A CARLEMAN TYPE THEOREM FOR PROPER HOLOMORPHIC EMBEDDINGS

Gregery T. Buzzard and Franc Forstneric

Abstract

In 1927, Carleman showed that a continuous, complex-valued function on the real line can be approximated in the Whitney topology by an entire function restricted to the real line. In this paper, we prove a similar result for proper holomorphic embeddings. Namely, we show that a proper $\mathcal{C}^r$ embedding of the real line into $\mathbf{C}^n$ can be approximated in the strong $\mathcal{C}^r$ topology by a proper holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^n$.

&1. Introduction

We denote the complex numbers by $\mathbf{C}$ and the real numbers by $\mathbf{R}$. Let $\mathbf{C}^n$ be the complex Euclidean space of dimension n with complex coordinates $z = (z_1, \ldots, z_n)$.

To motivate our main result we recall the Carleman approximation theorem [3], [10]: *For each continuous function $\lambda: \mathbf{R} \to \mathbf{C}$ and positive continuous function $\eta: \mathbf{R} \to (0, \infty)$ there exists an entire function f on $\mathbf{C}$ such that $|f(t) - \lambda(t)| < \eta(t)$ for all $t \in \mathbf{R}$.* This extends the Weierstrass approximation theorem in which the approximation takes place on compact intervals in $\mathbf{R}$. If λ is smooth, we can also approximate its derivatives by those of f. A more general result was proved by Arakelian [1] (see [13] for a simple proof).

Our main result is an extension of Carleman's theorem to *proper holomorphic embeddings* of $\mathbf{C}$ into $\mathbf{C}^n$ for $n > 1$:

1.1 Theorem. *Let $n > 1$ and $r \geq 0$ be integers. Given a proper embedding $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$ of class $\mathcal{C}^r$ and a continuous positive function $\eta: \mathbf{R} \to (0, \infty)$, there exists a proper holomorphic embedding $f: \mathbf{C} \hookrightarrow \mathbf{C}^n$ such that*

$$|f^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t), \quad t \in \mathbf{R}, \quad 0 \leq s \leq r.$$

If in addition $T = \{t_j\} \subset \mathbf{R}$ is discrete, there exists f as above such that

$$f^{(s)}(t) = \lambda^{(s)}(t), \quad t \in T, \quad 0 \leq s \leq r.$$

Buzzard's research at MSRI is supported in part by NSF grant DMS-9022140. Forstneric is supported in part by an NSF grant and by a grant from the Ministry of Science of the Republic of Slovenia.

Here $f^{(s)}(t)$ denotes the derivative of f order s, and similarly for λ.

We emphasize that, in general, one can not expect to extend λ to a holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^n$, even if λ is real-analytic. In this case λ will extend holomorphically to some open set in $\mathbf{C}$, but in general not to all of $\mathbf{C}$; and even if λ extends to $\mathbf{C}$, the extension need not be a proper map into $\mathbf{C}^n$. So our result is the best possible in this context.

Another motivation for Theorem 1.1 are the recent developments on embedding Stein manifolds in $\mathbf{C}^n$; see the papers [2], [6], [7], [8]. (For Stein manifolds and other topics in several complex variables mentioned here we refer the reader to Hörmander [11].) In these papers it was shown that a Stein manifold M which admits a proper holomorphic embedding in $\mathbf{C}^n$ for some $n > 1$ also admits an embedding $f\colon M \hookrightarrow \mathbf{C}^n$ whose image $f(M)$ contains a given discrete subset $E \subset \mathbf{C}^n$ (with prescribed finite order jet of $f(M)$ at any point of E), and such that $f(M)$ intersects the image of any entire map $g\colon \mathbf{C}^d \to \mathbf{C}^n$ of rank $d = n - \dim M$ at infinitely many points [6, Theorem 5.1]. This implies in particular that $f(M)$ cannot be mapped into any hyperplane in $\mathbf{C}^n$ by an automorphism of $\mathbf{C}^n$. In this context we recall that any Stein manifold M embeds in $\mathbf{C}^n$ for $n > (3\dim M + 1)/2$ according to Eliashberg and Gromov [4].

In light of these results it is a natural question whether one can in some sense prescribe the embedding $M \hookrightarrow \mathbf{C}^n$ on a positive dimensional submanifold $N \subset M$. Our result provides an affirmative answer in the simplest case when $M = \mathbf{C}$ and $N = \mathbf{R} \times \{i0\} \subset \mathbf{C}$. The details of our construction are considerable even in this simplest case, and the full scope of the method remains to be seen. If λ is of class $\mathcal{C}^\infty$, our method can be modified so that we approximate to increasingly high order on complements of compact subsets of $\mathbf{R}$. We shall not go into details of this. Another possible extension is to approximate a proper smooth embedding by a proper holomorphic embedding on a finite set of real lines (or certain other real curves) in $\mathbf{C}$.

A seemingly more difficult question is whether a result of this type holds when N is a *complex* submanifold of positive dimension in a complex manifold M. For instance, *does every proper holomorphic embedding $\lambda\colon \mathbf{C} \hookrightarrow \mathbf{C}^n$ for $n \geq 3$ extend to a proper holomorphic embedding $f\colon \mathbf{C}^2 \hookrightarrow \mathbf{C}^n$?*

One of the main tools in our construction is the following result from [9]. The case $r = 0$ was obtained earlier in [8]. This result can also be obtained by methods in [5].

