1}{\delta_\ell(P_k')}(P_k'^*\omega^\ell,\omega^{n-\ell}) = 1\ .$$

Finally, choose a subsequence $\{P_k''\}$ of $\{P_k'\}$ such that $\sum\frac{\delta_{\ell-1}(P_k'')}{\delta_\ell(P_k'')}<+\infty$, and apply Theorem 1.1 with $a_k=\delta_\ell(P_k'')$ to obtain (iii). □

Proof of Proposition 6.1: Let $\varphi\in\mathcal{D}^{n-\ell,n-\ell}(\mathbb{P}^n)$ be an arbitrary real form and choose a constant c_φ such that $-c_\varphi\omega^{n-\ell+1}\le dd^c\varphi\le c_\varphi\omega^{n-\ell+1}$. Then for all $W\in\mathbb{G}(\ell,m)$ such that $\mathrm{codim}P_k^{-1}(W)=\ell$, we have by (8),

$$|(P_k^*[W]-P_k^*\omega^\ell,\varphi)| = |(P_k^*\Lambda_W,dd^c\varphi)|\le c_\varphi m_k(W)\ .$$

The conclusion follows from the definition of $\mathcal{E}_\ell(\mathcal{P},\mathcal{A})$. □

Proof of Proposition 6.2: Let $P:\mathbb{P}^n\dashrightarrow\mathbb{P}^m$ be a rational map. We write

$$S_P = \left\{W\in\mathbb{G}(\ell,m) : \mathrm{codim}P^{-1}(W)<\ell\right\}\ ;$$

S_P is an algebraic subvariety (which is usually empty) of $\mathbb{G}(\ell,m)$ and thus is pluripolar. By (15) we can write

$$P^*\Lambda_W\wedge\omega_z^{n-\ell+1} = f(z,W)\omega_z^n\ ,$$

where

i) $0\le f(z,W)\le+\infty$,
ii) $f\in\mathcal{L}^1(\mathbb{P}^n\times\mathbb{G}(\ell,m))$,
iii) f is continuous on the set $U\overset{\mathrm{def}}{=}\mathbb{P}^n\times\mathbb{G}(\ell,m)\setminus I_Q$.

Here Q is the map given by equation (12) in Section 5. By the definition of the proximity function,

$$m_P^\ell(W) = \int_{\mathbb{P}_z^n}f(z,W)\omega_z^n\ ,\quad\text{for}\quad W\in\mathbb{G}(\ell,m)\setminus S_P\ .$$

Since

$$I_Q\cap(\mathbb{P}^n\times\{W\}) = \left[I_P\cup P^{-1}(W)\right]\times\{W\}\ ,$$

$m_P^\ell(W)\in[0,+\infty]$ is well defined by the above integral.

Let $(-m_P^\ell)^*$ be the upper-regularization of $(-m_P^\ell)$ given by

$$(-m_P^\ell)^*(W) = \limsup_{\xi\to W}[-m_P^\ell(\xi)] \ ,$$

for $W \in \mathbb{G}(\ell, m)$, and write

$$m_{P*}^\ell(W) = -(-m_P^\ell)^*(W) = \liminf_{\xi\to W} m_P^\ell(\xi) \ .$$

We claim that

$$m_P^\ell(W) = m_{P*}^\ell(W) \ \text{ for all } W \in \mathbb{G}(\ell, m) \setminus S_P \ . \tag{18}$$

To demonstrate (18), let $W_0 \in \mathbb{G}(\ell, m) \setminus S_P$, and let $\{W_\nu\}$ be a sequence of points converging to W_0 such that $m_P^\ell(W_\nu) \to m_{P*}^\ell(W_0)$. Since $f(z, W_\nu) \to f(z, W_0)$ for almost all $z \in \mathbb{P}^n$ (precisely, for $z \notin I_P \cup P^{-1}(W_0)$), we have by Fatou's Lemma,

$$m_P^\ell(W_0) = \int_{\mathbb{P}_z^n} f(z, W_0)\omega_z^n \le \liminf_{\nu\to\infty} \int_{\mathbb{P}_z^n} f(z, W_\nu)\omega_z^n = \liminf_{\nu\to\infty} m_P^\ell(W_\nu) = m_{P*}^\ell(W_0) \ .$$

Since by definition, $m_P^\ell(W_0) \ge m_{P*}^\ell(W_0)$, we obtain (18).

A function u on a complex manifold Y (with values in $\mathbb{R} \cup -\{\infty\}$) is said to be *quasi-plurisubharmonic* (*quasi-psh* for short) if u is locally equal to the sum of a $\mathcal{C}^\infty$ function and of a plurisubharmonic (psh) function, or equivalently, if $dd^c u$ is bounded below by a continuous real $(1,1)$-form. By Lemma 5.1, $-m_{P*}^\ell$ is quasi-psh.

