ON HOLOMORPHIC CURVES IN SEMI-ABELIAN VARIETIES

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Abstract. The algebraic degeneracy of holomorphic curves in a semi-Abelian variety omitting a divisor is proved (Lang's conjecture generalized to semi-Abelian varieties) by making use of the *jet-projection method* and the logarithmic Wronskian jet differential after Siu-Yeung. We also prove a structure theorem for the locus which contains all possible image of non-constant entire holomorphic curves in a semi-Abelian variety omitting a divisor.

Introduction

The purpose of this paper is to prove this result:

Main Theorem. *Let D be a non-zero algebraic effective reduced divisor of a semi-Abelian variety A over the complex number field $\mathbf{C}$. Let $f : \mathbf{C} \to A \setminus D$ be an arbitrary holomorphic mapping.*

(i) *The Zariski closure $X_0(f)$ of the image of f in A is a translate of a proper semi-Abelian subvariety of A, and $X_0(f) \cap D = \emptyset$.*

(ii) *In special, if D has a non-empty intersection with any translate of any positive dimensional semi-Abelian subvariety of A, then $f : \mathbf{C} \to A \setminus D$ is constant.*

Moreover, we prove a *structure theorem* for the locus (sometimes, called the exceptional set) of $A \setminus D$ which contains the images of all possible non-constant entire holomorphic curves in $A \setminus D$ (see Theorem (3.1) and Remark after it).

In the case where A is a semi-Abelian variety and D has two components which are homologous to each other, the algebraic degeneracy of a holomorphic curve $f : \mathbf{C} \to A \setminus D$ was proved by [N81]. In the case where A is an Abelian variety, Siu-Yeung [SY96] proved the Main Theorem (Lang's conjecture). Note that a generalization of the Main Theorem to the case of semi-Abelian varieties is already claimed in the introduction of [SY96], and that the lemmas we are going to prove are similar to those in [SY96]. The main difference is that instead of Siu-Yeung's elaborate Wronskian arguments ([SY96], Lemma (1.1) and Proof of Lemma (2.1)), we use a simpler and more direct "*jet-projection method*", the same idea as in [NO$\frac{84}{90}$], Chap. VI; there, a self-contained and detailed proof of Bloch's conjecture

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Research at MSRI supported in part by NSF grant #DMS 9022140.

is described. This proof, especially the proof of [NO$\frac{84}{90}$], Lemma (6.3.10), is an original one by the present author for the remaining part after [B26] and [O77] to finish the proof of Bloch's conjecture, and works as well for semi-Abelian varieties [N81]. M. Green gave a talk on a proof of [NO$\frac{84}{90}$], Lemma (6.3.10) based on Gauss' maps at Taniguchi Symposium, Katata/Kyoto, 1978; he did not publish it, but did [GG80] with P. Griffiths by making use of another idea based on Riemann-Roch and curvature methods (cf. also [K80] and [M96]); afterward, a gap in the proof of [GG80] was found; see [D96] and [DL96], where Dethloff and Lu also use the present jet-projection method.

It is worth noting that Ochiai [O77] proved the algebraic degeneracy of an entire holomorphic curve in an Abelian variety omitting two divisors, mutually linearly equivalent, and recognizing that this is the first link between Bloch's and Lang's conjectures.

In the course of the proof of the Main Theorem, we also need a generalization of theta functions for semi-Abelian varieties (see Lemma (2.1) and Remark after it).

In comparison with arithmetic on semi-Abelian varieties, the counter part in number theory was proved by Vojta [V95].

Acknowledgement. The present paper was written during the author's stay at Mathematical Sciences Research Institute, University of California, Berkeley for the special year program in Several Complex Variables, 1995/96. The author is grateful to the institute for the very active and stimulating circumstance, and to participating members, especially, Professors Y.-T. Siu, B. Shiffman, S. Lu, G. Dethloff, M. Ru, P.-M. Wong, P. Vojta, A. Huckleberry, F. Catanese, H. Tsuji, and S. Kobayashi.

§1. Jet space and translation invariance

Let A be a semi-Abelian variety of dimension n over $\mathbf{C}$; that is, A is a complex algebraic group of dimension n which is an extension of an Abelian variety A_0 by an algebraic torus $(\mathbf{C}^*)^t$ ($t \in \mathbf{Z}, \geqq 0$),

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \to 0. \tag{1.1}$$

(Cf. [I76].) Let $f : \mathbf{C} \to A$ be a holomorphic curve. Let $J_k(A)$ denote the k-th jet space over A with $k \geqq 0$ (where if $k = 0$, $J_0(A) = A$), and let $d^k f : \mathbf{C} \to J_k(A)$ denote the k-th jet lifting of f. (Cf. [NO$\frac{84}{90}$], Chap. VI.) Let $X_k(f)$ be the Zariski closure of $d^k f(\mathbf{C})$ in $J_k(A)$.

