A MIXED HODGE STRUCTURE ON A CR MANIFOLD

Takao Akahori

Abstract. The purpose of this work is to propose a mixed Hodge structure over a CR manifold. As you know, for a CR manifold, Kohn–Rossi cohomology is naturally introduced. However, the relation between Kohn–Rossi cohomology and De Rham cohomology is not so well understood, even in Tanaka's work. We discuss this point.

The purpose of this paper is to propose a mixed Hodge structure for CR manifolds. In our former papers([A1],[A2]), we studied deformation theory of CR manifolds, and constructed the versal family of deformations of CR structures. Later, we discussed the smoothness of the parameter space of this versal family, namely the analogy of Bogomolov theorem for compact Kaehler manifolds(cf. [A-M1],[A-M2]). However, our work is still not so definitive. And so, in order to investigate the versal family more precisely, we have to formulate our notion of a mixed Hodge structure for general CR manifolds. Our terminology is not the standard one. For example, algebraic geometrists already introduced the notin of mixed Hodge structures for algebraic spaces, and even for link structures(see [Dur]). However, for general CR-structures, the mixed Hodge structure has not been given. And the relation with Kohn–Rossi cohomology has never been discussed. In this paper, for general CR structures, we establish the notion of mixed Hodge structure, which is applicable to deformation theory of CR-structures. For this purpose, we use the same method which succeeded in deformation theory. Namely, we start with $\Gamma(M, (C\xi)^* \wedge \wedge^p({}^0T')^* \wedge^q ({}^0T'')^*)$(for the notation, see Sect.1). And consider $F^{p,q}$, which is defined as a subspace in the above space. And we show that this space admits d', d'' and show that our complex recovers the standard Kohn–Rossi cohomology. As you know, Kohn–Rossi cohomology doesn't depend on the choice of the supplement vector field. However, our mixed Hodge structure heavily relies on it.

We should note the recent Rumin's work on the contact geometry (cf.[Ru]). It is John M Lee who pointed out that our complex quite resembles Rumin's complex and also Rumin's work reminds him of our former work on E_j-strucutures about deformation theory of CR-structures(see [A1],[A2]). However, our point of view comes from deformation theory (see Sect.2 in this paper), naturally, it should lead to our Hodge theory(see [A-M2]). And as is mentioned already, our complex arises from deformation theory. So, we think that the view point is different from Rumin's one, and in order to stress this point, our work is worth publication.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

1. CR-structures and Kohn–Rossi cohomology group

Let N be a complex manifold. Let Ω be a strongly pseudo convex domain with smooth boundary $M = b\Omega$. Then, as is well knwon, over this boundary M, a CR structure is induced from N. Namely, we set

$$ {}^0T'' = C \otimes TM \cap T''N \mid_M . $$

Then, this ${}^0T''$, a subbundle of $C \otimes TM$ satisfies;

$$ {}^0T'' \cap \overline{{}^0T''} = 0, dim_C \frac{C \otimes TM}{{}^0T'' + \overline{{}^0T''}} = 1 \tag{1} $$

$$ [\Gamma(M, {}^0T''), \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T''). \tag{2} $$

This pair $(M, {}^0T'')$ is called a CR-structure. For this CR structure, we have $\overline{\partial}_b$-operator,

$$ \overline{\partial}_b; \Gamma(M, C) \to \Gamma(M, ({}^0T'')^*) $$

by: for $f \in \Gamma(M, C)$, $\overline{\partial}_b f(X) = Xf, X \in {}^0T''$. Like the standard exterior derivative d, we have $\overline{\partial}_b$-complex,

$$ 0 \to \Gamma(M, C) \xrightarrow{\overline{\partial}_b} \Gamma(M, ({}^0T'')^*) \xrightarrow{\overline{\partial}_b} \Gamma(M, \wedge^2({}^0T'')^*) \to $$

$$ \to \Gamma(M, \wedge^p({}^0T'')^*) \xrightarrow{\overline{\partial}_b} \Gamma(M, \wedge^{p+1}({}^0T'')^*) \to $$

Now we recall the Kohn–Rossi cohomology(cf.[4]). We set a C^∞ vector bundle decomposition

$$ C \otimes TM = {}^0T' + {}^0T'' + C\xi, $$

where $\xi_p \notin {}^0T'_p + {}^0T''_p$ for every point p of M, and ${}^0T'$ means $\overline{{}^0T''}$. We fix this and use the notation T' for ${}^0T' + C\xi$, that is to say,

$$ T' = {}^0T' + C\xi, and C \otimes TM = {}^0T'' + T' \tag{1.2.1} $$

We set

$$ C^{p,q} = \Gamma(M, \wedge^p(T')^* \wedge \wedge^q({}^0T'')^*). $$

Let d be the exterior differential operator. Then, not like the case complex manifolds,

$$ d : C^{p,q} \to C^{p+1,q} + C^{p,q+1} + C^{p+2,q-1}. $$

This fact was observed by Tanaka(cf.[6]). In fact, for u in $C^{p,q}$, for $X_i \in T'$, $Y_j \in {}^0T''$,

$$ \begin{aligned} (du)(X_1, .., X_{p+2}, Y_1, .., Y_{q-1}) &= \sum_j (-1)^{j+1} X_j u(X_1, ., \check{X}_j, ., X_{p+2}, Y_1, .., Y_{q-1}) \\ &+ \sum_i (-1)^{p+2+i+1} Y_i u(X_1, .., X_{p+2}, Y_1, ., \check{Y}_i, ., Y_{q-1}) \\ &+ \sum_{r<s} (-1)^{r+s} u([X_r, X_s], ., \check{X}_r, ., \check{X}_s, ., Y_1, .., Y_{q-1}) \\ &+ \sum_{r,s} (-1)^{r+s+p+2} u([X_r, Y_s], ., \check{X}_r, ., \check{Y}_s, .,) \\ &+ \sum_{r<s} (-1)^{r+s} u([Y_r, Y_s], X_1 .., X_{p+2}, \check{Y}_r, ., \check{Y}_s, ..). \end{aligned} $$

