FUCHSIAN GROUPS, QUASICONFORMAL GROUPS, CONICAL LIMIT SETS

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Abstract We construct examples showing that the normalized Lebesgue measure of the conical limit set of a uniformly quasiconformal group acting discontinuously on the disc may take any value between zero and one. This is in contrast to the cases of Fuchsian groups acting on the disc, conformal groups acting discontinuously on the ball in dimension three or higher, uniformly quasiconformal groups acting discontinuously on the ball in dimension three or higher, and discrete groups of biholomorphic mappings acting on the ball in several complex dimensions. In these cases the normalized Lebesgue measure is either zero or one.

1. Introduction

A classical result of Hopf says that the normalized one-dimensional Lebesgue measure of the conical limit set of any finitely generated Fuchsian group is either zero or one. This has been extended to infinitely generated Fuchsian groups, to conformal and to uniformly quasiconformal groups acting discontinuously on the unit ball in (real) dimension three or higher, and to discrete groups of complex hyperbolic isometries of the unit ball in several complex variables. In this article it is shown that this dichotomy does not hold for uniformly quasiconformal groups acting on the unit disc in the complex plane. For each λ with $0 \leq \lambda \leq 1$ we explicitly construct a quasiconformal group, acting on the disc, whose conical limit set has normalized one-dimensional Lebesgue measure equal to λ. The main step is to construct a doubling measure supported on the conical limit set of a certain Fuchsian group, where this conical limit set has Lebesgue measure zero.

A group G of homeomorphisms of the unit disc, in dimension two, or of the unit ball, in dimension greater than two, is said to act *discontinuously* if each point in the disc or ball has a neighbourhood U such that only finitely many of the images $g(U)$, $g \in G$, intersect U. The *limit set* $L(G)$ of such a group is the set of accumulation points of the orbit of the origin under the action of G. The same limit set is obtained using any point in the disc in place of the origin. The discontinuity of G implies that $L(G)$ is a subset of the unit circle or sphere. A point $x \in L(G)$ is a *conical limit point* of G if there is a sequence of orbit points which converges to x inside a Euclidean cone with vertex at x, axis perpendicular to the circle or sphere, and opening angle less than $\pi/2$, so that the sides of the cone are not tangent to the circle or sphere. The *conical limit set* $L_c(G)$ of G is the set of conical limit points of G.

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A *Fuchsian group* is a group of Möbius transformations which acts discontinuously on the unit disc $\mathbb{D}$ in the complex plane. For simplicity we assume that the group has no elliptic elements. Hopf [H] showed that the conical limit set of a finitely generated Fuchsian group has normalized Lebesgue measure zero or one. This depends on whether the Poincaré series for the group converges or diverges at the exponent 1:

$$|L_c(G)| = \begin{cases} 0, & \text{if } \sum_{g\in G}(1-|g(0)|) < \infty; \\ 1, & \text{if } \sum_{g\in G}(1-|g(0)|) = \infty; \end{cases} \tag{1.1}$$

the result also holds for infinitely generated Fuchsian groups.

For Fuchsian groups, and for groups of Möbius transformations acting discontinuously on the n-dimensional ball when $n \geq 3$, there are several dichotomies which are equivalent to the zero-one dichotomy for the normalized $(n-1)$-dimensional measure of the conical limit set. The equivalence of the following conditions was proved by Sullivan [S1, S2]:

1) $|L_c(G)| = 0$;
2) The Poincaré series converges at the exponent $n-1$: $\sum_{g\in G}(1-|g(0)|)^{n-1} < \infty$;
3) Green's function for the Laplace-Beltrami operator exists for the hyperbolic manifold $\mathbb{B}^n/G$; and
4) the geodesic flow on $\mathbb{B}^n/G$ is transient.

Otherwise, the conical limit set has full Lebesgue measure in the $(n-1)$-sphere; the Poincaré series diverges at the exponent $n-1$; there is no Green's function for $\mathbb{B}^n/G$ (that is, the integral of the heat kernel for $\mathbb{B}^n/G$ does not converge); and the geodesic flow on $\mathbb{B}^n/G$ is recurrent and completely ergodic. See Nicholls' book [N2] for further information. Analogous results hold in several complex variables [Kam1, Kam2, MW].

A homeomorphism $\varphi : \Omega \to \Omega'$ of complex domains is called K-*quasiconformal* if it is in the Sobolev class $W^{1,2}_{\text{loc}}$ and its directional derivatives satisfy

$$\max_{\alpha} |\partial_\alpha \varphi(z)| \leq K \min_{\alpha} |\partial_\alpha \varphi(z)| \tag{1.2}$$

for almost every $z \in \Omega$. Geometrically, this means that for almost every $z \in \Omega$, infinitesimal circles centred at z are mapped by φ to infinitesimal ellipses centred at $\varphi(z)$, whose eccentricities are uniformly bounded below by $1/K$. A *quasiconformal group* is a group of K-quasiconformal maps acting discontinuously on the disc, for some fixed $K \geq 1$.

One can define quasiconformal mappings in dimension $n \geq 3$ by generalizing the geometric definition above, using spheres and ellipsoids instead of circles and ellipses. Garnett, Gehring, and Jones [GGJ], and independently Tukia [T3], showed that results analogous to Hopf's on the measure of the conical limit set and the convergence of the Poincaré series at the exponent $n-1$ hold for uniformly quasiconformal groups when the dimension is at least three: If G is a group of K-quasiconformal mappings acting discontinuously on the n-dimensional unit ball, where $n \geq 3$, then

$$|L_c(G)| = \begin{cases} 0, & \text{if } \sum_{g\in G}(1-|g(0)|)^{n-1} < \infty; \\ 1, & \text{if } \sum_{g\in G}(1-|g(0)|)^{n-1} = \infty. \end{cases} \tag{1.3}$$

In this article we show that, in sharp contrast to the higher dimensional case, there is no zero-one dichotomy for the normalized Lebesgue measure of the conical limit set of a quasiconformal group acting on the two-dimensional unit disc.

Theorem 1.1. *For each number $\lambda \in [0,1]$, there is a quasiconformal group Γ acting on the unit disc such that the normalized one-dimensional Lebesgue measure of the conical limit set of Γ is λ.*

A *doubling measure* μ on the circle is a positive measure satisfying $\mu(\widetilde{I}) \leq c\,\mu(I)$, for some constant $c > 0$, whenever I is an arc of the circle and $\widetilde{I}$ is the arc with the same centre and twice the length. Equivalently, $\mu(I) \leq c'\mu(J)$ for some uniform constant $c' > 0$ whenever I and J are adjacent arcs of equal length.

Theorem 1.1 is a consequence of the following result:

Theorem 1.2. *There exist a Fuchsian group G and a doubling measure μ such that the conical limit set of G has one-dimensional Lebesgue measure zero, and μ is supported on the conical limit set of G.*

Our result is perhaps rather surprising given that Sullivan [S2] and independently Tukia [T1] have proven that every K-quasiconformal group acting on the disc is conjugate by a quasiconformal mapping to a Möbius group. The corresponding statement is false in higher dimensions [T2]. On the other hand, boundary values of quasiconformal homeomorphisms of the disc (in other words, quasisymmetric maps) have much less regularity than is true in higher dimensions. If $\varphi : \mathbb{B}^n \to \mathbb{B}^n$ is K-quasiconformal and $n \geq 3$, the restriction of φ to $\partial\mathbb{B}^n$ is also K-quasiconformal and hence lies in the Sobolev space $W^{1,n-1}(\partial\mathbb{B}^n)$. In particular $\varphi|_{\partial\mathbb{B}^n}$ takes sets of $(n-1)$-dimensional Lebesgue measure zero to sets of $(n-1)$-dimensional Lebesgue measure zero. Quasisymmetric mappings do not have this property. It is this lack of regularity for quasisymmetric mappings that we exploit to construct our examples.

Theorem 1.1 is an easy consequence of Theorem 1.2. This argument, carried out in Section 3, uses little more than the Beurling-Ahlfors theorem: A homeomorphism f of the circle is the restriction of a quasiconformal mapping of the disc to itself if and only if the distributional derivative $\frac{\partial f}{\partial \theta}$ on the circle is a doubling measure.

Theorem 1.2 gives some information about the possible geometry of limit sets and of sets which support doubling measures. Roughly speaking, the support of a doubling measure must be rather evenly distributed. We give a Fuchsian group whose conical limit set is evenly enough distributed to support a doubling measure, even though it has zero Lebesgue measure. (In fact, the Hausdorff dimension of this conical limit set is strictly less than one.) We construct a doubling measure that is tailored so its support is in this small set. Such constructions were first carried out by Kahane [K], and our method follows his philosophy, though the details are necessarily more complicated.

As José Fernández has pointed out to us, the Patterson-Sullivan measure associated to the group G is a natural candidate for the doubling measure of Theorem 1.2. Unfortunately, certain technical difficulties are encountered in this approach, and we have not yet understood how to overcome them.

It is interesting to note that the result of Theorem 1.2 cannot be achieved if the quasisymmetric map associated to the doubling measure μ is in the Teichmüller space of the group G, or if G is finitely generated. The first statement follows from the fact that the existence of Green's function is a quasiconformal invariant. If G is finitely generated and of the first kind, then $L_c(G)$ has full measure in the unit circle. If G is finitely generated and of the second kind, then $\partial\mathbb{D} \setminus L_c(G)$ contains an arc, and so $L_c(G)$ cannot support a doubling measure.

Section 4 contains a rather full outline and summary of our proof of Theorem 1.2. The rest of the paper is organized as follows.

In Section 2 we give some definitions and notation. In Section 3, we prove Theorem 1.1 as a corollary of Theorem 1.2. In Section 5 we establish some basic properties of the Fuchsian group G used in our proof of Theorem 1.2. In Section 6, we prove that a certain simple construction yields doubling measures which are supported on small sets. Sections 7 to 11 contain the construction of our doubling measure, up to the specification of some parameters. In Section 7 we fix a set of fundamental domains in the universal cover $\mathbb{D}$ of the Riemann surface $\mathbb{D}/G$. In Section 8 we define a collection of Whitney intervals in the Riemann surface, and establish some properties of its lift to the universal cover. In Section 9 we construct a grid of intervals in the circle, on which our doubling measure will be built. Section 10 contains estimates which control the geometric distortion caused by the covering map. In Section 11 we define a sequence of density functions which, by the lemma in Section 6, yields a doubling measure. In Section 12 we define some auxiliary functions and outline the rest of the proof. Sections 13 and 14 contain estimates of the expectation and second moment, with respect to the doubling measure, of the auxiliary functions. In Section 15, an argument involving a constrained random walk on the Riemann surface, together with the estimates of the previous two sections, proves that the measure is supported on the conical limit set of G.

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2. Definitions and Notation

The conical limit set is also known as the *radial* or *non-tangential* limit set. Hedlund [He] introduced conical limit points in connection with his study of horocyclic transitive points. The conical limit set may be characterized in terms of spherical caps. A *spherical cap* on a point x in the unit disc $\mathbb{D}$ is an arc of the unit circle $\partial\mathbb{D}$ of the form

$$\mathrm{Cap}\,(x,a) = \big\{y \in \partial\mathbb{D} \;\big|\; |y-x| \le a\,(1-|x|)\big\}, \tag{2.1}$$

where $1 < a < \infty$. A *non-tangential cone* on a point $y \in \partial\mathbb{D}$ is a subset of $\mathbb{D}$ of the form

$$\mathrm{Cone}\,(y,b) = \big\{x \in \mathbb{D} \;\big|\; |y-x| \le b\,(1-|x|)\big\}, \tag{2.2}$$

where $1 < b < \infty$. Clearly $y \in \mathrm{Cap}\,(x,a)$ if and only if $x \in \mathrm{Cone}\,(y,a)$. Non-tangential cones are comparable to the Euclidean cones discussed in the introduction, in the sense that each such Euclidean cone contains a non-tangential cone and is contained in a non-tangential cone. It follows that y is the non-tangential limit of $\{g_j(0)\}_{j=1}^{\infty}$ if and only if there is an a such that y lies in infinitely many spherical caps $\mathrm{Cap}\,(g_j(0),a)$. Taking a countable union over wider and wider opening angles of the cones, the conical limit set is given by

$$L_c(G) = \bigcup_{l=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \mathrm{Cap}\,(g_j(0),l). \tag{2.3}$$

A set K in the complex plane is *uniformly perfect* if there is a constant $c > 0$ such that for each $z_0 \in K$ and for all r such that $0 < r < \operatorname{diam} K$,

$$K \cap \{z \mid cr \leq |z - z_0| \leq r\} \neq \emptyset. \tag{2.4}$$

(Here $\operatorname{diam} K$ is the diameter of K.) In other words, there is an upper bound on the moduli of annuli lying in the complement of K.

The *Poincaré series* for a discrete group G is the series

$$\sum_{g \in G} (1 - |g(0)|)^s, \tag{2.5}$$

where s is positive. There is a critical exponent $\delta = \delta(G)$, called the *exponent of convergence*, such that the series converges for all $s > \delta$ and diverges for all $s < \delta$.

Let Ω be a domain in $\overline{\mathbf{C}}$, let E be a Borel subset of $\partial\Omega$, and let z be a point in Ω. The *harmonic measure at z of E in Ω* is the Perron solution $w(z) = w(z, E, \Omega)$ of the Dirichlet problem in Ω for the boundary values $\mathbf{1}_E$. As a function of z, $w(z, E, \Omega)$ is harmonic in Ω. For fixed z, $w(z, E, \Omega)$ is a probability measure on $\partial\Omega$; $w(z, E, \Omega)$ is the probability that a Brownian traveller from z first hits $\partial\Omega$ in the set E. Harmonic measure is monotonic in the domain Ω: if $z \in \Omega_1 \subset \Omega_2$ and $E \subset \partial\Omega_1 \cap \partial\Omega_2$, then $w(z, E, \Omega_1) \leq w(z, E, \Omega_2)$. It is also monotonic in E: if $E_1 \subset E_2 \subset \partial\Omega$, then $w(z, E_1, \Omega) \leq w(z, E_2, \Omega)$.

The hyperbolic metric on the disc $\mathbb{D}$ is given by the element of arclength $ds = \frac{2}{1-|z|^2}|dz|$. The Möbius transformations $g : \mathbb{D} \to \mathbb{D}$ are the isometries for this metric. They are of the form

$$g(z) = e^{i\theta} \, \frac{z - z_0}{1 - \overline{z}_0 z} \tag{2.6}$$

where $\theta \in [0, 2\pi]$ and $z_0 \in \mathbb{D}$. The hyperbolic geodesics are the *orthocircular arcs*: that is, arcs of circles which meet the unit circle $\partial\mathbb{D}$ at right angles. The hyperbolic distance from 0 to $a \in \mathbb{D}$ is

$$d_{\text{hyp}}(0, a) = \int_0^{|a|} \frac{2}{1 - |z|^2} \, |dz| = \log \left[\frac{1 + |a|}{1 - |a|} \right]. \tag{2.7}$$

Let Ω be a domain in $\overline{\mathbf{C}}$ whose universal covering space is the disc, and let $\pi : \mathbb{D} \to \Omega$ be the covering map. The hyperbolic metric on $\mathbb{D}$ can be projected via π to a metric on Ω given by $\lambda_\Omega(w)|dw| = \lambda_\Omega(\pi(z))|\pi'(z)||dz| = \frac{2}{1-|z|^2}|dz|$, where $w = \pi(z)$. The boundary $\partial\Omega$ of Ω is uniformly perfect if and only if there is a constant $c_\Omega > 0$ such that the function λ_Ω satisfies

$$\frac{c_\Omega}{\operatorname{dist}(w, \partial\Omega)} \leq \lambda_\Omega(w) \leq \frac{2}{\operatorname{dist}(w, \partial\Omega)}. \tag{2.8}$$

See [BP, Po2]. Here dist denotes Euclidean distance.

In dimension two, there is a connection between quasiconformal maps and doubling measures. Let $\varphi : \mathbf{C} \to \mathbf{C}$ be a K-quasiconformal map which preserves the upper half plane. Then φ maps $\mathbb{R}$ to itself, and the restriction f of φ to $\mathbb{R}$ is an increasing homeomorphism which satisfies

$$\frac{1}{M} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq M \tag{2.9}$$

with a constant M depending on K, for all $x \in \mathbb{R}$ and all $t > 0$. Functions with this property are called *M-quasisymmetric*. Quasisymmetry is a necessary and sufficient condition for an increasing homeomorphism f of $\mathbb{R}$ to be the boundary values of a quasiconformal mapping which preserves the upper half plane [BA]. Note that different quasiconformal mappings may have the same boundary values.

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism. Define a measure μ on $\mathbb{R}$ by setting $\mu([a, b]) = f(b) - f(a)$ for intervals $[a, b]$. Then f is M-quasisymmetric if and only if μ satisfies $1/M \leq \mu(I)/\mu(J) \leq M$ whenever I and J are adjacent intervals of equal length. In other words, f is M-quasisymmetric if and only if μ is doubling.

To summarize, there is a many-to-one correspondence between quasiconformal self-mappings of the upper half plane and doubling measures on the real line. There is a similar correspondence between quasiconformal self-mappings of the disc and doubling measures on the circle.

Some references for the material above are [N1] and [L].

Two quantities A and B are *comparable*, denoted $A \sim B$, if there is a constant $c > 0$ such that $\frac{1}{c}B \leq A \leq cB$. We also write $A \overset{c}{\sim} B$ if A and B are comparable with constant c. Throughout the paper we normalize Lebesgue measure so that the unit n-sphere has measure one.

In the figures we denote the unit circle $\partial\mathbb{D}$ by S^1.

3. Proof of Theorem 1.1 assuming Theorem 1.2

Fix a number λ with $0 \leq \lambda \leq 1$. Let μ be the measure given by Theorem 1.2, and take $c > 0$ such that $\mu(2I) \leq c\,\mu(I)$ for all arcs I in the unit circle $\partial\mathbb{D}$. Set $\nu(\cdot) = \lambda\mu(\cdot) + (1-\lambda)|\cdot|$, where $|\cdot|$ denotes normalized Lebesgue measure on $\partial\mathbb{D}$. Then ν is a doubling measure on the circle, since for each arc I in $\partial\mathbb{D}$,

$$\begin{aligned} \nu(2I) &= \lambda\,\mu(2I) + (1-\lambda)\,|2I| \\ &\leq c\,\lambda\,\mu(I) + 2\,(1-\lambda)\,|I| \\ &\leq \max\,(c,2)\,\nu(I). \end{aligned} \tag{3.1}$$

The measure ν assigns the fraction λ of its mass to the conical limit set of G.

Define $f : \partial\mathbb{D} \to \partial\mathbb{D}$ so that $\nu([a,b]) = f(b) - f(a)$ for all arcs $[a,b]$ in $\partial\mathbb{D}$. f is a quasisymmetric mapping of the circle to itself, and $|f(E)| = \nu(E)$ for any measurable subset E of the circle. The map f may be extended to a quasiconformal mapping φ of the unit disc onto itself [BA, DE] and further to a quasiconformal self-mapping of $\overline{\mathbf{C}}$. Conjugating the Fuchsian group G of Theorem 1.2 by φ yields a quasiconformal group $\Gamma = \varphi \circ G \circ \varphi^{-1}$ which acts on the disc.

It remains to show that the conical limit set of Γ is the image under φ of the conical limit set of the Fuchsian group G. This is a special case of a much more general fact; see [M] for example. We give a direct proof. It is sufficient to show that a quasiconformal map $\varphi : \mathbb{D} \to \mathbb{D}$ maps any non-tangential cone into another non-tangential cone, perhaps with larger opening angle. Since φ^{-1} is also quasiconformal, this implies that $L_c(\Gamma) = \varphi(L_c(G))$.

Define the *cross ratio* of any four distinct points α, β, γ, and δ in $\overline{\mathbf{C}}$ by

$$\tau = |\alpha, \beta, \gamma, \delta| = \frac{|\alpha - \gamma|}{|\alpha - \delta|} \cdot \frac{|\beta - \delta|}{|\beta - \gamma|}, \tag{3.2}$$

with the convention that if $\delta = \infty$,

$$|\alpha, \beta, \gamma, \infty| = \frac{|\alpha - \gamma|}{|\beta - \gamma|}. \tag{3.3}$$

Let $\varphi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be K-quasiconformal. Let $\varphi(\tau) = |\varphi(\alpha), \varphi(\beta), \varphi(\gamma), \varphi(\delta)|$. Väisälä has shown [V] that there is an increasing function $\Phi_K : (0,\infty) \to (0,\infty)$, depending only on K, such that $\varphi(\tau) \leq \Phi_K(\tau)$ for all τ.

Let z be a point in $\mathrm{Cone}\,(y,a) = \{z \in \mathbb{D} \mid |z-y| \leq a\,(1-|z|)\}$, where y is a point in $\partial\mathbb{D}$ and $1 < a < \infty$. Let w be the point in $\partial\mathbb{D}$ such that $1 - |\varphi(z)| = |\varphi(w) - \varphi(z)|$.

Suppose $\varphi^{-1}(\infty)=\infty$. Then

$$\begin{aligned}\frac{|\varphi(y)-\varphi(z)|}{1-|\varphi(z)|} &= \frac{|\varphi(y)-\varphi(z)|}{|\varphi(w)-\varphi(z)|}\\ &= |\varphi(y),\varphi(w),\varphi(z),\infty|\\ &\le \Phi_K(|y,w,z,\infty|)\\ &= \Phi_K\left(\frac{|y-z|}{|w-z|}\right)\\ &\le \Phi_K\left(\frac{|y-z|}{1-|z|}\right)\\ &\le \Phi_K(a).\end{aligned} \tag{3.4}$$

Here the second last inequality holds because $|w-z|\ge 1-|z|$ and Φ_K is increasing. Therefore $\varphi(z)$ lies in Cone $(\varphi(y),\Phi_K(a))$.

Now suppose $\varphi^{-1}(\infty)\ne\infty$. Let w and y be any two points in the unit circle. Since $\varphi^{-1}(\infty)$ is not in the closed unit disc, we have

$$|w-\varphi^{-1}(\infty)|\le\frac{2+d}{d}\,|y-\varphi^{-1}(\infty)| \tag{3.5}$$

where $d=\operatorname{dist}(\varphi^{-1}(\infty),\overline{\mathbb{D}})>0$. Then

$$\begin{aligned}\frac{|\varphi(y)-\varphi(z)|}{1-|\varphi(z)|} &= |\varphi(y),\varphi(w),\varphi(z),\infty|\\ &\le \Phi_K(|y,w,z,\varphi^{-1}(\infty)|)\\ &= \Phi_K\left(\frac{|y-z|}{|w-z|}\cdot\frac{|w-\varphi^{-1}(\infty)|}{|y-\varphi^{-1}(\infty)|}\right)\\ &\le \Phi_K\left(\frac{|y-z|}{1-|z|}\cdot\frac{2+d}{d}\right)\\ &\le \Phi_K(a\cdot\frac{2+d}{d}).\end{aligned} \tag{3.6}$$

Hence $\varphi(z)$ lies in Cone $(\varphi(y),\Phi_K(a\cdot\frac{2+d}{d}))$.

It follows that $L_c(\Gamma)=\varphi(L_c(G))$, and $|L_c(\Gamma)|=|\varphi(L_c(G))|=\nu(L_c(G))=\lambda$. In other words Γ is a quasiconformal group, acting on the unit disc, whose conical limit set supports the fraction λ of the mass of Lebesgue measure on the unit circle. This completes the proof of Theorem 1.1. $\square$

4. Outline of the proof of Theorem 1.2

Theorem 1.2. *There exist a Fuchsian group G and a doubling measure such that the conical limit set $L_c(G)$ of G has one-dimensional Lebesgue measure zero, and the doubling measure is supported on $L_c(G)$.*

In this section we present an outline of the proof of Theorem 1.2. Let $K \subset [0,1]$ be the classical ternary Cantor set, and let $\Omega = \overline{\mathbb{C}} \setminus K$ be the complement of K in the extended complex plane. Let G be the Fuchsian group, acting on the disc $\mathbb{D}$, which uniformizes Ω; in other words G is the covering group of Ω and $\mathbb{D}/G$ is conformally equivalent to Ω. Let $\pi : \mathbb{D} \to \Omega$ be the covering map, normalized so that $\pi(0) = \infty$. The conical limit set $L_c(G)$ is the set of non-tangential accumulation points of the orbit $\{g(0)\}_{g\in G}$. For this group G, the conical limit set has measure zero. (See Section 5.) Our aim is to construct a doubling measure which is supported on $L_c(G)$.

We begin with a model example of a doubling measure μ, obtained as the limit of a sequence of measures μ_n of the form $d\mu_n = F_n \cdots F_1\, dx$, such that μ is supported on a set of Lebesgue measure zero. This type of example is due to Kahane [K]. Divide the unit interval I_0 into 5 equal subintervals $I_{1,1}, \ldots, I_{1,5}$, $1 \le k \le 5$, numbered from left to right, and let

$$F_1(x) = \begin{cases} 1, & \text{for } x \in I_{1,1} \cup I_{1,5}; \\ 1/2, & \text{for } x \in I_{1,2} \cup I_{1,4}; \text{ and} \\ 2, & \text{for } x \in I_{1,3}. \end{cases} \tag{4.1}$$

Then

- The large fraction $2/5$ of the area under the graph of F_1 lies above the middle fifth $I_{1,3}$ of I_0.
- F_1 has mean value one on I_0;
- F_1 is a positive function whose values lie between $1/2$ and 2; and
- $F_1 \equiv 1$ on subintervals of length $|I_0|/5$ at each end of I_0.

Let $F_n(x) = F_{n-1}(5x \bmod 1)$, for $n \ge 2$. On each of the 5^{n-1} subintervals $I_{n-1,j}$ of I_0 of length $5^{-n+1}|I_0|$, F_n is a dilation of F_1, with the same properties on $I_{n-1,j}$ as F_1 has on I_0. We refer to the intervals $I_{n,j}$ as 5*-ary intervals*. The effect of F_n is to concentrate the large fraction $2/5$ of the mass assigned to each $I_{n-1,j}$ onto the middle fifth of $I_{n-1,j}$.

The measure μ_n given by $d\mu_n = F_n \cdots F_1\, dx$ satisfies $1/4 \le \mu_n(I)/\mu_n(J) \le 4$ for all adjacent 5-ary intervals I and J of equal length. Therefore μ_n is doubling. The measures μ_n converge weakly to a unique limit μ, which is also doubling. The support of this measure is a set E of Hausdorff dimension less than one, concentrated near the centres of the 5-ary intervals which appear in the construction. (More precisely, this model set E consists of those points x in I_0 such that the proportions of 0's, 1's, 2's, 3's, and 4's in the base five expansion of x are asymptotically equal to $1/5$, $1/10$, $2/5$, $1/10$, and $1/5$ respectively. Its Hausdorff dimension is $1 - \frac{1}{5}\frac{\log 2}{\log 5} \approx 0.914$.)

It is convenient to recast this construction in terms of a random walk on a tree. We use a 5-ary tree as a discrete model of the disc. Put a vertex at the origin, with five equally spaced edges emanating from it. Put a vertex at the end of each edge; call these the *children* of the original vertex. Add five more edges emanating from each new vertex.

Repeat, with the new edges always pointing outwards towards the circle. Each branch of the tree corresponds to a point of the circle; one can think of the tip of the branch landing somewhere on the circle.

We distinguish a proper subset V of vertices of the tree. Each vertex v in the tree has five children, $v_1, \ldots, v_5$; let V consist of all middle children v_3.

Consider a particle performing a random walk on the vertices of the tree, with the following constraints:

. the particle may jump from one vertex to another if and only if the second is a child of the first: in other words they are connected by an edge and the second vertex is *below* the first (i.e. closer to the circle); and
. when the particle is at the vertex v, it jumps to the children v_k, $1 \leq k \leq 5$, of v with *jump probabilities* p_k, $1 \leq k \leq 5$, respectively, where $p_1 = p_5 = 1/5$, $p_2 = p_4 = 1/10$, and $p_3 = 2/5$.

The particle always moves away from the origin, towards the tips of the branches.

We recover the 5-ary intervals by projecting the tree radially outwards onto the circle, so that each vertex corresponds to an interval in the circle. (The construction of the measure μ given above can be done on the unit circle instead of the unit interval.) Let the origin correspond to the whole circle. Project the five vertices which are the children of the origin to five intervals of equal length, which together cover the circle. And so on: for each vertex v, project the children of v to five equal subintervals of the interval corresponding to v. The collection of all these intervals forms a 5-adic *grid* of nested intervals in the circle; we refer to these intervals as *grid intervals.* We say a grid interval is in the n^{th} *layer* of the grid if it is the projection of a vertex which is separated from the origin by n edges in the tree.