Note that the group law of A canonically induces an additive action of A on $J_k(A)$.

Lemma (1.2). *If* $\dim X_k(f) > 0$, *then there exists a one-parameter subgroup of* A *which leaves* $X_k(f)$ *invariant.*

Proof. Since A is a semi-Abelian variety, we have a decomposition

$$J_k(A) = A \times \mathbf{C}^{nk}, \tag{1.3}$$

which is fixed from now on.

We take the l-th jet space $J_l(X_k(f))$ (resp. $J_l(J_k(A))$) of $X_k(f)$ (resp. $J_k(A)$), where $l = 1, 2, 3, \ldots$. Then $J_l(X_k(f)) \subset J_l(J_k(A))$, and we obtain the decomposition naturally induced from (1.3)

$$J_l(J_k(A)) = J_l(A \times \mathbf{C}^{nk}) = A \times \mathbf{C}^{nk} \times \mathbf{C}^{(n+nk)l}, \tag{1.4}$$

the natural projection to the second and third factors, and its restriction:

$$\begin{aligned} \psi_{k,l} &: J_l(J_k(A)) \to \mathbf{C}^{nk} \times \mathbf{C}^{(n+nk)l}, \\ \Psi_{k,l} = \psi_{k,l}|J_l(X_k(f)) &: J_l(X_k(f)) \to \mathbf{C}^{nk} \times \mathbf{C}^{(n+nk)l}. \end{aligned} \tag{1.5}$$

Let $d^l d^k f : \mathbf{C} \to J_l(X_k(f))$ be the l-the jet lifting of $d^k f$. Take a point $z_0 \in \mathbf{C}$ with $d^k f(z_0) \in X_k(f)_{\mathrm{reg}}$, where $X_k(f)_{\mathrm{reg}}$ denotes the set of regular points of $X_k(f)$. Set $y_l = d^l d^k f(z_0) \in J_l(X_k(f))$. Now, look at the kernel

$$\begin{aligned} \mathrm{Ker}\ d\Psi_{k,l}(y_l) \subset \mathbf{T}_{y_l}(J_l(X_k(f))) &\subset \mathbf{T}_{y_l}(J_l(J_k(A))) \\ &= \mathbf{T}_{f(z_0)}(A) \oplus \mathbf{T}_{\psi_{k,l}(y_l)}(\mathbf{C}^{nk} \times \mathbf{C}^{(n+nk)l}) \end{aligned} \tag{1.6}$$

(see (1.4)) of the differential $d\Psi_{k,l}$ of $\Psi_{k,l}$ at the point y_l, where $\mathbf{T}(*)$ stands for the holomorphic tangent space. By the definition of $\Psi_{k,l}$,

$$\mathrm{Ker}\ d\Psi_{k,l}(y_l) \subset \mathbf{T}_{f(z_0)}(A) \oplus O' \cong \mathbf{T}_{f(z_0)}(A), \tag{1.7}$$

where O' denotes the zero vector of $\mathbf{T}_{\psi_{k,l}(y_l)}(\mathbf{C}^{nk} \times \mathbf{C}^{(n+nk)l})$ in (1.6). By making use of the isomorphism in (1.7), $\mathrm{Ker}\ d\Psi_{k,l'}(y_{l'}) \subset \mathrm{Ker}\ d\Psi_{k,l}(y_l) \subset \mathbf{T}_{f(z_0)}(A)$ for $l' \geqq l$. Therefore, if $\cap_{l \geqq 1} \mathrm{Ker}\ d\Psi_{k,l}(y_l) = \{O\}$, $\mathrm{Ker}\ d\Psi_{k,l}(y_l) = \{O\}$ for some l. Then, by making use of Lemma on logarithmic derivative, we obtain an estimate of the order function $T_f(r)$ of f by those of $df, d^2 f, \ldots, d^{k+l} f$ which are small functions with respect to $T_f(r)$; this is a contradiction (cf. [NO$\frac{84}{90}$], Chap. VI for this argument). Thus, $\cap_{l \geqq 1} \mathrm{Ker}\ d\Psi_{k,l}(y_l) \neq \{O\}$. Let $v \in \cap_{l \geqq 1} \mathrm{Ker}\ d\Psi_{k,l} \setminus \{O\} \subset \mathbf{T}_{f(z_0)}(A) \setminus \{O\}$. We may regard v as a global holomorphic vector field on $J_k(A)$ through the trivialization (1.3). Then v is tangent to $X_k(f)$ at all points of the image of a neighborhood of z_0 by $d^k f$, and so it is tangent to $X_k(f)$ at all points of $X_k(f)$. Hence $X_k(f)$ is invariant by the action of the one-parameter subgroup generated by v. *Q.E.D.*