Because of $u \in C^{p,q}$, the first line must vanish. Similarly, the second line, the fourth line, the fifth line vanish. However, the third line may not vanish. Because the ${}^0T''$ part of $[X_r, X_s]$ might appear in general CR manifolds.(${}^0T''$ is integrable, but T' is not). With this in mind, we set; for u in $C^{p,q}$,

$$\overline{\partial}_b u = (du)_{\wedge^p(T')^* \wedge \wedge^{q+1}({}^0T'')^*}.$$

Here $(du)_{\wedge^p(T')^* \wedge \wedge^{q+1}({}^0T'')^*}$ means the $\wedge^p(T')^* \wedge \wedge^{q+1}({}^0T'')^*$ part of du. So, we have

$$C^{p,q} \xrightarrow{\overline{\partial}_b} C^{p,q+1},$$

$\overline{\partial}_b\overline{\partial}_b = 0$. And we have a cohomology group(Kohn–Rossi cohomology)

$$H^{p,q}(M) = \frac{Ker\overline{\partial}_b \cap C^{p,q}}{\overline{\partial}_b C^{p,q-1}}.$$

This coholomology group has the following mean. Let U be a tubular neighborhood of the boundary $M = b\Omega$ in N. Then,

$$H^{p,q}(M) \simeq H^q(U, \Omega_U^p), 1 \leq q \leq n-1.$$

Here $H^q(U, \Omega_U^p)$ means the $\overline{\partial}$ Dolbeault cohomology group over U. While, over a tubular neighborhood U, there is ∂ operator (U is a complex manifold), and we can discuss the mixed Hodge theory over U. The purpose of my work is to study a mixed Hodge theory over CR - manifolds. Of course, it is possible to introduce a kind of ∂_b operator. Namely, for $v \in C^{p,q}$, we set

$$\partial_b v = (dv)_{\wedge^{p+1}(T')^* \wedge \wedge^q({}^0T'')^*}.$$

Then, we have

$$C^{p,q} \xrightarrow{\partial_b} C^{p+1,q}.$$

However, as is shown, by Tanaka(cf.[6]), unfortunately, in this case, in general,

$$\partial_b\partial_b \neq 0$$

(because T' is not integrable). And so $(C^{p,q}, d)$ is not even a double complex.

2. Deformation theory of CR-structures

In this section, we recall deformation theory of CR-structures.

2.1.AlmostCRmanifolds

Let M be a C^∞ differentiable manifold with real dimension $2n-1$. Let E be a C^∞ subvector bundle of the complexfied tangent bundle $C \otimes TM$ satisfying;

$$E \cap \overline{E} = 0, dim_C \frac{C \otimes TM}{E + \overline{E}} = 1.$$

This pair (M,E) is called an almost CR-manifold or an almost CR-structure. Now let $(M, {}^0T'')$ be a CR manifold, Then, by using the C^∞ vector bundle decomposition

(1.2.1), for each point p of M, we have a homomorphism map from E to ${}^0T''_p$, a composition map of the inclusion of E to $C \otimes TM$ and the projection of $C \otimes TM$ to ${}^0T''$.

Definition2.1.1. Let $(M, {}^0T'')$ be a CR manifold. (M, E) is called an almost CR manifold with finite distance from $(M, {}^0T'')$ if and only if the above homomorhism map is an isomorphism map.

Then, we have

Proposition2.1.2. If (M, E) an almost CR manifold with finite distance from $(M, {}^0T'')$, then there is a $\phi \in \Gamma(M, T' \otimes ({}^0T'')^*)$ satisfying;

$$\begin{aligned} E =& {}^\phi T'' \\ =& \{X'; X' = X + \phi(X), X \in {}^0T''\} \end{aligned}$$

For the proof, see [A1].

This almost CR structure $(M, {}^\phi T'')$ is a CR structure if and only if ϕ satisfies the following non linear differential equation.

$$\overline{\partial}^{(1)}_{T'}\phi + R_2(\phi) + R_3(\phi) = 0.$$

For the notations, see [A1],[A2]. We recall $\overline{\partial}_{T'}$ - cohomology(so called Deformation complex). Namely, we set $\Gamma(M, T')$, consisting of T' valued global C^∞ sections on M, and consider a first order differential operator

$$\overline{\partial}_{T'}; \Gamma(M, T') \to \Gamma(M, T' \otimes ({}^0T'')^*)$$

defined as follows.

For Z in $\Gamma(M, {}^0T'')$, and for a u in $\Gamma(M, T')$,

$$[fZ, u] = f[Z, u] - u(f)Z.$$

So we take T'-component according to (1.2.1), then we have

$$[fZ, u]_{T'} = f[Z, u]_{T'}.$$

With this in mind, we set a first order differential operator by;

$$\overline{\partial}_{T'}u(X) = [X, u]_{T'} for X \in {}^0T''.$$

And like as for scalar valued differential forms, we can introduce differential operators

$$\overline{\partial}^{(i)}_{T'} : \Gamma(M, T' \otimes \wedge^i({}^0T'')^*) \to \Gamma(M, T' \otimes \wedge^{i+1}({}^0T'')^*)$$

and we have a differential complex

$$\begin{aligned} 0 \to \Gamma(M, T') \to \Gamma(M, T' \otimes ({}^0T'')^*) \to \Gamma(M, T' \otimes \wedge^2({}^0T'')^*) \to \\ \to \Gamma(M, T' \otimes \wedge^p({}^0T'')^*) \to \Gamma(M, T' \otimes \wedge^{p+1}({}^0T'')^*) \to \end{aligned}$$

2.2.$\mathbf{E_j}$structure

Let $(M, {}^0T'')$ be an orientable CR structure. Then we can introduce the Levi-form over M. Namely, for each point p of M, we set

$$L(X_p, Y_p) = -\sqrt{-1}[X', \overline{Y'}]_{C\otimes F},$$

where X'(resp. Y') is a C^∞ extension of X_p(resp. Y_p), and $[X', \overline{Y'}]_{C\otimes F}$ means the projection of $[X', \overline{Y'}]$ to $C \otimes F$ according to (1.2.1). If this form is positive or negative definite, then our CR structure $(M,{}^0 T'')$ is called strongly pseudo convex. Henceforth, we assume that our CR structure $(M,{}^0 T'')$ is strongly pseudo convex. In this section we introduce a subbundle E_j of $T' \otimes \wedge^j({}^0T'')$ and show that the cohomology group associated with $(\overline{\partial}^{(i)}_{T'}, \Gamma(M, T' \otimes \wedge^i({}^0T'')^*))$ can be reduced to the cohomology group associated with $(\overline{\partial}_i, \Gamma(M, E_i))$.

