NEVANLINNA THEORY AND RATIONAL POINTS

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Abstract S. Lang [L] conjectured in 1974 that a hyperbolic algebraic variety defined over a number field has only finitely many rational points, and its analogue over function fields. We discuss the Nevanlinna-Cartan theory over function fields of arbitrary dimension and apply it for Diophantine property of hyperbolic projective hypersurfaces (homogeneous Diophantine equations) constructed by Masuda-Noguchi [MN]. We also deal with the finiteness property of S-units points of those Diophantine equations over number fields.

Introduction

S. Lang [L] conjectured in 1974 that a hyperbolic algebraic variety defined over a number field has only finitely many rational points, and its analogue over function fields. For subvarieties of Abelian varieties the function field analogue was dealt with by M. Raynaud [R], and lately G. Faltings [F] proved this conjecture for subvarieties of Abelian varieties over number fields. On the other hand, the author [Nog5] proved the function field analogue in general case (cf. also [Nog1], [Nog2]). See Y. Imayoshi-H. Shiga [IS], M. Zaidenberg [Z], and M. Suzuki [Su1], [Su2] for non-compact versions of this result.

In the case of curves (Fermat, Catalan, Thue equations, etc.) defined over function fields, R.C. Mason [Ma1], J. Silverman [Si] and J. Mueller [Mu] obtained similar or more precise finiteness properties by making use of a different method which relies on the function field analogue of "abc-conjecture" of Masser-Oesterlé. The function field analogue of "abc-conjecture" was proved in more general form by R.C. Mason [Ma1], [Ma2], J. Voloch [Vo] and W. Brownawell-D. Masser [BrM]. They actually proved a version of "abc-conjecture" in several variables (to say, $abc\cdots$-conjecture), which is nothing but a special case of Nevanlinna-Cartan's second main theorem with truncated counting functions applied to algebraic case (see [C], (3), and §2).

Here we discuss the Nevanlinna-Cartan theory over function fields of arbitrary dimension (cf. J. Wang [W1], [W2] and [W3] for related results), and apply it for Diophantine property of hyperbolic projective hypersurfaces (homogeneous Diophantine equations) constructed by [MN]. We also deal with the finiteness property of S-units points of those Diophantine equations over number fields.

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1. Function field case

We deal with the Nevanlinna theory over function fields of an arbitrary dimension by making use of the method of Cartan [C] and Nochka [N], which affirmatively proved Cartan's conjecture (cf. also [Ch] and [Fu2]). Let k be an algebraically closed field of characteristic 0, let R be a smooth projective algebraic variety of dimension N over k, and let K denote the rational function field of R. To use analytic definitions as well, we set $k = \mathbf{C}$.

We fix a class ω of an ample line bundle (a Hodge metric form) on R. Let D be a divisor on R, and define the counting function of D with respect to ω by

$$N(D;\omega) = \int_D \omega^{N-1}.$$

If σ is a meromorphic section of some line bundle and (σ) denotes the divisor determined by σ, then we write

$$N(\sigma;\omega) = N((\sigma);\omega).$$

Let $a_i \in K^*$, $1 \leqq i \leqq m$, and let $((a_j))$ denote the least common multiple of the polar divisors of a_i, $1 \leqq i \leqq m$. We define the height function of $(a_i)_{1 \leqq i \leqq m}$, by

$$\mathrm{ht}((a_j);\omega) = N(((a_i));\omega).$$

From $a_i \in K^*$, $1 \leqq i \leqq m$, we obtain a rational mapping $f = [1; a_1, \ldots, a_m] : R \to \mathbf{P}^m(\mathbf{C})$, and define the order function of f by

$$T(f;\omega) = \int_R f^*\Omega \wedge \omega^{N-1},$$

where Ω denotes the Fubini-Study Kähler form on $\mathbf{P}^m(\mathbf{C})$. The first main theorem is nothing but the Poincaré residue theorem:

(1.1) Theorem. *Let the notation be as above. Let L be the line bundle determined by (a_i), and let $\sigma \in \Gamma(R, L)$ be a holomorphic section. Then*

$$T(f;\omega) = \mathrm{ht}((a_j);\omega) = N(\sigma;\omega), \qquad 0 \leqq j \leqq m.$$

Next we have the "*second main theorem with truncated counting functions*":

(1.2) Theorem. *Let $f = [\sigma_0, \dots, \sigma_m] : R \to \mathbf{P}^m(\mathbf{C})$ be a reduced representaion of a rational mapping given by holomorphic sections σ_j of a line bundle. Let $H_1, \dots, H_q$ $(q \geqq m+1)$ be linear forms on $\mathbf{P}^m(\mathbf{C})$ in general position such that $f^*H_j \not\equiv 0$ for any j. Let r denote the rank of df at general point, and let l denote the dimension of the smallest linear subspace of $\mathbf{P}^m(\mathbf{C})$ containing $f(R)$. Then*

$$\begin{aligned}(q-l-1)T(f;\omega) \leqq & \sum_{i=1}^{q} N_{l-r+1}(f^*H_i;\omega) \\ & + \left\{ \frac{(l-r+1)(l-r+2)}{2} + r - 1 \right\} N(J;\omega).\end{aligned} \tag{1.3}$$

Here $N_{l-r+1}(H_i(x_j);\omega)$ is the truncated counting function of zeros of $H_i(x_j)$, and J is a divisor used to define the *generalized Wronskian bundle* $W = [pJ] \otimes L^{l+1}$, and is independent of f. Cf. J. T.-Y. Wang [W1], [W2], and [W3] for related results.

Theorem (1.2) plays an important role. Because of the truncated counting functions $N_{l-r+1}(H_i(x_j);\omega)$, we may consider inequality (1.3) as a version of "*abc*-conjecture" in several variables over function fields of an arbitrary dimension. (cf. [Ma2], [Vo], and [BrM], Corollary I). We derive a "ramification theorem" over function fields (Corollary (2.16)), and "Borel's theorem" over functions fields:

(1.4) Theorem. *Let $x_j \in K^*, 1 \leqq j \leqq s$, satisfy*

$$a_1 x_1^d + \cdots + a_s x_s^d = 0 \qquad (s \geqq 2).$$

Assume

$$d > s(s-2) + (s-1)^2 \mathrm{ht}((a_1, \dots, a_s);\omega) + \frac{(s-1)(s-2)}{2} N(J;\omega). \tag{1.5}$$

Then there is a disjoint decomposition $\{1, \dots, s\} = \cup_{\nu=1}^{l} I_\nu$ of indices such that

(i) $|I_\nu| \geqq 2$ *for all* ν;

(ii) *for arbitrary two indices* $j, k \in I_\nu$, *the ratio* x_j/x_k *is a constant.*

(iii) $\sum_{j \in I_\nu} a_j x_j^d = 0$ *for all* ν.

We use the above results to study the rational points of a Diophantine equation. Let $X \subset \mathbf{P}^{n-1}_K$ be a hypersurface defined over K by equation

$$a_1 M_1^d(z_1, \ldots, z_n) + \cdots + a_s M_s^d(z_1, \ldots, z_n) = 0, \tag{1.6}$$

where $a_j \in K^* = K \setminus \{0\}$ and $d \in \mathbf{Z}, > 0$. We set

$$Y(\mathrm{P}) = \Big\{(u_1, \ldots, u_n) \in \mathbf{P}^{n-1}(k); \sum_j a_j M_j^d(z_1, \ldots, z_n) M_j^d(u_1, \ldots, u_n) = 0, \\ \text{and } u_j = 0 \text{ if } z_j = 0\Big\}.$$

Then $Y(\mathrm{P})$ is a projective variety defined over k. Moreover, we set

$$\mathcal{R}(\mathrm{P}) = \big\{(z_1 u_1, \ldots, z_n u_n) \in \mathbf{P}^{n-1}(K); (u_1, \ldots, u_n) \in Y(\mathrm{P})\big\} \subset X(K).$$