We define the stabilizer $\mathrm{St}(X_k(f))$ to be the maximal semi-Abelian subvariety of A which leaves $X_k(f)$ invariant by translations. After taking the quotient by $\mathrm{St}(X_k(f))$, we apply Lemma (1.2) to conclude

Proposition (1.8). *Let the notation be as above.*
(i) $X_0(f)$ *is a translation of a semi-Abelian subvariety* B *of* A.
(ii) $X_k(f)$ *is invariant by the action of* B *on* $J_k(A)$ *for all* $k \geqq 0$.

§2. Proof of Main Theorem

By (1.1) we consider A as a $(\mathbf{C}^*)^t$-principal fiber bundle with projection $\rho : A \to A_0$. Using the natural compactification $(\mathbf{C}^*)^t \subset (\mathbf{P}^1_{\mathbf{C}})^t$, we have a compactification $\overline{A}$ of A with projection $\rho : \overline{A} \to A_0$, which is projective algebraic (cf. [N81] and [V95]). Since D be an algebraic effective reduced divisor on A, there is an effective reduced divisor $\overline{D}$ on $\overline{A}$ with $\overline{D} \cap A = D$.

Let $\pi : \mathbf{C}^n \to A$ be the universal covering of A with $\pi_1(A) = \Gamma$ ([I76]), which may be called an incomplete lattice or a semi-lattice.

Lemma (2.1). *Let the notation be as above. Then there is an entire function* $\theta(x)$ *on* $\mathbf{C}^n$ *such that* $\pi^* D = (\theta)$ *and for* $\gamma \in \Gamma$

$$\theta(x + \gamma) = e^{L_\gamma(x)} \theta(x), \qquad x \in \mathbf{C}^n,$$

where $L_\gamma(x)$ *is affine linear in* x.

Remark. In a glance this Lemma (2.1) seems to be already known and classical, but so far, asking several specialists, we could not find a reference stating the above lemma. So we give a proof, which is actually quite elementary.

Proof. We may assume that $\overline{D}$ is irreducible, and set $n_0 = \dim A_0$. If $\rho(\overline{D}) \neq A_0$, then $\dim \rho(\overline{D}) = n_0 - 1$, and $\rho^{-1}(\rho(\overline{D})) \cap A = D$. Then this is the case of the classical theta function on an Abelian variety (cf. [W58]).

Assume that $\rho(\overline{D}) = A_0$. Let $\{U_\lambda\}$ be an open covering of A_0 such that U_λ is a ball in $\mathbf{C}^{n_0}$ ($n_0 = \dim A_0$), and the restriction $A|U_\lambda$ is isomorphic to $U_\lambda \times (\mathbf{C}^*)^t$. We fix a multiplicative coordinate system $(z_1, \ldots, z_t)$ of $(\mathbf{C}^*)^t$. Then $D \cap (A|U_\lambda)$ is given by a zero set of a finite Laurent series in $(z_1, \ldots, z_t)$

$$\theta_\lambda = \sum_{\text{finite}} a_{\lambda l_1 \cdots l_t}(y) z_1^{l_1} \cdots z_t^{l_t},$$

where the coefficients $a_{\lambda l_1 \cdots l_t}(y)$ are holomorphic functions in U_λ. Then, on $(A|U_\lambda) \cap (A|U_\mu)$ we have

$$\theta_\lambda \cdot \theta_\mu^{-1} = a_{\lambda\mu}(y) z_1^{l_{\lambda\mu 1}} \cdots z_t^{l_{\lambda\mu t}},$$

where the decomposition of the right side is unique. Note that $\{a_{\lambda\mu}(y)\}$ forms a 1-cycle of non-vanishing holomorphic functions on A_0, and hence defines a holomorphic line bundle over A_0. Making use the classical theta function on A_0, we obtain the required entire function θ. *Q.E.D.*