We now let $\mathcal{P} = \{P_k\}$, $\mathcal{A} = \{a_k\}$ be as in Proposition 6.2 and suppose that $\sum \frac{\delta_{\ell-1}(P_k)}{a_k} < +\infty$. It suffices to show that $\mathcal{E}_\ell(\mathcal{P}, \mathcal{A})$ is pluripolar in an arbitrary affine open set

$$G' \stackrel{\text{def}}{=} \{W \in \mathbb{G}(\ell, m) : \lambda(W) \ne 0\}$$

given by a linear function $\lambda : E \to \mathbb{C}$. . We write $m_k = m_{P_k}^\ell$ and we consider the functions $u_k : G' \to \mathbb{R} \cup \{-\infty\}$ given by

$$u_k = v - \frac{m_{k*}}{\ell\delta_{\ell-1}(P_k)}$$

where

$$v(W) = \log \frac{|W|^2}{|\lambda(W)|^2} \ . \tag{19}$$

We assume that $\|\lambda\| = 1$, so $v(W) \ge 0$. By Lemma 5.1, u_k is plurisubharmonic on G'. Next, construct the series

$$u = \sum_{k=1}^\infty \frac{\delta_{\ell-1}(P_k)}{a_k} u_k \ . \tag{20}$$

To see that u is plurisubharmonic, represent it as a limit of the sequence

$$\tau_k = S \cdot v - \frac{1}{\ell} \sum_{j=1}^k \frac{m_{j*}}{a_j} \ ,$$

where

$$S=\sum_{k=1}^{\infty}\frac{\delta_{\ell-1}(P_k)}{a_k}\ .$$

Since $m_{k*}\geq 0$, $\{\tau_k\}$ is a decreasing sequence of plurisubharmonic functions on G', so the limit is either plurisubharmonic or identically $-\infty$. To see that the latter case is impossible, we average $u(W)$ over all W with respect to Haar probability measure σ on $\mathbb{G}(\ell,m)$. It is well known (e.g., see [Sh, Ch. 2, § 4, Th. 7]) that

$$\int\limits_{\mathbb{G}(\ell,m)}\Lambda_W(z)d\sigma(W)=c_{\ell,m}\omega^{\ell-1}(z) \tag{21}$$

for some constant $c_{\ell,m}$. Therefore, by Fubini's theorem,

$$\begin{aligned}\int\limits_{\mathbb{G}(\ell,m)} m_{P_k}(W)d\sigma(W) &= \int\limits_{\mathbb{G}(\ell,m)}\left(\int\limits_{\mathbb{P}^n}P_k^*\Lambda_W\wedge\omega^{n-l+1}\right)d\sigma(W)\\ &= c_{\ell,m}\int\limits_{\mathbb{P}^n}P_k^*\omega^{l-1}\wedge\omega^{n-\ell+1}=c_{\ell,m}\delta_{\ell-1}(P_k)\ .\end{aligned}$$

Hence

$$\begin{aligned}\int u(W)d\sigma(W) &\geq -\frac{1}{\ell}\sum_{k=1}^{\infty}\frac{1}{a_k}\int m_k(W)d\sigma(W)\\ &= -\frac{1}{\ell}\sum_{k=1}^{\infty}\frac{c_{\ell,m}\delta_{\ell-1}(P_k)}{a_k}\\ &= -\frac{c_{\ell,m}}{\ell}S>-\infty\ .\end{aligned}$$

Thus u is plurisubharmonic.

Finally, if $W\in\mathcal{E}_\ell(\mathcal{P},\mathcal{A})\setminus\bigcup_k S_{P_k}$, then by definition, for an infinite number of indices k we have

$$\varepsilon a_k<m_k(W)=m_{k*}(W)$$

for some $\varepsilon>0$, and therefore

$$\frac{\delta_{\ell-1}(P_k)}{a_k}u_k(W)\leq\frac{\delta_{\ell-1}(P_k)}{a_k}v(W)-\frac{\varepsilon}{\ell}\ ,$$

so $u(W)=-\infty$. Since the S_{P_k} are pluripolar, it follows that the set $\mathcal{E}_\ell(\mathcal{P},\mathcal{A})$ is pluripolar. □

7. Limit measures for iterates of rational maps

In this section we prove Theorem 1.3. We continue to use the notation from Section 6.