Each path which the particle may take, given by a sequence of vertices $v_1 \to v_2 \to \cdots$ in the tree, corresponds to a point x in the circle, and also to a nested, decreasing sequence of grid intervals $I_1 \supset I_2 \supset \cdots \ni x$ in the circle.

Clearly the jump probabilities $\{p_k\}$ and the functions $\{F_n\}$ defined on the grid intervals encode the same information. We can recover the functions $\{F_n\}$ as follows. Let v be a vertex which is $n-1$ edges away from the origin. Let v_k, $1 \leq k \leq 5$, be the children of v, let p_k be the probability of jumping from v to v_k, and let I and I_k be the grid intervals which are the projections of v and v_k respectively. Then, on I, F_n is simply the function which is constant on each I_k and satisfies

$$p_k = \mathrm{Prob}\,(v \to v_k) = \frac{\int_{I_k} F_n\,dx}{\int_I F_n\,dx} \tag{4.2}$$

for $1 \leq k \leq 5$.

In this setting, the measure μ which is the limit of the μ_n's defined by $d\mu_n = F_n \cdots F_1\,dx$ is the hitting measure for the random walk on the tree: the measure of a subset A of the circle is exactly the probability that a particle which starts from the origin will hit the circle somewhere in A. Note that the support of μ contains those points in the circle $\partial\mathbb{D}$ which correspond to paths in the tree which pass through infinitely many vertices from V.

As noted above, the support of μ is a small set E which is distributed in a very regular way. If we had a Fuchsian group whose conical limit set were exactly this set E, then we would have established Theorem 1.2. However, it is not apparent how to build a Fuchsian group with pre-determined conical limit set. Instead, we start with the particular Fuchsian group described above, and modify the Kahane-type construction to produce a doubling measure whose support E is contained in $L_c(G)$.

In Section 6 we generalize the Kahane-type construction, showing that one obtains a doubling measure under rather mild conditions on the functions $\{F_n\}$. In particular, the 5-adic grid of intervals in the example above may be replaced by one where each interval $I_{n-1,j}$ at the $(n-1)^{\text{th}}$ level is divided into finitely or infinitely many intervals $I_{n,k}$ at the n^{th} level, such that F_n is constant on each $I_{n,k}$. It suffices (Lemma 6.3) to assume that

. F_n has mean value one on each $I_{n-1,j}$ at the $(n-1)^{\text{th}}$ level;
. the functions F_n are positive and uniformly bounded away from zero and infinity; and
. $F_n \equiv 1$ on subintervals at each end of each interval $I_{n-1,j}$, whose lengths are at least some definite fraction (independent of n and j) of the length of $I_{n-1,j}$.

Then the measures μ_n given by $d\mu_n = F_n \cdots F_1 \, dx$ converge to a doubling measure μ.

The idea is now to distinguish a collection V of grid intervals in a new, less regular grid, such that points in $\partial\mathbb{D}$ which lie in infinitely many of the intervals in V are actually conical limit points, and to define the functions F_n so that they satisfy the three hypotheses above, are large on the intervals in V, and are small elsewhere. The result is a doubling measure which assigns full measure to $L_c(G)$.

In the setting of random walks on trees, our goal is to define a tree in the disc, in which the orbit points $\{g(0)\}_{g\in G}$ appear as a proper subset V of the vertices, and to define the jump probabilities for the edges in this tree so that the subset of paths which hit V infinitely often carries full measure. Here the measure on the space of paths is that which gives, to each family of paths with a common initial segment, measure equal to the product of the jump probabilities for the edges in that segment.

The branches of the tree which contain infinitely many vertices from V correspond to points in the circle which lie in the limit set of G. This is almost by definition: for each such point x in the circle there is a sequence of infinitely many orbit points which approaches x. When we implement this abstract procedure by specifying the vertices and edges in the tree, we will get something more: these points x will actually be *conical* limit points of G. This is shown in Section 15.

We make a minor simplification of the goal stated above. We impose on the particle the additional constraint that:

. whenever the particle reaches a vertex in V, it stops there.

Suppose the particle is started at some vertex v, with this stopping condition. Let $V(v)$ be the subset of vertices in V which can be reached from v. In other words, $V(v)$ consists of those $w \in V$ below v such that there are no other vertices from V between v and w. Then it suffices to define the jump probabilities so that, from each starting vertex v, the particle reaches $V(v)$ with probability one. We show below how to do this; then by removing the stopping condition we obtain the measure we require.

In the setting of the grid of intervals, this simplification means that it is sufficient to define the functions F_n so that, beginning with Lebesgue measure on any grid interval I, we obtain a doubling measure supported on the maximal grid intervals in I which correspond to orbit points $g(0)$ in the tree.

In the rest of this section we describe:

. a tree in the disc, in which the orbit points $\{g(0)\}_{g\in G}$ appear as a proper subset V of the vertices;
. the adjacencies in the tree (which pairs of vertices are connected by edges);
. a projection P from vertices to grid intervals in the circle;
. a definition of F_n (equivalently, of the jump probabilities); and
. a model calculation showing that this definition of F_n implies that the resulting measure μ is supported on $L_c(G)$.

We now implement the scheme described above, making extensive use of the well-understood geometry of the complement of the Cantor set K.

Let D be the closure of a simply connected domain in the disc such that $\pi(D)$ is either the upper or lower half plane. We call such a region D a *half fundamental domain* for Ω. The boundary of D consists of a closed subset $\partial D \cap \partial\mathbb{D}$ of the unit circle, and a union of disjoint arcs of circles which meet the unit circle at right angles. We call these *orthocircular arcs*. The covering map π maps $\partial D \cap \partial\mathbb{D}$ onto the Cantor set K, and it maps the orthocircular arcs in ∂D onto the components of $\overline{\mathbb{R}} \setminus K$. The normal fundamental domain $\mathcal{F}$ for Ω is the union of the two half fundamental domains whose boundaries contain the origin. Its images $\{g(\mathcal{F})\}_{g\in G}$ tile the disc. We use instead the tiling in which each fundamental domain $g(\mathcal{F})$, except $\mathcal{F}$ itself, is divided into the two half fundamental domains which it contains.

Divide each orthocircular arc in the boundaries of the half fundamental domains in this tiling into infinitely many intervals, all of comparable hyperbolic length. These intervals will be the vertices of the tree. A convenient way to divide the arcs is to make a Whitney decomposition of the components of $\overline{\mathbb{R}} \setminus K$, in the Riemann surface Ω, and then to pull it back to the disc via the branches of the inverse of the covering map.

To this end, let L be a component of $[0,1] \setminus K$. L is an open interval of length 3^{-k}, for some $k \geq 1$. Partition L into subintervals J so that there are 2^j J's of length $3^{-j}|L|$, for each $j \geq 1$, and so that each J satisfies the following Whitney condition:

$$|J| = 2\ \mathrm{dist}\,(J, K). \tag{4.3}$$

This can be done by putting the two J's of length $|J| = 3^{-1}|L|$ in the middle of L, and then adjoining the smaller J's at each end in decreasing order of size. The Euclidean length of each J is a negative power of 3.

For the component $L = \overline{\mathbb{R}} \setminus [0,1]$ of $\overline{\mathbb{R}} \setminus K$, we make a similar decomposition, except that there is one interval J_∞ which contains the point at infinity. Fix a small positive number σ, and let $J_\infty = \overline{\mathbb{R}} \setminus (-\sigma, 1+\sigma)$. Partition the intervals $(-\sigma, 0]$ and $[1, 1+\sigma)$ into J's satisfying (4.3) so that the length of each J is a negative power of 3 and the lengths decrease as the J's approach $[0,1]$. Make the convention that $|J_\infty| = 1/3$.

Let the vertices of the tree be the images I of these intervals J under all branches of the inverse of the covering map. Also, let V consist of those intervals I in the disc such that $\pi(I) = J_\infty$. The orbit $\{g(0)\}_{g\in G}$ is mapped by the covering map to the point at infinity. So each interval I such that $\pi(I) = J_\infty$ contains an orbit point, and each orbit point appears in the tree as a vertex in the set V.

The random walk on the tree in the disc can be pushed forward by the covering map to a random walk in $\Omega = \overline{\mathbf{C}} \setminus K$, with the intervals J as vertices. Our goal becomes: to define the jump probabilities for this random walk so that if the particle starts at any initial vertex, it reaches J_∞ with probability one.

The idea is that J_∞ is the largest interval (in the Euclidean sense), and we define the $\{p_k\}$ so that from any interval, the most likely outcome is that the particle jumps to a much larger interval (in the Euclidean sense).

For each interval J, we specify the intervals J' to which the particle may jump from J. The particle may jump only to the intervals J' in a particular segment E_J of $\overline{\mathbb{R}}$ near to J. See Figure 4 in Section 9 below. This segment E_J is much larger than J. We define E_J as follows for most grid intervals J, and we call these J *standard*. The Euclidean length of E_J is $|E_J| = 2 \cdot 3^N|J|$, where N is a fixed large integer, *independent of* J. E_J comprises one of the scaled copies of the Cantor set K, of length $3^N|J|$, which arise in the construction of K, together with an adjacent gap in K, also of length $3^N|J|$. By a *gap* in K, we mean an interval which is a component of $\overline{\mathbb{R}} \setminus K$. The segment E_J contains infinitely many intervals J'. Counting, we see that the two largest have Euclidean length $|J'| = 3^{N-1}|J|$, and there are 2^k intervals J' in E_J of length $|J'| = 3^{N-k}|J| = 2^{-1}\cdot 3^{-k}|E_J|$, for each $k \geq 1$.

For non-standard grid intervals J, the segments E_J take different forms. In particular, from some intervals J it is possible to jump directly to J_∞. (Full definitions of the segments E_J for standard and non-standard intervals are made in Section 9.)

There are now infinitely many intervals J' accessible from J. Pulling back to the disc, we obtain a tree in which each vertex has infinite valence. We define the adjacencies in the tree, in other words which pairs of vertices are joined by edges, as follows. Let I be a vertex in the tree, and let $J = \pi(I)$. Pull E_J back to the disc by the branch of π^{-1} which takes J to I. One can show (Section 10) that this pre-image $\pi^{-1}(E_J)$ lies roughly below I, that is, between I and the nearest part of the unit circle. The Whitney intervals J' in E_J are pulled back to vertices $I' = \pi^{-1}(J')$ in $\pi^{-1}(E_J)$. Let these I' be the children of I; put an edge between I and each such I'.

We also use the segments $\pi^{-1}(E_J)$ to define the grid intervals corresponding to the vertices of the tree. We define the projection P from the vertices to the grid intervals as follows. Since the endpoints of E_J lie in K, the endpoints of $\pi^{-1}(E_J)$ lie in the unit circle. Let $P(I)$ be the arc of the unit circle which has the same endpoints as $\pi^{-1}(E_J)$ and which lies below $\pi^{-1}(E_J)$; in other words $P(I)$ is the shorter arc of the unit circle between the endpoints of $\pi^{-1}(E_J)$. This interval $P(I)$ is the grid interval corresponding to the vertex I. The projection P is almost radial, but not quite.

We define some auxiliary functions X_i on the circle which keep track of the size of the Whitney intervals J along a path in Ω travelled by the particle. (See Section 12.) Let I_0 be a grid interval in the circle, corresponding to a vertex v_0 in the tree. Let x be a point in I_0. It corresponds to a path $v_0 \to v_1 \to v_2 \to \cdots$ in the tree; to a nested, decreasing sequence of grid intervals $I_0 \supset I_1 \supset I_2 \supset \cdots \ni x$ in the circle; and to a path $J_0 \to J_1 \to J_2 \to \cdots$ in Ω. Define

$$X_1(x) = \log_3 \left[\frac{|J_1|}{|J_0|} \right]. \tag{4.4}$$

The logarithm is simply in order to make X_1 integer-valued; recall that the Euclidean lengths of the intervals J are all powers of 3. X_1 is positive if and only if J_1 is larger than J_0. Define

$$X_i(x) = \begin{cases} \log_3 \left[\frac{|J_i|}{|J_{i-1}|} \right], & \text{if } J_1, \ldots, J_{i-1} \neq J_\infty; \\ 1, & \text{otherwise,} \end{cases} \tag{4.5}$$

for $i \geq 1$, with the convention that $|J_\infty| = 1/3$. This function measures the difference in size of the intervals where the particle is before and after the i^{th} jump, unless the particle has already reached J_∞. If the particle reaches J_∞, then it stays there, and all subsequent X_i's simply record a 1.

Let

$$S_k(x) = \sum_{i=1}^{k} X_i(x). \tag{4.6}$$

Then $S_k(x) \to +\infty$ if and only if the path in Ω which corresponds to x reaches J_∞ (and therefore stops there).

We wish to ensure that a particle starting from any vertex in the tree reaches V with probability one. Let v be a vertex, and let I_0 be the corresponding grid interval. Let $V(I_0)$ denote the union of the maximal grid intervals in I_0 which correspond to J_∞. These maximal grid intervals are the projections to the circle of those vertices v' in V which can be reached from v by a particle which stops when it first hits V. Then

$$V(I_0) = \{x \in I_0 \bigm| S_k(x) \to +\infty\}, \tag{4.7}$$

and our problem is reduced to defining the functions $\{F_n\}$ in such a way that $S_k(x) \to +\infty$ a.e. $d\mu$ on I_0, where μ is the limit of the measures μ_n, and $d\mu_n = F_n \cdots F_1 \, dx$.

To do this, it is sufficient to define the $\{F_n\}$ so that the following estimates hold:

$$EX_i = \frac{1}{\mu(I)} \int_I X_i(x) \, d\mu \geq c_1, \tag{4.8}$$

and

$$EX_i^2 = \frac{1}{\mu(I)} \int_I X_i^2(x) \, d\mu \leq c_2, \tag{4.9}$$

with uniform positive constants c_1 and c_2, for all grid intervals I in the $(n-1)^{\text{th}}$ layer of the grid, for all $n \geq 1$. Here X_i is the auxiliary function, defined in (4.5) above, for a particle which has not yet reached J_∞ and which makes its $(i-1)^{\text{th}}$ jump from the vertex corresponding to I. In other words, the expectations of the X_i's with respect to μ are uniformly large, and the second moments are uniformly bounded. Then an appeal to the strong law of large numbers for martingales (see Section 15) yields the result that $S_k(x) \to +\infty$ a.e. $d\mu$ on I_0, as required.

In the remainder of this section, we show how to define F_n for those grid intervals I which correspond to standard intervals J. (Full definitions of F_n are given in Section 11, for standard intervals, and in Section 14, for non-standard intervals.) We also give a model calculation for the estimate (4.8) on the expectation of X_i.

Let I be a grid interval, from the $(n-1)^{\text{th}}$ layer of the grid, whose corresponding interval J in $\overline{\mathbb{R}} \setminus K$ is standard. The grid intervals I' in I from the next (n^{th}) layer of the grid correspond to those vertices in the tree which are accessible from the vertex coresponding to I. Therefore these I' are exactly those grid intervals which correspond to the intervals J' in E_J in the Riemann surface Ω. In other words, we pull back the segment E_J from $\overline{\mathbb{R}}$ to the disc via the appropriate branch of the inverse of the covering map, and then project it to the interval I in the circle using the projection P defined above.

The key idea is that the geometry of the segment E_J is not distorted very much by this operation. In fact, we now assume that it is not distorted at all, so that the n^{th} layer of grid intervals in I has exactly the same structure as E_J. Namely, one half of I contains an exact scaled copy $\widetilde{K}$ of the ternary Cantor set K, of length $|I|/2$, and the other half of I corresponds to a gap in K. Again, we are assuming that the intervals I' in $I \setminus \widetilde{K}$ are exactly Whitney intervals with $|I'| = 2\,\mathrm{dist}\,(I', \widetilde{K})$, just like the Whitney intervals J' in E_J. In particular, $I \setminus \widetilde{K}$ contains 2^k intervals I' of length $|I'| = 2^{-1} \cdot 3^{-k}|I|$, for each $k \geq 1$.

Much of the paper (especially in Sections 8, 10, 11, 13, and 14) is devoted to proving estimates which control the distortion of the segments E_J by $P \circ \pi^{-1}$. These estimates rely on the fact that on the boundary of a chord-arc domain, one can use harmonic measure to estimate arclength (via A_∞-equivalence). For the remainder of this section, however, we simply assume that there is no distortion.

Let B_1 denote the union of the two largest grid intervals I' in I from the n^{th} layer of the grid; they satisfy $|I'| = |I|/6$. Fix a small positive number ε, independent of I. Define the function F_n on I so that:

- $F_n \equiv (1-\varepsilon)\,|I|/|B_1|$ on B_1;
- $F_n \equiv 1$ on small segments I_l and I_r at each end of I, where $|I_l|/|I|$ and $|I_r|/|I|$ are fixed quantities independent of I, and the endpoints of I_l and I_r are endpoints of intervals from the n^{th} layer of the grid; and
- $F_n \equiv \delta$ on the rest of I, where δ is a small number chosen to ensure that F_n has mean value 1 on I.

The force of the first condition is that $\mu(B_1)/\mu(I) = 1-\varepsilon$; the effect of F_n is to concentrate almost all the mass of I onto B_1. This is because of the way that μ is constructed: the μ-measure of an interval from the n^{th} layer of the grid is simply its μ_n-measure, since all

F_l with $l > n$ have mean value 1 on the grid interval. So

$$\begin{aligned}
\frac{\mu(B_1)}{\mu(I)} &= \frac{\mu_n(B_1)}{\mu_{n-1}(I)} \\
&= \frac{\int_{B_1} F_n(x)\cdots F_1(x)\,dx}{\int_I F_{n-1}(x)\cdots F_1(x)\,dx} \\
&= (1-\varepsilon)\,\frac{|I|}{|B_1|}\,\frac{\int_{B_1} F_{n-1}(x)\cdots F_1(x)\,dx}{\int_I F_{n-1}(x)\cdots F_1(x)\,dx} \\
&= 1-\varepsilon,
\end{aligned} \tag{4.10}$$

since the product $F_{n-1}\cdots F_1$ is constant on I.

In Sections 11 and 14 we show that these functions F_n are uniformly bounded away from zero and infinity, which is a hypothesis of Lemma 6.3 on doubling measures.

The second condition above is another hypothesis of Lemma 6.3. However, in order to simplify the model calculation below, we temporarily forget that $F_n = 1$ near the ends of I, and assume that $F_n = \delta$ on all of $I \setminus B_1$.

Let

$$\begin{aligned}
B_k &= \{x \in I \mid X_i(x) = N-k\} \\
&= \bigcup \{I' \subset I \mid |I'| = 2^{-1}\cdot 3^{-k}|I|\},
\end{aligned} \tag{4.11}$$

for $k \geq 1$. We know that B_k consists of exactly 2^k intervals I', each of length $2^{-1}\cdot 3^{-k}|I|$, and so

$$|B_k| = \frac{1}{2}\left(\frac{2}{3}\right)^k |I|. \tag{4.12}$$

The function X_i takes the constant value $N-k$ on B_k, for $k \geq 1$. The function F_n is also constant on each B_k; we are assuming that $F_n \equiv \delta$ on $I \setminus B_1$, and forgetting that actually $F_n = 1$ on the parts of some B_k's which are very near the endpoints of I. We know that $\mu(B_k) = \mu_n(B_k)$, and $\mu(I) = \mu_{n-1}(I)$. Therefore, in computing the ratio $\mu(B_k)/\mu(I)$, we need only take account of the value of F_n on B_k; the F_l's with $l \neq n$ are irrelevant. We see that $\mu(B_k)/\mu(I)$ is just $\delta\,|B_k|/|I|$, if $k \geq 2$. Therefore

$$\begin{aligned} EX_i &= \frac{1}{\mu(I)} \int_I X_i(x)\,d\mu \\ &= \frac{1}{\mu(I)} \sum_{k=1}^{\infty} \int_{B_k} X_i(x)\,d\mu \\ &= \sum_{k=1}^{\infty} (N-k)\,\frac{\mu(B_k)}{\mu(I)} \\ &= (N-1)\,\frac{\mu(B_1)}{\mu(I)} + \delta \sum_{k=2}^{\infty} (N-k)\,\frac{|B_k|}{|I|} \\ &\geq (N-1)\,(1-\varepsilon) - \delta \sum_{k=2}^{\infty} k\,\frac{|B_k|}{|I|} \\ &= (N-1)\,(1-\varepsilon) - \delta \sum_{k=2}^{\infty} k\,\frac{1}{2}\left(\frac{2}{3}\right)^k . \end{aligned} \tag{4.13}$$

The last series converges, so we can ensure that $EX_i \geq c_1 > 0$ by choosing N sufficiently large with respect to δ.

When we correct for the fact that $F_n = 1$, not δ, near the endpoints of I, we get some other terms involving similar convergent series, and we can still make the whole argument work by choosing N large. More crucially, we prove estimates (see Sections 10, 13, and 14, especially Lemmas 13.1 and 14.1) showing that, even allowing for the distortion of lengths produced by the inverse of the covering map and by the projection from the disc to the circle, the ratio $|B_k|/|I|$ is controlled by β^k, where β is smaller than 1 and independent of I, so that the relevant series converge and a calculation like the one above can be done.

The estimate $EX_i^2 \leq c_2$ is similar. The estimates (4.8) and (4.9) are established in Section 13 for standard intervals, and in Section 14 for non-standard intervals.

5. Choice of the Fuchsian group

Let $K \subset [0,1]$ be the classical ternary Cantor set. Let $\Omega = \overline{\mathbf{C}} \setminus K$ be the complement of K in the extended complex plane. Ω is an infinitely connected planar Riemann surface. Since $K = \partial\Omega$ contains more than two points, the universal covering space of Ω is the unit disc $\mathbb{D}$, and Ω is conformally equivalent to $\mathbb{D}/G$, where G is a Fuchsian group. We choose this infinitely generated Fuchsian group G as our example.

In this section we show that the conical limit set $L_c(G)$ of G has one-dimensional Lebesgue measure zero, and moreover the Hausdorff dimension of $L_c(G)$ satisfies $1/2 \leq \dim(L_c(G)) < 1$. The remainder of the paper is devoted to the construction of a doubling measure which is supported on $L_c(G)$.

Since K has positive logarithmic capacity, Green's function exists for Ω, and the result cited in the introduction on the measure of the conical limit set of a Fuchsian group implies that $|L_c(G)| = 0$. Fernández [F] has shown that if the boundary $\partial\Omega$ of a planar domain Ω is uniformly perfect, then the exponent of convergence of the corresponding Fuchsian group is strictly less than one. The Cantor set K is uniformly perfect. For Fuchsian groups the exponent of convergence is equal to the Hausdorff dimension of the conical limit set [Pa1, S1]. (See also [BJ], where this is proved in much greater generality.) We may therefore conclude that in fact $\dim(L_c(G)) < 1$.

A Fuchsian group G is of *fully accessible type* if there is a measurable fundamental set $B \subset \partial\mathbb{D}$ for the action of G on $\partial\mathbb{D}$. In other words, B contains no two G-equivalent points, and $\partial\mathbb{D}$ is a.e. equal to the union of the G-images of B: $\sum_{g\in G} |g(B)| = 1$. G is of *accessible type* if there is a measurable set $B \subset \partial\mathbb{D}$, containing no two G-equivalent points, such that $\sum_{g\in G} |g(B)| > 0$.

The normal fundamental domain $\mathcal{F}_0$ (see Section 7) for $\Omega = \overline{\mathbf{C}} \setminus K$ has the property that $|\partial\mathcal{F}_0 \cap \partial\mathbb{D}| = 0$. We outline the well-known proof. The covering map π takes $\partial\mathcal{F}_0$ to $[0,1]$. Since $\partial\mathcal{F}_0$ is a rectifiable curve, π preserves sets of zero length. (This is the F. and M. Riesz theorem; see [Po3].) Since K has Lebesgue measure zero, $\partial\mathcal{F}_0 \cap \pi^{-1}(K) = \partial\mathcal{F}_0 \cap \partial\mathbb{D}$ has measure zero.

This implies that the group G is not of accessible type [Po1]. On the other hand, Patterson [Pa2] showed that if $\delta(G) < 1/2$ then G is of fully accessible type. It follows that our group G must have $\delta(G) = \dim(L_c(G)) \geq 1/2$.

6. A Lemma on Doubling Measures

In this section we follow the philosophy first laid out by Kahane [K] to construct doubling measures on an interval I_0. Our measure is the limit of a sequence of measures whose densities are step functions. In Lemma 6.3 we show that mild conditions on these step functions ensure that the limit measure is doubling.

Simple constructions of this type can yield doubling measures which are supported on very small sets. In particular there are examples of such measures which are supported on sets of arbitrarily small, but positive, Hausdorff dimension. Such a measure is equivalent to the derivative of a quasisymmetric mapping that takes a set of small Hausdorff dimension to a set of full one-dimensional Lebesgue measure.

The two interrelated ingredients of this construction are a grid of nested subintervals of I_0 and a sequence of suitable density functions.

Definition 6.1. A *grid* of subintervals of an interval I_0 is a collection $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ of subintervals of I_0 satisfying

i) $\mathcal{H}_0 = I_0$;

ii) for each $n \geq 0$, the intervals in $\mathcal{H}_n$ have disjoint interiors, and $|I_0 \setminus \bigcup_{I \in \mathcal{H}_n}| = 0$; and

iii) for each $n \geq 1$, for each interval $J \in \mathcal{H}_n$ there is an interval $I \in \mathcal{H}_{n-1}$ such that $J \subset I$.

The collections $\mathcal{H}_n$ are called the *layers* of intervals in the grid $\mathcal{H}$.

Definition 6.2. A function F defined on a interval I is a *step function* if it is constant on each interval in some finite or infinite partition of I. We say F is (δ, η)*-suitable for* I if F is a step function satisfying

i) $\frac{1}{|I|} \int_I F(x)\, dx = 1$;

ii) $0 < \delta \leq F(x) \leq 1/\delta$ for all $x \in I$; and

iii) $F(x) \equiv 1$ on subintervals I_l and I_r of I such that I_l has the same left endpoint as I, I_r has the same right endpoint as I, $|I_l| \geq \eta |I|$, and $|I_r| \geq \eta |I|$.

See Figure 1.

Fix an interval I_0, and numbers δ and η such that $0 < \delta \leq 1$ and $0 < \eta \leq 1/2$. We construct a sequence $\{\mu_n\}$ of measures on I_0, whose densities are of the form $d\mu_n = F_n(x) \cdots F_1(x)\, dx$. We define the functions F_n simultaneously with a related grid $\mathcal{H}$ of subintervals of I_0. Each F_n is (δ, η)-suitable for each interval from the $(n-1)^{\text{th}}$ layer $\mathcal{H}_{n-1}$ of the grid.

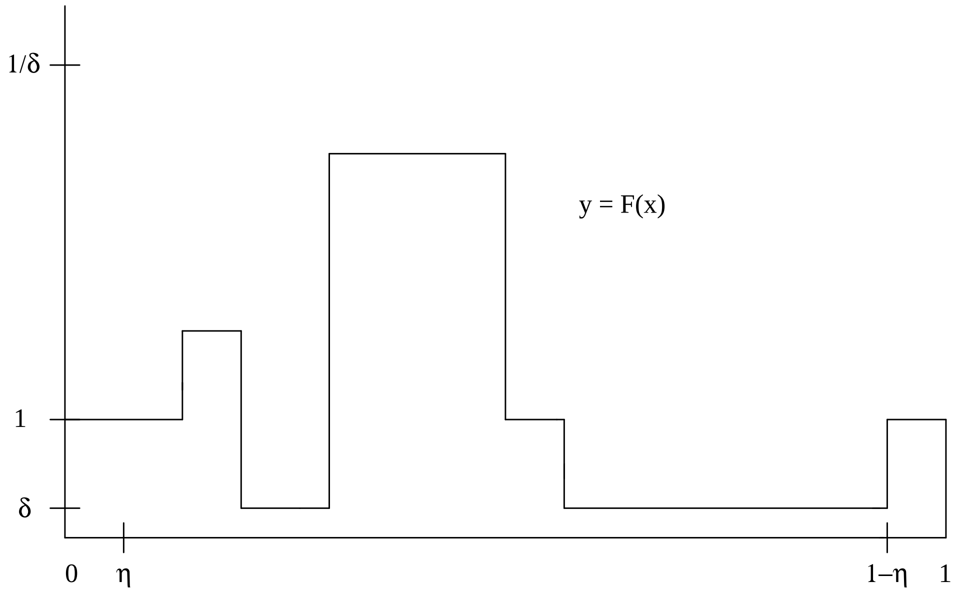

Figure 1. $F(x)$ is (δ, η)-suitable for $I = [0, 1]$.