We take $\theta(x)$ in Lemma (2.1), which may be called the theta function associated with D. Choose and fix a linear coordinate system $(x_1, \ldots, x_n)$ of $\mathbf{C}^n$, so that the lifting $\tilde{f} : \mathbf{C} \to \mathbf{C}^n$ of f is expressed as

$$\tilde{f}(z) = (f_1(z), \ldots, f_{n'}(z), 0, \ldots, 0)$$

with entire functions, $f_1(z), \ldots, f_{n'}(z)$, which are linearly independent over $\mathbf{C}$. We define an algebraic jet subbundle $J_k(A)'$ of $J_k(A)$ $(k = 0, 1, \ldots)$ by equations

$$d^i x_j = 0, \qquad 1 \leqq i \leqq k, \quad n' + 1 \leqq j \leqq n$$

(cf. the notion of directed manifolds in [D96]). Then, $X_{n'+1}(f) \subset J_{n'+1}(A)'$, and after Siu-Yeung [SY96] we define the logarithmic jet differential on $J_{n'+1}(A)'$

$$\Theta = \begin{vmatrix} d \log \theta & dx_1 & \cdots & dx_{n'} \\ \vdots & \vdots & \vdots & \vdots \\ d^{n'+1} \log \theta & d^{n'+1} x_1 & \cdots & d^{n'+1} x_{n'} \end{vmatrix}, \tag{2.2}$$

which is well-defined by Lemma (2.1). Denote by the same Θ the restriction of Θ over $X_{n'+1}(f)$:

$$\Theta : X_{n'+1}(f) \to \mathbf{C},$$

which is a rational function with logarithmic poles on fibers over D. Then, taking the derivatives of Θ, we have

$$d^l \Theta : J_l(X_{n'+1}(f)) \to \mathbf{C}, \qquad l = 0, 1, 2, \ldots.$$

Set

$$\begin{aligned} \Lambda_{n'+1,l} &= \Psi_{n'+1,l} \times (\Theta \circ p_{0,l}, d\Theta \circ p_{1,l}, \ldots, d^l \Theta \circ p_{l,l}) \\ &: J_l(X_{n'+1}(f)) \to \mathbf{C}^{n(n'+1)} \times \mathbf{C}^{(n+n(n'+1))l} \times \mathbf{C}^{l+1}, \end{aligned}$$

where $p_{j,l} : J_l(X_{n'+1}(f)) \to J_j(X_{n'+1}(f))$, $0 \leqq j \leqq l$, are the canonical projections. Then we apply the same argument as in the proof of Lemma (1.2) for $\Lambda_{n'+1,l}$ in place of $\Psi_{k,l}$ to deduce

Lemma (2.3). *If* $\dim X_{n'+1}(f) > 0$, *then there is a non-zero holomorphic vector field* v' *on* A *such that* $X_k(f)$, $k \geqq 0$, *are invariant by the translations generated by* v', *and* $v'\Theta \equiv 0$ *on* $X_{n'+1}(f)$.

Now, assume that $f : \mathbf{C} \to A \setminus D$ is not constant. Then, $\dim X_{n'+1}(f) > 0$ for some $n' \geq 1$, and we may use Lemma (2.3). Since $v'\Theta(d^{n'+1} f(z)) \equiv 0$ in z, there are complex numbers $c_1, \ldots, c_{n'}$ such that

$$dv' \log \theta(f(z)) + c_1 df_1(z) + \cdots + c_{n'} df_{n'}(z) \equiv 0.$$

Therefore

$$dv' \log \theta + c_1 dx_1 + \cdots + c_{n'} dx_{n'} \equiv 0$$

on $X_1(f)$. By Lemma (2.3), $X_1(f)$ is invariant by the translations $\mathrm{Exp}(tv')$, $t \in \mathbf{C}$, and so $dv'v' \log \theta \equiv 0$ on $X_1(f)$. Thus, $dv'v' \log \theta(f(z)) \equiv 0$, and then

$$v'v' \log \theta \equiv \text{constant on } X_0(f). \tag{2.4}$$

Note that by Proposition (1.8) and the assumptions, $X_0(f)$ is a translation of a proper semi-Abelian subvariety B of positive dimension, and that v' is tangent to B at all points of B. It follows from (2.4) that $X_0(f) \setminus D$ and $X_0(f) \cap D$ are invariant by the translations generated by v'.

Suppose that $X_0(f) \cap D \neq \emptyset$, and set $D' = X_0(f) \cap D$. By a translation we may assume that $B = X_0(f)$. Then D' is a non-zero divisor of B. Let $\mathrm{St}(D') \subset B$ be the stabilizer of D' in B. Let $g : \mathbf{C} \to (X_0/\mathrm{St}(D')) \setminus (D'/\mathrm{St}(D'))$ be the composition of f and the quotient mapping by the action of $\mathrm{St}(D')$. Then g is not algebraically degenerate. We apply again the above proved for g to deduce the algebraic degeneracy of g. This is a contradiction. So we have proved (i) and (ii).