First we set a subspace Γ_i of $\Gamma(M, T' \otimes \wedge^i({}^0T'')^*)$ by;

$$\Gamma_i = \{u; u \in \Gamma(M, {}^0T' \otimes \wedge^i({}^0T'')^*), (\overline{\partial}^{(i)}_{T'}u)_{C\otimes F} = 0\},$$

where $(\overline{\partial}^{(i)}_{T'}u)_{C\otimes F}$ denotes the projection from $T' \otimes \wedge^i({}^0T'')$ to $C \otimes F \wedge {}^i({}^0T'')$ according to (1.2.1). Then we have

Proposition2.2.1. There is a subbundle E_i of $T' \otimes \wedge^i({}^0T'')^*$ to $C \otimes F \wedge^i ({}^0T'')^*$ satisfying;

$$\Gamma_i = \Gamma(M, E_i).$$

Proof. We show that the mapping $(\overline{\partial}^{(i)}_{T'}u)_{C\otimes F}$ from $\Gamma(M,{}^0 T' \otimes \wedge^i({}^0T'')^*)$ to $\Gamma(M, C \otimes F \wedge^{i+1} ({}^0T'')^*)$ is linear with respect to C^∞ funtions. In fact, for any C^∞ function f and for any section u in $\Gamma(M,{}^0 T' \wedge \otimes({}^0T'')^*)$, we obtain

$$\begin{aligned}
&(\overline{\partial}^{(i)}_{T'}fu)_{C\otimes F}(X_1, ., X_{i+1})\\
&\quad=(\sum_j(-1)^j[X_j, fu(X_1, ., \check{X}_j, ., X_{i+1})]_{T'}\\
&\quad+\sum_{\alpha<\beta}(-1)^{\alpha+\beta}fu([X_\alpha, X_\beta], X_1, ., \check{X}_\alpha, ., \check{X}_\beta, ., X_{i+1}))_{C\otimes F}\\
&\quad=\sum_j(-1)^j[X_j, fu(X_1, ., \check{X}_j, ., X_{i+1})]_{C\otimes F}.
\end{aligned}$$

As u is an element in $\Gamma(M,{}^0 T' \otimes \wedge^i({}^0T'')^*)$, this leads

$$\begin{aligned}
&(\overline{\partial}^{(i)}_{T'}fu)_{C\otimes F}(X_1, ., X_{i+1})\\
&\quad=\sum_j(-1)^j[X_j, fu(X_1, ., \check{X}_j, ., Xi + 1)]_{C\otimes F}\\
&\quad=f\sum_j(-1)^j[X_j, u(X_1, ., \check{X}_j, ., X_{i+1})]_{C\otimes F}\\
&\quad=f(\overline{\partial}^{(i)}_{T'}u)_{C\otimes F}(X_1, ., X_{i+1}) forall\Gamma(M,{}^0 T'').
\end{aligned}$$

Therefore for each point p of M, we can deffine a vector space $E_{i,p}$ by;

$$E_{i,p} = \{u_p; u \in^0 T' \otimes \wedge^i({}^0T'')^*, (\overline{\partial}^{(i)}_{T'}u)_{C\otimes F}(p) = 0\}.$$

First, we prove that $dim_C E_{i,p}$ is independent of p. To do this, it is enough to show that the following vector bundle sequence is exact.

$$
{}^0T' \otimes \wedge^i({}^0T'')^* \to C \otimes F \otimes \wedge^{i+1}({}^0T'')^* \to 0 \tag{2.3.1}
$$
$$
u \to (\overline{\partial}^{(1)}_{T'} u)_{C\otimes F}
$$

For each point p in M, we take a system of moving frame $\{e_j\}_{j=1,.,n-1}$ of ${}^0T''$ in a neighborhood of p and F such that

$$
[e_i, \overline{e}_j]_{C\otimes F} = \sqrt{-1}\delta_{i,j}F.
$$

Then for any $F \otimes e^*_{k_1} \wedge e^*_{k_2} \wedge ... \wedge e^*_{k_{i+1}} (k_1 < k_2 < ... < k_{i+1})$, we set

$$
u = \overline{e}_{k_1} \otimes e^*_{k_2} \wedge ... \wedge e^*_{k_{i+1}}.
$$

By a simple calculation, we obtain

$$
(\overline{\partial}^{(1)}_{T'} u)_{C\otimes F} = F \otimes e^*_{k_1} \wedge ... \wedge e^*_{k_{i+1}}.
$$

Therefore the above mapping $(\overline{\partial}^{(i)}_{T'} u)_{C\otimes F}$ is surjective. Thus the sequence (2.2.1) is exact and $dim_C E_{i,p}$ is indepent of p. And so, $\cup_{p\in M} E_{i,p}$ is a vector bundle on M. By E_j, we write this vector bundle. Then, we have

$$
\Gamma_i = \Gamma(M, E_i).\mathbf{Q.E.D}
$$

We see this vector bundle more precisely.