Let μ_0 be Lebesgue measure on I_0. Let F_1 be any function which is (δ, η)-suitable for I_0. Define μ_1 by $d\mu_1 = F_1(x)\,dx$. Since F_1 has mean value one on I_0, μ_1 has the same total mass as μ_0. Take any partition of I_0 into subintervals J such that F_1 is constant on each J. Note that we allow F_1 to take the same value on adjacent J's; the J's need not be maximal. Also, there may be finitely or infinitely many J's. Let $\mathcal{H}_1$, the first layer of intervals in the grid, be the collection of subintervals in this partition.

We define the measures μ_n inductively for $n \geq 2$. Suppose we have already defined functions $F_1, \ldots, F_{n-1}$, measures $\mu_1, \ldots, \mu_{n-1}$, and the layers $\mathcal{H}_1, \ldots, \mathcal{H}_{n-1}$ of the grid. In particular, F_{n-1} is constant on each interval $I \in \mathcal{H}_{n-1}$. Let F_n be any function which is (δ, η)-suitable for each interval $I \in \mathcal{H}_{n-1}$. Define μ_n by

$$d\mu_n = F_n(x)\,d\mu_{n-1} = F_n(x) \cdots F_1(x)\,dx. \tag{6.1}$$

Again, the fact that F_n has mean value one implies that μ_n has the same total mass as μ_0. To define the next layer $\mathcal{H}_n$ of the grid, we partition each $I \in \mathcal{H}_{n-1}$ into subintervals J such that F_n is constant on each J. Let $\mathcal{H}_n$ be the collection of all these subintervals J.

Note that the collection $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ of subintervals of I_0 does form a grid according to the definition above: each interval $J \in \mathcal{H}_n$ is contained in some $I \in \mathcal{H}_{n-1}$, and each layer $\mathcal{H}_n$ is a partition of I_0, so $|I_0 \setminus \bigcup_{I \in \mathcal{H}_n} I\,| = 0$ and the intervals in $\mathcal{H}_n$ have disjoint interiors. Also, on each $J \in \mathcal{H}_n$, μ_n is a constant multiple of Lebesgue measure: $d\mu_n = (\mu_n(J)/|J|)\,dx$ on J. Furthermore, the total mass assigned to an interval $J \in \mathcal{H}_n$ does not change after the n^{th} step: $\mu_n(J) = \mu_{n+1}(J) = \cdots$.

Lemma 6.3. *Fix an interval I_0 and numbers δ and η with $0 < \delta \leq 1$ and $0 < \eta \leq 1/2$. Let $\{\mu_n\}$ be a sequence of measures and $\mathcal{H}$ a grid of subintervals of I_0 as described above, so that for each $n \geq 1$, $d\mu_n = F_n(x) \cdots F_1(x)\,dx$ and $F_n(x)$ is (δ, η)-suitable for each interval*

$I \in \mathcal{H}_{n-1}$. Then the sequence $\{\mu_n\}$ converges weakly to a measure μ on I_0 which has the same total mass as μ_0 and which is doubling with a constant depending only on δ and η.

Proof. The μ_n are a sequence of measures, all with the same finite total mass, on a compact interval I_0. Therefore there is a subsequence converging weakly to a measure μ on I_0 with the same mass. It is an exercise to see that this limit measure is unique; in other words the full sequence μ_n converges to μ. The μ-measure of any interval $I \in \mathcal{H}_n$ is given by $\mu(I) = \mu_n(I)$.

Let J and K be adjacent intervals, of equal length, contained in I_0. If, for each n, the function F_n is constant on $J \cup K$, then $\mu(J) = \mu(K)$. Otherwise, let m be the first integer such that F_m is not constant on $J \cup K$. In particular,

$$\mu_m(J) \overset{1/\delta}{\sim} \mu_{m-1}(J) = \mu_{m-1}(K) \overset{1/\delta}{\sim} \mu_m(K). \tag{6.2}$$

We show that $\mu(J) \sim \mu_m(J)$ and $\mu(K) \sim \mu_m(K)$.

We may assume that F_m is not constant on J. Let $x \in J$ be a point where the value of F_m changes. Then for each $n \geq m+1$ there is a neighbourhood of x on which $F_n \equiv 1$.

Let J_r be the part of J to the right of x. If J_r does not contain any whole interval from $\mathcal{H}$, then $F_n \equiv 1$ on J_r for all $n \geq m+1$, and so $\mu(J_r) = \mu_m(J_r)$. Otherwise, let

$$\mathcal{L}_m = \bigcup_j \{I_{m,j} \in \mathcal{H}_m \mid I_{m,j} \subset J_r\}. \tag{6.3}$$

Then $\mu(\mathcal{L}_m) = \mu_m(\mathcal{L}_m)$. (Therefore if J_r consists only of whole intervals from $\mathcal{H}_m$, in other words if $J_r = \mathcal{L}_m$, then $\mu(J_r) = \mu_m(J_r)$ and we are done.)

The function F_{m+1} is constant on each interval in $\mathcal{H}_{m+1}$. Suppose there is an interval $I_{m+1,j} \in \mathcal{H}_{m+1}$ which meets $J_r \setminus \mathcal{L}_m$ and on which $F_{m+1} \neq 1$. The interval $I_{m,k} \in \mathcal{H}_m$ which contains $I_{m+1,j}$ is adjacent to $\mathcal{L}_m$, or has x as its left endpoint in case $\mathcal{L}_m$ is empty, and it contains $J_r \setminus \mathcal{L}_m$. Let I be the largest subinterval of $I_{m,k}$, with the same left endpoint as $I_{m,k}$, on which $F_{m+1} \equiv 1$. I is a union of intervals in $\mathcal{H}_{m+1}$, $|I| \geq \eta\,|I_{m,k}|$, and $I \subset J_r \setminus \mathcal{L}_m \subset I_{m,k}$. Then

$$\begin{aligned}
\mu(J_r \setminus \mathcal{L}_m) &\geq \mu_{m+1}(I) \\
&= \mu_m(I) \\
&= \frac{\mu_m(I_{m,k})}{|I_{m,k}|}\,|I| \\
&\geq \eta\,\mu_m(I_{m,k}) \\
&\geq \eta\,\mu_m(J_r \setminus \mathcal{L}_m),
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
\mu(J_r \setminus \mathcal{L}_m) &\leq \mu_m(I_{m,k}) \\
&\leq \frac{\mu_m(I_{m,k})}{|I_{m,k}|}\,\eta^{-1}\,|I| \\
&= \eta^{-1}\,\mu_m(I) \\
&\leq \eta^{-1}\,\mu_m(J_r \setminus \mathcal{L}_m).
\end{aligned} \tag{6.5}$$

Therefore $\mu(J_r \setminus \mathcal{L}_m) \overset{1/\eta}{\sim} \mu_m(J_r \setminus \mathcal{L}_m)$, and so $\mu(J_r) \overset{1/\eta}{\sim} \mu_m(J_r)$.

On the other hand, suppose there is no interval in $\mathcal{H}_{m+1}$ which meets $J_r \setminus \mathcal{L}_m$ and on which $F_{m+1} \neq 1$. Then $F_{m+1} \equiv 1$ on $J_r \setminus \mathcal{L}_m$. If $J_r \setminus \mathcal{L}_m$ contains an interval $I_{m+1,j}$ from $\mathcal{H}_{m+1}$ on which $F_{m+1} \equiv 1$, then the μ-measure of $I_{m+1,j}$ is not only equal to its μ_{m+1}-measure but also to its μ_m-measure. Let

$$\mathcal{L}_{m+1} = \bigcup_j \left\{ I_{m+1,j} \in \mathcal{H}_{m+1} \,\middle|\, I_{m+1,j} \subset J_r \setminus \mathcal{L}_m;\, F_{m+1} \equiv 1 \text{ on } I_{m+1,j} \right\}. \tag{6.6}$$

be the union of all such intervals. $\mathcal{L}_{m+1}$ is an interval to the right of the interval $\mathcal{L}_m$ and adjacent to it, if both are non-empty. Now $\mu(\mathcal{L}_{m+1}) = \mu_m(\mathcal{L}_{m+1})$. If $J_r = \mathcal{L}_m \cup \mathcal{L}_{m+1}$, then $\mu(J_r) = \mu_m(J_r)$ and we are done. Otherwise, we have reduced the problem to estimating the μ-measure of $J_r \setminus (\mathcal{L}_m \cup \mathcal{L}_{m+1})$.

Apply this argument to $J_r \setminus \bigcup_{n=m}^{m+i} \mathcal{L}_n$ for $i \geq 1$, where $\mathcal{L}_{m+i}$ is defined inductively by

$$\mathcal{L}_{m+i} = \bigcup_j \left\{ I_{m+i,j} \in \mathcal{H}_{m+i} \,\middle|\, I_{m+i,j} \subset J_r \setminus \left(\bigcup_{n=m}^{m+i-1} \mathcal{L}_n \right);\, F_{m+i} \equiv 1 \text{ on } I_{m+i,j} \right\}. \tag{6.7}$$

Suppose that for some integer $i \geq 1$ there is an interval $I_{m+i+1,j} \in \mathcal{H}_{m+i+1}$ which meets $J_r \setminus \bigcup_{n=m}^{m+i} \mathcal{L}_n$ and on which $F_{m+i+1} \neq 1$. Let l be the first such integer. As in the case $l = 1$ above, let $I_{m+l,k}$ be the interval from $\mathcal{H}_{m+l}$ which contains $I_{m+l+1,j}$ and let I be the largest subinterval of $I_{m+l,k}$ on which $F_{m+l+1} \equiv 1$. Then $I \subset J_r \setminus \bigcup_{n=m}^{m+l} \mathcal{L}_n \subset I_{m+l,k}$ and so, by the analogues of (6.4) and (6.5), $\mu(J_r) \overset{1/\eta}{\sim} \mu_m(J_r)$, and we are done.

On the other hand, suppose there is no such integer $l \geq 1$. If the intervals $\mathcal{L}_{m+i}$, $i \geq 1$, exhaust J_r, then $\mu(J_r) = \mu_m(J_r)$, and we are done. Otherwise, $J_r \setminus \bigcup_{n=m}^{\infty} \mathcal{L}_n$ is a non-empty interval which has the same right endpoint as J_r and which contains no whole interval from $\mathcal{H}$. So $F_{m+i} \equiv 1$ on $J_r \setminus \bigcup_{n=m}^{\infty} \mathcal{L}_n$ for all $i \geq 1$. Therefore $\mu(J_r \setminus \bigcup_{n=m}^{\infty} \mathcal{L}_n) = \mu_m(J_r \setminus \bigcup_{n=m}^{\infty} \mathcal{L}_n)$, and so $\mu(J_r) = \mu_m(J_r)$. This is the last possible case; we have shown that $\mu(J_r) \overset{1/\eta}{\sim} \mu_m(J_r)$ in all cases.

The same argument shows that $\mu(J_l) \overset{1/\eta}{\sim} \mu_m(J_l)$, where J_l is the part of J to the left of x. Therefore $\mu(J) \overset{1/\eta}{\sim} \mu_m(J)$.

If the function F_m is not constant on the adjacent interval K then $\mu(K) \overset{1/\eta}{\sim} \mu_m(K)$ by the argument above. Also, if $F_n \equiv 1$ on K for all $n \geq m+1$, then $\mu(K) \overset{1/\eta}{\sim} \mu_m(K)$. Otherwise, either F_n is constant (although not necessarily $\equiv 1$) on K for each $n \geq m+1$, or we let $l \geq m+1$ be the first integer such that F_l is not constant on K. In these cases it is enough to estimate the number of integers $n \geq m+1$ for which $F_n \equiv$ constant $\neq 1$ on K. We show that this number is bounded by a constant $c = c(\eta)$. Then in the first case $\mu(K) \overset{c'}{\sim} \mu_m(K)$, and in the second $\mu(K) \overset{1/\eta}{\sim} \mu_l(K) \overset{c'}{\sim} \mu_m(K)$, where the constants c' depend on c and on δ.

Suppose there are at least k integers $n_1 < \cdots < n_k$ greater than m for which $F_{n_j} \equiv$ constant $\neq 1$ on the interval K. We may assume that K is to the right of J. For each j with $1 \leq j \leq k$, there is an interval $I_j \in \mathcal{H}_{n_j}$ which contains K and whose left endpoint lies in J. These intervals are nested: $I_1 \supset \cdots \supset I_k$. Let L_j (respectively R_j) be the largest subinterval of I_j, with the same left (respectively right) endpoint as I_j, on which $F_{n_j} \equiv 1$, and let $M_j = I_j \setminus (L_j \cup R_j)$. Then $|M_j| \leq (1-2\eta)|I_j|$, and $I_j \subset M_{j-1}$ for $2 \leq j \leq k$. See Figure 2.

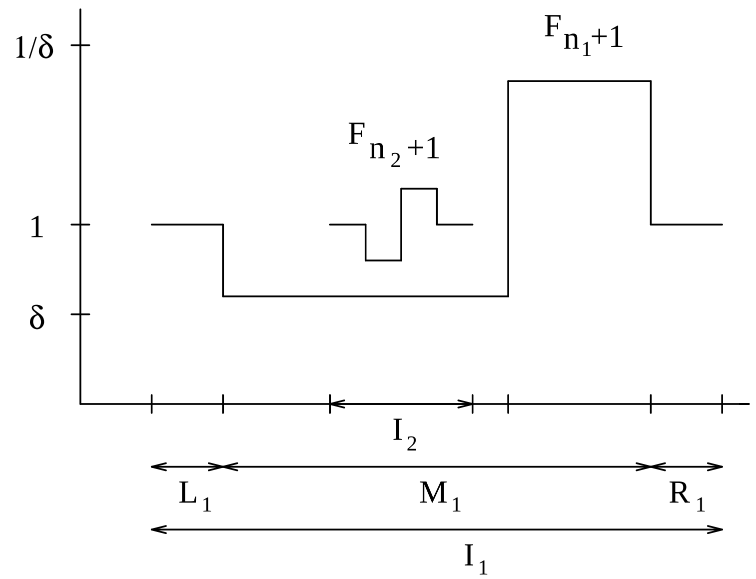

Figure 2. Graphs of $F_{n_1+1}(x)$ and $F_{n_2+1}(x)$.

Also, $L_1 \subset J$ and $\eta\,|I_1| \leq |L_1| \leq |J|$. So

$$|K| \leq |M_k| \leq (1-2\eta)\,|I_k| \leq (1-2\eta)\,|M_{k-1}|. \tag{6.8}$$

After k steps we see that

$$|K| \leq (1-2\eta)^k\,|I_1| \leq (1-2\eta)^k\,\eta^{-1}\,|J| = (1-2\eta)^k\,\eta^{-1}\,|K|. \tag{6.9}$$

Therefore $k \leq \log\eta / \log(1-2\eta)$.

We have shown that $\mu(J) \overset{c}{\sim} \mu(K)$, where c depends only on δ and η, for every pair J, K of adjacent intervals of equal length. In other words μ is a doubling measure, and the doubling constant depends only on δ and η. $\square$

7. Fundamental domains for Ω

We begin the construction of the doubling measure by describing the fundamental domains for $\Omega = \overline{\mathbf{C}} \setminus K$, where $K \subset [0,1]$ is the classical ternary Cantor set. G is the Fuchsian group such that $\mathbb{D}/G$ is conformally equivalent to Ω. G acts as the covering group of Ω on the universal covering space $\mathbb{D}$ of Ω. Let $\pi : \mathbb{D} \to \Omega$ be the covering map; it is a many-to-one conformal mapping which takes each G-orbit in $\mathbb{D}$ to a single point in Ω. Normalize π, by composition with a Möbius transformation of the disc, so that $\pi(0) = \infty$.

A *fundamental domain* for Ω is a domain $\mathcal{F}$ in $\mathbb{D}$ such that $\overline{\mathcal{F}}$ contains at least one point from every G-orbit, and $\mathcal{F}$ contains no two G-equivalent points. The domain $\Omega = \overline{\mathbf{C}} \setminus K$ is a *Denjoy domain*; that is, it is the complement in $\overline{\mathbf{C}}$ of a closed linear set. Denjoy domains have fundamental domains which have a particularly simple form; namely, they are *orthocircular*. This means that the boundary is of the form $\partial\mathcal{F} = E \cup \bigcup_{n=1}^{\infty} A_n$, where E is a closed subset of the unit circle $\partial\mathbb{D}$, and the A_n's (called *orthocircular arcs*) are disjoint arcs of circles which meet $\partial\mathbb{D}$ at right angles. The covering map takes the set $\partial\mathcal{F} \cap \partial\mathbb{D}$ to $\partial\Omega$, and it takes the orthocircular arcs comprising $\partial\mathcal{F} \setminus \partial\mathbb{D}$ to the components of $\overline{\mathbb{R}} \setminus \partial\Omega$. The images $\{g(\mathcal{F})\}_{g \in G}$ are orthocircular domains which tile the disc. (See [RR]; they consider the case where the boundary $\partial\Omega$ has positive length, but a Denjoy domain Ω with $|\partial\Omega| = 0$ also has an orthocircular fundamental domain $\mathcal{F}$, with $|\partial\mathcal{F} \cap \partial\mathbb{D}| = 0$.)

We refer to the images in the disc of the upper and lower half planes under the branches of π^{-1} as *half fundamental domains* for $\Omega = \overline{\mathbf{C}} \setminus K$. Each half fundamental domain D is orthocircular. To fix ideas, suppose that $\pi(D)$ is the upper half plane. The boundary $\overline{\mathbb{R}}$ of the upper half plane lifts via π^{-1} to the boundary of D. The Cantor set K lifts to the closed subset $\partial D \cap \partial\mathbb{D}$ of the circle, and the open intervals which are the components of $\overline{\mathbb{R}} \setminus K$ lift to the orthocircular arcs in the boundary of D.

We now fix a tiling of the disc by half fundamental domains for Ω, together with one fundamental domain for Ω. Since $\pi(0) = \infty$, one of the preimages under π of the open interval $\overline{\mathbb{R}} \setminus [0,1]$ is a diameter γ of $\mathbb{D}$. Let D be the preimage of the upper half plane whose boundary contains γ. Let D' be the reflection of D in γ. D' is a preimage of the lower half plane. Then $D \cup \gamma \cup D'$ is a fundamental domain for Ω; it contains the origin; and it is symmetric with respect to the diameter γ. We denote this fundamental domain by $\mathcal{F}_0$; it is, by symmetry, the normal fundamental domain for Ω. Its image under the covering map is $\pi(\mathcal{F}_0) = \overline{\mathbf{C}} \setminus [0,1]$. Each open interval in $[0,1] \setminus K$ lifts to two orthocircular arcs in $\partial\mathcal{F}_0$.

Tile the remainder of the disc, $\mathbb{D} \setminus \mathcal{F}_0$, by the half fundamental domains which are the G-images of D and D'. These half fundamental domains are indexed in a natural way by the number of orthocircular boundary arcs separating them from the origin. Write $\Omega_{1,j}$ for any half fundamental domain which is adjacent to $\mathcal{F}_0$; that is, which is separated from the origin 0 by a single orthocircular arc. Similarly, write $\Omega_{n,j}$ for a half fundamental domain which is separated from 0 by n orthocircular arcs in the boundaries of the half fundamental domains in the fixed tiling of $\mathbb{D}$.

A note on orientation: we think of the direction from a point in the disc towards the unit circle as *down*, and of the direction towards the origin as *up*. Terms like *below* and *above* are to be understood in the same way; *below* means closer to the unit circle. Given a half fundamental domain $\Omega_{n,j}$, we refer to the large orthocircular arc, denoted $A_{n,j}$, which separates $\Omega_{n,j}$ from 0 as the *upper part of the boundary of* $\Omega_{n,j}$, and to $\partial\Omega_n \setminus A_{n,j}$ as the *lower part of the boundary of* $\Omega_{n,j}$.

Let $\mathcal{A}_1 = \partial\mathcal{F}_0 \setminus \partial\mathbb{D}$ be the union of the orthocircular arcs in the boundary of $\mathcal{F}_0$. Each arc $A_{1,j} \subset \mathcal{A}_1$ is the upper part of some $\partial\Omega_{1,j}$. For each $n \geq 2$, let $\mathcal{A}_n$ be the union, over all j, of the orthocircular arcs in the lower parts of the boundaries of the $\Omega_{n-1,j}$:

$$\mathcal{A}_n = \left[\bigcup_j \partial\Omega_{n-1,j} \setminus \partial\mathbb{D}\right] \setminus \mathcal{A}_{n-1}. \tag{7.1}$$

The union $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ is the collection of all orthocircular boundary arcs which appear in the tiling of $\mathbb{D} \setminus \mathcal{F}_0$ by half fundamental domains.

We show in Section 10 that the orthocircular boundary arcs are uniformly hyperbolically separated: the hyperbolic distance between any two of these arcs is greater than a fixed positive constant. It follows from this observation that the boundaries of the half fundamental domains are chord-arc curves, with a uniform chord-arc constant. We will not need this fact, but we will need to show (Sections 13 and 14) that certain subsets of the half fundamental domains are chord-arc with uniform chord-arc constants.

8. Whitney decompositions

In this section we make a Whitney decomposition of the components of $\overline{\mathbb{R}} \setminus K$, and show that it lifts via π^{-1} to a Whitney decomposition of the orthocircular arcs in the boundaries of the half fundamental domains in $\mathbb{D}$. The intervals in this lifted decomposition are the vertices in the tree described in Section 4.

The Cantor set K is formed by removing the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ from $[0,1]$, then removing the open middle thirds $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from the closed intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ which remain, and so on. We refer to the closed intervals $[0,1]$, $\left[0, \frac{1}{3}\right]$, $\left[\frac{2}{3}, 1\right]$, $\left[0, \frac{1}{9}\right]$, $\left[\frac{2}{9}, \frac{1}{3}\right]$, ... which appear in this procedure as *closed construction intervals* of K.

A *Whitney decomposition* of an open interval L in $\mathbb{R}$ is a partition of L into closed intervals J, with disjoint interiors, such that the Euclidean length of each J is comparable to the Euclidean distance from J to the nearest endpoint of L. In other words, there is a constant $c > 0$ such that $|J| \overset{c}{\sim} \operatorname{dist}(J, \mathbb{R} \setminus L)$ for each J. This also implies that for each point x in such an interval J, the distance from x to $\mathbb{R} \setminus L$ is comparable to the distance from J to $\mathbb{R} \setminus L$. The intervals J in a Whitney decomposition are called *Whitney intervals.* We will also deal with Whitney decompositions of orthocircular arcs; these are defined analogously, so that the length of each Whitney interval in the arc is comparable to its distance from the nearest end of the arc.

Write $\overline{\mathbb{R}} \setminus K$ as a countable union of disjoint open intervals; let L denote any one of these intervals. We now fix a Whitney decomposition of each component L of $\overline{\mathbb{R}} \setminus K$.

First consider a component $L = (a, b)$ of $[0,1] \setminus K$. L is an open interval of length 3^{-k}, for some $k \geq 1$. Partition L into subintervals of length 3^{-n-k}, $n \geq 1$, as follows. For

$n \geq 1$, let

$$\begin{aligned} J_n &= \left[b - 3^{-n+1} \frac{|L|}{2},\, b - 3^{-n} \frac{|L|}{2} \right], \qquad \text{and} \\ J_{-n} &= \left[a + 3^{-n} \frac{|L|}{2},\, a + 3^{-n+1} \frac{|L|}{2} \right]. \end{aligned} \tag{8.1}$$

Then J_1 is the interval whose left endpoint is the midpoint of L and which extends two-thirds of the way towards the right endpoint of L; for $n \geq 2$, J_n is the interval whose left endpoint is the right endpoint of J_{n-1} and which extends two-thirds of the way towards the right endpoint of L; and for $n \geq 1$, J_{-n} is the reflection of J_n in the midpoint of L. The intervals $\{J_{\pm n}\}_{n\geq 1}$ form a Whitney decomposition of L, $|J_{\pm n}| = 2\,\mathrm{dist}\,(J_{\pm n}, K)$, and $|J_{\pm n}| = 3^{-n}\,|L| = 3^{-n-k}$.

Let N be a large integer; its value will be fixed below. It is convenient to amalgamate the $2N$ intervals $J_{-N}, \ldots, J_N$ which are closest to the midpoint of L into a single interval: Let

$$J_c = J_{-N} \cup \cdots \cup J_N. \tag{8.2}$$

This interval has length $|J_c| = (1 - 3^{-N})|L|$, and its distance from K is $2^{-1} \cdot 3^{-N}|L|$, so it satisfies $|J_c| = 2\,(3^N - 1)\,\mathrm{dist}\,(J_c, K)$.

The intervals $\{J_{\pm n}\}_{n\geq N+1}$ and J_c form a Whitney decomposition of L, with constant $2\,(3^N - 1)$.

Now consider the remaining component $L = \overline{\mathbb{R}} \setminus [0,1]$. Fix the small number $\sigma = (2 \cdot 3^{N+2})^{-1}$; the reasons for this particular choice will become apparent later. Let

$$J_\infty = \overline{\mathbb{R}} \setminus (-\sigma, 1 + \sigma). \tag{8.3}$$

J_∞ is an interval of large but finite hyperbolic length, which contains the point at infinity. For $n \geq N + 3$, let

$$\begin{aligned} J_n &= \left[-\frac{1}{2}\frac{1}{3^{n-1}},\, -\frac{1}{2}\frac{1}{3^n} \right], \qquad \text{and} \\ J_{-n} &= \left[1 + \frac{1}{2}\frac{1}{3^n},\, 1 + \frac{1}{2}\frac{1}{3^{n-1}} \right]. \end{aligned} \tag{8.4}$$

Then $J_{N+3} = [-\sigma, -\sigma/3]$, and for $n \geq N + 4$, J_n is the interval whose left endpoint is the right endpoint of J_{n-1} and which extends two-thirds of the way towards 0. For $n \geq N+3$, J_{-n} is the reflection of J_n through the midpoint of $[0, 1]$. The intervals $\{J_{\pm n}\}_{n\geq N+3}$ form a Whitney decomposition of $[-\sigma, 0) \cup (1, 1 + \sigma]$, in which $|J_{\pm n}| = 3^{-n} = 2\,\mathrm{dist}\,(J_{\pm n}, K)$.

Make the convention that

$$|J_\infty| = 1/3. \tag{8.5}$$

With this convention, the intervals $\{J_{\pm n}\}_{n\geq N+3}$ and J_∞ form a Whitney decomposition of $L = \overline{\mathbb{R}} \setminus [0, 1]$, with constant $2 \cdot 3^{N+3}$.

We use the phrase *the Whitney decomposition of* $\overline{\mathbb{R}} \setminus K$ to denote the collection of Whitney intervals $\{J_{\pm n}\}_{n\geq N+1}$ and J_c in the components of $[0, 1] \setminus K$, together with the

Whitney intervals $\{J_{\pm n}\}_{n \geq N+3}$ and J_∞ in $\overline{\mathbb{R}} \setminus [0,1]$. This decomposition will remain fixed for the rest of the construction. We call the intervals $\{J_{\pm n}\}$ *standard* and the intervals J_c and J_∞ *non-standard.*

The inverse π^{-1} of the covering map lifts the components of $\overline{\mathbb{R}} \setminus K$ to the orthocircular arcs $\bigcup_n \mathcal{A}_n$ which appear in the boundaries of the half fundamental domains in our tiling of $\mathbb{D}$. π^{-1} lifts the Whitney decomposition of $\overline{\mathbb{R}} \setminus K$ to a decomposition of $\bigcup_n \mathcal{A}_n$ into closed intervals with disjoint interiors.

The main result of this section is that this is a Euclidean Whitney decomposition of the orthocircular arcs $\bigcup_n \mathcal{A}_n$.

Lemma 8.1. *The Whitney decomposition of $\overline{\mathbb{R}} \setminus K$ lifts via π^{-1} to a Whitney decomposition of the orthocircular arcs $\bigcup_n \mathcal{A}_n$.*

Proof. Since $\partial\Omega = K$ is uniformly perfect, there is a constant $c_\Omega > 0$ such that the function λ_Ω satisfies

$$\frac{c_\Omega}{\operatorname{dist}(w, K)} \leq \lambda_\Omega(w) \leq \frac{2}{\operatorname{dist}(w, K)} \tag{8.6}$$

for all w in Ω, where the element of hyperbolic arclength in Ω is $ds = \lambda_\Omega(w)\,|dw|$. Combined with the relation $|J| = 2\operatorname{dist}(J, K)$ for standard Whitney intervals J, this implies that $\ell_{\text{hyp}}(J) \sim 1$ with a uniform constant for all standard J in $\overline{\mathbb{R}} \setminus K$. Similarly, since the ratio $|J_c|/\operatorname{dist}(J_c, K)$ is the same for all non-standard intervals J_c, $\ell_{\text{hyp}}(J_c) \sim 1$ for all such J_c. Also, the hyperbolic length of J_∞ depends only on the constant σ.