1.2 Proposition. *Let $K \subset\subset \mathbf{C}^n$ $(n \geq 2)$ be a compact, polynomially convex set, and let $C \subset \mathbf{C}^n$ be a smooth embedded arc of class $\mathcal{C}^\infty$ which is attached to K in a single point of K. Given a $\mathcal{C}^\infty$ diffeomorphism $F\colon K \cup C \to K \cup C' \subset \mathbf{C}^n$ such that F is the identity on $(K \cup C) \cap U$ for some open neighborhood U of K, and given numbers $r \geq 0$, $\epsilon > 0$, there exist a neighborhood W of K and an automorphism $\Phi \in \mathrm{Aut}\mathbf{C}^n$ satisfying*

$$\|\Phi - \mathrm{Id}\,\|_{\mathcal{C}^r(W)} < \epsilon, \qquad \|\Phi - F\|_{\mathcal{C}^r(C)} < \epsilon.$$

(Here Id *denotes the identity map.) Moreover, for each finite subset* $Z \subset K \cup C$ *we can choose* Φ *such that it agrees with the identity to order* r *at each point* $z \in Z \cap K$ *and* $\Phi|_C$ *agrees with* F *to order* r *at each point of* $Z \cap C$.

The same result holds with any finite number of disjoint hairs attached to K.

We explain our notation. Δ_ρ is the closed disk in $\mathbf{C}$ of radius ρ and center 0. $\mathbf{B}$ is the open unit ball in $\mathbf{C}^n$ with center 0, and $R\mathbf{B}$ is the ball of radius R. For a set $A \subset \mathbf{C}^n$ and $\rho > 0$, let $A + \rho\overline{\mathbf{B}} = \{a + z : a \in A, |z| \le \rho\}$. We identify $\mathbf{C}$ and $\mathbf{R}$ with their images in $\mathbf{C}^n$ under the embedding $\zeta \to (\zeta, 0, \ldots, 0)$. For $1 \le j \le n$ we denote by π_j the coordinate projection $\pi_j(z_1, \ldots, z_n) = z_j$.

In the proof we shall use special automorphisms of $\mathbf{C}^n$ of the form

$$\Psi(z) = z + f(\pi z)v, \quad z \in \mathbf{C}^n,$$

where $v \in \mathbf{C}^n$, $\pi: \mathbf{C}^n \to \mathbf{C}^k$ is a linear map for some $k < n$ with $\pi v = 0$ (in most cases $k = 1$), and f is an entire function on $\mathbf{C}^k$. An automorphism of this form is called a *shear*; clearly $\Psi^{-1}(z) = z - f(\pi z)v$.

&2. Some lemmas.

The following is standard, e.g., [12, prop. 2.15.4].

2.1 Lemma. *Let* $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$ *be a* $\mathcal{C}^\infty$ *proper embedding. Then there exists a continuous* $\eta: \mathbf{R} \to (0, \infty)$ *such that if* $\gamma: \mathbf{R} \to \mathbf{C}^n$ *with* $|\gamma^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t)$ *for all* $t \in \mathbf{R}$, $s = 0, 1$, *then* γ *is a proper embedding.*

Recall that a compact set $A \subset\subset \mathbf{C}^n$ is polynomially convex if for each $z \in \mathbf{C}^n \backslash A$ there is a holomorphic polynomial P on $\mathbf{C}^n$ such that $|P(z)| > \max\{|P(w)| : w \in A\}$. We refer the reader to [11] for properties of such sets.

2.2 Lemma. *Let* $A \subset \mathbf{C}^n$ *be compact and polynomially convex and* $\rho > 0$. *Let* $I \subset \mathbf{R}$ *be an interval whose endpoints lie in* $\mathbf{C}^n \backslash (A \cup \Delta_\rho)$, *and let* $r, \epsilon > 0$. *Then there exists an automorphism* $\Psi(z) = z + g(z_1)e_2$ *of* $\mathbf{C}^n$ *such that*

(i) $|\Psi(z) - z| < \epsilon$ *for* $z \in \Delta_\rho$,

(ii) $\|\Psi|_{\mathbf{R}}(t) - t\|_{\mathcal{C}^r(I)} < \epsilon$, *and*

(iii) $\Psi(t) \notin A$ *for* $t \in \overline{\mathbf{R}\backslash I}$.

If $Z \subset I$ *is finite, we can choose* Ψ *as above so that* $g^{(s)}(t) = 0$ *for* $t \in Z$, $0 \le s \le r$.

Proof. Let $\mu_1 < \mu_2$ denote the endpoints of I in $\mathbf{R}$, and let $\Gamma_j = \{(\mu_j, \zeta, 0, \ldots, 0) : \zeta \in \mathbf{C}\}$ for $j = 1, 2$. Let $R > \max\{|\mu_1|, |\mu_2|\} + 1$ such that $A \subset R\mathbf{B}$. Consider the set $E_j = A \cap \Gamma_j$. Since A is polynomially convex, E_j is polynomially convex in Γ_j and hence $\Gamma_j \backslash E_j$ is connected. Since the endpoints of I lie in $\Gamma_j \backslash E_j$, there exists a smooth curve $\gamma_j: [0, 1] \to \Gamma_j \backslash E_j$ with $\gamma_j(0) = (\mu_j, 0, \ldots, 0)$ and $|\pi_2\gamma_j(1)| > R + 1$ for $j = 1, 2$.