Theorem2.2.2. $E_0 = 0$. And there is a following differential subcomplex.

$$
\begin{aligned}
0 \to \Gamma(M, E)_1) \xrightarrow{\overline{\partial}_1} \Gamma(M, E_2) \xrightarrow{\overline{\partial}_2} \Gamma(M, E_3) \\
\xrightarrow{\overline{\partial}_{i-1}} \Gamma(M, E_i) \xrightarrow{\overline{\partial}_i} \Gamma(M, E_{i+1})
\end{aligned}
$$

where $\overline{\partial}_i$ means the restriction of $\overline{\partial}^{(i)}_{T'}$ to $\Gamma(M, E_i)$.

Proof. By the definition of E_0, we have

$$
E_0 = \{u; u \in {}^0T', [u, X]_{C\otimes F} = 0 forall X \in {}^0T''\}.
$$

Since the Levi form is non-degenerate, we obtain

$$
E_0 = 0.
$$

Next we prove

$$
\overline{\partial}^{(i)}_{T'} \Gamma(M, E_i) \subset \Gamma(M, E_{i+1}).
$$

In fact for all $u \in \Gamma(M, E_i)$, it follows

$$
\overline{\partial}^{(i)}_{T'} u \in \Gamma(M, {}^0T' \otimes \wedge^{i+1}({}^0T'')^*)
$$

from the relation $(\overline{\partial}^{(i)}_{T'}u)_{C\otimes F} = 0$(definition of E_i). And

$$(\overline{\partial}^{(i+1)}_{T'}(\overline{\partial}^{(i)}_{T'}u))_{C\otimes F} = 0(from\overline{\partial}^{(i+1)}_{T'}\overline{\partial}^{(i)}_{T'} = 0).$$

These considerations leads to that;for any $u \in \Gamma(M, E_i)$, we have

$$\overline{\partial}^{(i)}_{T'}u \in \Gamma(M, E_{i+1}).$$

This completes the proof. **Q.E.D.**

As for E_1, we prove that $\Gamma(M, E_1)$ has sufficiently many sections. We impose the Levi-metric over M and we form the adjoint operator $\overline{\partial}^*_{T'}$ of $\overline{\partial}_{T'}$. We set the Laplacian

$$\Box_{T'} = \overline{\partial}_{T'}\overline{\partial}^*_{T'} + \overline{\partial}^*_{T'}\overline{\partial}_{T'}.$$

As is usual in the theory of harmonic forms, we obtain the harmonic space $\mathbf{H}^{(1)}_{\mathbf{T}'}$ in $\Gamma(M, T' \otimes ({}^0T'')^*)$ (we assume that $dim_R M \geq 7$). We introduce a differential operator $\mathcal{L}$ from $\Gamma(M, T' \otimes ({}^0T'')^*)$ to $\Gamma(M,{}^0T' \otimes ({}^0T'')^*)$ as folows; for each ϕ, $\Gamma(M, T' \otimes ({}^0T'')^*)$, we put

$$\mathcal{L}\phi = \phi - \overline{\partial}_{T'}\theta_\phi,$$

where θ_ϕ is an element of $\Gamma(M, {}^0T')$ defined by;

$$[X, \theta_\phi]_{C\otimes F} = \phi(X)_{C\otimes F} for any X \in \Gamma(M, {}^0T'').$$

Then we have the following theorem.

Theorem2.2.3. The mapping $\mathcal{L}\mid_{\mathbf{H}^{(1)}_{\mathbf{T}'}}$, being restricted to $\mathbf{H}^{(1)}_{\mathbf{T}'}$, is injective and

$$\mathcal{H} \subset \Gamma(M, E_1)$$

holds, where $\mathcal{H}$ denotes $\mathcal{L}(\mathbf{H}^{(1)}_{\mathbf{T}'})$.

Proof. Injectivity is obvious. So it suffices to show

$$\mathcal{H} \subset \Gamma(M, E_1).$$

That is to say, for all $\phi \in \mathbf{H}^{(1)}_{\mathbf{T}'}$,

$$\mathcal{L}\phi \in \Gamma(M, {}^0T' \otimes ({}^0T'')^*)$$

and

$$(\overline{\partial}^{(1)}_{T'}(\mathcal{L}\phi))_{C\otimes F}) = 0.$$

By the definition of $\mathcal{L}$, we have that; for ϕ in $\Gamma(M, T' \otimes ({}^0T'')^*)$

$$\mathcal{L}\phi in \Gamma(M,{}^0T' \otimes ({}^0T'')^*)$$

and for ϕ in $Ker\overline{\partial}^{(1)}_{T'}$, obviously,

$$\mathcal{L}\phi in Ker\overline{\partial}^{(1)}_{T'}.$$

Therefore our theorem follows. **Q.E.D.**

Especially by Theorem 2.2.3, we have that the injection $i; \mathcal{H} \hookrightarrow Ker\overline{\partial}_{T'}^{(1)}$ induces the surjective map

$$\mathcal{H} \to Ker\overline{\partial}_{T'}^{(1)}/Im\overline{\partial}_{T'}.$$

As for $E_i (2 \leq i \leq n-1)$, we have the following theorem.

Theorem2.2.4. The injection induces the isomorphism map

$$\tau : Ker\overline{\partial}_i/Im\overline{\partial}_{i-1} \to Ker\overline{\partial}_{T'}^{(i)}/Im\overline{\partial}_{T'}^{(i-1)},$$

where $2 \leq i \leq n-1$.