These observations imply that the intervals I in the preimage of the Whitney decomposition of $\overline{\mathbb{R}} \setminus K$ are all of comparable hyperbolic length, because the covering map π preserves hyperbolic length. So $1/r \leq \ell_{\text{hyp}}(I) \leq r$ for all I, where the constant r depends on the uniformly perfect constant of K, on the constant of the Whitney decomposition of $\overline{\mathbb{R}} \setminus K$, and on the choice of σ in the definition of J_∞.

It follows that for each I,

$$\frac{1}{2r}(1 - |z_1|) \leq |I| \leq 2re^r\,(1 - |z_1|), \tag{8.7}$$

where z_1 is the endpoint of I closest to $\partial\mathbb{D}$. Therefore the intervals I form a Whitney decomposition of the orthocircular arcs $\bigcup_n \mathcal{A}_n$. $\square$

9. Construction of palm leaves and the grid of intervals

In this section we construct a grid $\mathcal{H}$ of subintervals of the unit circle, of the kind described in Section 6. We already have a collection of Whitney intervals which form a Whitney decomposition of the orthocircular arcs $\bigcup_n \mathcal{A}_n$ in the boundaries of the half fundamental domains in the tiling of the unit disc. The idea is to project these Whitney intervals onto the unit circle, using a projection map P defined below. The projections of the intervals in $\mathcal{A}_n$ form the n^{th} layer $\mathcal{H}_n$ of the grid. Heuristically, the projection P is almost radial projection. Let $\widehat{\ }$ denote the inverse of the projection P: given a grid interval I, $\widehat{I}$ denotes the unique Whitney interval in $\bigcup_n \mathcal{A}_n$ such that $P(\widehat{I}) = I$.

Recall that $\mathcal{A}_1 = \partial\mathcal{F}_0 \setminus \partial\mathbb{D}$ is the collection of orthocircular arcs in the boundary of the fundamental domain $\mathcal{F}_0$ which contains the origin; and that for $n \geq 2$, $\mathcal{A}_n$ consists of the collection of orthocircular boundary arcs, from the tiling of $\mathbb{D}\setminus\mathcal{F}_0$ by half fundamental domains, which are separated from 0 by exactly $n-1$ other orthocircular boundary arcs.

In order to define the projection P, we make a construction in $\Omega = \overline{\mathbf{C}} \setminus K$ and lift it to the disc via π^{-1}. An outline follows. Given any Whitney interval $\widehat{I} \subset \mathcal{A}_n$, we must specify which of the intervals $\widehat{I}_{n+1,k} \subset \mathcal{A}_{n+1}$ should project via P to subintervals of $P(\widehat{I})$. (In terms of the tree described in Section 4, whose vertices are the Whitney intervals in $\bigcup_n \mathcal{A}_n$, we are now specifying the adjacencies between vertices; in other words which pairs of vertices are connected by edges.) Let $A_{n,j}$ be the orthocircular arc containing $\widehat{I}$, and let $\Omega_{n,j}$ be the half fundamental domain below $A_{n,j}$. The covering map π maps $\widehat{I}$ to a Whitney interval J in $\overline{\mathbb{R}} \setminus K$; it maps $A_{n,j}$ to the component L of $\overline{\mathbb{R}} \setminus K$ which contains $J = \pi(\widehat{I})$; it maps $\Omega_{n,j}$ to either the upper or lower half plane; and it maps $\partial\Omega_{n,j}$ to $\overline{\mathbb{R}}$. See Figure 3.

To each interval J in the Whitney decomposition of L we associate an interval $E_J \subset \overline{\mathbb{R}} \setminus L$, in such a way that the E_J for all J in L have disjoint interiors and their union is $\overline{\mathbb{R}} \setminus L$. (The rest of this section contains precise definitions of these intervals E_J.) These intervals E_J are of a certain form; in particular, for most J the Euclidean length $|E_J|$ is a large fixed multiple of $|J|$, and the endpoints of E_J always lie in the Cantor set K.

Let $\widetilde{I}$ be the segment of $\partial\Omega_{n,j}$ such that $\pi(\widetilde{I}) = E_J$. $\widetilde{I}$ lies in the lower part of the boundary of $\Omega_{n,j}$, and its endpoints lie in the unit circle. Let $P(\widehat{I})$ be the arc of the unit circle which has the same endpoints as $\widetilde{I}$ and which lies below $\widetilde{I}$. With this definition the intervals $\widehat{I}_{n+1,k} \subset \mathcal{A}_{n+1}$ whose projections $P(\widehat{I}_{n+1,k})$ are contained in $P(\widehat{I})$ are precisely those intervals $\widehat{I}_{n+1,k}$ which are contained in $\widetilde{I}$.

The projections $\{P(\widehat{I}_{n,j})\}_j$ of the Whitney intervals $\widehat{I}_{n,j}$ in $\mathcal{A}_n$ constitute the n^{th} layer $\mathcal{H}_n$ of the grid of subintervals of the circle. With the definition of P outlined above, each interval $P(\widehat{I}_{n+1,k})$ in $\mathcal{H}_{n+1}$ is contained in some $P(\widehat{I}_{n,j})$ from the previous layer $\mathcal{H}_n$. Also, since the intervals E_J have pairwise disjoint interiors for all J in each L, the $P(\widehat{I}_{n,j})$ at the n^{th} level have pairwise disjoint interiors. Finally, by the remarks in Section 5 above, the part of $\partial\mathbb{D}$ below the orthocircular arcs in $\mathcal{A}_n$ has full Lebesgue measure in $\partial\mathbb{D}$ for each $n \geq 1$, so $\bigcup_{\widehat{I}\subset\mathcal{A}_n} P(\widehat{I})$ has full Lebesgue measure in $\partial\mathbb{D}$ for each $n \geq 1$. In other words, the union of the intervals $P(\widehat{I})$ in $\mathcal{H}_n$ covers $\partial\mathbb{D}$, up to a set of measure zero. Thus the layers $\mathcal{H}_n$ of intervals form a grid $\mathcal{H}$ of subintervals of $\partial\mathbb{D}$, according to the definition in Section 6.

If $\widehat{R}$ is a segment of the boundary of some half fundamental domain $\Omega_{n,j}$, consisting of a collection of whole Whitney intervals $\{\widehat{I}_{n+1,k}\}_k$, possibly together with a subset E of $\partial\Omega_{n,j} \cap \partial\mathbb{D}$, then we define $P(\widehat{R})$ to be $E \cup \bigcup_k P(\widehat{I}_{n+1,k})$.

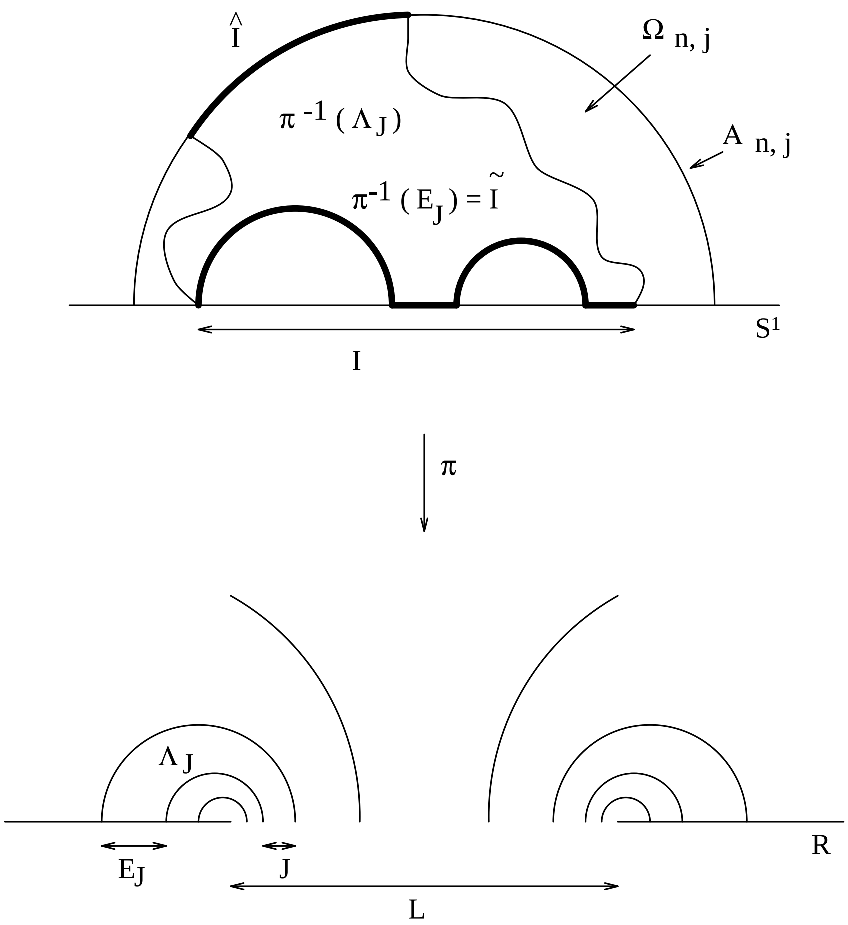

Figure 3. Fan of palm leaves Λ_J based at the Whitney intervals $J = \pi(\widehat{I})$ in L; and a preimage in $\mathbb{D}$ of one leaf Λ_J.

We now give the details of the construction of the grid $\mathcal{H}$ of intervals, in particular defining the intervals E_J associated to the Whitney intervals J in $\overline{\mathbb{R}} \setminus K$. Let $N >> 1$ be the same large integer as in the previous section. Until the last part of the paper we regard N as fixed, and we make all our geometric constructions using this fixed value.

Let L be a component of $\overline{\mathbb{R}} \setminus K$. We define a decomposition of the upper half plane into infinitely many regions Λ_J, indexed by the Whitney intervals J in L. See Figure 3. We refer to these regions as *palm leaves based at the Whitney intervals J in L*, and to the collection of these regions as a *fan of palm leaves, based at L*. The boundary of the leaf Λ_J

based at the Whitney interval J consists of two intervals in $\overline{\mathbb{R}}$, J itself and E_J, together with two non-intersecting semicircles which join the endpoints of J to the endpoints of E_J and which meet $\mathbb{R}$ at right angles. We call J the *base* and E_J the *tip* of the leaf Λ_J. The union over all J in L of the intervals E_J covers $\overline{\mathbb{R}} \setminus L$.

Once the entire construction of the fans of palm leaves is complete, we reflect them in the real axis to obtain analogous fans in the lower half plane.

We distinguish two cases: when the interval L is a component of $[0,1] \setminus K$, and when $L = \overline{\mathbb{R}} \setminus [0,1]$.

Case 1: Let L be a component of $\overline{\mathbb{R}} \setminus K$ which lies in $[0,1]$ and has length $|L| = 3^{-l}$ for some integer $l \geq 1$. Write $L = (a, a+3^{-l})$. Let $K_l = [a-3^{-l}, a]$ and $K_r = [a+3^{-l}, a+2\cdot 3^{-l}]$ be the closed construction intervals of K of length $|K_l| = |K_r| = |L| = 3^{-l}$ such that K_l is immediately to the left of L and K_r is immediately to the right of L. The Whitney intervals in L are enumerated from left to right as $\ldots, J_{-N-2}, J_{-N-1}, J_c, J_{N+1}, J_{N+2}, \ldots$ Recall that $|J_{\pm n}| = 3^{-n}|L|$, for $n \geq N+1$, and $|J_c| = (1-3^{-N})|L|$.

We refer to the Whitney intervals $J_{\pm n}$ in L such that $n \geq N+1$ as *standard* intervals. For each standard interval J we define a leaf Λ_J based at J such that the tip E_J of the leaf has length $|E_J| = 2 \cdot 3^N |J|$. We begin with the standard Whitney interval J_{N+1}, which has length $|J_{N+1}| = 3^{-N-1}|L|$ and is in the right half of L.

Let $E_{J_{N+1}}$ be the closed interval which has the same right endpoint as K_r and whose length is two-thirds the length of K_r. $E_{J_{N+1}}$ consists of a closed construction interval of K and the closure of an adjacent open interval in $[0,1] \setminus K$ of the same length. Now $|E_{J_{N+1}}| = \frac{2}{3}|K_r| = \frac{2}{3}|L| = \frac{2}{3} \cdot 3^{N+1}|J_{N+1}| = 2 \cdot 3^N |J_{N+1}|$. Join the left endpoint of J_{N+1} to the right endpoint of $E_{J_{N+1}}$ by a semicircle in the upper half plane. Join the right endpoint of J_{N+1} to the left endpoint of $E_{J_{N+1}}$ in the same way. Let the leaf $\Lambda_{J_{N+1}}$ be the region in the upper half plane bounded by J_{N+1}, $E_{J_{N+1}}$, and the two semicircles. See Figure 4.

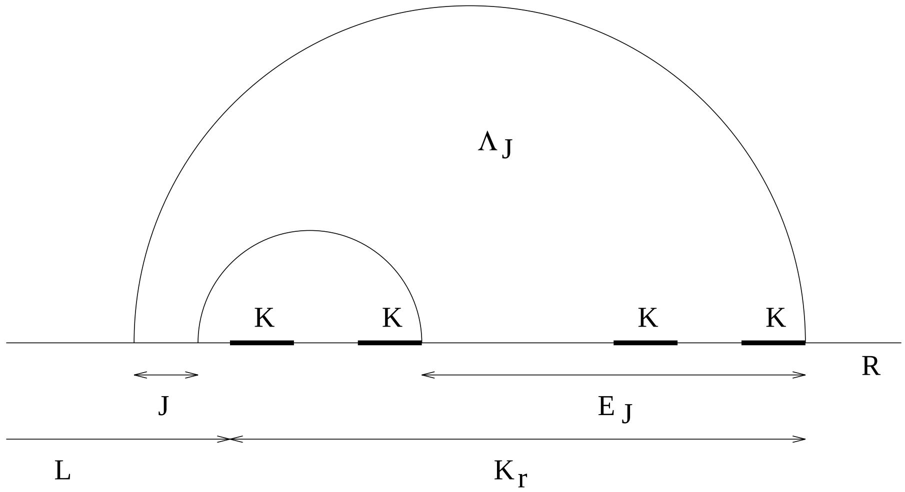

Figure 4. Standard palm leaf Λ_J for $J = J_{N+1}$.
Not to scale: $|E_J| = 2 \cdot 3^N |J|$ and $N >> 1$.

Consider the next standard Whitney interval J_{N+2}; it has length $3^{-N-2}|L|$ and is immediately to the right of J_{N+1}. Construct the leaf $\Lambda_{J_{N+2}}$ and its tip $E_{J_{N+2}}$ exactly as for J_{N+1}: Let $E_{J_{N+2}}$ be the right two-thirds of $K_r \setminus E_{J_{N+1}}$, and join the endpoints of J_{N+2} to those of $E_{J_{N+2}}$ by semicircles. The leaf $\Lambda_{J_{N+2}}$ bounded by J_{N+2}, $E_{J_{N+2}}$, and these semicircles is a copy of $\Lambda_{J_{N+1}}$, shrunk by a factor of one-third. The smaller semicircle in the boundary of $\Lambda_{J_{N+1}}$ is the larger semicircle in the boundary of $\Lambda_{J_{N+2}}$. Clearly $|E_{J_{N+2}}| = 2 \cdot 3^N |J_{N+2}|$.

Repeat this construction for the intervals J_{N+k}, $k \geq 1$, obtaining a sequence of leaves $\Lambda_{J_{N+k}}$ based at J_{N+k} which satisfy $|E_{J_{N+k}}| = 2 \cdot 3^N |J_{N+k}|$. The leaves $\Lambda_{J_{N+k}}$ are all congruent to each other via dilations by powers of three. The intervals $E_{J_{N+k}}$, $k \geq 1$, have disjoint interiors and their union is K_r, the closed construction interval of length $|L|$ to the right of L.

Define leaves $\Lambda_{J_{-N-k}}$, $k \geq 1$, for the standard intervals at the left end of L, by reflecting the leaves $\Lambda_{J_{N+k}}$ through the perpendicular bisector of L. The leaf $\Lambda_{J_{-N-k}}$ based at J_{-N-k} is the mirror image of the leaf $\Lambda_{J_{N+k}}$ based at J_{N+k}. The intervals $E_{J_{-N-k}}$, $k \geq 1$, cover K_l, the closed construction interval of length $|L|$ to the left of L.

We refer to the leaves we have just constructed for the standard intervals $J_{\pm(N+k)}$, $k \geq 1$, as *standard* leaves, and to their tips as *standard* E_J's. We also refer to any grid interval I such that $J = \pi(\widehat{I}\,)$ is standard as a *standard* grid interval. For any standard leaf Λ_J, $|E_J| = 2 \cdot 3^N |J|$; the endpoints of E_J lie in the Cantor set K; and E_J consists of a closed construction interval of length $3^N|J|$ together with the closure of an adjacent open interval in $\overline{\mathbb{R}} \setminus K$, also of length $3^N|J|$.

It remains to define the leaf for the central interval $J_c = J_{-N} \cup \cdots \cup J_N$. Let $E_{J_c} = \overline{\mathbb{R}} \setminus (K_l \cup L \cup K_r)$. Define the leaf Λ_{J_c} to be the region in the upper half plane bounded by J_c, E_{J_c}, and the two non-intersecting semicircles joining the endpoints of J_c to those of E_{J_c}. The endpoints of E_{J_c} lie in the Cantor set K. Notice that when $L = \left(\frac{1}{3}, \frac{2}{3}\right)$, E_{J_c} is exactly $\overline{\mathbb{R}} \setminus [0,1]$, since the tips of the leaves for the standard intervals in $\left(\frac{1}{3}, \frac{2}{3}\right)$ take up all of $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. For smaller intervals L, E_{J_c} contains $\overline{\mathbb{R}} \setminus [0,1]$ and part of $[0,1]$. We use the term *non-standard* to refer to J_c, Λ_{J_c}, E_{J_c}, and any grid interval I such that $\pi(\widehat{I}\,) = J_c$.

To summarize: given an open interval L which is a component of $[0,1] \setminus K$, we have defined a fan of palm leaves Λ_J based at the Whitney intervals J in L, such that:

- the Λ_J's tile the upper half plane;
- the tips E_J of the leaves are intervals whose union is $\overline{\mathbb{R}} \setminus L$;
- the E_J have disjoint interiors;
- the endpoints of each E_J lie in the Cantor set K;
- $|E_J| = 2 \cdot 3^N |J|$ for all standard J, i.e. for all but the central interval J_c in L; and
- each standard E_J consists of a closed construction interval of K and the closure of an adjacent open interval in $\overline{\mathbb{R}} \setminus K$ of the same length.

Case 2: L is the component $\overline{\mathbb{R}} \setminus [0,1]$ of $\overline{\mathbb{R}} \setminus K$. Then L consists of the large interval $J_\infty = \overline{\mathbb{R}} \setminus (-\sigma, 1+\sigma)$ and Whitney intervals $\{J_{\pm n}\}_{n \geq N+3}$ in $[-\sigma, 0) \cup (1, 1+\sigma]$, where $\sigma = (2 \cdot 3^{N+2})^{-1}$.

For the interval $J_\infty = \overline{\mathbb{R}} \setminus (-\sigma, 1+\sigma)$, define $E_{J_\infty} = \left[\frac{1}{9}, \frac{8}{9}\right]$. Join the endpoints of J_∞ to the endpoints of $\left[\frac{1}{9}, \frac{8}{9}\right]$ by non-intersecting semicircles in the upper half plane, and let the leaf Λ_{J_∞} be the region in the upper half plane bounded by J_∞, E_{J_∞}, and the two semicircles. See Figure 5(a).

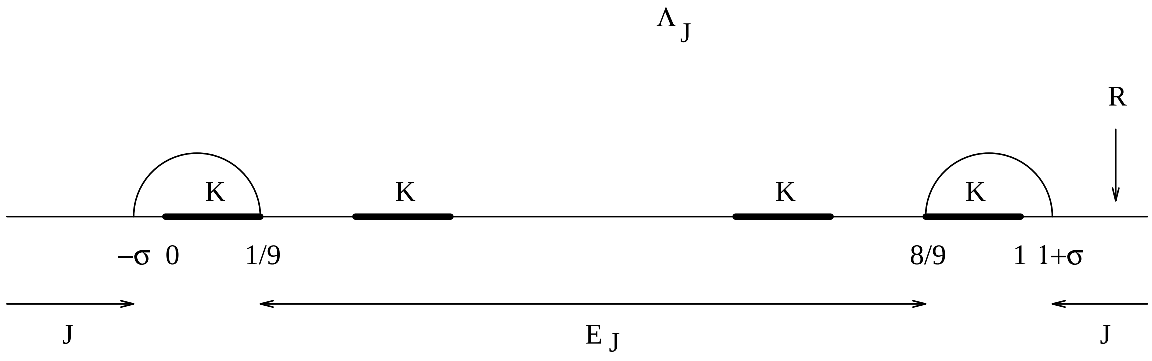

Figure 5(a). Non-standard palm leaf Λ_{J_∞} for $J = J_\infty$.

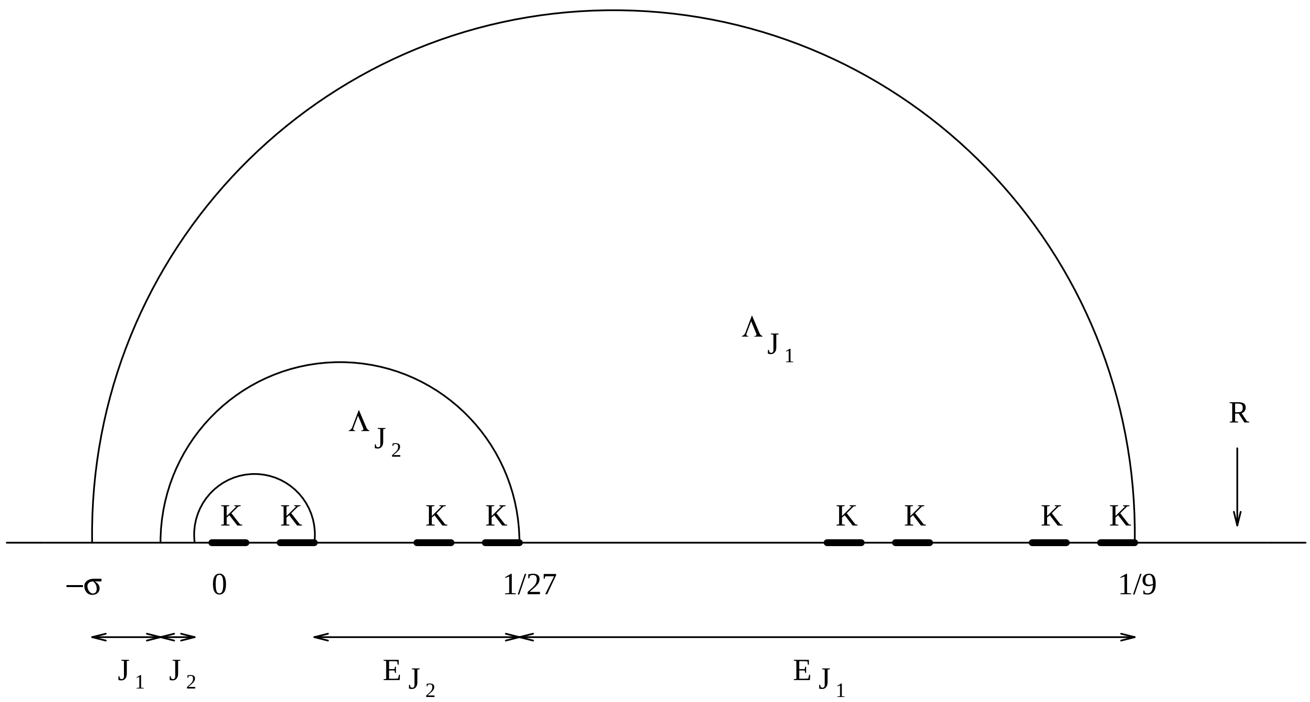

Figure 5(b). Standard palm leaves for $J_1 = J_{N+3}$, $J_2 = J_{N+4}$ in $[-\sigma, 0)$.

The Whitney intervals in $[-\sigma, 0)$ are enumerated from left to right as $J_{N+3}, J_{N+4}, \ldots,$ in order of decreasing size. See Figure 5(b). For $J_{N+3} = [-\sigma, -\sigma/3]$, let $E_{J_{N+3}} = [3^{-3}, 3^{-2}]$. Form the leaf $\Lambda_{J_{N+3}}$ by joining the endpoints of J_{N+3} to those of $E_{J_{N+3}}$

by two non-intersecting semicircles which meet $\mathbb{R}$ at right angles. Then $|J_{N+3}| = \frac{2}{3}\sigma = 3^{-(N+3)} = (2\cdot 3^N)^{-1}|E_{J_{N+3}}|$. So $\Lambda_{J_{N+3}}$ and $E_{J_{N+3}}$ are of the standard form.

Similarly, for $J_{N+k} = [-\sigma/3^{k-3}, -\sigma/3^{k-2}]$, $k \geq 3$, let $E_{J_{N+k}} = [3^{-k}, 3^{-k+1}]$ and define $\Lambda_{J_{N+k}}$ as usual as the region in the upper half plane bounded by J_{N+k}, $E_{J_{N+k}}$, and the two non-intersecting semicircles joining the endpoints of J_{N+k} to those of $E_{J_{N+k}}$. Then $|E_{J_{N+k}}| = 2\cdot 3^N|J_{N+k}|$ for $k \geq 3$; the J_{N+k}'s are standard intervals with standard $E_{J_{N+k}}$'s and $\Lambda_{J_{N+k}}$'s; the $E_{J_{N+k}}$'s cover $[0, \frac{1}{9}]$, and the $\Lambda_{J_{N+k}}$'s tile the half-disc bounded by $[-\sigma, \frac{1}{9}]$ and the semicircle in the upper half plane which joins $-\sigma$ to $\frac{1}{9}$. See Figure 5(b).

The Whitney intervals in $(1, 1+\sigma]$ are enumerated from right to left in order of decreasing size, as $J_{-(N+3)}, J_{-(N+4)}, \ldots$ Define leaves $\Lambda_{J_{-(N+k)}}$ for $J_{-(N+k)}$, $k \geq 3$, by reflecting the leaves $\Lambda_{J_{N+k}}$ through the line $x = \frac{1}{2}$. The leaf $\Lambda_{J_{-(N+k)}}$ based at J_{N+k} is the mirror image of the leaf $\Lambda_{J_{N+k}}$ based at J_{N+k}. All these leaves and their $E_{J_{N+k}}$'s are standard. The $E_{J_{N+k}}$, $k \geq 3$, cover $[\frac{8}{9}, 1]$.

In the case $L = \overline{\mathbb{R}} \setminus [0,1]$ we have defined standard leaves Λ_J for all the Whitney intervals J in L, except for $J_\infty = \overline{\mathbb{R}} \setminus (-\sigma, 1+\sigma)$, which has a non-standard leaf. For all intervals $J \subset L$, the endpoints of E_J lie in the Cantor set K. The palm leaves have the same properties as those summarized at the end of Case 1.

For each component L of $\overline{\mathbb{R}} \setminus K$, reflect the fan of palm leaves based at L through the real axis, obtaining a fan of palm leaves, also based at L, which tiles the lower half plane.

10. Distortion estimates for Whitney intervals

Let $\widehat{I}$ be a Whitney interval in the upper orthocircular boundary arc $A_{n,j}$ of a half fundamental domain $\Omega_{n,j}$. Let $I = P(\widehat{I})$ be the corresponding grid interval, and let $\widetilde{I}$ be the segment of $\partial\Omega_{n,j}$, below $A_{n,j}$, with the same endpoints as I. The purpose of this section is to show that the Euclidean lengths of $\widehat{I}$, $\widetilde{I}$, and I are comparable to each other, with constants which are uniform for all $\widehat{I}$.

We prove a preliminary lemma.

Lemma 10.1. *The orthocircular arcs in $\bigcup_n \mathcal{A}_n$ are uniformly hyperbolically separated.*

Proof. It is sufficient to prove that the components of $\overline{\mathbb{R}} \setminus K$ are uniformly hyperbolically separated, since these components lift via π^{-1} to $\bigcup_n \mathcal{A}_n$, and the conformal map π^{-1} is a hyperbolic isometry.