This completes the proof of the Main Theorem.

§3. Translates of semi-Abelian subvarieties in $A \setminus D$

Let A and D be as in the Main Theorem. Let X be an algebraic subset of A, and let $E(X, D)$ denote the set of all translates of semi-Abelian subvarieties of A which are contained in X and have no intersection with D.

Theorem (3.1). *Let the notation be as above. Then $E(X, D)$ is an algebraic subset such that it decomposes to irreducible components E_i with* $\dim \mathrm{St}(E_i) > 0$.

Remark. In the case where A is an Abelian variety and $D = \emptyset$, this was proved by Kawamata [K80], and when A is a semi-Abelian variety and $D = \emptyset$, it was proved by [N81]. Vojta [V95] generalized it to the case of A and D as in the above Theorem (3.1). By making use of the results of §§1 and 2, we are going to give another proof, which is totally different to Vojta's [V95]. Abramovich [A94] gave also a proof for the case of $D = \emptyset$, which works over fields of arbitrary characteristic $\geqq 0$. The way of Abramovich [A94] referring to the result in [N81] would be misleading.

Proof. Let $\pi : \mathbf{C}^n \to A$ be the universal covering of A with semi-lattice Γ. Let $\mathbf{P}(\mathbf{C}^n)$ be the projective space of 1-dimensional linear subspaces $[v]$ $(= \mathbf{C}v)$ with $v \in \mathbf{C}^n \setminus \{O\}$. Set

$$\begin{aligned} &X' = X \setminus D \\ &\mathcal{E} = \{(x, [v]) \in X' \times \mathbf{P}(\mathbf{C}^n); x + \mathbf{C}v \subset X'\}, \\ &\mu : (x, [v]) \in \mathcal{E} \to x \in X'. \end{aligned}$$

Since D is a Cartier divisor, $\mathcal{E}$ is closed in $X' \times \mathbf{P}(\mathbf{C}^n)$, so that μ is proper. We use $\Psi_{0,l}$ in (1.5) with taking $X_0(f) = X$, and θ in (2.4). It follows from the proofs of Lemmas (1.2) and (2.3) that $\mathcal{E}$ is the set of points $(x, [v]) \in X' \times \mathbf{P}(\mathbf{C}^n)$ defined by algebraic equations

$$v \in \cap_{l \geqq 1} \mathrm{Ker}\, d\Psi_{0,l}(x), \quad \text{and} \quad v^k \log \theta(x) = 0, \quad k \geqq 3,$$

where v is identified with a holomorphic vector field on A and v^k stands for the k-th derivation in the direction v. Thus $\mathcal{E}$ is algebraic, and so is $E(X, D) = \mu(\mathcal{E})$ in X'.

By making use of the countability of semi-Abelian subgroups of A and the Baire's category theorem we infer that $\dim \mathrm{St}(E_i) > 0$ for every irreducible component E_i of $E(X, D)$ (cf. the proof of [N81], Lemma (4.1)). *Q.E.D.*

Example. 1) Let A be an Abelian variety and let D be a divisor. Then $E(A, D) \subsetneqq A$ if and only if D is ample; moreover, if D is ample, $E(A, D) = \emptyset$ (cf., e.g., [W58]).

2) Let $A = (\mathbf{C}^*)^t$ with coordinates $(z_1, \ldots, z_t)$ and let D be defined by

$$z_1 + \cdots + z_t = 1.$$

Let $f : \mathbf{C} \to A \setminus D$ be a holomorphic curve. It follows from Borel's theorem that after a change of order of coordinates

$$\begin{array}{ccc} f_1 \sim \cdots \sim f_{j_1} \sim 1 & \text{and} & f_1 + \cdots + f_{j_1} = 1 \\ f_{j_1+1} \sim \cdots \sim f_{j_2} & \text{and} & f_{j_1+1} + \cdots + f_{j_2} = 0 \\ \cdots & \cdots & \cdots \\ f_{j_k+1} \sim \cdots \sim f_n & \text{and} & f_{j_k+1} + \cdots + f_n = 0, \end{array}$$

where $f_1 \sim f_2$ means the constancy of f_1/f_2. Thus $E(A, D)$ consists of this kind of translates of subgroups.

See [N81] and [V95] for more examples.

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