Proof. First, we prove that the above map is surjective. For this purpose, it suffices to prove that for all ϕ in $Ker\overline{\partial}_{T'}^{(i)}$, there is an element θ_ϕ of $\Gamma(M, {}^0T' \otimes \wedge^{i-1}({}^0T'')^*)$ satisfying;

$$\phi - \overline{\partial}_{T'}^{(i-1)}\theta_\phi \in \Gamma(M, E_i).$$

ϕ being in $Ker\overline{\partial}_{T'}^{(i)}$, we obtain

$$\phi - \overline{\partial}_{T'}^{(i-1)}\theta_\phi \in \Gamma(M, E_i).$$

Therefore the surjectivity is proved. Next, we prove that the above map is in injective. If ψ is in $Ker\overline{\partial}_i$ satisfying $\psi = \overline{\partial}_{T'}^{(i-1)}\phi$, there is an element θ_ϕ in $\Gamma(M, {}^0T' \otimes \wedge^{(i-2)}({}^0T'')^*)$ such that

$$(\psi)_{C\otimes F} = (\overline{\partial}_{T'}^{(i-2)}\theta_\phi)_{C\otimes F}$$

by the same argument as above(we assume $i \geq 2$). Of course $\phi - \overline{\partial}_{T'}^{(i-2)}\theta_\phi$ is in $\Gamma(M, E_i)$. And

$$\psi = \overline{\partial}_{T'}^{(i-1)}(\phi - \overline{\partial}_{T'}^{(i-2)}\theta_\phi)$$

holds. So we have our theorem. **Q.E.D.**

3. $F^{p,q}$ complex and Mixed Hodge structure

We start with

$$D^{p,q}(M) = \Gamma(M, (C\xi)^* \wedge \wedge^{p-1}({}^0T')^* \wedge \wedge^q({}^0T'')^*), if p \geq 1,$$

$$D^{0,q}(M) = 0.$$

And consider

$$F^{p,q}(M) = \{u : u \in D^{p,q}(M), (du)_{\wedge^p({}^0T')^* \wedge \wedge^{q+1}({}^0T'')^*} = 0\},$$

where $(du)_{\wedge^p({}^0T')^*\wedge\wedge^{q+1}({}^0T'')^*}$ means the $\wedge^p({}^0T')^*\wedge\wedge^{q+1}({}^0T'')^*$ part of du. We see this $F^{p,q}$ more precisely. For this, we compute $(du)_{\wedge^p({}^0T')^*\wedge\wedge^{q+1}({}^0T'')^*}$. For $X_i \in {}^0T'$,$Y_j \in {}^0T''$,

$$\begin{aligned}
&(du)(X_1,..,X_p,Y_1,..,Y_{q+1})\\
&=\sum_j(-1)^{i+1}X_ju(X_1,.,\check{X}_i,.,X_p,Y_1,..,Y_{q+1})\\
&+\sum_j(-1)^{p+j+1}Y_iu(X_1,..,X_p,Y_1,.,\check{Y}_j,.,Y_{q+1})\\
&+\sum_{r<s}(-1)^{r+s}u([X_r,Y_s],X_1,.,\check{X}_r,.,\check{X}_s,.,Y_1,..,Y_{q+1})\\
&+\sum_{r,s}(-1)^{r+p+s}u([X_r,Y_s],X_1,.,\check{X}_r,.,X_p,Y_1,.,\check{Y}_s,.,Y_{q+1})\\
&+\sum_{r<s}u([Y_r,Y_s],X_1,..,X_p,Y_1,.,\check{Y}_r,.,\check{Y}_s,.,Y_{q+1}).
\end{aligned}$$

So the condition $(du)_{\wedge^p({}^0T')^*\wedge\wedge^{q+1}({}^0T'')^*}=0$ becomes

$$d\theta\wedge u=0,$$

where θ is a real one form defined by;

$$\theta(\xi)=1,\theta\mid_{{}^0T'+{}^0T''}=0.$$

We set

$$F^k=\sum_{p+q=k}F^{p,q}.$$

Now we introduce d',d'' operators by;

for u in $F^{p,q}$,

$$d'u=(du)_{(C\xi)^*\wedge\wedge^p({}^0T')^*\wedge\wedge^q({}^0T'')^*},$$

for u in $F^{p,q}$,

$$d''u=(du)_{(C\xi)^*\wedge\wedge^{p-1}({}^0T')^*\wedge\wedge^{q+1}({}^0T'')^*}.$$

Then, our theorem is;

Theorem3.1.

$$\begin{aligned}
&d'F^{p,q}\subset F^{p+1,q},\\
&d''F^{p,q}\subset F^{p,q+1},\\
&dF^k\subset F^{k+1}\\
&d'd'=d''d''=0,\\
&d'd''+d''d'=0\\
&d=d'+d''.
\end{aligned}$$

Namely our (F^k,d',d'') is a double complex. Then, we have three cohomology groups which were observed in [A-M2]. The first one is

$$\frac{Kerd''\cap F^{p,q}}{d''F^{p,q-1}}.$$

The second one is

$$\frac{Ker d' \cap Ker d'' \cap F^{p,q}}{Ker d' \cap d'' F^{p,q-1}}$$

because of

$$d''(Ker d') \subset Ker d'.$$

The third one is

$$\frac{d' \cap Ker d'' \cap F^{p,q}}{d' F^{p-1,q} \cap d'' F^{p,q-1}}$$

because of

$$d'' d' = d'(-d'').$$

In fact, we have the following theorem.

Theorem3.2. If $(M, {}^0T'')$ is strongly pseudo convex with $dim_R M = 2n-1$, as for d'' - operator,

if $p+q=n$,

$$Ker d'' \to \frac{Ker \overline{\partial}_b \cap C^{p,q}}{\overline{\partial}_b C^{p,q-1}} \to 0,$$

if $p+q \geq n+1$,

$$\frac{Ker d''}{Im d''} \simeq \frac{Ker \overline{\partial}_b \cap C^{p,q}}{\overline{\partial}_b C^{p,q-1}},$$

and as for d -operator,

if $k=n$,

$$Ker d \cap F^k \to \frac{Ker d}{Im d} \to 0,$$

if $k \geq n+1$,

$$\frac{Ker d \cap F^k}{d F^{k-1}} \simeq \frac{Ker d}{Im d}.$$

So our double complex (F^k, d', d'') recovers the standard Kohn–Rossi cohomology of degree $\geq n+1$, and so we can discuss a mixed Hodge theory over CR manifolds. The proof of Theorem 3.2 is very like in the proof for E_j bundles(cf.[2]). We consider a bundle map from $(C\xi)^* \wedge \wedge^{l-1}({}^0T')^* \wedge \wedge^{s-1}({}^0T'')^*$ to $\wedge^l({}^0T')^* \wedge \wedge^s({}^0T'')^*$, defined by :

$$(C\xi)^* \wedge \wedge^{l-1}({}^0T')^* \wedge \wedge^{s-1}({}^0T'')^* \to \wedge^l({}^0T')^* \wedge \wedge^s({}^0T'')^*$$

$$u \mapsto (du)_{\wedge^l({}^0T')^* \wedge \wedge^s({}^0T'')^*}.$$

The key lemma is as follows.