Recall that the hyperbolic metric on $\Omega = \overline{\mathbf{C}} \setminus K$, given by $\lambda_\Omega(w)|dw|$, satisfies

$$\frac{c_\Omega}{\operatorname{dist}(w, K)} \leq \lambda_\Omega(w) \leq \frac{2}{\operatorname{dist}(w, K)} \tag{10.1}$$

where $c_\Omega > 0$ depends only on the uniformly perfect constant of the Cantor set K.

Let L and L' be components of $\overline{\mathbb{R}} \setminus K$, with $|L| \leq |L'|$. Let γ be an arc joining L to L'. Then the Euclidean length of γ is at least $|L|$, since between L and L' there is a closed construction interval of K of length at least $|L|$.

If each point in γ is within distance $2|L|$ of K, then

$$\begin{aligned}
\ell_{\text{hyp}}(\gamma) &= \int_\gamma \lambda_\Omega(w)|dw| \\
&\geq c_\Omega \int_\gamma \frac{|dw|}{\operatorname{dist}(w,K)} \\
&\geq c_\Omega \int_\gamma \frac{|dw|}{2|L|} \\
&\geq c_\Omega/2.
\end{aligned} \tag{10.2}$$

If some point in γ is not within distance $2|L|$ of K, then there is a segment γ' of γ of length $|L|$, which has one endpoint at distance $2|L|$ from K, and which stays within distance $2|L|$ of K. Then

$$\begin{aligned}
\ell_{\text{hyp}}(\gamma) &\geq \ell_{\text{hyp}}(\gamma') \\
&= \int_{\gamma'} \lambda_\Omega(w)\,|dw| \\
&\geq c_\Omega \int_{\gamma'} \frac{|dw|}{\operatorname{dist}(w,K)} \\
&\geq \frac{c_\Omega}{2|L|}\,|L| \\
&= \frac{c_\Omega}{2}.
\end{aligned} \tag{10.3}$$

Therefore the hyperbolic distance between any two orthocircular arcs in $\bigcup_n \mathcal{A}_n$ is at least $c_\Omega/2$. $\square$

Lemma 10.2. *Let $I \in \mathcal{H}$ be any grid interval. Then $|\widehat{I}\,| \overset{c}{\sim} |\widetilde{I}\,| \overset{\frac{\pi}{2}}{\sim} |I|$, and the constant $c > 0$ is independent of I.*

We split the proof into several sublemmas, showing that $|\widehat{I}\,| \leq c\,|I|$ for all standard and non-standard grid intervals, and then that $|\widehat{I}\,| \geq c\,|I|$ for all standard and non-standard grid intervals. As usual, c denotes constants which may change from line to line.

Note that if $A_{n,j}$ is any orthocircular arc, then its length is comparable with constant $\pi/2$ to the length of the arc of $\partial\mathbb{D}$ below $A_{n,j}$ which has the same endpoints as $A_{n,j}$. For each $I \in \mathcal{H}$, the endpoints of $\widetilde{I}$ are the endpoints of I, and they lie in $\partial\mathbb{D}$. Therefore $\widetilde{I}$ consists of a subset of $\partial\mathbb{D}$ together with a union of whole orthocircular arcs, and so $|\widetilde{I}\,| \sim |I|$ with constant $\pi/2$.

Lemma 10.3. *$|\widehat{I}\,| \leq c\,|I|$ for all standard intervals $I \in \mathcal{H}$.*

Proof. Let $A_{n,j}$ be the orthocircular arc containing $\widehat{I}$, and let $\Omega_{n,j}$ be the half fundamental domain below $A_{n,j}$. Without loss of generality, assume that $\pi(\Omega_{n,j})$ is the upper half plane. Let $\mathcal{F}$ be the fundamental domain which consists of $\Omega_{n,j}$, the arc $A_{n,j}$, and the half fundamental domain immediately above $A_{n,j}$. Let $L = \pi(A_{n,j})$. Then $\pi(\mathcal{F})$ is the union of the open upper and lower half planes together with the component L of $\overline{\mathbb{R}} \setminus K$; its boundary is $\partial\pi(\mathcal{F}) = \overline{\mathbb{R}} \setminus L$.

Let $J = \pi(\widehat{I})$; J is a standard Whitney interval in $\overline{\mathbb{R}} \setminus K$. As usual, let E_J be the tip of the standard leaf based at J; $E_J = \pi(\widetilde{I})$. See Figure 6.

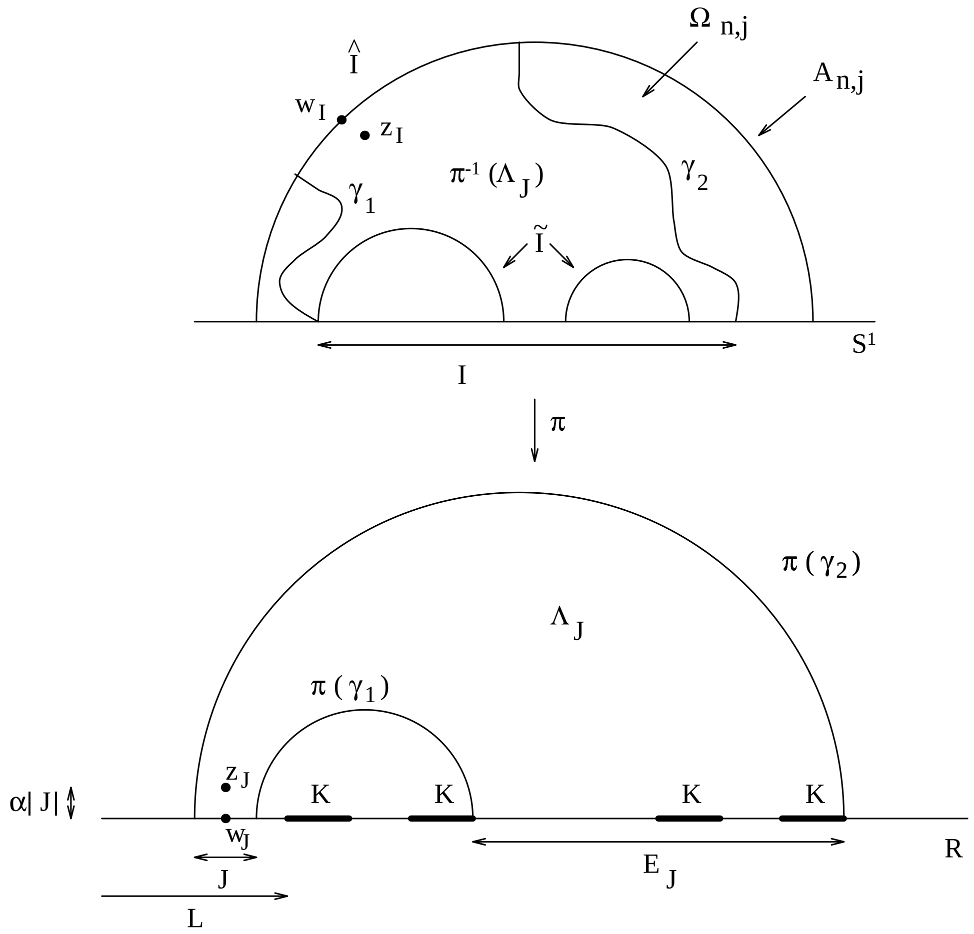

Figure 6. Standard grid interval I; definition of z_J.

Let w_J be the midpoint of J and let w_I be the point in $\widehat{I}$ such that $\pi(w_I) = w_J$. Let $\alpha \leq |J|/2$ be a small number, which will be fixed later. Let z_J be the point in the upper half plane, directly above the midpoint w_J of J, such that $|z_J - w_J| = \alpha\,|J|$. Let z_I be the point in $\Omega_{n,j}$ such that $\pi(z_I) = z_J$.

The hyperbolic distance from z_I to w_I is bounded away from zero and infinity, uniformly for all $\widehat{I}$. The Euclidean ball of radius $|J|/2$ centred at w_J, which contains z_J and J, lies in the set $\{z \mid \operatorname{dist}(J,K) \leq \operatorname{dist}(z,K) \leq 3\operatorname{dist}(J,K)\}$, since $|J| = 2\operatorname{dist}(J,K)$. It follows that

$$\begin{aligned} d_{\text{hyp}}(z_I, w_I) &= d_{\text{hyp}}(z_J, w_J) \\ &\le |z_J - w_J| \, \frac{2}{\operatorname{dist}(J, K)} \\ &= \alpha \, |J| \, \frac{2}{|J|/2} \\ &= 4\alpha. \end{aligned} \tag{10.4}$$

Also,

$$\begin{aligned} d_{\text{hyp}}(z_J, w_J) &\ge |z_J - w_J| \, \frac{c_\Omega}{3 \operatorname{dist}(J, K)} \\ &= \frac{2}{3} \, c_\Omega \alpha. \end{aligned} \tag{10.5}$$

So $\frac{2}{3} c_\Omega \alpha \le d_{\text{hyp}}(z_I, w_I) \le 4\alpha$ for all I. This implies that the Euclidean distance from z_I to w_I is less than $c(\alpha)\,(1 - |w_I|)$, where $c(\alpha)$ depends only on α and decreases to zero with α.

Since z_I and w_I are close in the hyperbolic metric, harmonic measure on $\partial\mathcal{F}$ with basepoint z_I is close to harmonic measure on $\partial\mathcal{F}$ with basepoint w_I. Let $u(z) = \omega(z, E, \mathcal{F})$, where E is any Borel subset of $\partial\mathcal{F}$. The function $u(z)$ is harmonic in $\mathcal{F}$. The hyperbolic distance from w_I to $\partial\mathcal{F}$ is at least $c_\Omega/2$, by Lemma 10.1. The hyperbolic ball of radius $c_\Omega/2$ centred at w_I contains a Euclidean ball of radius $c'_\Omega\,(1 - |w_I|)$, centred at w_I, where c'_Ω depends only on c_Ω. The Euclidean distance from z_I to w_I decreases to zero with the hyperbolic distance from z_I to w_I. Choose the number α, where $|z_I - w_I| = \alpha\,|J|$ and $|z_I - w_I| \le c(\alpha)\,(1 - |w_I|)$, small enough that the Euclidean distance from z_I to w_I is less than $(c'_\Omega/3)\,(1 - |w_I|)$. By Harnack's inequality,

$$\frac{1}{2} u(w_I) \le u(z_I) \le 2u(w_I). \tag{10.6}$$

In particular, setting $E = \widetilde{I}$, we obtain $\omega(z_I, \widetilde{I}, \mathcal{F}) \overset{2}{\sim} \omega(w_I, \widetilde{I}, \mathcal{F})$ for all I.

After these preliminaries, we now use estimates on harmonic measure based at z_I and at w_I to show that $|\widetilde{I}\,| \le c\,(1 - |w_I|)$. This is sufficient to establish Lemma 10.3, since $|\widehat{I}\,| \sim 1 - |w_I|$ with a constant independent of I (by Lemma 8.1).

For standard intervals J, the shapes of the leaves Λ_J and the position of z_J within Λ_J are all identical, up to dilations and reflections. Specifically, $|E_J| = 2 \cdot 3^N |J|$; the distance from J to E_J is also a fixed multiple of $|J|$; and z_J is always at height $\alpha|J|$ above the midpoint of J. Therefore the harmonic measure of E_J in the upper half plane $\mathbf{U}$, taken with respect to the basepoint z_J, is constant for all standard J: $\omega(z_J, E_J, \mathbf{U}) = c > 0$.

Let $\mathcal{F}_{\widetilde{I}}$ be the unit disc without the fundamental domains below $\widetilde{I}$. Let γ be the orthocircular arc joining the endpoints of $\widetilde{I}$, and let $\mathcal{F}_\gamma$ be the disc without the "bite" below γ. So $\Omega_{n,j} \subset \mathcal{F} \subset \mathcal{F}_\gamma \subset \mathcal{F}_{\widetilde{I}}$. Then

$$\begin{aligned} c &= \omega(z_J, E_J, \mathbf{U}) \\ &= \omega(z_I, \widetilde{I}, \Omega_{n,j}) \\ &\leq \omega(z_I, \widetilde{I}, \mathcal{F}) \\ &\leq 2\,\omega(w_I, \widetilde{I}, \mathcal{F}) \\ &\leq 2\,\omega(w_I, \widetilde{I}, \mathcal{F}_{\widetilde{I}}) \\ &= 2\left[1 - \omega(w_I, \partial\mathcal{F}_{\widetilde{I}} \setminus \widetilde{I}, \mathcal{F}_{\widetilde{I}})\right] \\ &\leq 2\left[1 - \omega(w_I, \partial\mathcal{F}_{\widetilde{I}} \setminus \widetilde{I}, \mathcal{F}_\gamma)\right] \\ &= 2\left[1 - \omega(w_I, \partial\mathcal{F}_\gamma \setminus \gamma, \mathcal{F}_\gamma)\right] \\ &= 2\,\omega(w_I, \gamma, \mathcal{F}_\gamma). \end{aligned} \tag{10.7}$$

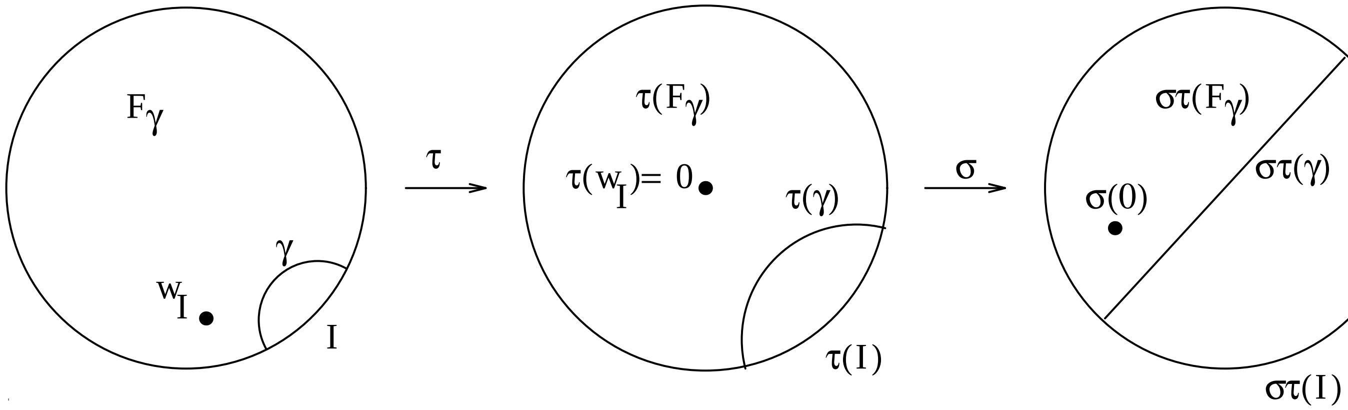

Figure 7. Möbius transformations $\tau, \sigma : \mathbb{D} \to \mathbb{D}$.

We have used the conformal invariance of harmonic measure; the observations made above; and the fact that harmonic measure $\omega(z, E, \Omega)$ is monotonic in the domain Ω: if $z \in \Omega \subset \Omega'$ and $E \subset \partial\Omega \cap \partial\Omega'$, then $\omega(z, E, \Omega) \leq \omega(z, E, \Omega')$.

We assumed here that w_I was not on or below the orthocircular arc γ which joins the endpoints of I. Otherwise, $|I| \geq |\gamma|/2 \geq 1 - |w_I|$ and we are done.

Let $\tau : \mathbb{D} \to \mathbb{D}$ be the Möbius transformation $\tau(z) = (z - w_I)/(1 - \overline{w}_I z)$ which takes w_I to 0. See Figure 7. Then

$$\begin{aligned} \omega(w_I, \gamma, \mathcal{F}_\gamma) &= \omega(0, \tau(\gamma), \tau(\mathcal{F}_\gamma)) \\ &= 2\,\omega(0, \tau(I), \mathbb{D}) \\ &= \pi^{-1}|\tau(I)|. \end{aligned} \tag{10.8}$$

The second step can be justified by mapping $\tau(\gamma)$ to a diameter of $\mathbb{D}$ via a Möbius transformation $\sigma : \mathbb{D} \to \mathbb{D}$ (Figure 7). Consider the probability that a Brownian traveller from $\sigma(0)$ in $\mathbb{D}$ first hits $\partial\mathbb{D}$ somewhere on $\sigma \circ \tau(I)$. By symmetry, this is exactly half the probability that the traveller first hits the boundary of the half disc $\sigma \circ \tau(\mathcal{F}_\gamma)$ somewhere on the diameter $\sigma \circ \tau(\gamma)$. That is,

$$\begin{aligned} \omega(0, \tau(I), \mathbb{D}) &= \omega(\sigma(0), \sigma \circ \tau(I), \mathbb{D}) \\ &= 2^{-1}\, \omega(\sigma(0), \sigma \circ \tau(\gamma), \sigma \circ \tau(\mathcal{F}_\gamma)) \\ &= 2^{-1}\, \omega(0, \tau(\gamma), \tau(\mathcal{F}_\gamma)), \end{aligned} \tag{10.9}$$

as required.

We have shown that $c \leq 2\,\omega(w_I, \gamma, \mathcal{F}_\gamma)| \leq 2\pi^{-1}|\tau(\gamma)|$. Now $\tau'(z) = (1 - |w_I|^2)/(1 - w_I z)^2$, and so

$$\begin{aligned} |\tau(I)| &\leq |I| \max_{z \in I} |\tau'(z)| \\ &\leq |I| \, \frac{2}{1 - |w_I|}. \end{aligned} \tag{10.10}$$

It follows that $|I| \geq c\,(1 - |w_I|)$, where the constant c is independent of I, and therefore $|I| \geq c\,|\widehat{I}\,|$, with c independent of I. $\square$

Lemma 10.4. *$|\widehat{I}\,| \leq c\,|I|$ for all non-standard intervals $I \in \mathcal{H}$.*

Proof. We use the same proof as for Lemma 10.3 above. The key point is to choose a suitable point z_J in the upper half plane $\mathbf{U}$ for each J.

Let J be a non-standard Whitney interval. J is of one of two types: either J is the central Whitney interval J_c in an open component L of $[0,1] \setminus K$, or $J = J_\infty$. In the first case, define z_J as in the proof of Lemma 10.3, with the same α. See Figure 8(a). The leaves Λ_J for these intervals are identical up to dilations, so the harmonic measure of E_J in $\mathbf{U}$, taken from z_J, is uniformly bounded below for all these J. (The bound is not necessarily the same as that in Lemma 10.3.) Also, z_J is close enough to w_J in the hyperbolic metric that the other estimates on harmonic measure hold with the same constants as in Lemma 10.3.

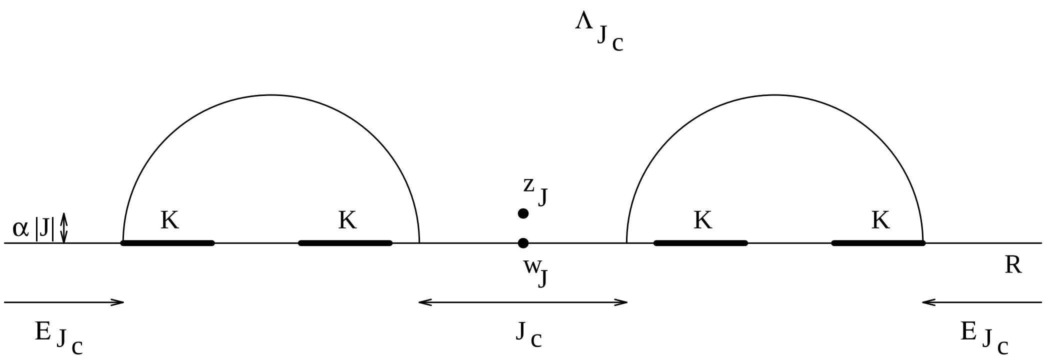

Figure 8(a). Definition of z_J for non-standard $J = J_c$.

For $J = J_\infty$, fix a point $w_J \in J_\infty$ which is far from $[0,1]$. See Figure 8(b).

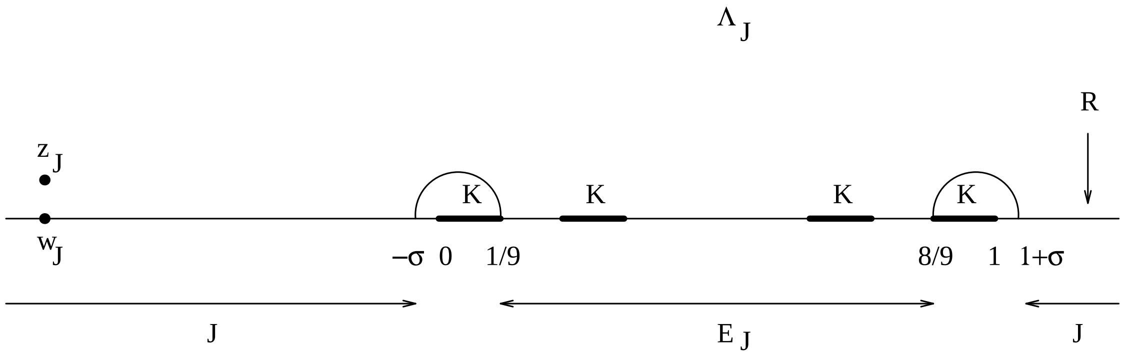

Figure 8(b). Definition of z_J for non-standard $J = J_\infty$.

Let z_J be a point in $\mathbf{U}$, directly above w_J, which is close to w_J in the hyperbolic metric. Then the harmonic measure of E_J in $\mathbf{U}$, taken from z_J, is a positive constant, not necessarily the same as the lower bound in the earlier cases. Choose z_J sufficiently close to w_J that the other harmonic measure estimates hold with the same constants as in the proof of Lemma 10.3. □

Lemma 10.5. $|\widehat{I}\,| \geq c\,|I|$ *for all standard intervals* $I \in \mathcal{H}$.

Proof. We use the same notation as in the proof of Lemma 10.3. In particular, w_I is the point in $\widehat{I}$ such that $w_J = \pi(w_I)$ is the midpoint of the Whitney interval $J = \pi(\widehat{I}\,)$. Without loss of generality assume that w_I is in the left half of the orthocircular arc $A_{n,j}$ containing $\widehat{I}$. Let a and b be the left and right endpoints, respectively, of I. See Figure 9.

It is enough to show that $|b - w_I| \leq c\,(1 - |w_I|)$ with a constant c independent of I. For I is trapped between b and the left endpoint a' of $A_{n,j}$, which satisfies $|a' - w_I| \leq 2\,(1 - |w_I|)$. So then

$$\begin{aligned} |I| &\leq |a' - b| \\ &\leq |a' - w_I| + |w_I - b| \\ &\leq c\,(1 - |w_I|) \\ &\leq c\,|\widehat{I}\,| \end{aligned} \tag{10.11}$$

(using Lemma 8.1), with a constant c independent of I.

Let B be the shorter segment of $\partial\mathcal{F}$ between the right endpoint b of I and the right endpoint b' of $A_{n,j}$. See Figure 9. The following calculation is justified below:

$$\begin{aligned} |w_I - b| &\leq c\,\mathrm{dist}\,(w_I, B) \\ &\leq c\,\frac{\mathrm{dist}\,(w_I, \partial\mathcal{F})}{\omega(w_I, B, \mathcal{F})^2} \\ &\leq c\,\mathrm{dist}\,(w_I, \partial\mathcal{F}) \\ &\leq c\,(1 - |w_I|), \end{aligned} \tag{10.12}$$

where dist denotes Euclidean distance, and c denotes constants independent of I.

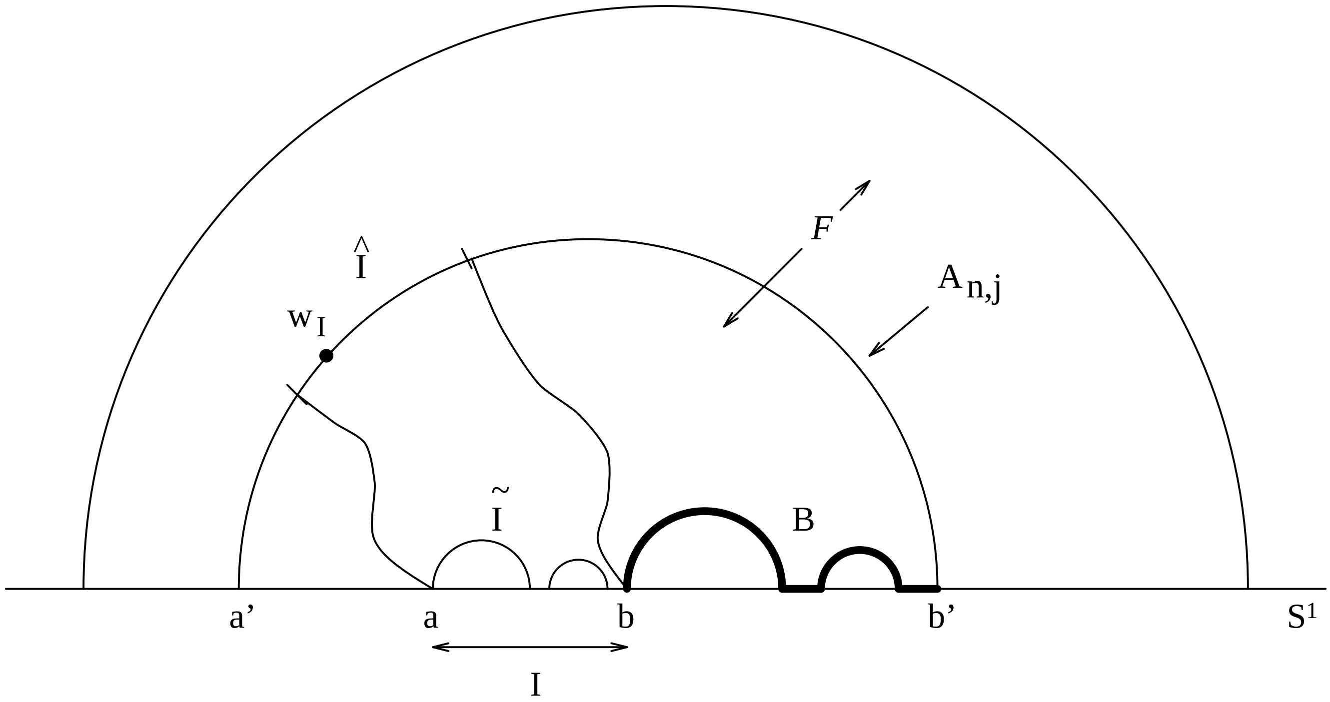

Figure 9. $\widetilde{I}$ is near $\widehat{I}$; definition of B.

The Euclidean distance from w_I to B is $|w_I - b|$, unless there is a point in an orthocircular arc in B which lies closer to w_I than b does. If $|b - w_I| \leq 10\,(1 - |w_I|)$ the lemma is proved. If not, B is far to the right of w_I and the distance from w_I to any point in B is at least $\frac{1}{2}|w_I - b|$, justifying the first line of (10.12).

By Beurling's Lemma [A], there is a constant $C > 0$ such that for each point z_0 in $\mathcal{F}$, and for all $M > 0$,

$$\omega(z_0, \{z \mid |z - z_0| \geq M \operatorname{dist}(z_0, \partial\mathcal{F})\}, \mathcal{F}) \leq CM^{-\frac{1}{2}}. \tag{10.13}$$

Setting $z_0 = w_I$ and $M = \operatorname{dist}(w_I, B)/\operatorname{dist}(w_I, \partial\mathcal{F})$, we find

$$\begin{aligned} \omega(w_I, B, \mathcal{F}) &\leq \omega(w_I, \{z \mid |z - w_I| \geq \operatorname{dist}(z_0, B)\}, \mathcal{F}) \\ &\leq C \left[\frac{\operatorname{dist}(w_I, \partial\mathcal{F})}{\operatorname{dist}(w_I, B)}\right]^{\frac{1}{2}}, \end{aligned} \tag{10.14}$$

and so

$$\operatorname{dist}(w_I, B) \leq C \frac{\operatorname{dist}(w_I, \partial\mathcal{F})}{\omega(w_I, B, \mathcal{F})^2}, \tag{10.15}$$

which is the second line of (10.12).