Since A is compact, there exists $\delta > 0$ such that $\gamma_j([0,1]) + 3\delta\overline{\mathbf{B}} \subset \mathbf{C}^n \backslash A$. Let $\pi_2(z) = z_2$. Let $K = \{x + iy \in \mathbf{C}: \mu_1 - \delta/2 \le x \le \mu_2 + \delta/2,\ |y| \le \rho + 1\}$. Define a function $h: K \cup [-R, R] \to \mathbf{C}$ by

$$h(t) = \begin{cases} \pi_2\gamma_1(1) & \text{if } t \in [-R, \mu_1 - 2\delta]; \\ \pi_2\gamma_1((\mu_1 - \delta - t)/\delta) & \text{if } t \in [\mu_1 - 2\delta, \mu_1 - \delta]; \\ \pi_2\gamma_2((t - \mu_2 - \delta)/\delta) & \text{if } t \in [\mu_2 + \delta, \mu_2 + 2\delta]; \\ \pi_2\gamma_2(1) & \text{if } t \in [\mu_2 + 2\delta, R]; \\ 0 & \text{otherwise.} \end{cases}$$

Choose η, $0 < \eta < \min\{\epsilon, \delta\}$. By Mergelyan's theorem [14, p.386] there is an entire function g on $\mathbf{C}$ such that $|h(z) - g(z)| < \eta$ for $z \in K \cup [-R, R]$. The shear $\Psi(z) = z + g(z_1)e_2$ then satisfies (i) and (iii). Since $I \subset \mathrm{Int}K$, Cauchy's estimates imply that it also satisfies (ii) provided that $\eta > 0$ is chosen sufficiently small. The last condition on g is a trivial addition to Mergelyan's theorem. ♠

2.3 Lemma. *Let $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$ be a proper, $\mathcal{C}^\infty$ embedding, $K \subset \mathbf{C}^n$ compact, $\epsilon > 0$, and $r \in \mathbf{Z}_+$. Let $Z \subset \mathbf{R}$ be finite, and suppose $\lambda(t) \in \mathbf{C} = \mathbf{C} \times \{0\}^{n-1}$ for each $t \in Z$. Then there exists a shear $\Psi(z) = z + h(z_1)v$ for some $v \in \mathbf{C}^n$ with $\pi_1 v = 0$ such that*

(i) $\Psi(\mathbf{C}) \cap \lambda(\mathbf{R}) = \lambda(Z)$,

(ii) $|\Psi(z) - z| < \epsilon$ *for* $z \in K$, *and*

(iii) $\Psi(z) = z + O(|z - \lambda(t)|^{r+1})$ *as* $z \to \lambda(t)$, *for all* $t \in Z$.

Proof. Let $Z = \{t_j\}_{j=1}^s$. Define a polynomial on $\mathbf{C}$ by $h(\zeta) = \Pi_{1 \le j \le s}(\zeta - \pi_1\lambda(t_j))^{r+1}$. Consider the map $\Phi: \mathbf{C} \times \mathbf{C}^{n-1} \to \mathbf{C}^n$ given by

$$\Phi(z_1, \alpha_2, \ldots, \alpha_n) = (z_1, 0, \ldots, 0) + h(z_1)(0, \alpha_2, \ldots, \alpha_n).$$

Clearly Φ is an automorphism of $(\mathbf{C} \backslash \lambda(Z)) \times \mathbf{C}^{n-1}$. Let $\Delta_{R,j}$ denote the closed disk of radius R in $\mathbf{C}$ with center $\pi_1\lambda(t_j)$ for $j = 1, 2, \ldots, s$. Choose $R > 0$ such that the discs $\Delta_{R,j}$ for $1 \le j \le s$ are pairwise disjoint. Choose a ρ, $0 < \rho < R$, such that ρ^2 is a regular value of $\mu_j(t) = |\pi_1\lambda(t) - \pi_1\lambda(t_j)|^2$ ($t \in \mathbf{R}$) for each $j = 1, 2, \ldots, s$.

Let $M_\rho = \mathbf{C} \backslash \cup_{1 \le j \le s} \mathrm{int}\Delta_{\rho,j}$. Let $\Phi_\rho = \Phi|_{M_\rho \times \mathbf{C}^{n-1}}$ and $\partial\Phi_\rho = \Phi|_{\partial M_\rho \times \mathbf{C}^{n-1}}$. A simple check shows that Φ_ρ and $\partial\Phi_\rho$ are transverse to $\lambda(\mathbf{R})$. Hence by the transversality theorem, there exists a set $A_\rho \subset \mathbf{C}^{n-1}$ of full measure such that for each $\alpha = (\alpha_2, \ldots, \alpha_n) \in A_\rho$, $\Phi(M_\rho, \alpha) = \{\Phi(z_1, \alpha): z_1 \in M_\rho\}$ and $\lambda(\mathbf{R})$ are transverse, hence disjoint by dimension considerations.