Lemma3.3. The above map

$$(C\xi)^* \wedge \wedge^{l-1}({}^0T')^* \wedge \wedge^{s-1}({}^0T'')^* \to \wedge^l({}^0T')^* \wedge \wedge^s({}^0T'')^*,$$

is surjective if $l+s \geq n$.

Proof. For $D^{p,q}(M) = \Gamma(M, (C\xi)^* \wedge \wedge^{p-1}({}^0T')^* \wedge \wedge^q({}^0T'')^*)$, we have an operator

$$L : D^{p,q}(M) \to D^{p+1,q+1}(M)$$

by $u \mapsto d\theta \wedge u$.

Then, just like the case hermitian manifolds, we obtain the adjoint operator of L, Λ, and we can introduce the notion of *primitive forms*. Namely, for u in $D^{p,q}(M)$, u is called a primitive form if and only if

$$\Lambda u = 0.$$

For primitive forms, we have two fact.

Fact.1. For v in $D^{p,q}(M)$, v is primitive if and only if

$$\begin{aligned} L^r v = 0, wherer =& max(n-1-(p-1+q), 0) \\ =& max(n-p-q, 0). \end{aligned}$$

Fact.2. For u in $D^{p,q}(M)$, u has the following unique decomposition.

$$u = u_0 + Lu_1 + ... + L^k u_k, wherek = [\frac{p-1+q}{2}].$$

The proof is the same as in hermitian manifolds. So we omit this.

Now we see Lemma 3.3. In order to show Lemma 3.3, it suffices to show that; if $l+s \geq n$, then the following map is surjective.

$$L : (C\xi)^* \wedge \wedge^{l-1}({}^0T')^* \wedge \wedge^{s-1}({}^0T'')^* \to (C\xi)^* \wedge \wedge^l({}^0T')^* \wedge \wedge^s({}^0T'')^*$$

by $u \mapsto d\theta \wedge u$, because of the computation of

$$(du)_{\wedge^l({}^0T')^* \wedge \wedge^s({}^0T'')^*}.$$

While for v in $(C\xi)^* \wedge \wedge^l({}^0T')^* \wedge \wedge^s({}^0T'')^*$ $(l+s \geq n)$, v is primitive if and only if $v = 0$(by Fact 1). And by Fact 2, we have : for u in $(C\xi)^* \wedge \wedge^l({}^0T')^* \wedge \wedge^s({}^0T'')^*$,

$$u = u_0 + Lu_1 + ... + L^k u_k, wherek = [\frac{l+s}{2}],$$

and u_i are primitive. So in our case, u_0 must be zero. Hence

$$\begin{aligned} u =& Lu_1 + ... + L^k u_k \\ =& L(u_1 + ... + L^{k-1} u_k). \end{aligned}$$

Therefore we have the surjectivity.

4. Estimates

By the same method as in [A1], we show the following a priori estimate. For this, we put the Levi metric on $F^{p,q}$ and consider the adjoint operator of $d''(resp.d')$, $d''^*(resp.d'^*)$.

Theorem4.1. If $2n+1 \geq k = p+q \geq n+1$ and $n-1 \geq p, q \geq 2$, then our complexes (F^k, d), $(F^{p,q}, d')$, $(F^{p,q}, d'')$ are subelliptic.

In order to prove our theorem, we must prepare several facts. On $\Gamma(M, (C\xi)^* \wedge \wedge^{p-1}({}^0T')^* \wedge \wedge^q({}^0T'')^*)$, we we can introduce the formal adjoint operators, δ'' of d'', and δ' of d' as in for $\Gamma(M, \wedge^p({}^0T')^* \wedge \wedge^q({}^0T'')^*)$(see [T]). By using this operator, we compute the adjoint operator of d'' for $F^{p,q}$.

Lemma4.2. The adjoint operator of d'' in $F^{p,q}$ becomes ; for u in $F^{p,q}$,

$$d''^* u = \delta'' u - \frac{1}{p+q-n+1} \Lambda L \delta'' u.$$

Proof. First, we show that; for u in $F^{p,q}$,

$$\delta'' u - \frac{1}{p+q-n+1} \Lambda L \delta'' u \in F^{p,q-1}$$

For the proof, it suffices to show ; for v in $F^{p,q-1}$, u in $F^{p,q}$,

$$\begin{aligned} &(d''v, u) \\ &= (v, (\delta'' - \frac{1}{p+q-n+1} \Lambda L \delta'')u) \end{aligned}$$

Q.E.D..

Now we establish an a priori estimate for $\|d''u\|^2 + \|d''^* u\|^2$ for u in $F^{p,q}$. For this, we recall several Kaehler identities on $\Gamma(M, (C\zeta)^* \wedge \wedge^p({}^0T')^* \wedge \wedge^q({}^0T'')^*)$.