Assume that $P : \mathbb{P}^n - \to \mathbb{P}^n$ is as in Theorem 1.3 and write $\lambda = \delta_n(P)$, $\delta = \delta_{n-1}(P)$. Let

$$h = \frac{1}{\lambda} P^* \omega^n \in \mathcal{D}'^{n,n}(\mathbb{P}^n).$$

By the definition of the topological degree $\delta_n(P)$, $\int_{\mathbb{P}^n} h = 1$.

Claim: If f is a quasi-plurisubharmonic function on $\mathbb{P}^n$, then

$$\int_{\mathbb{P}^n} fh > -\infty \ . \tag{22}$$

(Note that $h \geq 0$; since f is bounded above, the claim is equivalent to saying that fh is $\mathcal{L}^1$.) To verify the claim, we again consider the commutative diagram (3). We then have

$$\int fh = \frac{1}{\lambda} \int_{\widetilde{G}} (f \circ \pi_1) \pi_2^* \omega^n > -\infty$$

since $f \circ \pi_1$ is quasi-plurisubharmonic on $\widetilde{G}$ and hence is in $\mathcal{L}^1$, verifying (22).

Choose $a > 0$ such that $\delta < a < \lambda$. By Lemma 4.6, $\delta_{n-1}(P^k) \leq \delta^k$ and thus

$$\sum \frac{\delta_{n-1}(P^k)}{a^k} = S < +\infty.$$

Applying Theorem 1.1 with $\ell = n$, $\mathcal{P} = \{P_k\}$, where P_k is the k-th iterate of P, and $\mathcal{A} = \{a^k\}$, we conclude that

$$\frac{m_k(W)}{a^k} \to 0 \tag{23}$$

for points W of $\mathbb{P}^n$ outside a pluripolar set.

Since pluripolar sets have Lebesgue measure zero, (23) is valid for a.a. $W \in \mathbb{P}^n$. Let

$$U \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{m_k}{a^k} = Sv - u,$$

where v, u are given by (19),(20). By the proof of Proposition 6.2, u is psh and hence $-U$ is quasi-plurisubharmonic. Therefore, by (22),

$$\int_{\mathbb{P}^n} Uh < +\infty \tag{24}$$

Let $\varphi \in \mathcal{D}^0(\mathbb{P}^n)$ be arbitrary. Since $\int_{\mathbb{P}^n} h = 1$,

$$\begin{aligned}\frac{1}{a^k}(P_k^* h - P_k^*\omega^n, \varphi) &= \frac{1}{a^k}\int (P_k^*[W] - P_k^*\omega^n, \varphi) h(W) \\ &= -\frac{1}{a^k}\int (P_k^*\Lambda_W, dd^c\varphi) h(W)\ .\end{aligned}$$

As in the proof of Proposition 6.1, $|(P_k^*\Lambda_W, dd^c\varphi)| \le c_\varphi m_k(W)$, and we conclude that

$$|\frac{1}{a^k}(P_k^*\Lambda_W, dd^c\varphi)| \le c_\varphi U(W)\ . \tag{25}$$

By (23), (24) and (25), we can let $k \to +\infty$ and apply Lebesgue's dominated convergence theorem to conclude that

$$\frac{1}{a^k}(P_k^* h - P_k^*\omega^n, \varphi) \to 0 \tag{26}$$

as $k \to +\infty$.

We note that

$$\frac{1}{\lambda^{k+1}} P_{k+1}^*\omega^n = \frac{1}{\lambda^k} P_k^* h,$$

as an identity of currents with $\mathcal{L}^1_{\rm loc}$ coefficients.

Hence by (26)

$$\left(\frac{1}{\lambda^{k+1}} P_{k+1}^*\omega^n - \frac{1}{\lambda^k} P_k^*\omega^n, \varphi\right) = \frac{1}{\lambda^k}\left(P_k^* h - P_k^*\omega^n, \varphi\right) \le \left(\frac{a}{\lambda}\right)^k.$$

for k sufficiently large. Therefore the sequence $\{(\frac{1}{\lambda^k}P_k^*\omega^n, \varphi)\}$ is Cauchy. Since φ is arbitrary, it follows that $\frac{1}{\lambda^k}P_k^*\omega^n$ converges to a measure μ.

Let $\mu_k = \frac{1}{\lambda^k}P_k^*\omega^n$. Since the μ_k are probability measures, so is μ. The last conclusion of Theorem 1.3 then follows from Theorem 1.1. □

Acknowledgements

Part of the work on this paper was done while both authors were at the Mathematical Sciences Research Institute in Berkeley. This work was started when the first author was visiting Johns Hopkins University. The authors would like to thank these organizations for their hospitality and support.

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