Let $A = \cap_{j=1}^\infty A_{1/j}$. Then $A \subset \mathbf{C}^{n-1}$ has full measure, and for each $\alpha \in A$ we see that $\Phi(\mathbf{C} \backslash \lambda(Z), \alpha)$ and $\lambda(\mathbf{R})$ are disjoint. Finally, choose $\alpha \in A$ such that $|h(z_1)\alpha| < \epsilon$ for $z_1 \in \pi_1(K)$, and let $\Psi(z) = z + h(z_1)\alpha$. Then $\Psi(z_1, 0, \ldots, 0) = \Phi(z_1, \alpha_2, \ldots, \alpha_n)$, and Ψ satisfies the conclusions of the lemma. ♠

2.4 Lemma. *Let $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$ be a $\mathcal{C}^\infty$ embedding, $f: \mathbf{C} \hookrightarrow \mathbf{C}^n$ a proper holomorphic embedding, and $I \subset \mathbf{R}$ a closed interval with $f|_I = \lambda|_I$. Let $K \subset \mathbf{C}^n$ be compact and* *polynomially convex, $a, r, \epsilon > 0$, and $T \subset \mathbf{R}$ discrete. Suppose that $\lambda(t), f(t) \notin K$ for $t \in \overline{\mathbf{R}\backslash I}$. Then there exists $\Phi \in \mathrm{Aut}\mathbf{C}^n$ such that if $g = \Phi \circ f$, then*

(i) $|g^{(s)}(t) - \lambda^{(s)}(t)| < \epsilon$ *for* $t \in [-a, a]$, $0 \leq s \leq r$,

(ii) $g^{(s)}(t) = \lambda^{(s)}(t)$ *for* $t \in T \cap [-a, a]$, $0 \leq s \leq r$, *and*

(iii) $|\Phi(z) - z| < \epsilon$ *for* $z \in K$.

Proof. We may assume that $I \subset (-a, a)$. Let I_1, I_2 be the two connected components of $\{\zeta \in I \colon f(\zeta) \in \mathbf{C}^n\backslash K\}$ containing the respective endpoints of I, and let $I_0 = I\backslash(I_1 \cup I_2)$. Let A be the polynomial hull of $K \cup f(I_0)$. Then A is the union of $K \cup f(I_0)$ and the bounded connected components of $\overline{f(\mathbf{C})\backslash(K \cup f(I_0))}$. Note that $f(I_1)$ and $f(I_2)$ lie in $f(\mathbf{C})\backslash A$ since $f(t) \notin K$ for all $t \in \overline{\mathbf{R}\backslash I}$.

Let $L = A \cup f([-a, a])$. Then $C = \overline{L\backslash A}$ is the union of two embedded arcs, each containing an endpoint of $f([-a, a])$. Define F on L by $F(z) = z$ if $z \in A$, and $F(z) = \lambda f^{-1}(z)$ if $z \in f([-a, a])$. Then F is a $\mathcal{C}^\infty$ diffeomorphism of L which extends as the identity map on $(A \cup C) \cap U$ for some neighborhood U of A. Apply proposition 1.2 to get $\Phi \in \mathrm{Aut}\mathbf{C}^n$ such that $|\Phi(z) - z| < \epsilon$ for $z \in K$ and such that $g = \Phi \circ f$ satisfies (i) and (ii). ♠

&3. Proof of theorem 1.1.

Choose a smooth cutoff function χ on $\mathbf{R}$ such that $\chi(t) = 1$ for $|t|$ small and $\operatorname{supp} \chi \subset (-1, 1)$. Define the constant $C = C_r > 1$ such that $\|\chi h\|_{\mathcal{C}^r} \leq C\|h\|_{\mathcal{C}^r}$ for each $h \in \mathcal{C}^r(\mathbf{R})$. We fix such C for the entire proof.

By approximation we may assume that $\lambda \colon \mathbf{R} \hookrightarrow \mathbf{C}^n$ in theorem 1.1 is a proper $\mathcal{C}^\infty$ embedding. Decreasing η if necessary we may also assume that η satisfies lemma 2.1 for λ and $\eta(t) < 1/2$ for all $t \in \mathbf{R}$.

We use an inductive procedure to obtain a sequence of proper holomorphic embeddings $f_k \colon \mathbf{C} \hookrightarrow \mathbf{C}^n$ such that $f = \lim_{k\to\infty} f_k$ exists on $\mathbf{C}$ and satisfies theorem 1.1. Each f_k will be a restriction to $\mathbf{C} = \mathbf{C} \times \{0\}^{n-1}$ of a holomorphic automorphism of $\mathbf{C}^n$. The next map f_{k+1} will be of the form $f_{k+1} = \Phi_{k+1} \circ f_k \circ \Psi_{k+1}$ for suitably chosen $\Phi_{k+1}, \Psi_{k+1} \in \mathrm{Aut}\mathbf{C}^n$.