Kaehleridentities.

$$\begin{aligned} &[\Lambda, d''] = -\sqrt{-1}\delta' \\ &[\Lambda, d'] = \sqrt{-1}\delta'' \\ &[\delta'', L] = \sqrt{-1}d' \\ &[\delta', L] = -\sqrt{-1}d'' \\ &\delta' d'' + d'' \delta' = 0 \\ &\delta'' d' + d' \delta'' = 0 \\ &[L, \Lambda] = k - (n-1), where k = p + q \end{aligned}$$

With these equalities, we establish an a priori estimte. For $u \in F^{p,q}$,

$$\begin{aligned} \|\Lambda d'u\|^2 =& (\Lambda d'u, \Lambda d'u) \\ =& (L\Lambda d'u, d'u), \\ \|L\delta''u\|^2 =& \|d'u\|^2 \end{aligned}$$

By the way, because of $(L\Lambda - \Lambda L)v = (k-(n-1))v$ for k-form v,

$$L\Lambda d'u = (k-n+1)d'u.$$

So,

$$\begin{aligned}
\|d''^{*}u\|^2 =&\|\delta''u-\frac{1}{p+q-n+1}\Lambda L\delta''u\|^2\\
=&\|\delta''u\|^2+\|\frac{1}{p+q-n+1}\Lambda L\delta''u\|^2-2Re(\delta''u,\frac{1}{p+q-n+1}\Lambda L\delta''u)\\
=&\|\delta''u\|^2+\frac{1}{(p+q-n+1)^2}\|\Lambda L\delta''u\|^2-\frac{2}{p+q-n+1}(L\delta''u,L\delta''u)\\
=&\|\delta''u\|^2+\frac{1}{p+q-n+1}\|d'u\|^2-\frac{2}{p+q-n+1}\|d'u\|^2\\
=&\|\delta''u\|^2-\frac{1}{p+q-n+1}\|d'u\|^2
\end{aligned} \tag{4.1}$$

Hence

$$\begin{aligned}
&\|d''u\|^2+\|d''^{*}u\|^2\\
=&\|d''u\|^2+\|\delta''u\|^2-\frac{1}{p+q-n+1}\|d'u\|^2\\
\geq&\|d''u\|^2+\|\delta''u\|^2-\frac{1}{p+q-n+1}\{\|d'u\|^2+\|\delta'u\|^2\}\\
\geq&\sum_{I,J}\{(\sum_{k\notin J}\|e_ku_{I,J}\|^2+\sum_{j\in J}\|\overline{e}_ju_{I,J}\|^2)\\
&-\frac{1}{p+q-n+1}(\sum_{l\notin I}\|\overline{e}_lu_{I,J}\|^2+\sum_{i\in I}\|e_iu_{I,J}\|^2)\}, where\mid I\mid=p-1,\mid J\mid=q\\
\geq&\frac{p+q-n}{p+q-n+1}\sum_{I,J}\{\sum_{k\notin J}\|e_ku_{I,J}\|^2+\sum_{j\in J}\|\overline{e}_ju_{I,J}\|^2\}\\
&+\frac{1}{p+q-n+1}\{\sum_{k\notin J}\|e_ku_{I,J}\|^2+\sum_{j\in J}\|\overline{e}_ju_{I,J}\|^2\\
&-\sum_{l\notin I}\|\overline{e}_lu_{I,J}\|^2-\sum_{i\in I}\|e_iu_{I,J}\|^2\}
\end{aligned}$$

While

$$\begin{aligned}
J^c=I\cap J^c+I^c\cap J^c, J=I\cap J+I^c\cap J\\
I^c=I^c\cap J^c+I^c\cap J, I=I\cap J^c+I\cap J
\end{aligned}$$

And so

$$\begin{aligned}
\sum_{k\notin J}\|e_ku_{I,J}\|^2&=\sum_{k\in I\cap J^c}\|e_ku_{I,J}\|^2+\sum_{k\in I^c\cap J^c}\|e_ku_{I,J}\|^2\\
\sum_{j\in J}\|\overline{e}_ju_{I,J}\|^2&=\sum_{j\in I\cap J^c}\|\overline{e}_ku_{I,J}\|^2+\sum_{j\in I^c\cap J^c}\|\overline{e}_ju_{I,J}\|^2\\
\sum_{l\notin I}\|\overline{e}_lu_{I,J}\|^2&=\sum_{l\in I^c\cap J^c}\|\overline{e}_lu_{I,J}\|^2+\sum_{l\in I^c\cap J}\|\overline{e}_lu_{I,J}\|^2
\end{aligned}$$

$$\sum_{i\in I}\|e_k u_{I,J}\|^2 = \sum_{i\in I\cap J^c}\|e_i u_{I,J}\|^2 + \sum_{i\in I\cap J}\|e_i u_{I,J}\|^2$$

By using this, the above becomes

$$\begin{aligned}
&\geq \frac{p+q-n}{p+q-n+1}\sum_{I,J}\{\sum_{k\notin J}\|e_k u_{I,J}\|^2 + \sum_{j\in J}\|\overline{e}_j u_{I,J}\|^2\} \\
&+\frac{1}{p+q-n+1}\{\sum_{l\in I\cap J}\|\overline{e}_l u_{I,J}\|^2 - \sum_{l\in I\cap J}\|e_l u_{I,J}\|^2 \\
&+\sum_{m\in I^c\cap J^c}\|e_m u_{I,J}\|^2 - \sum_{m\in I^c\cap J^c}\|\overline{e}_m u_{I,J}\|^2\}
\end{aligned} \tag{4.2}$$

While

$$\begin{aligned}
\mid I\mid + \mid J\mid - \mid I\cap J\mid =& \mid I\cup J\mid \\
=& n-1- \mid I^c\cap J^c\mid
\end{aligned}$$

So

$$\begin{aligned}
\mid I\cap J\mid =& p-1+q-n+1+ \mid I^c\cap J^c\mid \\
=& p+q-n+ \mid I^c\cap J^c\mid .
\end{aligned}$$