The harmonic measure of B in $\mathcal{F}$, measured from w_I, is bounded below by a positive constant independent of I. With z_I as in the proof of Lemma 10.3,

$$\begin{aligned} \omega(w_I, B, \mathcal{F}) &\geq 2^{-1}\,\omega(z_I, B, \mathcal{F}) \\ &= 2^{-1}\,\omega(z_J, \pi(B), \pi(\mathcal{F})) \\ &\geq 2^{-1}\,\omega(z_J, \pi(B), \mathbf{U}), \end{aligned} \tag{10.16}$$

by (10.6) and since $\pi(\mathcal{F})$ contains $\mathbf{U}$. $\pi(B)$ is one of the two components of $\overline{\mathbb{R}} \setminus (L \cup E_J)$, where L is the open component of $\overline{\mathbb{R}} \setminus K$ which contains J. Again, since the picture of $\mathbf{U}$ with J, z_J, and E_J is invariant up to dilations and reflections for all standard J, the harmonic measure of $\pi(B)$ in $\mathbf{U}$, taken from z_J, is bounded below by a constant $c > 0$ independent of J, for all standard J. This justifies the third line of (10.12). The last line of (10.12) is true because $\widehat{I}$ is a Whitney interval in $A_{n,j}$. $\square$

Lemma 10.6. $|\widehat{I}\,| \geq c\,|I|$ *for all non-standard intervals* $I \in \mathcal{H}$.

Proof. We use the same method as for Lemma 10.5. Define z_J for each non-standard J as in the proof of Lemma 10.4. The interval $\pi(B)$ is one of the two components of $\overline{\mathbb{R}} \setminus (L \cup E_J)$. See Figures 8(a) and 8(b). By the usual arguments we may conclude that the harmonic measure of B in $\mathcal{F}$, taken from w_I, is bounded below by a positive constant independent of I. The rest of the proof applies without change. $\square$

This completes the proof of Lemma 10.2: $|\widehat{I}\,| \sim |\widetilde{I}\,| \sim |I|$ for all grid intervals $I \in \mathcal{H}$, with constants which are independent of I.

11. Definition of the density functions $F_n \cdots F_1$

In this section we define the functions F_n whose products $F_n \cdots F_1$ give the densities of the measures μ_n. Here we define the F_n on all standard grid intervals $I \in \mathcal{H}_{n-1}$, $n \geq 1$, up to the values of certain parameters, Q and ε, which will be fixed later. We show that these functions F_n are (δ, η)-suitable for each standard $I \in \mathcal{H}_{n-1}$, with constants δ and η independent of I and n. For non-standard intervals $I \in \mathcal{H}_{n-1}$, we define the functions F_n and prove that they are (δ, η)-suitable in Section 14.

We first prove a lemma giving a rough estimate of the distortion caused by the composition $P \circ \pi^{-1}$ of a branch of the inverse π^{-1} of the covering map with the almost radial projection P of the disc onto the circle.

Lemma 11.1. *For each standard Whitney interval J in $\overline{\mathbb{R}} \setminus K$, let S_J be a segment of E_J, and let A_1 and A_2 be the components (possibly empty) of $E_J \setminus S_J$. Suppose that $|S_J|/|E_J| = c_1$, $|A_1|/|E_J| = c_2$, and $|A_2|/|E_J| = 1 - c_1 - c_2$, where c_1 and c_2 are constants independent of J. Suppose also that S_J is a union of whole Whitney intervals, possibly together with a subset of K. Let I be any standard grid interval such that $\pi(\widehat{I}\,) = J$, and let $\widehat{R_I}$ be the segment of $\widehat{I}$ such that $\pi(\widehat{R_I}) = S_J$. Let $R_I = P(\widehat{R_I}) = P \circ \pi^{-1}(S_J)$. Then $|R_I| \geq c\,|I|$, where c is a constant independent of I and J.*

Proof. Without loss of generality, assume that π maps the half fundamental domain below the orthocircular arc containing $\widehat{I}$ to the upper half plane. As in the proof of Lemma 10.3, let w_J be the midpoint of J, and let z_J be the point in the upper half plane, directly above w_J, such that $|z_J - w_J| = \alpha|J|$, where α is a small fixed number independent of J. Let w_I be the point in $\widehat{I}$ such that $\pi(w_I) = w_J$. See Figure 6 in Section 10.

The harmonic measure of S_J in the upper half plane, taken from z_J, is the same for all J, since S_J is always in the same place in E_J. The proof of Lemma 10.3, applied to S_J instead of to E_J, shows that $|R_I| \geq c\,(1 - |w_I|)$, where c is a constant depending on c_1 and c_2. By Lemmas 8.1 and 10.2, $1 - |w_I|$ is comparable to $|I|$ with a constant independent of I and J, which proves the lemma. ☐

We have shown that for all standard intervals $I \in \mathcal{H}$, if S_J is a fixed segment of E_J, where $J = \pi(\widehat{I}\,)$, then $|R_I|/|I| = |P \circ \pi^{-1}(S_J)|/|P \circ \pi^{-1}(E_J)|$ is uniformly bounded away from zero. The same result holds for non-standard intervals I, if S_J is defined so that its length is a fixed multiple of $|J|$ and so that it is always in the same position in E_J.

On each interval I in the grid $\mathcal{H}$ of subintervals of the unit circle we define a function F which is (δ, η)-suitable for I, where the constants δ and η are independent of I. Recall from Section 6 the definition of (δ, η)-suitable: F has mean value one on I; $0 < \delta \leq F(x) \leq 1/\delta$ on I; and $F \equiv 1$ on subintervals I_l and I_r of I, at the left and right ends respectively of I, such that $|I_l| \geq \eta|I|$ and $|I_r| \geq \eta|I|$.

The idea is as follows. Divide each E_J into five segments $S_1, \ldots, S_5$ whose lengths are prescribed fractions of $|E_J|$. Pull these back to $\widetilde{I}$ via π^{-1}, then project them via P to subintervals $R_j = P \circ \pi^{-1}(S_j)$, $1 \leq j \leq 5$, of I. In Lemma 11.1 we gave bounds on the distortion in length caused by $P \circ \pi^{-1}$. Each ratio $|R_j|/|I|$ is uniformly bounded away from zero and infinity for all I. Define F to be constant on each R_j, in such a way that F is large on R_3, identically equal to one on R_1 and R_5, and small enough on $R_2 \cup R_4$

to ensure that F has mean value one on I. See Figure 10. We show that F is uniformly bounded away from zero and infinity for all I.

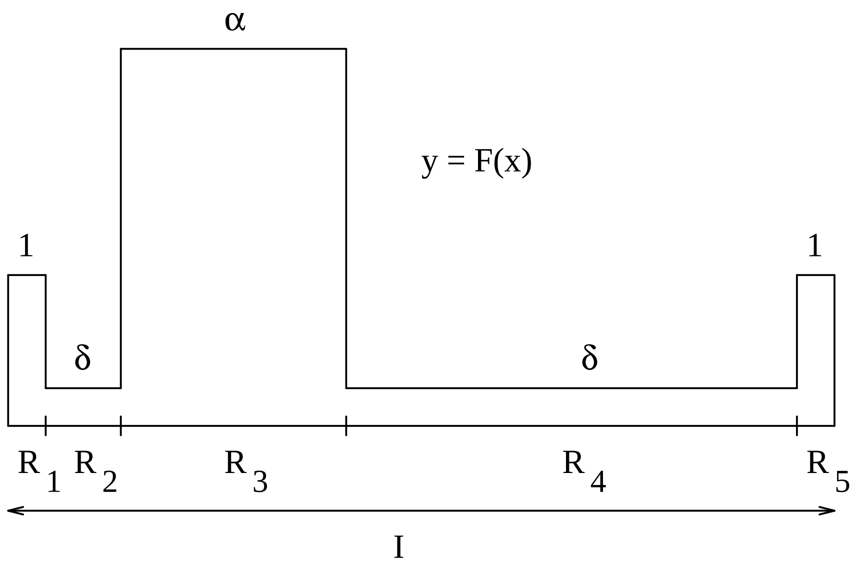

Figure 10. Graph of F; $\alpha = (1-\varepsilon)\,|I|/|R_3|$.

Now we define the segments S_j and R_j for a standard grid interval I. Let $\widehat{I}$ be the Whitney interval in $\bigcup_n \mathcal{A}_n$ such that $I = P(\widehat{I})$, and let $J = \pi(\widehat{I})$. Without loss of generality we describe the case when E_J is to the right of J. Divide E_J into five segments S_j, $1 \le j \le 5$, numbered from left to right, as follows. The left half of E_J is an open component of $\overline{\mathbb{R}} \setminus K$ of length $3^N|J|$. Let S_3 be the central Whitney interval J_c in this open interval. The length of S_3 is $|S_3| = \frac{1}{2}(1-3^{-N})|E_J|$.

Let Q be a large integer; its value will be fixed below. Let S_1 be the interval with the same left endpoint as E_J and with length $|S_1| = \frac{3}{2}3^{-Q}|J|$. S_1 is a union of whole Whitney intervals. Let S_5 be the interval with the same right endpoint as E_J and with length $|S_5| = 3^{-Q}|J|$. S_5 is a closed construction interval of the Cantor set K. Let S_2 be the interval between S_1 and S_3, and let S_4 be the interval between S_3 and S_5.

Now pull back these intervals S_j to $\widetilde{I}$ via the appropriate branch of π^{-1}. Then project them to I via P, obtaining five intervals $R_j = P \circ \pi^{-1}(S_j)$, $1 \le j \le 5$, whose union is I. By Lemma 11.1, there are positive constants c_j, $1 \le j \le 5$, such that $|R_j| \ge c_j\,|I|$ for all standard intervals I. These constants are less than one; they depend on N and Q but are independent of J and I.

Let ε be a positive number, small enough that $(1-\varepsilon)|I|/|R_3| > 1$; its exact value will be fixed below. Define a step function F on I by

$$F(x) = \begin{cases} 1, & x \in R_1 \cup R_5; \\ (1-\varepsilon)\frac{|I|}{|R_3|}, & x \in R_3; \\ \delta, & x \in R_2 \cup R_4; \end{cases} \tag{11.1}$$

where $\delta = (\varepsilon|I| - |R_1 \cup R_5|)/|R_2 \cup R_4|$ is chosen so that $\frac{1}{|I|}\int_I F = 1$. See Figure 10.

This function F takes its maximum on R_3, where $F(x) \equiv (1-\varepsilon)|I|/|R_3| \le 1/c_3$; this upper bound is independent of I. The subintervals R_1 and R_5 at each end of I, on which $F \equiv 1$, each have length at least $\eta|I|$, where $\eta = \min(c_1, c_5)$ is independent of I.

If $d\mu_n = F_n \cdots F_1\, dx$, and each F_m is defined on each $I \in \mathcal{H}_{m-1}$, $m \ge 1$, according to (11.1), then most of the mass $\mu_{n-1}(I)$ is concentrated onto the subinterval R_3 of I:

$$\begin{aligned} \mu_n(R_3) &= \int_{R_3} F_n(x)\, d\mu_{n-1}(x) \\ &= (1-\varepsilon)\frac{|I|}{|R_3|}\, \mu_{n-1}(R_3) \\ &= (1-\varepsilon)\frac{|I|}{|R_3|} \cdot \frac{|R_3|}{|I|}\, \mu_{n-1}(I) \\ &= (1-\varepsilon)\, \mu_{n-1}(I). \end{aligned} \tag{11.2}$$

The third step is valid because μ_{n-1} is a constant multiple of Lebesgue measure on I.

To show that the function F defined in (11.1) is (δ_0, η)-suitable for I, with constants δ_0 and η independent of I, it only remains to give a uniform lower bound for the value δ of F on $R_2 \cup R_4$. Observe that

$$\delta = \frac{\varepsilon|I| - |R_1 \cup R_5|}{|R_2 \cup R_4|} = \left(\varepsilon - \frac{|R_1 \cup R_5|}{|I|}\right) \frac{|I|}{|R_2 \cup R_4|} \ge \varepsilon - \frac{|R_1 \cup R_5|}{|I|}.$$

We would like to ensure that $|R_1 \cup R_5|/|I| \le \varepsilon/2$, say, by choosing $|S_1 \cup S_5|/|E_J|$ sufficiently small (that is, by choosing Q sufficiently large). Unfortunately, an application of Lemma 11.1 to $S_J = E_J \setminus (S_1 \cup S_5)$ does not guarantee that this can be done, since the lemma does not give a good estimate on the size of the constant c, nor on its behaviour as $c_1 \to 0$. However, the sharper estimate in Lemma 13.1 below, based on the A_∞-equivalence of harmonic measure and arclength on the boundary of a chord-arc domain, does imply that $|R_1 \cup R_5|/|I|$ goes to zero with $|S_1 \cup S_5|/|E_J|$. Therefore, we may choose a large Q so that $|S_1 \cup S_5|/|E_J| = \frac{5}{2} \cdot \frac{1}{2} \cdot 3^{-Q-N}$ is sufficiently small that $|R_1 \cup R_5|/|I| \le \varepsilon/2$. Then $\delta \ge \varepsilon/2$.

We have shown that the function F defined in (11.1) is (δ_0, η)-suitable for I, where the constants δ_0 and η depend on ε, N, and Q, but are independent of I, for all standard grid intervals $I \in \mathcal{H}$.

In Section 14 we define the function F for non-standard intervals $I \in \mathcal{H}$, and show that it is (δ_0, η)-suitable for those I.

12. Definition of auxiliary functions X_i

Let I_0 be an interval from the l^{th} layer $\mathcal{H}_l$ of the grid, such that $\pi(\widehat{I_0})$ is not J_∞. For each point $x \in I_0$ and for each $i \geq 0$, let $I_i(x)$ be the unique interval in $\mathcal{H}_{l+i}$ which contains x. The intervals $I_i(x)$ are nested: $I_0(x) \supset I_1(x) \supset I_2(x) \supset \cdots \supset I_i(x) \supset \cdots \ni x$. Let $\widehat{I_i}(x)$ be the Whitney interval in the disc such that $P(\widehat{I_i}(x)) = I_i(x)$. Let $J_i(x) = \pi(\widehat{I_i}(x))$. We have associated to the point $x \in I_0 \subset \partial\mathbb{D}$ a sequence $\{J_i(x)\}_{i \geq 0}$ of Whitney intervals in $\overline{\mathbb{R}} \setminus K$.

We now define auxiliary functions $X_i(x)$ which keep track of the lengths $|J_i(x)|$ of these intervals. Make the convention that

$$|J_c| = |L|/3 \tag{12.1}$$

when J_c is the non-standard central Whitney interval in a component L of $[0,1] \setminus K$. Let

$$X_1(x) = \log_3 \left[\frac{|J_1(x)|}{|J_0|} \right] \tag{12.2}$$

for $x \in I_0$. For $i \geq 2$, let

$$X_i(x) = \begin{cases} \log_3 \left[\frac{|J_i(x)|}{|J_{i-1}(x)|} \right], & \text{if } J_1(x), \ldots, J_{i-1}(x) \neq J_\infty; \\ 1, & \text{otherwise}, \end{cases} \tag{12.3}$$

for $x \in I_0$ and $i \geq 1$. Let

$$S_k(x) = \sum_{i=1}^{k} X_i(x), \tag{12.4}$$

for $k \geq 1$. If none of $J_1(x), \ldots, J_{k-1}(x)$ is the large interval $J_\infty = \overline{\mathbb{R}} \setminus (-\sigma, 1+\sigma)$, then $S_k(x) = \log_3\big(|J_k(x)|/|J_0|\big)$. Note that X_i and S_k are integer-valued.

Let $V(I_0)$ denote the collection of those grid intervals I in I_0 which satisfy $\pi(\widehat{I}\,) = J_\infty$, and which are maximal with respect to this property. In other words, $V(I_0)$ is the collection of the maximal grid intervals I in I_0 which are images of J_∞ under branches of $P \circ \pi^{-1}$.

Observe that $V(I_0) = \{x \in I_0 \mid S_k(x) \to +\infty\}$. For if x is contained in some interval $I = I_N(x)$, say, which is in $V(I_0)$, then $X_i(x) = 1$ for all $i \geq N$, and so $S_k(x)$ tends to infinity. And if x is not contained in any interval I in $V(I_0)$, then no $J_i(x)$ is J_∞, and so $S_k(x)$ remains bounded above, by $\log_3\big[(9|J_0|)^{-1}\big]$, for all k. (Recall that $J_0 = \pi(\widehat{I_0})$ is fixed in this discussion, and that the largest Whitney interval other than J_∞ is the J_c in $\left(\frac{1}{3}, \frac{2}{3}\right)$, which has length $1/9$ by our convention.)

In the remainder of the paper we show that for appropriate choices of the parameters in the definitions of the functions F_n, the measures μ_n given by $d\mu_n = F_n \cdots F_1\, dx$ converge to a measure μ whose restriction to I_0 is supported on $V(I_0)$. In Sections 13 and 14 we show that for all grid intervals $I \subset I_0$, the mean of X_i on I (where i is such that $J_{i-1} = \pi(\widehat{I}\,)$) with respect to μ is uniformly large, and the second moment is uniformly small. In Section 15 we conclude that $S_k \to +\infty$ a.e. $(d\mu)$ on I_0; that is, $\mu(V(I_0)) = \mu(I_0)$.

Finally, also in Section 15, we observe that this implies the existence of a doubling measure μ supported on the set S of points which lie in infinitely many grid intervals corresponding to J_∞. Heuristically, the grid intervals corresponding to J_∞ are near preimages $\widehat{I_\infty}$ of J_∞ under π. Each preimage $\widehat{I_\infty}$ contains an orbit point $g(0)$, $g \in G$, since the covering map π is normalized so that the orbit of 0 is $\pi^{-1}(\infty)$, and $\infty \in J_\infty$. We prove that S is contained in the conical limit set of G, which completes the proof of Theorem 1.2.

13. Estimates $EX_i \geq c_1$ and $EX_i^2 \leq c_2$ for standard intervals

The main result of this section (Lemma 13.4) is that on each standard grid interval $I \in \mathcal{H}$, the function X_i satisfies $\mu(I)^{-1} \int_I X_i \, d\mu \geq c_1$ and $\mu(I)^{-1} \int_I X_i^2 \, d\mu \leq c_2$, where c_1 and c_2 are positive constants independent of I. Here $X_i = \log_3\big(|J_i(x)|/|J_{i-1}(x)|\big)$ is the auxiliary function, defined in Section 12, for a particle which has not yet reached J_∞ and which makes its i^{th} jump from $J = \pi(\widehat{I}\,)$. We begin by establishing an estimate, sharper than that in Section 11, on the distortion in length caused by the map $P \circ \pi^{-1}$.

As usual, for a standard grid interval I with $I = P(\widehat{I}\,)$, let $J = \pi(\widehat{I}\,)$, and let $E_J = \pi(\widetilde{I}\,)$ be the tip of the leaf Λ_J based at J. See Figure 11.

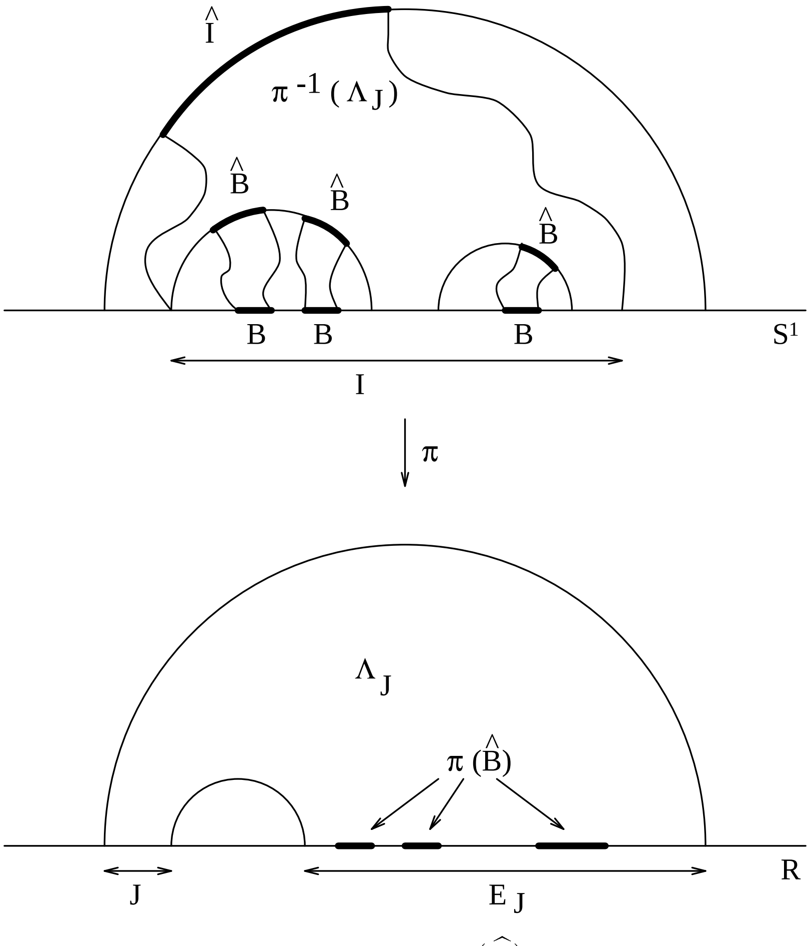

Figure 11. B is in I; $\pi(\widehat{B})$ is in E_J.

Lemma 13.1. *Let $I \in \mathcal{H}_{n-1}$, $n \geq 1$, be a standard grid interval. Let $B \subset I$ be a union of grid intervals from $\mathcal{H}_n$. Let $\widehat{B}$ be the subset of $\widetilde{I}$ such that $P(\widehat{B}) = B$. There are positive constants c and α such that*

$$\frac{|B|}{|I|} \leq c \left[\frac{|\pi(\widehat{B})|}{|E_J|} \right]^{\alpha}. \tag{13.1}$$

The constants c and α are independent of B, I, and n, but they depend on the large number N such that $|E_J| = 2 \cdot 3^N |J|$ for standard intervals.

To prove this lemma, we use Lemmas 13.2 and 13.3 below and the following remarks.

A Jordan curve Γ is said to be *chord-arc* if for all points x and y on Γ the Euclidean length $|x - y|$ of the chord between x and y is comparable, with a uniform constant, to the Euclidean arclength of the shorter arc of Γ between x and y. In other words, there is a constant $c > 0$ such that

$$|x - y| \leq \ell_\Gamma(x, y) \leq c\,|x - y| \tag{13.2}$$

for all x, y on Γ. A domain D is a *chord-arc domain* if its boundary is a chord-arc curve.

The leaves Λ_J are chord-arc domains with chord-arc constant independent of J.

If D is a bounded chord-arc domain, then arclength on its boundary ∂D and harmonic measure based at any point z in D are *A_∞-equivalent*. This means that there are positive constants c_1, c_2, α_1, and α_2 such that whenever S is a segment of ∂D and E is a Borel subset of S, then

$$\frac{|E|}{|S|} \leq c_1 \left[\frac{\omega(z, E, D)}{\omega(z, S, D)} \right]^{\alpha_1} \quad \text{and} \quad \frac{\omega(z, E, D)}{\omega(z, S, D)} \leq c_2 \left[\frac{|E|}{|S|} \right]^{\alpha_2}. \tag{13.3}$$

See [JK]. Moreover, if D is bounded, then for all points $z \in D$ which satisfy

$$\operatorname{dist}(z, \partial D) \geq C \operatorname{diam}(D), \tag{13.4}$$

the constants c_1, c_2, α_1, and α_2 depend only on C and the chord-arc constant of ∂D.

In the situation of Lemma 13.1, consider the domain $D = \pi^{-1}(\Lambda_J)$ whose boundary consists of $\widehat{I}$, $\widetilde{I}$, and two arcs γ_1 and γ_2 which project via π to the two semicircles in the boundary of the leaf Λ_J. We prove (Lemma 13.2) that ∂D is chord-arc, with a constant independent of I, and (Lemma 13.3) that the inequality (13.4) holds, with a constant independent of I, for the point $z_I \in D$ defined in Section 10. Then we prove Lemma 13.1.

Lemma 13.2. *The boundary ∂D of D is chord-arc, with a chord-arc constant which is independent of I.*

Proof. We follow the proof of a similar result by González [G]. The boundary of D consists of $\widehat{I}$, γ_1, γ_2, the orthocircular arcs in $\widetilde{I}$, and the subset $\widetilde{I} \cap \partial\mathbb{D}$ of the unit circle. It is sufficient to show that:

1. Wherever two of $\widehat{I}$, γ_1, γ_2, and $\widetilde{I}$ meet, the angle they form is bounded below by some $\theta_0 > 0$ which is independent of I; and
2. The components of $\partial\Lambda_J \setminus K$ (which are $J = \pi(\widehat{I})$; the semicircles $\pi(\gamma_1)$ and $\pi(\gamma_2)$; and the open intervals in $E_J \setminus K$) are chord-arc, in the hyperbolic metric, with a chord-arc constant which is independent of I.

Then the components of $\partial D \setminus \partial\mathbb{D}$ are also chord-arc in the hyperbolic metric; they are chord-arc in the Euclidean metric; and the whole boundary ∂D is chord-arc in the Euclidean metric with a constant depending only on the chord-arc constant of Λ_J.

1. $\widehat{I}$ meets γ_1 and γ_2 at right angles, because J meets $\pi(\gamma_1)$ and $\pi(\gamma_2)$ at right angles and π^{-1} is conformal.

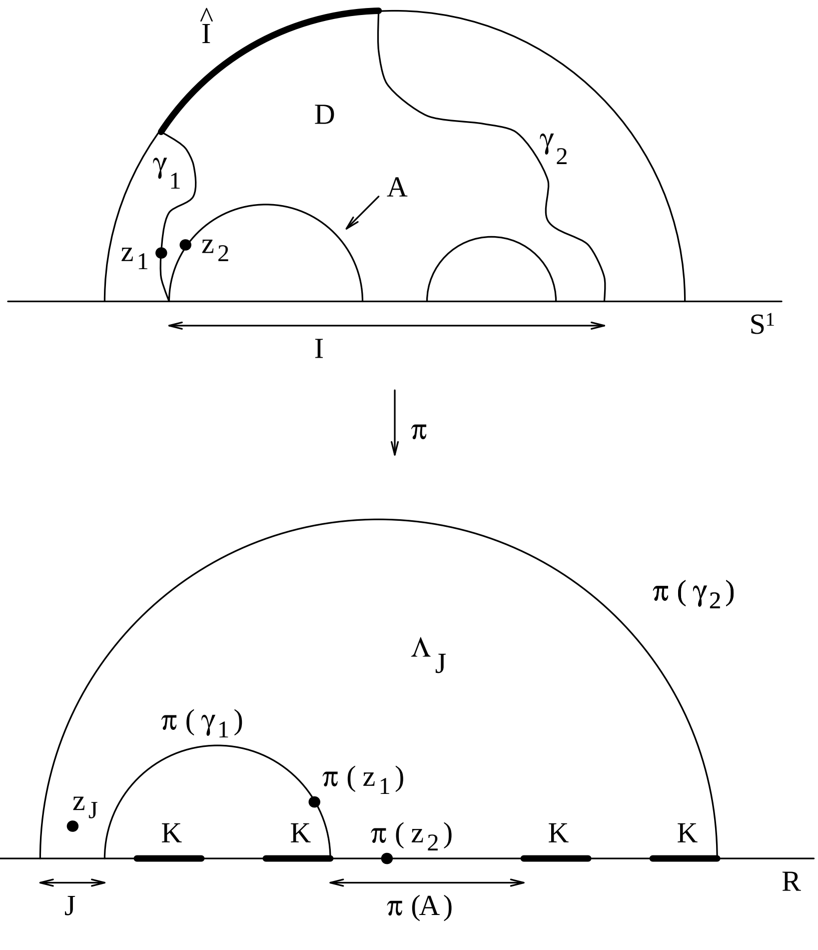

Figure 12. Λ_J and $D = \pi^{-1}(\Lambda_J)$ are chord-arc domains.

Let γ_1 be the arc in ∂D such that $\pi(\gamma_1)$ is the smaller semicircle in $\partial\Lambda_J$, and let A be the orthocircular arc in the lower part of ∂D which meets γ_1. See Figure 12. We show that the hyperbolic distance between any two points $z_1 \in \gamma_1$ and $z_2 \in A$ is uniformly bounded away from zero; this implies that γ_1 and A do not form a cusp at their common endpoint but make a positive angle there. Take a path from $\pi(z_1)$ to $\pi(z_2)$ in Λ_J. Let γ be the segment of this path which starts at $\pi(z_1)$ and has length $\mathrm{Im}(z_1)/2$. Then

$$\begin{aligned} d_{\mathrm{hyp}}(z_1, z_2) &= d_{\mathrm{hyp}}(\pi(z_1), \pi(z_2)) \\ &\geq c_\Omega \int_\gamma \frac{|dw|}{\mathrm{dist}\,(w, K)} \\ &\geq c_\Omega \, \frac{1}{\frac{3}{2}\,\mathrm{Im}\,\pi(z_1)} \cdot \frac{\mathrm{Im}\,\pi(z_1)}{2} \\ &\geq c_\Omega/3. \end{aligned} \tag{13.5}$$

So the angle formed by γ_1 and A is bounded below by some $\theta_0 > 0$, independent of I.