We will describe the case $k = 1$ after the inductive step is given. Recall that Δ_k is the closed disc in $\mathbf{C} = \mathbf{C} \times \{0\}^{n-1}$ with center 0 and radius k, and $\mathbf{B}$ is the unit ball in $\mathbf{C}^n$. For the induction at step k, suppose we have the following:

(a) closed balls $B_j = R_j\overline{\mathbf{B}} \subset \mathbf{C}^n$ with $R_j \geq \max\{j + 1, R_{j-1} + 1\}$, $j = 1, \ldots, k$,

(b) automorphisms $\Phi_1, \ldots, \Phi_k$ of $\mathbf{C}^n$ with $|\Phi_j(z) - z| < 2^{-j}$ for $z \in B_{j-1}$, $j = 2, \ldots, k$,

(c) numbers $\epsilon_j > 0$ such that $\epsilon_1 < 2^{-1}$ and $\epsilon_j < \epsilon_{j-1}/2 < 2^{-j}$ for $j = 1, \ldots, k$,

(d) automorphisms $\Psi_1, \ldots, \Psi_k$ of $\mathbf{C}^n$ of the form $\Psi_j(z) = z + g_j(z_1)e_2 + h_j(z_1)v_j$, where $\pi_1(v_j) = 0$ and $|\Psi_j(z) - z| < \epsilon_j$ for $|z| \leq j$,

(e) closed intervals $I_j = [-a_j, a_j]$, $j = 1, \ldots, k$, with $a_j > \max\{a_{j-1} + 2, j + 2\}$, and

(f) numbers $0 < \delta_j < C^{-1} \inf\{\eta(t) \colon t \in I_j\}$, $j = 1, \ldots, k$, such that the automorphism

$$f_k = \Phi_k \circ \cdots \Phi_1 \circ \Psi_1 \circ \cdots \circ \Psi_k \in \mathrm{Aut}\mathbf{C}^n$$

(whose restriction to $\mathbf{C} = \mathbf{C} \times \{0\}^{n-1}$ provides an embedding $\mathbf{C} \hookrightarrow \mathbf{C}^n$) satisfies:

(1_k) $f_k(\Delta_j + \epsilon_k \overline{\mathbf{B}}) \subset \mathrm{Int} B_j$ for $j = 1, \ldots, k$,

(2_k) $|f_k^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t)$ for $t \in I_k$ and $0 \le s \le r$,

(3_k) $|f_k^{(s)}(t) - \lambda^{(s)}(t)| < \delta_k$ for $t \in I_k \backslash (-a_k + 1, a_k - 1)$ and $0 \le s \le r$,

(4_k) $f_k^{(s)}(t) = \lambda^{(s)}(t)$ for $t \in T \cap I_k$, $0 \le s \le r$,

(5_k) $f_k(\mathbf{C}) \cap \lambda(\mathbf{R}) = \lambda(T \cap I_k)$,

(6_k) $|\lambda(t)| > R_k + 1$ for $|t| \ge a_k - 1$,

(7_k) $|f_k(t)| > R_k$ for $|t| \ge a_k - 1$.

We will now show how to obtain these hypotheses at step $k+1$. Let I_k^1 and I_k^2 be the two connected components of the set $\{\zeta \in I_k \backslash \Delta_{k+1} : |f_k(\zeta)| > R_k\}$ containing the respective endpoints of the interval I_k. Let $I_k^0 = I_k \backslash (I_k^1 \cup I_k^2)$ be the middle interval. By (7_k) we have $I_k^0 \subset (-a_k + 1, a_k - 1)$.

Let K_k be the polynomial hull of the set $B_k \cup f_k(\Delta_{k+1} \cup I_k^0)$. Since $f_k(\mathbf{C})$ is a complex submanifold of $\mathbf{C}^n$ and B_k is polynomially convex, it is seen easily that K_k is contained in $B_k \cup f_k(\mathbf{C})$, and it is the union of $B_k \cup f_k(\Delta_{k+1} \cup I_k^0)$ and the bounded connected components of the complement $f_k(\mathbf{C}) \backslash (B_k \cup f_k(\Delta_{k+1} \cup I_k^0))$ (see Lemma 5.4 in [6]). Note that (7_k) and (e) imply that $f_k(\mathbf{R} \backslash (-a_k + 1, a_k - 1)) \subset \mathbf{C}^n \backslash K_k$.

Choose $R_{k+1} > R_k + 1$ such that $K_k \subset (R_{k+1} - 1)\mathbf{B}$, and let $B_{k+1} = R_{k+1}\overline{\mathbf{B}}$. Choose $a_{k+1} > a_k + 2$ to get (6_{k+1}). We now want to approximate λ on the larger interval $I_{k+1} = [-a_{k+1}, a_{k+1}]$ by the image of the next embedding $\mathbf{C} \hookrightarrow \mathbf{C}^n$ (to be constructed). In order to apply lemma 2.4 we first approximate λ as follows:

3.1 Lemma. *There exists a proper $\mathcal{C}^\infty$ embedding $\lambda_k : \mathbf{R} \hookrightarrow \mathbf{C}^n$ satisfying*

(i) $\lambda_k = f_k$ *on* $[-a_k + 1, a_k - 1]$,

(ii) $\lambda_k = \lambda$ *on* $\mathbf{R} \backslash I_k$,

(iii) $|\lambda_k^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t)$ *for* $t \in I_k \backslash (-a_k + 1, a_k - 1)$, $0 \le s \le r$,

(iv) $\lambda_k^{(s)}(t) = \lambda^{(s)}(t)$ *for* $t \in T$, $0 \le s \le r$, *and*

(v) $\lambda_k(t) \notin K_k$ *when* $|t| \ge a_k - 1$.