(1.7) Main Theorem. *Let the notation be as above. Assume that* $\{M_j(z_1, \ldots, z_n)\}_{j=1}^s$ *is n-admissible ([MN]).*

(i) *Assume that* $d > s(s-2)$. *Then the heights* $\mathrm{ht}((z_i); \omega)$ *of points of* $X(K)$ *are bounded.*

(ii) *Assume (4) for* d. *Then all points of* $X(K)$ *are defined over* k*; that is, all points* $x = (x_1, \ldots, x_n) \in X(K)$ *are represented by* $x_j \in k$.

(iii) *Assume that*

$$d > s!(s!-2) + \frac{(s!-1)(s!-2)}{2} N(J; \omega).$$

Then there are only finitely many rational points $\mathrm{P}_\mu \in X(K)$, $\mu = 1, \ldots, \mu_0$ $(< \infty)$ *such that*

$$X(K) = \bigcup_{\mu=1}^{\mu_0} \mathcal{R}(\mathrm{P}_\mu).$$

In the proof of (iii) we use the result of E. Bombieri-J. Mueller [BM] in a form generalized by Theorem (1.2).

2. Number field case–S-unit points

In this section we deal with X defined by (1.6) over a number field F. We say that a point $(z_i) \in \mathbf{P}^{n-1}(F)$ is *S-units point* if $z_i \in \mathcal{O}_S^*$ or $z_i = 0$, denote by $X(\mathcal{O}_S^*)$ the set of S-units point of $X \subset \mathbf{P}^{n-1}_F$. By making use the same idea as in the proof of the Main Theorem (1.7) we then apply Borel's Theorem for S-units to prove

(2.1) Theorem. *Let $\{M_j(z_1, \ldots, z_n)\}_{j=1}^s$ be an n-admissible set of monomials, and let X be defined by (1.6) with an arbitrary $d \geq 1$. Then $X(\mathcal{O}_S^*)$ is a finite set.*

Remark. (i) Mahler ([M], p. 724, Folgerung 2) proved Theorem (2.1) in the case of $n = s = 3$.

(ii) Let $[F; \mathbf{Q}]$ denote the extension degree and let $|S|$ be the cardinality of S. By making use of the bound obtained by Győry [G1, G2], we have

$$|X(\mathcal{O}_S^*)| \leqq (2^n - 1)(2^{35} s^2)^{(s-1)^3 |S|}.$$

(iii)(ASMT-conjecture) We give an analogue of Theorem (2.1) over number fields, which is an extension of Schmidt's linear subspace theorem. This may be called "*Arithmetic Second Main Theorem-conjecture*" by its nature. Let F and S be as above. We define the truncated counting function $N_\lambda(H(x_i))$ over the places of F outside S. Let H_j, $1 \leqq j \leqq q$, be linear forms in general position on $\mathbf{P}_F^m$. Then for an arbitrary $\epsilon > 0$ there are finitely many hyperplanes E_ν such that for $(x_i) \in \mathbf{P}^m(F) \setminus \cup E_\nu$

$$(q - m - 1 - \epsilon)\mathrm{ht}((x_i)) \leqq N_m(H(x_i)).$$

(iv) The above ASMT-conjecture implies the finiteness of $X(F)$. In this respect the works of Khoai-Tu [KT] and Sarnak-Wang [SW] are of interest; especially Sarnak-Wang [SW] showed that some hyperbolic smooth hypersurfaces of $\mathbf{P}^4$ and $\mathbf{P}^5$ constructed by [MN] defined over $\mathbf{Q}$ has infinitely many rational points over the p-adic number filed $\mathbf{Q}_p$ for every prime p, and that its Brauer-Manin group known as an obstruction for the Hasse principle is vanishing.

(v) See [MN] for a number of examples.

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