The same reasoning applied to γ_2 shows that γ_2 makes a positive angle, uniformly bounded below, with $\partial\mathbb{D}$.

2. J and the open intervals in $E_J \setminus K$ are geodesic arcs in $\Omega = \overline{\mathbf{C}} \setminus K$, so they are chord-arc in the hyperbolic metric. To show that the semicircle $\pi(\gamma_1)$ is hyperbolically chord-arc, take points z and ζ in $\pi(\gamma_1)$. Let γ be the segment of $\pi(\gamma_1)$ between z and ζ, and let L be the chord between z and ζ. It is enough to consider z and ζ in the part of $\pi(\gamma_1)$ near (in the Euclidean metric) to an endpoint of $\pi(\gamma_1)$. Then γ is almost vertical, and

$$\ell_{\mathrm{hyp}}(\gamma) \sim \int_\gamma \frac{|dw|}{\mathrm{dist}\,(w, K)} \sim \int_\gamma \frac{|dw|}{\mathrm{Im}\,w} \tag{13.6}$$

is comparable to

$$\ell_{\mathrm{hyp}}(L) \sim \int_L \frac{|dw|}{\mathrm{dist}\,(w, K)} \sim \int_L \frac{|dw|}{\mathrm{Im}\,w} \tag{13.7}$$

with a uniform constant.

This completes the proof of Lemma 13.2. □

Lemma 13.3. *For each standard grid interval $I \in \mathcal{H}$, let z_I be the point in D, near $\widehat{I}$, defined in Section 10. There is a constant $C > 0$ independent of I such that*

$$\mathrm{dist}\,(z_I, \partial D) \geq C\,\mathrm{diam}\,(D). \tag{13.8}$$

Proof. We compare both sides of (13.8) to $1 - |w_I|$, where w_I is the point in $\widehat{I} \subset \partial D$, near z_I, defined in Section 10. (See Figure 6.)

The hyperbolic distance from z_I to ∂D is uniformly bounded away from zero. To see this, consider a geodesic arc from $z_J = \pi(z_I)$ to $\partial\Lambda_J$. Let γ be the segment of this geodesic which has z_J as one endpoint and which has Euclidean length $\alpha|J|/2$. Then

$$\begin{aligned} d_{\mathrm{hyp}}(z_I, \partial D) &= d_{\mathrm{hyp}}(\pi(z_I), \pi(\partial D)) \\ &= d_{\mathrm{hyp}}(z_J, \partial\Lambda_J) \\ &\geq c_\Omega \int_\gamma \frac{|dz|}{\mathrm{dist}\,(z, K)} \\ &\geq c_\Omega \cdot \frac{1}{\frac{3}{2}\alpha|J| + \frac{1}{2}|J|} \cdot \frac{\alpha|J|}{2} \\ &= c\,(\Omega, \alpha). \end{aligned} \tag{13.9}$$

The hyperbolic ball of radius $c\,(\Omega, \alpha)$ centred at z_I contains a Euclidean ball of radius $c'(1-|z_I|)$, centred at z_I. Here c' is independent of I. Since $(1-|z_I|) \sim (1-|w_I|)$, with a constant independent of I, we conclude that

$$\operatorname{dist}(z_I, \partial D) \geq c\,(1-|w_I|), \tag{13.10}$$

where c is independent of I.

To estimate $\operatorname{diam}(D)$, it is sufficient to show that

$$\operatorname{dist}(x, \widehat{I}\,) \leq c\,(1-|w_I|) \tag{13.11}$$

for all $x \in \partial D$. For if x and y are in ∂D, then

$$|x-y| \leq \operatorname{dist}(x, \widehat{I}\,) + \operatorname{dist}(y, \widehat{I}\,) + |\widehat{I}\,|, \tag{13.12}$$

and we saw in Section 10 that $|\widehat{I}\,| \sim 1-|w_I|$. We showed (13.11) for $x \in \widetilde{I}$, in the proof of Lemma 10.5. We prove (13.11) for $x \in \gamma_1$; the same argument works for $x \in \gamma_2$.

Let a and $\widehat{a}$ be the endpoints of γ_1 which lie in I and $\widehat{I}$ respectively. Using the fact that γ_1 is chord-arc,

$$\begin{aligned} \operatorname{dist}(x, \widehat{I}\,) &\leq |x-\widehat{a}| \\ &\leq \ell_{\gamma_1}(x, \widehat{a}) \\ &\leq \ell_{\gamma_1}(a, \widehat{a}) \\ &\leq c\,|a-\widehat{a}| \\ &\leq |I| + |\widehat{I}\,| + \operatorname{dist}(I, \widehat{I}\,). \end{aligned} \tag{13.13}$$

Now $|I| \leq c\,(1-|w_I|))$ by Lemma 8.1; $|\widehat{I}\,| \leq c\,|I|$ by Lemma 10.3, and $\operatorname{dist}(I, \widehat{I}\,) \leq c\,(1-|w_I|))$ by the proof of Lemma 10.5. So we have established (13.11), which together with (13.10) proves Lemma 13.3. ◻

Proof of Lemma 13.1. By Lemma 13.2, Lemma 13.3, and the remarks after the statement of Lemma 13.1, arclength on ∂D and harmonic measure at z_I in D are A_∞-equivalent with constants depending only on the chord-arc constant of Λ_J and the constant C from Lemma 13.3. Let c_1 and α_1 be such that

$$\frac{|E|}{|S|} \leq c_1 \left[\frac{\omega(z_I, E, D)}{\omega(z_I, S, D)}\right]^{\alpha_1} \tag{13.14}$$

for all Borel subsets E of segments S of ∂D. The leaves Λ_J are also chord-arc, and $d_{\text{hyp}}(z_J, \partial\Lambda_J) \geq c > 0$, so we may choose c_2 and α_2 independent of I such that

$$\frac{\omega(z_J, E, \Lambda_J)}{\omega(z_J, S, \Lambda_J)} \leq c_2 \left[\frac{|E|}{|S|}\right]^{\alpha_2} \tag{13.15}$$

for all Borel subsets E of segments S of $\partial\Lambda_J$.

By the distortion estimates in Section 10, $|\widetilde{I}\,| \sim |I|$ and $|B| \sim |\widehat{B}|$. We use (13.14) and (13.15) on the sets $\widehat{B} \subset \widetilde{I} \subset \partial D$ and $\pi(\widehat{B}) \subset E_J \subset \partial\Lambda_J$:

$$\begin{aligned} \frac{|B|}{|I|} &\le c\,\frac{|\widehat{B}|}{|\widetilde{I}\,|} \\ &\le c\,c_1 \left[\frac{\omega(z_I, \widehat{B}, D)}{\omega(z_I, \widetilde{I}, D)}\right]^{\alpha_1} \\ &= c\,c_1 \left[\frac{\omega(\pi(z_I), \pi(\widehat{B}), \pi(D))}{\omega(\pi(z_I), \pi(\widetilde{I}\,), \pi(D))}\right]^{\alpha_1} \\ &= c\,c_1 \left[\frac{\omega(z_J, \pi(\widehat{B}), \Lambda_J)}{\omega(z_J, E_J, \Lambda_J)}\right]^{\alpha_1} \\ &\le c\,c_1 \left[c_2 \left(\frac{|\pi(\widehat{B})|}{|E_J|}\right)^{\alpha_2}\right]^{\alpha_1}, \end{aligned} \tag{13.16}$$

which establishes the right hand inequality in (13.1). A similar argument, again using (13.3), proves the left hand inequality in (13.1). The constants are independent of I and B, but they depend on the N such that $|E_J| = 2 \cdot 3^N |J|$, since the chord-arc constant of Λ_J depends on this N. $\square$

Lemma 13.4. *There are positive constants c_1 and c_2 such that for all $n \ge 1$, for every standard grid interval $I \in \mathcal{H}_{n-1}$,*

$$EX_i = \frac{1}{\mu(I)} \int_I X_i(x)\, d\mu \ge c_1; \tag{13.17}$$

and

$$EX_i^2 = \frac{1}{\mu(I)} \int_I X_i^2(x)\, d\mu \le c_2. \tag{13.18}$$

Here $X_i(x) = \log_3\big(|J_i(x)|/|J|\big)$ is the auxiliary function for a particle which makes its i^{th} jump from $J = \pi(\widehat{I}\,)$ and which has not yet reached J_∞.

Proof. Let L be the component of $[0,1] \setminus K$ which contains $J = \pi(\widehat{I}\,)$.

We begin by slightly modifying some definitions, in order to simplify the calculations. In each component L' of $E_J \setminus K$, we amalgamated the $2N$ Whitney intervals in the centre of L' into a single interval $J_c = J_{-N} \cup \cdots \cup J_N$. We now temporarily assume that we have retained the individual intervals $J_{-N}, \ldots, J_N$ instead. We also change the definitions of the region R_3 and of the function F_n. Set $S_3 = J_{-1} \cup J_1$ in the component L' of $[0,1] \setminus K$ which is the left half of E_J, and let $R_3 = P \circ \pi^{-1}(S_3)$. (In Section 11, we had $S_3 = J_{-N} \cup \cdots \cup J_N \subset L'$. The regions R_1 and R_5 are unchanged; the regions R_2 and R_4 become larger since they now include $J_{-N} \cup \cdots \cup J_{-2}$ and $J_2 \cup \cdots \cup J_N$ respectively.)

Define

$$F_n(x) = \begin{cases} 1, & x \in R_1 \cup R_5; \\ (1-\varepsilon)\frac{|I|}{|R_3|}, & x \in R_3; \\ \delta, & x \in R_2 \cup R_4, \end{cases} \tag{13.19}$$

using the new definition of R_3. (At the end of the proof we show that the estimates with these new definitions imply the estimates for the original functions F_n and X_i.)

With these assumptions, we begin with (13.17). Let $I \in \mathcal{H}_{n-1}$ be a standard interval. Let $J = \pi(\widehat{I})$. For each point x in I, let $I_i(x)$ be the interval in $\mathcal{H}_n$ which contains x. I, n, and i are fixed throughout the proof. Let $J_i(x) = \pi(\widehat{I}_i(x))$. Then

$$X_i(x) = \log_3 \left[\frac{|J_i(x)|}{|J|} \right]. \tag{13.20}$$

Notice that to compute EX_i or EX_i^2, we need only deal with F_n. The functions F_l for $l \neq n$ are irrelevant, since for $l < n$ F_l is constant on I, and for $l > n$ F_l has mean value one on each subset of I where X_i is constant.

Let

$$\begin{aligned} B_k &= \{x \in I \mid X_i(x) = N - k\} \\ &= \{x \in I \mid |J_i(x)| = 3^{N-k}|J|\} \\ &= \{x \in I \mid |J_i(x)| = 2^{-1} \cdot 3^{-k}|E_J|\}, \end{aligned} \tag{13.21}$$

for $k \geq 1$. These B_k's are exactly the sets on which the integrand X_i in (13.17) is constant. The union of the B_k's is I. We estimate (13.17) by integrating over each B_k separately (see (13.27) below); before that we need some preliminaries.

Let $\widehat{B_k}$ be the subset of $\widetilde{I}$ such that $P(\widehat{B_k}) = B_k$. Then $\pi(\widehat{B_k})$ is the union of all Whitney intervals in E_J of length $2^{-1} \cdot 3^{-k}|E_J|$.

We count the number of Whitney intervals of this length in E_J. The segment E_J contains one component of $\overline{\mathbb{R}} \setminus K$ of length $2^{-1}|E_J|$, and it contains 2^j components of $\overline{\mathbb{R}} \setminus K$ of length $2^{-1} \cdot 3^{-j-1}|E_J|$, for each $j \geq 0$. Within each component L of $\overline{\mathbb{R}} \setminus K$ there are two Whitney intervals of size $3^{-l}|L|$, for each $l \geq 1$. Therefore E_J contains 2^k Whitney intervals of length $2^{-1} \cdot 3^{-k}|E_J| = 3^{N-k}|J|$, for each $k \geq 1$. Hence

$$|\pi(\widehat{B_k})| = 2^k \cdot 2^{-1} \cdot 3^{-k}|E_J| = \frac{1}{2}\left(\frac{2}{3}\right)^k |E_J|, \tag{13.22}$$

for $k \geq 1$.

Therefore, by Lemma 13.1, there are positive constants c and α, independent of I but dependent on N, such that

$$\frac{|B_k|}{|I|} \leq c \left[\frac{|\pi(\widehat{B_k})|}{|E_J|} \right]^\alpha = c \left[\frac{1}{2}\left(\frac{2}{3}\right)^k \right]^\alpha = c\,\beta^k, \tag{13.23}$$

where $\beta = (2/3)^\alpha$ is strictly less than one.

We also estimate the sizes of $B_k \cap R_1$ and $B_k \cap R_5$, for $k \geq 1$. The sets R_j and S_j, $1 \leq j \leq 5$, were defined in Section 11. $S_1 = \pi(\widehat{R_1})$ is the segment at one end of E_J, contained in $E_J \setminus K$, such that $|S_1| = \frac{3}{2} \cdot 3^{-Q}|J|$. Q is a large integer, independent of I. The largest Whitney interval in S_1 has length $3^{-Q}|J| = 2^{-1} \cdot 3^{-Q-N}|E_J|$. The segment S_1 contains exactly one Whitney interval of length $2^{-1} \cdot 3^{-k}|E_J|$, in other words one Whitney interval which lies in $\pi(\widehat{B_k})$, for each $k \geq Q+N$. By Lemma 13.1,

$$\begin{aligned} \frac{|B_k \cap R_1|}{|I|} &\leq c \left[\frac{|\pi(\widehat{B_k} \cap \widehat{R_1})|}{|E_J|}\right]^{\alpha} \\ &= c \left[\frac{|\pi(\widehat{B_k}) \cap S_1|}{|E_J|}\right]^{\alpha} \\ &= c \left[\frac{2^{-1} \cdot 3^{-k}\, |E_J|}{|E_J|}\right]^{\alpha} \\ &= c\,(3^{-\alpha})^k, \end{aligned} \tag{13.24}$$

for $k \geq Q+N$. The left hand side is zero when $0 \leq k \leq Q+N-1$.

$S_5 = \pi(\widehat{R_5})$ is the segment at the other end of E_J such that $|S_5| = 3^{-Q}|J|$. S_5 is a closed construction interval of the Cantor set K. The largest Whitney intervals in S_5 are two intervals of length $3^{-2} \cdot 3^{-Q}|J| = 2^{-1} \cdot 3^{-Q-N-2}|E_J|$. S_5 contains $2^{k-Q-N-1}$ Whitney intervals of size $2^{-1} \cdot 3^{-k}|E_J|$, for each $k \geq Q+N+2$. By Lemma 13.1,

$$\begin{aligned} \frac{|B_k \cap R_5|}{|I|} &\leq c \left[\frac{|\pi(\widehat{B_k}) \cap S_5|}{|E_J|}\right]^{\alpha} \\ &= c \left[\frac{2^{k-Q-N-1} \cdot 2^{-1} \cdot 3^{-k}|E_J|}{|E_J|}\right]^{\alpha} \\ &= c\,(2^{-Q-N})^{\alpha}(2/3)^{k\alpha} \\ &= c\,(2^{-\alpha})^{Q+N}\beta^k, \end{aligned} \tag{13.25}$$

for $k \geq Q+N+2$. The left hand side vanishes for $0 \leq k \leq Q+N+1$.

The set B_1 is exactly R_3, and so $F_n \equiv (1-\varepsilon)\,|I|/|B_1|$ on B_1. Also, on I, μ_{n-1} is given by $d\mu_{n-1} = (\mu_{n-1}(I)/|I|)\,dx$. Therefore

$$\begin{aligned} \mu(B_1) &= \mu_n(B_1) \\ &= \int_{B_1} F_n(x)\, d\mu_{n-1}(x) \\ &= (1-\varepsilon)\,\frac{|I|}{|B_1|}\,\frac{\mu_{n-1}(I)}{|I|}\,|B_1| \\ &= (1-\varepsilon)\,\mu_{n-1}(I) \\ &= (1-\varepsilon)\,\mu(I). \end{aligned} \tag{13.26}$$

After these preliminaries we can estimate EX_i:

$$
\begin{aligned}
EX_i &= \frac{1}{\mu(I)} \int_I \log_3 \left[\frac{|J_i(x)|}{|J|}\right] d\mu(x) \\
&= \sum_{k=1}^{\infty} (N-k)\, \frac{\mu(B_k)}{\mu(I)} \\
&= (N-1)\, \frac{\mu(B_1)}{\mu(I)} + \sum_{k=2}^{\infty} (N-k)\, \frac{\mu(B_k)}{\mu(I)} \\
&\geq (N-1)\,(1-\varepsilon) + \sum_{k=N}^{\infty} (N-k)\, \frac{\mu(B_k)}{\mu(I)} \qquad (13.27) \\
&\geq (N-1)\,(1-\varepsilon) - \sum_{k=N}^{\infty} k\, \frac{\mu(B_k)}{\mu(I)} \\
&= (N-1)\,(1-\varepsilon) - \sum_{k=N}^{\infty} k\, \frac{\mu_n(B_k)}{\mu_{n-1}(I)} \\
&\geq (N-1)\,(1-\varepsilon) - \sum_{k=N}^{\infty} \frac{k}{\mu_{n-1}(I)} \int_{B_k} F_n(x)\, d\mu_{n-1}(x).
\end{aligned}
$$

The last series converges. To show this, we split each B_k into three pieces, $B_k \cap (R_2 \cup R_4)$, $B_k \cap R_1$, and $B_k \cap R_5$, and estimate the sums over the three types of terms.

First, since $F_n \equiv \delta$ on $R_2 \cup R_4$,

$$
\begin{aligned}
\sum_{k=N}^{\infty} \frac{k}{\mu_{n-1}(I)} \int_{B_k \cap (R_2 \cup R_4)} & F_n(x)\, d\mu_{n-1}(x) \\
&= \sum_{k=N}^{\infty} k\, \delta\, \frac{\mu_{n-1}\big(B_k \cap (R_2 \cup R_4)\big)}{\mu_{n-1}(I)} \\
&= \delta \sum_{k=N}^{\infty} k\, \frac{|B_k \cap (R_2 \cup R_4)|}{|I|} \\
&\leq \delta \sum_{k=N}^{\infty} k\, \frac{|B_k|}{|I|} \qquad (13.28) \\
&\leq \delta\, c\,(N) \sum_{k=N}^{\infty} k\, \beta^k \\
&\leq \delta\, \frac{c\,(N)}{(1-\beta)^2},
\end{aligned}
$$

which can be made arbitrarily small by choosing ε small (which implies δ small) in the definition of F_n. Here the second equality holds because, on I, μ_{n-1} is a constant multiple of Lebesgue measure, and the second last line holds by (13.23).

Similarly, since $F_n \equiv 1$ on R_1,

$$\begin{aligned}
&\sum_{k=N}^{\infty} \frac{k}{\mu_{n-1}(I)} \int_{B_k \cap R_1} F_n(x)\, d\mu_{n-1}(x) \\
&\qquad = \sum_{k=N}^{\infty} k\, \frac{\mu_{n-1}(B_k \cap R_1)}{\mu_{n-1}(I)} \\
&\qquad = \sum_{k=N}^{\infty} k\, \frac{|B_k \cap R_1|}{|I|} \\
&\qquad = \sum_{k=N+Q}^{\infty} k\, \frac{|B_k \cap R_1|}{|I|} \\
&\qquad \le c\,(N) \sum_{k=N+Q}^{\infty} k\, \left(3^{-\alpha}\right)^k,
\end{aligned} \tag{13.29}$$

which can be made arbitrarily small by choosing Q large. Here the last line holds by (13.24).

Finally, since $F_n \equiv 1$ on R_5,

$$\begin{aligned}
&\sum_{k=N}^{\infty} \frac{k}{\mu_{n-1}(I)} \int_{B_k \cap R_5} F_n(x)\, d\mu_{n-1}(x) \\
&\qquad = \sum_{k=N}^{\infty} k\, \frac{\mu_{n-1}(B_k \cap R_5)}{\mu_{n-1}(I)} \\
&\qquad = \sum_{k=N}^{\infty} k\, \frac{|B_k \cap R_5|}{|I|} \\
&\qquad = \sum_{k=Q+N+2}^{\infty} k\, \frac{|B_k \cap R_5|}{|I|} \\
&\qquad \le c(N)\, (2^{-\alpha})^{Q+N} \sum_{k=Q+N+2}^{\infty} k\, \beta^k \\
&\qquad \le \frac{c(N)\, (2^{-\alpha})^{Q+N}}{(1-\beta)^2},
\end{aligned} \tag{13.30}$$

which decreases to zero as Q increases. Here the second last inequality holds by (13.25).

Once N is fixed, we can make (13.28), (13.29), and (13.30) arbitrarily small by choosing ε sufficiently small and Q sufficiently large. Therefore we can make the last term in (13.27) arbitrarily small. So for some positive constant c_1 depending on N, we can ensure that $EX_i \ge c_1$ for all standard intervals I in $\mathcal{H}_{n-1}$, for all $n \ge 1$.

We can estimate the second moment more simply. We showed in Section 11 that there is an upper bound $1/\delta_0$ for F_n on I, independent of I and n. Then

$$\begin{aligned}
EX_i^2 &= \frac{1}{\mu(I)} \int_I \left\{ \log_3 \left[\frac{|J_i(x)|}{|J|} \right] \right\}^2 d\mu(x) \\
&= \sum_{k=1}^{\infty} (N-k)^2 \, \frac{\mu(B_k)}{\mu(I)} \\
&= \sum_{k=1}^{\infty} (N-k)^2 \, \frac{\mu_n(B_k)}{\mu_{n-1}(I)} \\
&= \sum_{k=1}^{\infty} (N-k)^2 \, \frac{1}{\mu(I)} \int_{B_k} F_n(x)\, d\mu_{n-1}(x) \\
&\leq \frac{1}{\delta_0} \sum_{k=1}^{\infty} (N-k)^2 \, \frac{\mu_{n-1}(B_k)}{\mu_{n-1}(I)} \\
&= \frac{1}{\delta_0} \sum_{k=1}^{\infty} (N-k)^2 \, \frac{|B_k|}{|I|} \\
&\leq \frac{1}{\delta_0} \sum_{k=1}^{\infty} (N^2+k^2) \, \frac{|B_k|}{|I|} \\
&\leq \frac{c(N)}{\delta_0} \sum_{k=1}^{\infty} (N^2+k^2) \, \beta^k \\
&\leq \frac{c(N)}{\delta_0} \left[\frac{N^2}{(1-\beta)} + \frac{1}{(1-\beta)^3} \right],
\end{aligned} \tag{13.31}$$

using (13.23) in the second last line. In the sixth line we used the fact that μ_{n-1} is a constant multiple of Lebesgue measure on I. Again, once N, Q, and ε are fixed, this estimate gives a uniform upper bound c_2 on EX_i^2 for all standard intervals $I \in \mathcal{H}_{n-1}$, for all $n \geq 1$.

It remains to prove the same estimates for our original functions X_i and F_n. Write Y_i and G_n for the modified versions of X_i and F_n defined at the start of the proof and used in the calculations (13.19)–(13.31) above.

To estimate EX_i^2, define $Z_i(x) = N^2 + k^2$ on B_k, $k \geq 1$, where $B_k = \{x \in I \mid Y_i(x) = N - k\}$ as above. A comparison of Y_i, X_i, and Z_i shows that on each B_k,

$$X_i^2 \leq (N+k)^2 \leq 2\,(N^2+k^2) = 2\,Z_i. \tag{13.32}$$

The calculation (13.31) shows that the mean value of $Z_i G_n$ on I is at most c_2. Also, $F_n \sim G_n$ with constants independent of I, since they are both bounded above and below by positive constants independent of I. Therefore

$$\begin{aligned} EX_i^2 &= \frac{1}{|I|}\int_I X_i^2 F_n\,dx \\ &\le c\,\frac{1}{|I|}\int_I 2\,Z_i G_n\,dx \\ &\le 2\,c\,c_2, \end{aligned} \tag{13.33}$$

for all I.

Finally, we estimate EX_i. Let G_b (respectively F_b) be the maximum value of G_n (respectively F_n). Then $G_b \sim F_b$ with constants independent of I, by a comparison of harmonic measures in Λ_J. Let

$$I_+ = \{x \in I \bigm| F_n(x) = F_b\} \tag{13.34}$$

and

$$I_s = \{x \in I \bigm| F_n(x) = \delta'\}, \tag{13.35}$$

where δ' is the minimum value of F_n on I. In the next calculation we neglect the region where $Y_i \equiv 1$. Then, since $X_i \ge Y_i$ on I,

$$\begin{aligned} EX_i &= \frac{1}{|I|}\int_I X_i F_n\,dx \\ &\ge \frac{1}{|I|}\int_I Y_i F_n\,dx \\ &\ge \frac{1}{|I|}\int_{I_+} Y_i F_b\,dx - \frac{1}{|I|}\int_{I_s} Y_i F_n\,dx \\ &\ge c\,\frac{1}{|I|}\int_{I_+} Y_i G_b\,dx - \frac{1}{|I|}\int_{I_s} Y_i F_n\,dx \\ &\ge c\,\frac{N-1}{2} - \delta'\,\frac{1}{|I|}\int_{I_s} Y_i\,dx; \end{aligned} \tag{13.36}$$

by (13.27). We showed above that $\frac{1}{|I|}\int_{I_s} Y_i\,dx$ is bounded above by a constant independent of I. Therefore, by choosing δ' small enough in the definition of F_n, we may ensure that EX_i is bounded below by a positive constant independent of I.

This completes the proof of Lemma 13.4. □

14. Estimates $EX_i \geq c_1$ and $EX_i^2 \leq c_2$ for non-standard intervals

Let $I \in \mathcal{H}$ be any non-standard grid interval such that $J_c = \pi(\widehat{I}\,)$ is the non-standard central interval in some component L of $[0,1] \setminus K$. In this section we prove (Lemma 14.3) the estimates $\mu(I)^{-1} \int_I X_i \, d\mu \geq c_1$ and $\mu(I)^{-1} \int_I X_i^2 \, d\mu \leq c_2$, where c_1 and c_2 are positive constants independent of I. Here $X_i(x) = \log_3\bigl(|J_i(x)|/|J_c|\bigr)$ is the auxiliary function for a particle which makes its i^{th} jump from $J_c = \pi(\widehat{I}\,)$ and which has not yet reached J_∞. We begin with an estimate, analogous to Lemma 13.1, on the distortion in length caused by the map $P \circ \pi^{-1}$.

Lemma 14.1. *Let $I \in \mathcal{H}_{n-1}$, $n \geq 1$, be a non-standard grid interval such that $J_c = \pi(\widehat{I}\,)$ is the central interval in some component L of $[0,1] \setminus K$. Fix a number $\lambda > 1$, and let q be any positive integer. Let B and T be unions of grid intervals from $\mathcal{H}_n$ such that $B \subset T \subset I$ and $\pi(\widehat{T}) \subset \{w \mid \lambda^q \leq \operatorname{dist}(w, J_c) \leq \lambda^{q+1}\}$. There are constants c and α, independent of q, B, T, I, and n, such that*

$$\frac{|B|}{|T|} \leq c \left[\frac{|\pi(\widehat{B})|}{|\pi(\widehat{T})|}\right]^\alpha. \tag{14.1}$$

The proof is essentially the same as for standard intervals, but we need a stronger version of A_∞-equivalence because $\partial\Lambda_J$ is unbounded. Namely, if Ω is a chord-arc domain, and $z \in \Omega$, then there are positive constants c_1, c_2, α_1, and α_2 such that

$$\frac{|E|}{|S|} \leq c_1 \left[\frac{\omega(z,E,\Omega)}{\omega(z,S,\Omega)}\right]^{\alpha_1} \quad \text{and} \quad \frac{\omega(z,E,\Omega)}{\omega(z,S,\Omega)} \leq c_2 \left[\frac{|E|}{|S|}\right]^{\alpha_2} \tag{14.2}$$

whenever S is a segment of $\partial\Omega$ satisfying

$$S \subset \{w \mid \lambda^q \leq \operatorname{dist}(w, z) \leq \lambda^{q+1}\} \tag{14.3}$$

for any positive integer q and for a fixed $\lambda > 1$, and E is a Borel subset of S. The constants c_1, c_2, α_1, and α_2 depend on the chord-arc constant of $\partial\Omega$ and on the constant λ, but not on q.