Proof. We define the cutoff function χ_k on $\mathbf{R}$ using χ, so that $\chi_k = 1$ on $[-a_k + 1, a_k - 1]$, $\chi_k = 0$ on $\mathbf{R} \backslash I_k$, and $\|\chi_k h\|_{\mathcal{C}^r} < C\|h\|_{\mathcal{C}^r}$ as before. Let

$$\hat{\lambda}_k(t) = f_k(t)\chi_k(t) + \lambda(t)(1 - \chi_k(t)), \quad t \in \mathbf{R}.$$

By lemma 2.1, (3_k), (4_k), and choice of η and δ_k, we see that (i)–(iv) are satisfied for $\hat{\lambda}_k$ in place of λ_k.

To obtain (v) we use a transversality argument to perturb $\hat{\lambda}_k$ on $I_k\backslash(-a_k+1, a_k-1)$. First note that if $|t|>a_k$, then $|\hat{\lambda}_k(t)| = |\lambda(t)| > R_k+1$ by (6_k), so $\hat{\lambda}_k(t) \notin B_k$. Also, by (5_k), we see that $\hat{\lambda}_k(t) \notin f_k(\mathbf{C})$, so $\hat{\lambda}_k(t) \notin K_k$. Next, if $t \in T\cap(I_k\backslash(-a_k+1, a_k-1))$, then by (4_k), (7_k), and (e) we see that $\hat{\lambda}_k(t) = f_k(t) \notin K_k$. Hence there exists a neighborhood V of $T\cap(I_k\backslash(-a_k+1, a_k-1))$ such that $\hat{\lambda}_k(\overline{V})\cap K_k = \emptyset$.

Thus we need only perturb $\hat{\lambda}_k$ on $I_k\backslash(V\cup(-a_k+1, a_k-1))$ to get (v). Note that if $t\in I_k\backslash(-a_k+1, a_k-1)$, then from (6_k) and (2_k) we see that $|\hat{\lambda}_k(t)| > R_k+1/2$, so $\hat{\lambda}_k(t)\notin B_k$. Finally, a simple transversality argument implies that we can make an arbitrarily small $\mathcal{C}^\infty$ perturbation of $\hat{\lambda}_k$ to avoid $f_k(\mathbf{C})$, and hence we get λ_k with $\lambda_k = \hat{\lambda}_k$ outside $I_k\backslash(V\cup(-a_k+1, a_k-1))$ and λ_k satisfying (i)–(v). ♠

Now we can use lemma 2.4 to approximate λ_k, hence to approximate λ. Set

$$\delta_{k+1} = \min\{\eta(t) : t\in I_{k+1}\}/(2C),$$
$$\sigma_{k+1} = \min\{\eta(t) - |\lambda_k^{(s)}(t) - \lambda^{(s)}(t)| : t\in I_{k+1}, 0\le s\le r\} > 0.$$

Choose $\epsilon>0$ so small that

$$\epsilon < \min\{2^{-(k+1)}, \delta_{k+1}, \sigma_{k+1}\}, \qquad f_k(\Delta_j + \epsilon_k\overline{\mathbf{B}}) + \epsilon\overline{\mathbf{B}} \subset \mathrm{Int}B_j, \quad 1\le j\le k.$$

Apply lemma 2.4 with $\lambda=\lambda_k$, $f=f_k$, $I=[-a_k+1, a_k-1]$, $K=K_k$, $a=a_{k+1}$, r and T unchanged, and ϵ as above. This provides $\Phi_{k+1}\in\mathrm{Aut}\mathbf{C}^n$ and $G=\Phi_{k+1}\circ f_k\in\mathrm{Aut}\mathbf{C}^n$ satisfying

$$\begin{cases} |\Phi_{k+1}(z) - z| < \epsilon & \text{for } z\in K_k, \text{ hence on } B_k; \\ |G^{(s)}(t) - \lambda_k^{(s)}(t)| < \epsilon & \text{for } t\in I_{k+1},\ 0\le s\le r; \\ G^{(s)}(t) = \lambda_k^{(s)}(t) & \text{for } t\in T\cap I_{k+1},\ 0\le s\le r. \end{cases}$$

In particular, (2_{k+1})–(4_{k+1}) hold with G in place of f_{k+1}.

Since $f_k(\Delta_{k+1})\subset K_k\subset(R_{k+1}-1)\mathbf{B}$, we can choose $\epsilon'_{k+1}<\epsilon_k$ small enough that (1_{k+1}) holds with G in place of f_{k+1} and ϵ'_{k+1} in place of ϵ_{k+1}, and such that if $\psi\in\mathrm{Aut}\mathbf{C}^n$ with $\|\psi(t)-t\|_{\mathcal{C}^r(I_{k+1})} < \epsilon'_{k+1}$, then (2_{k+1}) and (3_{k+1}) hold with $G\circ\psi$ in place of f_{k+1}. Let $\epsilon_{k+1} = \epsilon'_{k+1}/2$. Then with G in place of f_{k+1}, we have (1_{k+1})–(4_{k+1}), (6_{k+1}), and $G(-a_{k+1}), G(a_{k+1})\notin B_{k+1}$ by (6_{k+1}) and (2_{k+1}).