Especially,

$$\mid I^c\cap J^c\mid \leq \mid I\cap J\mid \ (by p+q-n\geq 1).$$

And

$$\mid I\cap J\mid \leq (p+q-n)\mid J^c\mid + \mid I^c\cap J^c\mid$$

These mean that there is an injective map

$$\kappa; I\cap J\mapsto (p+q-n)J^c\cup(I^c\cap J^c)$$

satisfying

$$\kappa(I\cap J)\supset I^c\cap J^c,$$

where $(p+q-n)J^c$ means the disjoint $(p+q-n)$'s union of J^c. Hence

$$\begin{aligned}
&\|\overline{e}_l u_{I,J}\|^2 + \|e_{\kappa(l)}u_{I,J}\|^2 \\
\sim&\|e_l u_{I,J}\|^2 + \|\overline{e}_{\kappa(l)}u_{I,J}\|^2
\end{aligned}$$

In the case $\mid J^c\mid = 1$,

$$\|d''u\|^2 + \|d''^*u\|^2 \gtrapprox \|d'u\|^2.$$

By (4.1) with (4.2), we have

$$\|d''u\|^2 + \|d''^*u\|^2 + \|u\|^2 \gtrapprox \|d''u\|^2 + \|\delta''u\|^2 + \|u\|^2 \gtrapprox \|u\|'^2$$

In the case $\mid J^c\mid > 1$, we directly have

$$\|d''u\|^2 + \|d''^*u\|^2 + \|u\|^2 \gtrapprox \|u\|'^2.$$

5. Finiteness

In Sect.3, we introduced three groups. In this section, we show
Theorem5.1. If $2n+1 \geq p+q \geq n+1$ and $n-2 \geq p, q \geq 2$,

$$\frac{Kerd' \cap Kerd'' \cap F^{p,q}}{Kerd' \cap d''F^{p,q-1}} \backsimeq \mathbf{H_{d''}} \cap Kerd'$$

and

$$\frac{d'F^{p-1,q} \cap Kerd'' \cap F^{p,q}}{d'F^{p-1,q} \cap d''F^{p,q-1}} \backsimeq \mathbf{H_{d''}} \cap d'F^{p-1,q}.$$

keyequality.

$$d'\Box_{d''}u = \frac{p+q-n+2}{p+q-n+1}d''d''^{*}(d'u) + \frac{p+q-n+3}{p+q-n+2}d''^{*}d''(d'u)$$

$$for u \in F^{p,q}.$$

We show this.

$$\begin{aligned}
&d'\{d''(\delta''u - \frac{1}{p+q-n+1}\Lambda L\delta''u)\} \\
=&d''\{-d'(\delta''u - \frac{1}{p+q-n+1}\Lambda L\delta''u)\} \\
=&d''\{\delta''d'u + \frac{1}{p+q-n+1}d'\Lambda L\delta''u\}
\end{aligned}$$

While

$$d'\Lambda - \Lambda d' = -\sqrt{-1}\delta''.$$

Therefore

$$\begin{aligned}
d'\Lambda L\delta''u =&\Lambda d'L\delta''u - \sqrt{-1}\delta''L\delta''u(by\delta''L - L\delta'' = -\sqrt{-1}d') \\
=&-\sqrt{-1}\delta''L\delta''u \\
=&\delta''d'.
\end{aligned}$$

So we have

$$\begin{aligned}
d'd''d''^{*}u =&\frac{p+q-n+2}{p+q-n+1}d''\delta''^{*}d'u \\
=&\frac{p+q-n+2}{p+q-n+1}d''(\delta''d'u - \frac{1}{p+q-n+2}\Lambda L\delta''d'u) \\
=&\frac{p+q-n+2}{p+q-n+1}d''d''^{*}d'u.
\end{aligned}$$

By the same way, we have

$$d'd''^{*}d''u = \frac{p+q-n+3}{p+q-n+2}d''^{*}d''d'u.$$

The correspondence is as follows.

$$\frac{Kerd' \cap Kerd'' \cap F^{p,q}}{Kerd' \cap d''F^{p,q-1}} \to \mathbf{H}_{d''} \cap Kerd'$$

for $u \to \mathbf{H}_{d''}u$.

And

$$\frac{d'F^{p-1,q} \cap Kerd'' \cap F^{p,q}}{d'F^{p-1,q} \cap d''F^{p,q-1}} \to \mathbf{H}_{d''} \cap d'F^{p-1,q}$$

for $u \to \mathbf{H}_{d''}u$.

Lemma5.2. If $p+q \geq n+1$, then for u in $F^{p,q}$,

$$d'\mathbf{H}_{d''}u = \mathbf{H}_{d''}d'u.$$

Proof. For u in $F^{p,q}$,

$$d'u = \mathbf{H}_{d''}d'u + \Box_{d''}N_{d''}d'u.$$

And

$$u = \mathbf{H}_{d''}u + \Box_{d''}N_{d''}u.$$

While by the key equality,

$$d'\Box_{d''}u = \frac{p-1+q-n+2}{p-1+q-n+1}d''d''^*(d'u) + \frac{p-1+q-n+3}{p-1+q-n+2}d''^*d''(d'u)$$

We take $v = \mathbf{H}_{d''}u$ and put v in the place of u, then

$$\begin{aligned} d'\Box_{d''}\mathbf{H}_{d''}u =& \frac{p-1+q-n+2}{p-1+q-n+1}d''d''^*(d'u) + \frac{p-1+q-n+3}{p-1+q-n+2}d''^*d''(d'u) \\ 0 =& \frac{p-1+q-n+2}{p-1+q-n+1}d''d''^*(d'u) + \frac{p-1+q-n+3}{p-1+q-n+2}d''^*d''(d'u) \end{aligned}$$

Therefore $d'\mathbf{H}_{d''}u$ is a harmonic form. This means that

$$\mathbf{H}_{d''}d'u = d'\mathbf{H}_{d''}u$$

by taking the harmonic part.

Q.E.D.

Even for the case $p+q=n$, we can introduce a harmonic operator by;

$$foruinF^{p,q},$$

$$\mathbf{H}_{d''}u = u - d''^*N_{d''}d''u.$$

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Department of Mathematics, Himeji Institute of Technology, 2167 Shosha, Himeji, Hyogo 671-22, Japan