This follows from the case of bounded chord-arc domains. By dilations and translations we may assume that $z = i$, and that $-i$ is at least distance ε from Ω. Consider the Möbius transformation $\tau : w \mapsto (w+i)^{-1}$. This maps Ω to a bounded chord-arc domain. Harmonic measure is invariant under Möbius transformations. Also, for each positive integer q, the map τ scales everything in the annulus in (14.3), in particular E and S, by approximately the same factor. The estimates in (14.2) follow.

Consider the domain $D = \pi^{-1}(\Lambda_{J_c})$, where I is a non-standard interval as above. The analogues of Lemmas 13.2 and 13.3 hold, and Lemma 14.1 follows in the same way as Lemma 13.1 follows from Lemmas 13.2 and 13.3.

Notice that, although the interval I is no longer standard, arclength and harmonic measure are still A_∞-equivalent on ∂D, since D is a bounded chord-arc domain. Therefore, for any segment S of ∂D and Borel subset E of S, we still have

$$\frac{|E|}{|S|} \leq c_1 \left[\frac{\omega(z_J, \pi(E), \Lambda_{J_c})}{\omega(z_J, \pi(S), \Lambda_{J_c})}\right]^{\alpha_1}, \tag{14.4}$$

and the analogous inequality in the other direction, both with constants independent of I. We will frequently use this fact, combined with direct estimates of harmonic measures in Λ_{J_c}. We refer to such an argument as a *comparison of harmonic measure on the domain side*.

We will need the next lemma, which follows immediately from our decomposition of $\overline{\mathbb{R}} \setminus K$ into Whitney intervals.

Lemma 14.2. *Let A be a subinterval of $[0,1]$ of length 3^{-l}, $l \geq 0$. Then*

a) A contains a Whitney interval of length at least $3^{-2}|A|$; and

b) A meets at most 2^k Whitney intervals of length $3^{-k}|A|$, for $k \geq 1$.

The main result of this section is:

Lemma 14.3. *There are positive constants c_1 and c_2 such that for all $n \geq 1$, for every non-standard grid interval $I \in \mathcal{H}_{n-1}$ such that $\pi(\widehat{I}\,) = J_c$ is the central Whitney interval in some component L of $[0,1] \setminus K$,*

$$EX_i = \frac{1}{\mu(I)} \int_I X_i(x)\, d\mu \geq c_1; \tag{14.5}$$

and

$$EX_i^2 = \frac{1}{\mu(I)} \int_I X_i^2(x)\, d\mu \leq c_2. \tag{14.6}$$

Here $X_i(x) = \log_3\big(|J_i(x)|/|J_c|\big)$ is the auxiliary function for a particle which has not yet reached J_∞ and which makes its i^{th} jump from $J_c = \pi(\widehat{I}\,)$.

Proof. Let J_c be the central Whitney interval, $J_c = J_{-N} \cup \cdots \cup J_N$, in some component L of $[0,1] \setminus K$. Make the convention that $|J_c| = |L|/3$. The tips of the leaves for the standard intervals in L cover K_l and K_r, the closed construction intervals of K immediately to the left and right of L. So the tip E_{J_c} of the leaf for J_c is $\overline{\mathbb{R}} \setminus (K_l \cup L \cup K_r)$. See Figure 13.

To simplify the calculations, we again assume, as in the proof of Lemma 13.4, that in each component L of $E_{J_c} \setminus K$ we have not amalgamated the $2N$ central Whitney intervals but have retained them individually. As before, the result follows from this simplified version.

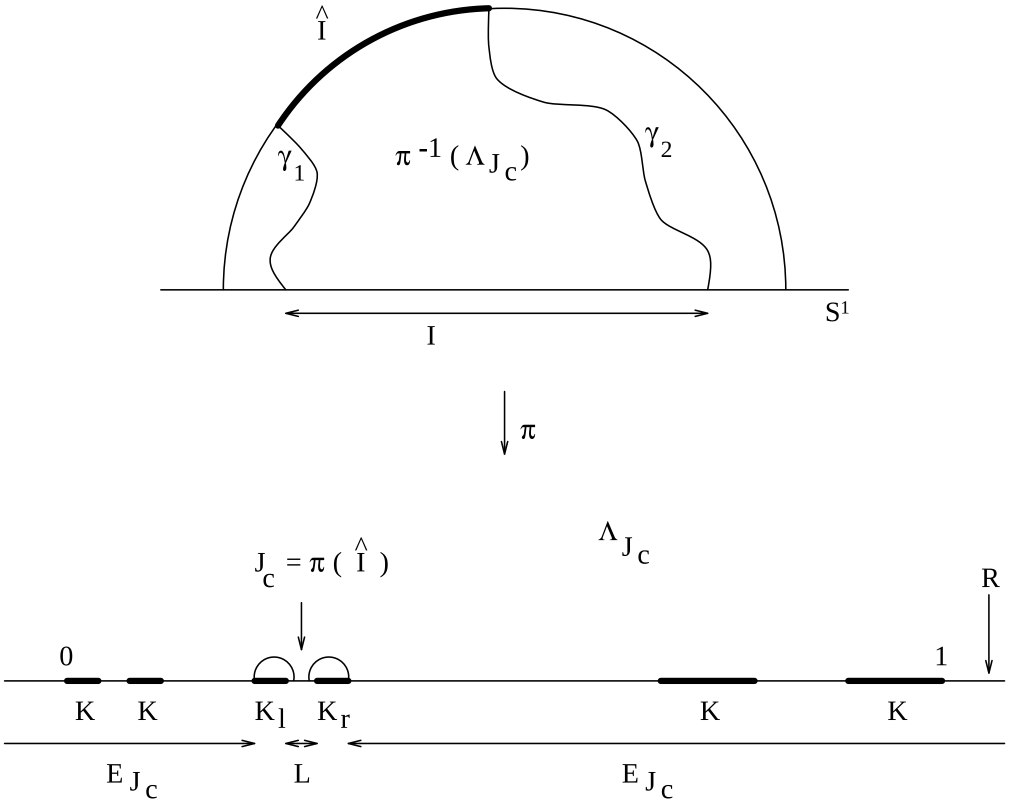

Figure 13. A non-standard central interval J_c.

We divide E_{J_c} into subsets V_j, $j \geq 1$, of length $|V_j| = 3^{j+10}|J_c|$. Let each V_j have two components of equal length, one on each side of J_c. Put the components of V_1 as close as possible to J_c, that is, on each side of and adjacent to $K_l \cup L \cup K_r$. Put the components of V_2 on each side of and adjacent to $V_1 \cup K_l \cup L \cup K_r$, and so on. For each $j \geq 1$, let T_j be the subset of I such that $\pi(\widehat{T_j}) = V_j$.

Fix $j \geq 1$, and for each $k \geq 1$ define the subset B_k of T_j by

$$\begin{aligned} B_k &= \{x \in T_j \bigm| |J_i(x)| = 3^{-k}\,|V_j|\} \\ &= \{x \in T_j \bigm| |J_i(x)| = 3^{-k} \cdot 3^{j+10}|J_c|\} \\ &= \{x \in T_j \bigm| X_i(x) = j + 10 - k\}. \end{aligned} \tag{14.7}$$

By Lemma 14.2, the set $\pi(\widehat{B_k})$ meets at most 2^k Whitney intervals of size $3^{-k}|V_j|$, so

$$|\pi(\widehat{B_k})| \leq \left(\frac{2}{3}\right)^k |V_j|. \tag{14.8}$$

By Lemma 14.1,

$$\frac{|B_k|}{|T_j|} \leq c\left[\frac{|\pi(\widehat{B_k})|}{|V_j|}\right]^\alpha = c\left[\left(\frac{2}{3}\right)^k\right]^\alpha = c\,\beta^k, \tag{14.9}$$

where $\beta = (2/3)^\alpha$ is strictly less than one, and c and β are independent of j.

In fact, Lemma 14.1 in the form stated above may not apply, since B_k and T_j need not be unions of whole grid intervals. However, the difficulty is only that at either end of T_j there may be grid intervals which meet T_j (and hence some B_k) but are not contained in T_j (or B_k). The distortion estimates in Section 10, and the fact that grid intervals are comparable in size to their neighbours, imply that Lemma 14.1 can be extended to cover this case.

We first prove the estimate on EX_i^2. In estimating X_i^2 we can neglect the function F_n, since the F_n we use will be bounded above by a constant independent of I. So it is enough to find a bound independent of I for

$$\frac{1}{|I|}\int_I X_i^2\,dx = \sum_{j=1}^{\infty} \frac{|T_j|}{|I|}\frac{1}{|T_j|}\int_{T_j}\left\{\log_3\left[\frac{|J_i(x)|}{|J_c|}\right]\right\}^2 dx. \tag{14.10}$$

First, for each j,

$$\begin{aligned}\frac{1}{|T_j|}\int_{T_j}\left\{\log_3\left[\frac{|J_i(x)|}{|J_c|}\right]\right\}^2 &= \sum_{k=1}^{\infty}\{j+10-k\}^2\,\frac{|B_k|}{|T_j|}\\ &\le c\sum_{k=1}^{\infty}\left\{j^2+(k-10)^2\right\}\,\beta^k\\ &\le c'j^2+c''.\end{aligned} \tag{14.11}$$

Also, $|T_j|/|I|$ decays exponentially in j:

$$\begin{aligned}\frac{|T_j|}{|I|} &\le c\,\frac{|\widehat{T_j}|}{|\widetilde{I}\,|}\\ &\le c\,c_1\left[\frac{\omega\big(z_I,\widehat{T_j},\pi^{-1}(\Lambda_{J_c})\big)}{\omega\big(z_I,\widetilde{I},\pi^{-1}(\Lambda_{J_c})\big)}\right]^{\alpha_1}\\ &= c\,c_1\left[\frac{\omega(z_J,V_j,\Lambda_{J_c})}{\omega(z_J,E_{J_c},\Lambda_{J_c})}\right]^{\alpha_1}\\ &\le c\left[\omega\left(z_J,V_j,\mathbf{U}\right)\right]^{\alpha_1},\end{aligned} \tag{14.12}$$

with constants independent of j. Now $\pi\cdot\omega(z_J,V_j,\mathbf{U})$ is the angle subtended at z_J by V_j. By elementary trigonometry, this angle is less than $c\,3^{-j}$, with c independent of j. Therefore

$$\frac{|T_j|}{|I|}\le c\,3^{-\alpha_1 j}. \tag{14.13}$$

Hence

$$\frac{1}{|I|}\int_I X_i^2\,dx \le c\sum_{j=1}^{\infty} 3^{-\alpha_1 j}(c'j^2+c''). \tag{14.14}$$

The right hand side is finite and independent of I.

It remains to prove the estimate on EX_i. Fix $\varepsilon > 0$, and $j \geq 2$. By Lemma 14.2, B_2 is not empty. (For those $V_j \not\subset [0,1]$, B_2 may be empty; part a) of Lemma 14.2 need not hold as stated because of the amalgamation of the large Whitney intervals far from $[0,1]$ to form J_∞. However our argument can be modified to cover this case.) Let $B = B_1 \cup B_2$, and define F_n on T_j by

$$F(x) = F_n(x) = \begin{cases} (1-\varepsilon)\,\frac{|T_j|}{|B|}, & x \in B; \\ \delta, & x \in T_j \setminus B, \end{cases} \tag{14.15}$$

where $\delta = \varepsilon|T_j|/|T_j \setminus B|$ is chosen so that F has mean value one on T_j. A comparison of harmonic measure shows that $|T_j|/|B|$ is uniformly bounded for all j and I. Therefore F is uniformly bounded above and below by positive constants independent of j and I.

On B, $X_i \geq j+8$. Then

$$\begin{aligned} \frac{1}{\mu(T_j)}\int_{T_j} X_i\,d\mu &= \frac{1}{|T_j|}\int_{T_j} X_i F\,dx \\ &\geq (j+8)(1-\varepsilon) + \delta\sum_{k=3}^{\infty}(j+10-k)\,\frac{|B_k|}{|T_j|} \\ &\geq (j+8)(1-\varepsilon) - \delta\sum_{k=j+10}^{\infty}(k-j-10)\,\frac{|B_k|}{|T_j|} \\ &\geq (j+8)(1-\varepsilon) - \delta\, c(N)\sum_{k-j=10}^{\infty}(k-j-10)\,\beta^k. \end{aligned} \tag{14.16}$$

The sum is finite and independent of j. Hence, by choosing δ (or equivalently ε) small enough in the definition of F, we may ensure that the mean of X_iF with respect to μ on T_j is at least some positive constant c_1, independent of j.

To ensure that $F \equiv 1$ on suitable intervals at each end of I, we define F slightly differently on T_1. Specifically, let I_l and I_r be subintervals at each end of I. Then $\pi(\widehat{I_l})$ and $\pi(\widehat{I_r})$ are subintervals of V_1, one on each side of J_c and as close as possible to J_c; in other words, adjacent to $K_l \cup L \cup K_r$. Choose the lengths of I_l and I_r so that

$$\frac{|\pi(\widehat{I_l})|}{|T_1|} = \frac{|\pi(\widehat{I_r})|}{|T_1|} = \frac{3}{2}\cdot 3^{-Q}, \tag{14.17}$$

for some large integer Q. The harmonic measure of $\pi(\widehat{I_l}) \cup \pi(\widehat{I_r})$ in Λ_{J_c} as seen from z_J is a positive constant independent of I, since the domains Λ_{J_c} for different I are all Euclidean dilations of each other. By a comparison of harmonic measure, there is an $\eta > 0$ such that $|I_l|$ and $|I_r| \geq \eta\,|I|$ for all I. Now let

$$F(x) = \begin{cases} (1-\varepsilon)\frac{|T_1|}{|B|}, & x \in B; \\ 1, & x \in I_l \cup I_r; \\ \delta, & x \in T_1 \setminus (B \cup I_l \cup I_r). \end{cases} \tag{14.18}$$

Again, we choose δ so that F has mean value one on T_1. Then

$$\delta = \left(\varepsilon - \frac{|I_l \cup I_r|}{|T_1|}\right) \frac{|T_1|}{|T_1 \setminus (B \cup I_l \cup I_r)|} \geq \varepsilon/2, \tag{14.19}$$

say, if Q is chosen large enough that $|I_l \cup I_r|/|T_1| \leq \varepsilon/2$. Therefore F is bounded above and below on T_1 by positive constants independent of I.

Notice that with this new definition of F on T_1, we can still make the mean value of $X_i F$ on T_1 greater than c_1. For, just as for standard intervals, harmonic measure estimates show that $|B_k \cap (I_l \cup I_r)|/|T_1|$ decays exponentially in k, and so the contribution from $I_l \cup I_r$ to the mean value goes to zero as Q goes to infinity.

This completes the proof of Lemma 14.3. □

15. μ is supported on $L_c(G)$

Define the functions F_n as in Section 11 on all standard intervals $I \in \mathcal{H}_{n-1}$, $n \geq 1$, and as in Section 14 on all non-standard intervals $I \in \mathcal{H}_{n-1}$, $n \geq 1$, such that $J_c = \pi(\widehat{I}\,)$ is the non-standard central Whitney interval in any component of $[0,1] \setminus K$. Define $F_n \equiv 1$ on the remaining grid intervals $I \in \mathcal{H}_{n-1}$, $n \geq 1$; these are exactly those $I \in \mathcal{H}$ such that $\pi(\widehat{I}\,) = J_\infty$. We have shown that the first two types of these functions are (δ, η)-suitable for all $I \in \mathcal{H}_{n-1}$, $n \geq 1$, where δ and η are constants independent of I and n. This is also true when $F_n \equiv 1$ on $I \in \mathcal{H}_{n-1}$. Therefore, by Lemma 6.3, the measures μ_n defined by $d\mu_n = F_n(x) \cdots F_1(x)\, dx$ converge to a doubling measure μ on the circle.

Consider the random walk on the tree discussed in Section 4, where the vertices in the tree are the Whitney intervals $\widehat{I}$ in the boundary arcs $\bigcup_n \mathcal{A}_n$ of the half fundamental domains in our tiling of the disc; V is the subset of vertices such that $\pi(\widehat{I}\,) = J_\infty$ (in other words, those intervals which contain orbit points $g(0)$, $g \in G$); and the probabilities of jumps between adjacent vertices are determined by the functions F_n defined above.

In this section, we first use the estimates $EX_i \geq c_1$ and $EX_i^2 \leq c_2$ established in Sections 13 and 14 to show that a particle starting from any vertex v reaches $V(v)$ with probability one. Here $V(v)$ is the set of vertices $w \in V$ below v such that there are no other vertices from V between w and v. We conclude that the doubling measure μ is supported on the set S of points which lie in infinitely many grid intervals I such that $\pi(\widehat{I}\,) = J_\infty$. Finally, we show that the conical limit set of G contains this set S, which establishes Theorem 1.2.

Let v be a vertex in the tree, and $I_0 \in \mathcal{H}_s$ the corresponding grid interval. Let $V(I_0)$ be the collection of maximal grid intervals $I \subset I_0$ such that $\pi(\widehat{I}\,) = J_\infty$. We show that the restriction of μ to I_0 is supported on $V(I_0)$.

Recall that

$$X_i(x) = \begin{cases} \log_3 \left[\frac{|J_i(x)|}{|J_{i-1}(x)|}\right], & \text{if } J_1(x), \ldots, J_{i-1}(x) \neq J_\infty; \\ 1, & \text{otherwise}, \end{cases} \tag{15.1}$$

that $S_k(x) = \sum_{i=1}^k X_i(x)$, and that $V(I_0) = \{x \in I_0 \bigm| S_k(x) \to +\infty\}$. It is enough to show that $S_k(x) \to +\infty$ almost surely on I_0 with respect to μ.

Let $E_I X_i$ denote the mean value of X_i on I with respect to μ. We know that $|E_I X_i| < \infty$ for all grid intervals $I \in \mathcal{H}$. This is because $|E_I X_i| \leq E_I |X_i| \leq E_I X_i^2 < c_2$, since X_i is integer-valued. Define new functions $\widetilde{X}_i(x)$ on I_0 which have mean zero on each $I \in \mathcal{H}_{s+i-1}$ which lies in I_0:

$$\begin{aligned}\widetilde{X}_1(x) &= X_1(x) - E_{I_0} X_1;\\ \widetilde{X}_i(x) &= \sum_{\substack{I \in \mathcal{H}_{s+i-1}\\ I \subset I_0}} \left(X_i(x) - E_I X_i\right) \chi_I(x),\end{aligned} \tag{15.2}$$

for $i \geq 1$, and let

$$\begin{aligned}\widetilde{S}_k(x) &= \sum_{i=1}^{k} \widetilde{X}_i(x)\\ &= \sum_{i=1}^{k} \big(X_i(x) - E_{I_{i-1}} X_i\big)\end{aligned} \tag{15.3}$$

where $x \in I_{i-1}$, $i \geq 1$.

Then $\{\widetilde{S}_k\}$ is a martingale with respect to μ.

Also, $E\widetilde{X}_i^2$ is bounded by EX_i^2, which is bounded by c_2. Therefore, by the strong law of large numbers for martingales [Fel], $\widetilde{S}_k/k \to 0$ almost surely on I_0 with respect to μ.

Now let x be a point in I_0 such that $\widetilde{S}_k(x)/k \to 0$, and let $I_i \in \mathcal{H}_{s+i}$ be the interval containing x, for $i \geq 0$. Then

$$\begin{aligned}\frac{S_k(x)}{k} &= \frac{1}{k} \sum_{i=1}^{k} X_i(x)\\ &= \frac{1}{k} \sum_{i=1}^{k} \Big(\widetilde{X}_i(x) + E^{d\mu}_{I_{i-1}} X_i\Big)\\ &\geq \frac{1}{k}\, \widetilde{S}_k(x) + c_1\\ &\geq c_1/2,\end{aligned} \tag{15.4}$$

for all k sufficiently large that $|\widetilde{S}_k(x)/k\,| \leq c_1/2$. Therefore $S_k(x) \geq c_1 k/2$ for all such k, and so $S_k(x) \to +\infty$. Thus $S_k \to +\infty$ almost surely with respect to μ on I_0. Since $V(I_0)$ is exactly the set of points x in I_0 such that $S_k \to +\infty$, this implies that $\mu(V(I_0)) = \mu(I_0)$.

Since a particle starting from any vertex v reaches $V(v)$ with probability one, it is now clear that a particle starting anywhere in the tree will pass through infinitely many vertices in V, with probability one. (Note that we have now defined the only jump probabilities which were missing; these are the probabilities of the jumps from vertices in the subset V, and they are determined by the functions $F_n \equiv 1$ on the corresponding grid intervals.) Thus we have shown that the doubling measure μ is supported on the set S of points in the circle which lie in infinitely many grid intervals $I = P(\widehat{I}\,)$ such that $\pi(\widehat{I}\,) = J_\infty$.

It remains to show that S is contained in the conical limit set of G. We show that there is a uniform constant l such that for each orbit point $g_j(0) \neq 0$, the Whitney interval $\widehat{I_j}$ which contains $g_j(0)$ projects to a grid interval $P(\widehat{I_j}\,)$ which lies in the spherical cap $\mathrm{Cap}\,(g_j(0), l)$ on $g_j(0)$. Then each point $x \in S$ lies in infinitely many of these caps, and therefore in the conical limit set.

By definition, the tip of the leaf based at J_∞ is $E_{J_\infty} = [1/9, 8/9]$. For each $g_j \in G\backslash\{\mathrm{id}\}$ let A_j be the orthocircular arc in $\bigcup_n \mathcal{A}_n$ which contains $g_j(0)$, and let Ω_j be the half fundamental domain below A_j. Let $\mathcal{F}_j$ be the fundamental domain consisting of Ω_j and the half fundamental domain above A_j. Let $\widehat{E_j}$ be the segment of $\partial\Omega_j$, below $g_j(0)$, such that $\pi(\widehat{E_j}) = E_{J_\infty}$. The endpoints of $\widehat{E_j}$ lie in $\partial\mathbb{D}$. Then $P(\widehat{I_j}\,) = E_j$ is the arc of $\partial\mathbb{D}$ below $\widehat{E_j}$ (with the same endpoints as $\widehat{E_j}$). See Figure 14.

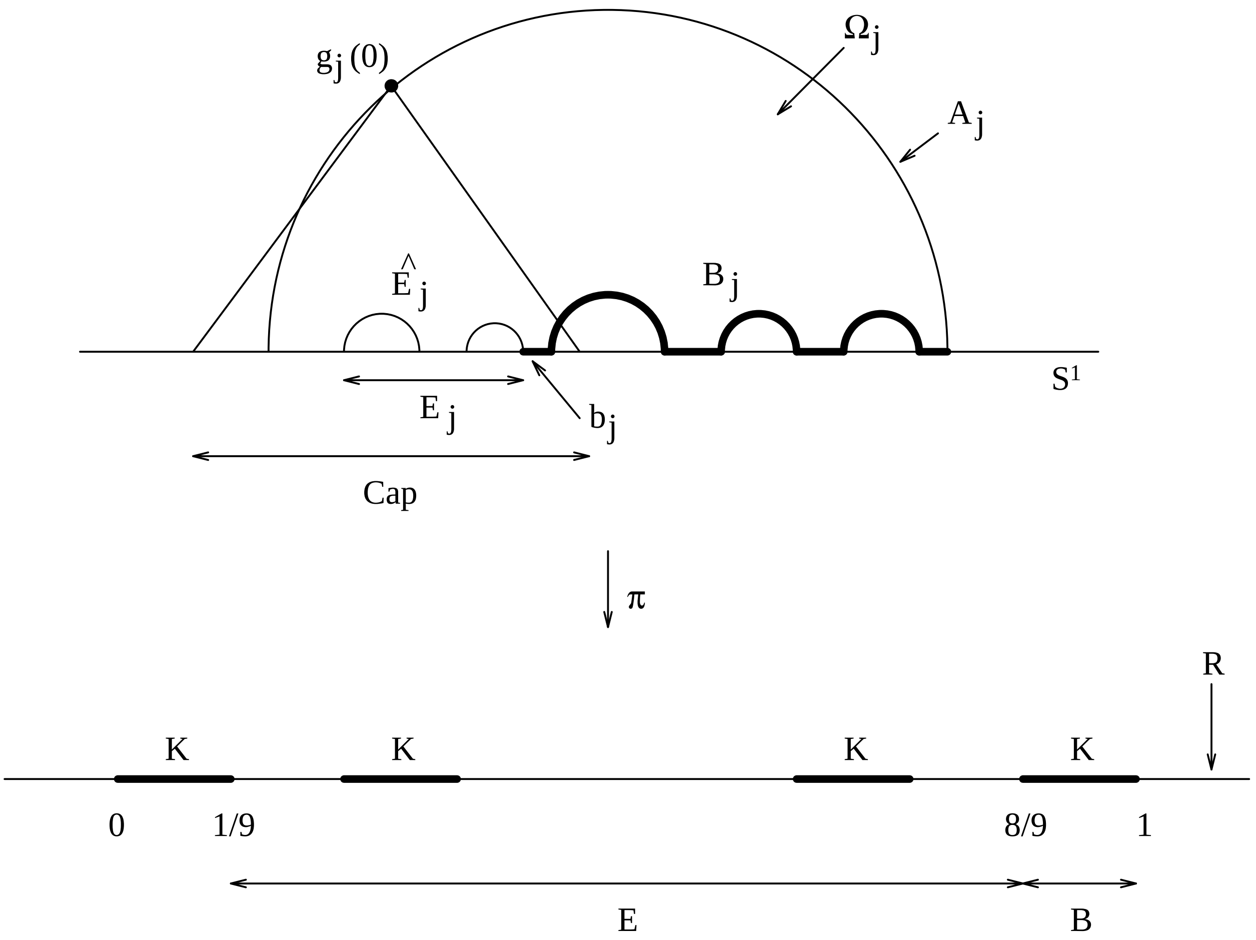

Figure 14. $P(\widehat{I_j}\,) = E_j$ lies in Cap $= \mathrm{Cap}\,(g_j(0), l)$; $E = E_{J_\infty}$.

Lemma 15.1. *There is a constant $l > 0$ such that for all j,*

$$P(\widehat{I_j}\,) \subset \mathrm{Cap}\,(g_j(0), l). \tag{15.5}$$

Proof. By definition,

$$\operatorname{Cap}(g_j(0), l) = \{z \in \partial\mathbb{D} \mid \text{dist } (z, g_j(0)) \le l\,(1 - |g_j(0)|)\}.$$

Let $z_j = g_j(0)$. Let $b_j \in \partial\mathbb{D}$ be the endpoint of $P(\widehat{I_j}\,)$ which is furthest from z_j. We show that $|z_j - b_j|/(1 - |z_j|) \le c$.

If b_j satisfies $|b_j - z_j| \le 10\,(1 - |z_j|)$, we are done. If not, let B_j be the component of $\pi^{-1}([0,1] \setminus E_{J_\infty})$ which has b_j as an endpoint; and let $B = \pi(B_j)$. See Figure 14 for one possible configuration.

First, $|z_j - b_j| \le c \operatorname{dist}(z_j, B_j)$. For if a is any point in $B_j \cap \partial\mathbb{D}$, then $|z_j - b_j| \le |z_j - a|$, while if a is in $B_j \setminus \partial\mathbb{D}$ then a lies in one of the orthocircular arcs in $\partial\mathcal{F}_j$, and $|z_j - a| \ge c\,|z_j - b_j|$.

By Beurling's Lemma (see also (10.12)),

$$\begin{aligned}
\operatorname{dist}(z_j, B_j) &\le c\,\frac{\operatorname{dist}(z_j, \partial\mathcal{F}_j)}{\omega(z_j, B_j, \mathcal{F}_j)^2} \\
&\le c\,\frac{\operatorname{dist}(z_j, \partial\mathcal{F}_j)}{\omega(\pi(z_j), \pi(B_j), \pi(\mathcal{F}_j))^2} \\
&= c\,\frac{\operatorname{dist}(z_j, \partial\mathcal{F}_j)}{\omega(\infty, B, \overline{\mathbf{C}} \setminus K)^2} \\
&\le c \operatorname{dist}(z_j, \partial\mathcal{F}_j) \\
&\le c\,(1 - |z_j|).
\end{aligned} \tag{15.6}$$

□

We have shown that each point in $\partial\mathbb{D}$ which is at the end of a path which passes through infinitely many orbit points is actually a conical limit point of G; in particular, that the orbit points on the path not only accumulate at the endpoint but lie in some non-tangential cone based at the endpoint. Therefore the set S lies in the conical limit set of G.

This completes the proof of Theorem 1.2.

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