Next we want to obtain (7_{k+1}). We do this using lemma 2.2 to change the embedding so that the image of $\mathbf{R}\backslash I_{k+1}$ misses B_{k+1} while leaving the embedding essentially unchanged on $\Delta_{k+1}\cup I_{k+1}$. Apply lemma 2.2 with $A=G^{-1}(B_{k+1})$, $\rho=k+1$, $I=I_{k+1}$, r unchanged, $Z=T\cap I_{k+1}$ and $\epsilon=\epsilon_{k+1}/2$. This gives a shear

$$\psi_{k+1}(z) = z + g_{k+1}(z_1)e_2$$

with

$$\begin{cases} |\psi_{k+1}(z) - z| < \epsilon_{k+1}/2, \quad z\in\Delta_{k+1}; \\ \|\psi_{k+1}|_{\mathbf{R}}(t) - t\|_{\mathcal{C}^r(I_{k+1})} < \epsilon_{k+1}/2; \\ g_{k+1}^{(s)}(t) = 0, \quad t\in T\cap I_{k+1}, 0\le s\le r; \\ \psi_{k+1}(t)\notin G^{-1}(B_{k+1}), \quad t\in\overline{\mathbf{R}\backslash I_{k+1}}. \end{cases}$$

Let $H = G\psi_{k+1}$. Then with H in place of f_{k+1}, we have (1_{k+1})–(4_{k+1}), (6_{k+1}), and (7_{k+1}).

For the final correction, we use lemma 2.3 to obtain (5_{k+1}) while maintaining the other properties. Let $R > a_{k+1}$ be such that $A = G^{-1}(B_{k+1}) \subset R\mathbf{B}$. Let $\delta > 0$ be such that

$$\psi_{k+1}\left([-R,R]\backslash(-a_{k+1},a_{k+1}) + \delta\overline{\mathbf{B}}\right) \cap A = \emptyset,$$

and such that if $\theta \in \mathrm{Aut}\mathbf{C}^n$, with $|\theta(z) - z| < \delta$ on $R\overline{\mathbf{B}}$, then

$$\|\psi_{k+1}\theta|_{\mathbf{R}}(t) - t\|_{\mathcal{C}^r(I_{k+1})} < \epsilon_{k+1}. \tag{1}$$

Apply lemma 2.3 with λ replaced by $H^{-1}\lambda$, $K = R\overline{\mathbf{B}}$, r unchanged, $Z = T \cap I_{k+1}$, and $\epsilon = \min\{\delta, \epsilon_{k+1}/2\}$. This gives a shear $\theta_{k+1}(z) = z + h_{k+1}(z_1)v_{k+1}$ with $\pi_1 v_{k+1} = 0$ such that

$$\begin{cases} |\theta_{k+1}(z) - z| < \min\{\delta, \epsilon_{k+1}/2\}, \quad z \in \Delta_{k+1}; \\ \theta_{k+1}(\mathbf{C}) \cap H^{-1}\lambda(\mathbf{R}) = H^{-1}\lambda(T \cap I_{k+1}); \\ h_{k+1}^{(s)}(t) = 0, \quad t \in T \cap I_{k+1},\ 0 \le s \le r, \end{cases}$$

and such that (1) holds with $\theta = \theta_{k+1}$. Also, by the choice of R and δ,

$$\psi_{k+1}\theta_{k+1}(\overline{\mathbf{R}\backslash I_{k+1}}) \cap A = \emptyset.$$

Taking $\Psi_{k+1} = \psi_{k+1}\theta_{k+1}$ and

$$f_{k+1} = H \circ \theta_{k+1} = \Phi_{k+1} \circ f_k \circ \Psi_{k+1}$$

we obtain (5_{k+1}) and preserve the remaining hypotheses. Hence we obtain (1_{k+1})–(7_{k+1}). Note that $(k+1)\mathbf{B} \subset B_{k+1}$ so we also obtain (a)–(f), thus finishing the inductive step.

The case $k = 1$ is similar to the general step. First apply proposition 1.2 with $K = \emptyset$, $C = [-3,3] \subset \mathbf{C}$, $F = \lambda$, $\epsilon = C^{-1}\inf\{\eta(t): t \in [-3,3]\}$, and $Z = T \cap [-3,3]$ to get $\phi_1^1 \in \mathrm{Aut}\mathbf{C}^n$ satisfying the conclusions of that proposition. Choose $R_1 \ge 2$ such that $\phi_1^1(\Delta_1) \subset (R_1 - 1)\mathbf{B}$, choose $a_1 > 4$ to get (6_1), and let $I_1 = [-a_1, a_1]$. Choose δ_1 to satisfy (f) for $j = 1$.

Define a proper $\mathcal{C}^\infty$ embedding λ_0 as in lemma 3.1 so that (i)–(v) are satisfied with λ_0 in place of λ_k, ϕ_1^1 in place of f_k, 3 in place of a_k, $[-3,3]$ in place of I_k, and $\phi_1^1(\Delta_1)$ in place of K_k. Apply lemma 2.4 with $\lambda = \lambda_0$, $f = \phi_1^1$, $I = [-2,2]$, $K = \phi_1^1(\Delta_1)$, $a = a_1$, $\epsilon = \delta_1$, and T and r unchanged. This gives $\phi_1^2 \in \mathrm{Aut}\mathbf{C}^n$ such that

$$|\phi_1^2(z) - z| < \delta_1 \le 1/2, \quad z \in \phi_1^1(\Delta_1),$$

and such that $\Phi_1 = \phi_1^2\phi_1^1$ satisfies

$$\begin{cases} \|\Phi_1 - \lambda_0\|_{\mathcal{C}^r(I_1)} < \epsilon; \\ \Phi_1^{(s)}(t) = \lambda_0^{(s)}(t), \quad t \in T \cap I_1,\ 0 \le s \le r. \end{cases}$$

As before, we can apply lemmas 2.2 and 2.3 to obtain $\epsilon_1 > 0$ and a shear Ψ_1 such that the hypotheses (1_1)–(7_1) hold for $f_1 = \Phi_1\Psi_1$, and (a)–(f) hold for $k = 1$. This completes the base case.

To finish the proof of theorem 1.1, note that

$$\Psi_1 \cdots \Psi_k(z) = z + \sum_{j=1}^{k} (g_j(z_1)e_2 + h_j(z_1)v_j)$$

and that (c) implies

$$|g_j(z_1)e_2 + h_j(z_1)v_j| < 2^{-j}, \quad |z_1| < j.$$

Hence this sum converges uniformly on compacts to a shear $\Psi(z) = z + G(z_1)$ for some holomorphic map $G\colon \mathbf{C} \to \{0\} \times \mathbf{C}^{n-1}$.

By proposition 4.2 in [6], the composition $\Phi_k \circ \cdots \circ \Phi_1$ converges locally uniformly to a biholomorphic map from a domain Ω onto $\mathbf{C}^n$, and

$$\Omega = \cup_{k=1}^{\infty} (\Phi_k \cdots \Phi_1)^{-1}(B_{k-1}).$$

We claim that $\Psi(\mathbf{C} \times \{0\}) \subset \Omega$. Let $k > 1$. By (1_k) we have

$$\Psi_1 \cdots \Psi_k(\Delta_{k-1} + \epsilon_k \overline{\mathbf{B}}) \subset (\Phi_k \cdots \Phi_1)^{-1}(B_{k-1}).$$

Since $|\Psi_j(z) - z| < \epsilon_j$ on Δ_{k-1} for $j \geq k$, and $\sum_{j=k+1}^{\infty} \epsilon_j < \epsilon_k$ by (c), we see that

$$\lim_{m\to\infty} \Psi_{k+1} \cdots \Psi_m(z) \in \Delta_{k-1} + \epsilon_k \overline{\mathbf{B}}$$

for each $z \in \Delta_{k-1}$. Hence

$$\Psi(\Delta_{k-1}) \subset (\Phi_k \cdots \Phi_1)^{-1}(B_{k-1}) \subset \Omega, \qquad k > 1,$$

so the claim holds. In particular, $\Phi\Psi\colon \mathbf{C} \to \mathbf{C}^n$ is a proper holomorphic embedding.

Finally, using the conditions (1_k)–(7_k), we see that $\Phi\Psi\colon \mathbf{C} \hookrightarrow \mathbf{C}^n$ is a proper holomorphic embedding with the desired properties. ♠

References.

1. Arakelian, N. U.: Uniform approximation on closed sets by entire functions (Russian). *Izv. Akad. Nauk SSSR* **28**, 1187–1206 (1964)
2. Buzzard, G., Fornæss, J.E.: An embedding of $\mathbf{C}$ into $\mathbf{C}^2$ with hyperbolic complement. *Math. Ann.*, to appear

3. Carleman, T.: Sur un theorem de Weierstrass. *Arkiv Mat., Astr. och Fysik* **20B**, no. 4 (1927)
4. Eliashberg, Y., Gromov, M.: Embeddings of Stein manifolds of dimension n into the affine space of dimension $3n/2+1$. *Ann. of Math.* **136**, 123–135 (1992)
5. Forstneric, F.: Approximation by automorphisms on smooth submanifolds of $\mathbf{C}^n$. *Math. Ann.* **300**, 719–738 (1994)
6. Forstneric, F.: Interpolation by holomorphic automorphisms and embeddings in C^n. Preprint, 1996
7. Forstneric, F., Globevnik, J., Rosay, J.-P.: Non straightenable complex lines in $\mathbf{C}^2$. *Arkiv Mat.* **34** (1996)
8. Forstneric, F., Globevnik, J., Stensønes, B.: Embedding holomorphic discs through discrete sets. *Math. Ann.* **304**, (1995)
9. Forstneric, F., Løw, E.: Global holomorphic equivalence of certain smooth submanifolds in $\mathbf{C}^n$. Preprint, 1996
10. Gaier, D.: *Lectures on Carleman Approximation.* Birkhäuser: Boston 1987
11. Hörmander, L.: *An Introduction to Complex Analysis in Several Variables, 3rd ed.* North Holland: Amsterdam 1990
12. Narasimhan, R.: *Analysis on real and complex manifolds*, 3rd printing. North-Holland: Amsterdam 1985
13. Rosay, J.-P., Rudin, W.: Arakelian's approximation theorem. *Amer. Math. Monthly* **96**, 432–434 (1989)
14. Rudin, W.: *Real and Complex Analysis.* McGraw Hill: New York 1970

Gregery T. Buzzard, Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

Franc Forstneric, Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA