RIGIDITY OF IRREDUCIBLE HERMITIAN SYMMETRIC SPACES OF THE COMPACT TYPE UNDER KÄHLER DEFORMATION

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Let S be a Hermitian symmetric space of the compact type. It follows from Bott [Bo] that the complex structure of S is infinitesimally rigid. In the special case where $S = \mathbb{P}^1 \times \mathbb{P}^1$ is a product of two Riemann spheres, it is well-known that S can be holomorphically deformed to any Hirzebruch surface Σ_a with a even. In the case where S is irreducible, following Kodaira-Spencer [KS, 1958] it was however conjectured that S is rigid under deformation as a complex manifold. While this was affirmed in the case of the projective space $\mathbb{P}^n$ by Siu [S1, 1992] and in the case of the hyperquadric Q^n, $n \geq 3$, by Hwang [H1, 1995], the general case remains open. In the present article we consider the special case of Kähler deformations and prove the rigidity of S under such deformations. More precisely, we prove

Theorem 1. *Let S be an irreducible Hermitian symmetric space of the compact type. Let $\pi : X \to \triangle$ be a regular family of compact complex manifolds over the unit disk $\triangle$. Suppose $X_t := \pi^{-1}(t)$ is biholomorphic to S for $t \neq 0$ and the central fiber X_0 is Kähler. Then, X_0 is also biholomorphic to S.*

Since the second Betti number of S is 1, the Kähler condition on X_0 is equivalent to the assumption that X_0 is projective-algebraic. We have the following equivalent algebro-geometric formulation of our result (cf. (5.2)).

Theorem 1′. *Let S be an irreducible Hermitian symmetric space of the compact type. Let $\rho : X \to Z$ be a smooth proper morphism between two connected algebraic varieties over $\mathbb{C}$. Suppose for some point y on Z, the fiber X_y is biregular to S. Then, for any point z on Z, the fiber X_z is biregular to S.*

Let M be a compact Kähler manifold homeomorphic to the projective space $\mathbb{P}^n$ (resp. the hyperquadric Q^n, $n \geq 3$) such that the canonical line bundle is not ample. Then, by the work of Hirzebruch-Kodaira [HK, 1957] (resp. Brieskorn [B, 1964]) using the Riemann-Roch formula, it was well-known that M is biholomorphic to $\mathbb{P}^n$ (resp. to Q^n, $n \geq 3$). Theorem 1 for $\mathbb{P}^n$ and Q^n ($n \geq 3$) already follows from these early results. The difficulty of our Theorem 1 lies therefore in

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the case of other model spaces S, for which there do not exist algebro-geometric characterizations in terms of holomorphic line bundles. As such our difficulties are quite distinct from those encountered in Siu [S1] and Hwang [H1, 2], where the major obstacle was the lack of the Kähler condition on the central fiber.

For a historical overview on the general problem of rigidity (or nonrigidity) of complex structures of Hermitian symmetric spaces of the compact type, and more specifically on the problem of deformation (non)rigidity of such spaces, we refer the reader to Siu [S2]. Regarding our Theorem 1, Mabuchi [Ma] kindly provided the first author with a preprint of his, explaining a proof using Euler vector fields. As far as we are aware, it has so far not been possible to complete his scheme of proof except in some special cases such as Grassmannians of rank 2.

Our method of proof is based on two ingredients: the analytic ingredient of holomorphic G-structures and the algebro-geometric ingredient of deformation of rational curves. Hermitian symmetric spaces of the compact type S are compactifications of Euclidean spaces, obtained canonically using the Harish-Chandra Embedding Theorem. For S irreducible and of rank ≥ 2, S admits an integrable (holomorphic) $K^{\mathbb{C}}$-structure for some proper complex Lie subgroup $K^{\mathbb{C}}$ of the general linear group. Here a $K^{\mathbb{C}}$-structure refers to a holomorphic reduction of the holomorphic frame bundle from the general linear group to $K^{\mathbb{C}}$. Integrability refers to the possibility of realizing such a reduction by means of holomorphic local coordinates on S. By a result of Ochiai [Oc], a simply-connected compact complex manifold with an integrable $K^{\mathbb{C}}$-structure is biholomorphic to the model space S. This may be taken as our point of departure, reducing our Theorem 1 to proving the convergence of the integrable $K^{\mathbb{C}}$-structures on X_t, $t \neq 0$, to one on the central fiber X_0. As integrability is a closed condition, the problem is reduced to verifying the convergence of $K^{\mathbb{C}}$-structures.

Here enters the deformation theory of rational curves. On the model space S there exists at every $s \in S$ a homogeneous complex submanifold $\mathcal{C}_s \subset \mathbb{P}(T_s(S))$, called the cone at s, consisting of all directions tangent to minimal rational curves. The collection of $\mathcal{C}_s$ then gives a holomorphic fiber bundle $\mathcal{C} \mapsto S$ whose fibers are isomorphic to a model $\mathcal{C}_o$. The holomorphic $K^{\mathbb{C}}$-structure of S can be recovered from the bundle of cones, $K^{\mathbb{C}}$ being isomorphic to the group of linear transformations inducing an isomorphism on $\mathcal{C}_o$. We are thus led to examine the limit of the bundles of cones over X_t, as $t \mapsto 0$. To prove Theorem 1 it suffices to show that the limit is a bundle of cones modeled on $\mathcal{C}_o$ such that for $x \in X_0$, the embedding $\mathcal{C}_x \subset \mathbb{P}T_x(X_0)$ is isomorphic to the standard embedding $\mathcal{C}_o \subset \mathbb{P}T_o(S)$.

Remarkably the standard cone $\mathcal{C}_o \subset \mathbb{P}T_0(S)$ is itself a Hermitian symmetric space of the compact type, of rank ≤ 2, irreducible except in the case of Grassmannians of rank ≥ 2. This entitles us to an argument by induction. For the model space S, $\mathcal{C}_s$ is isomorphic to the Chow space $\mathcal{M}_s$ of minimal rational curves marked at s. From the Kähler condition on X_0 we deduce that the limiting normalized Chow space $\mathcal{M}_x$ is smooth at a generic point $x \in X_0$. For S different from Grassmannians, $\mathcal{M}_x$ is by induction a Hermitian symmetric space. Embedding $\mathcal{M}_x \hookrightarrow \mathbb{P}(V_x)$ projectively, there is a linear map $V_x \to T_x(X_0)$ inducing a rational map $\Phi_x : \mathcal{M}_x \to \mathcal{C}_x$. Our main task is reduced to proving that Φ_x is an isomorphism. The same reduction is valid in the remaining case of Grassmannians, for which the cones are products of two projective spaces, as follows. From cohomological considerations we can exclude the possibility such as the deformation of $\mathbb{P}^1 \times \mathbb{P}^1$ into non-trivial Hirzebruch surfaces. The point is to prove that limits of direct factors of the cone cannot decompose. As deformations of such direct factors arise from deformations of projective subspaces of Grassmannians, we are led to proving that limits of such projective spaces cannot decompose, which follows from the cohomological structure of Grassmannians and the Kähler condition on X_0.

The main difficulty in proving Theorem 1 is to rule out the possibility that the cones $\mathcal{C}_x \subset \mathbb{P}T_x(X_0)$ are linearly degenerate. In this case we have a meromorphic distribution W on X_0, defined at generic points as the linear span of vectors tangent to minimal rational curves. If this distribution is integrable where holomorphic, we show that leaves of the associated foliation can be compactified to subvarieties of X_0, and obtain a contradiction to the fact that the Picard number of X_0 is 1, using the deformation theory of rational curves.

Our theorem is therefore finally reduced to another problem of integrability, one for which we need to verify the Frobenius condition for the distribution W defined algebro-geometrically using minimal rational curves. We further reduce the verification to finding at a generic point x of X_0 a certain set of integral surfaces Σ such that $\Lambda^2 T_x(\Sigma)$ generates $\Lambda^2 W_x$. Such integral surfaces can be obtained as pencils of minimal rational curves, and the linear-algebra statement can be verified using a uniform description of cones as Zariski closures of graphs of quadratic maps on Euclidean spaces. Alternatively it can also be verified using descriptions of second exterior powers of isotropy representations (cf. Onishchik-Vinberg [OV]).

In §1 we collect relevant basic facts about G-structures arising from irreducible Hermitian symmetric spaces of rank ≥ 2, including a discussion on integrability. We also prove a Hartogs-type extension result on holomorphic G-structures for G reductive, allowing us in what follows to concentrate our discussion at generic points of the central fiber. In §2 we describe the cones of minimal rational curves on the model spaces S, noting that a model cone $\mathcal{C}_o$ is itself Hermitian symmetric, irreducible except in the case of Grassmannians. We observe the generic smoothness of normalized Chow spaces of marked minimal rational curves on the central fiber X_0, and prove that generic minimal rational curves on X_0 are immersed and of the standard type (i.e., direct factors of normal bundles are of degree 1 or 0). In §3 we prove that in the case of deformations of Grassmannians, generic limits of normalized Chow spaces of marked minimal rational curves are biholomorphic to the standard cone. In §4 we consider distributions W on Fano manifolds spanned at generic points by tangents to standard minimal rational curves. We prove in particular that W is integrable whenever the varieties of tangent lines to cones are linearly non-degenerate in $\mathbb{P}\Lambda^2 W$ at generic points. In §5 we apply results of §3 and §4 to our central fiber X_0 to deduce the linear non-degeneracy of cones at generic points, completing the proofs of Theorems 1 and $1'$.

Table of Contents

§1 G-structures associated to Hermitian symmetric spaces

(1.1) Let n be a positive integer. Fix an n-dimensional complex vector space V and let M be any n-dimensional complex manifold. In what follows all bundles are understood to be holomorphic. The frame bundle $\mathcal{F}(M)$ is a principal $GL(V)$-bundle with the fiber at x defined as $\mathcal{F}(M)_x = \mathrm{Isom}(V, T_x(M))$, the set of linear isomorphisms from V to the holomorphic tangent space at x. Let $G \subset GL(V)$ be any complex Lie subgroup. A (holomorphic) G-structure is a G-principal sub-bundle $\mathcal{G}(M)$ of $\mathcal{F}(M)$. For $G \neq GL(V)$ we say that $\mathcal{G}(M)$ defines a holomorphic reduction of the tangent bundle to G.

Let $\varphi_\alpha : U_\alpha \to V$ be a chart on M. In terms of Euclidean coordinates we identify $\mathcal{F}(U_\alpha)$ with the product $GL(V) \times U_\alpha$. We say that a G-structure $\mathcal{G}(M)$ on M is integrable if and only if there exists an atlas of charts $\{\varphi_\alpha : U_\alpha \to V\}$ such that the restriction $\mathcal{G}(U_\alpha)$ of $\mathcal{G}(M)$ to U_α is the product $G \times U_\alpha \subset GL(V) \times U_\alpha$.

(1.2) Let (S, h) be a Hermitian symmetric space of the compact type and G be the identity component of the isometry group of (S, h). (The meaning of G here is not to be confused with that appearing in "G-structure", where G is used as a generic symbol for groups.) Let $G^{\mathbb{C}}$ be the identity component of the group of automorphisms of S. Then, $G^{\mathbb{C}}$ is a semisimple complex Lie group and $G \subset G^{\mathbb{C}}$ is a compact real form. Let $K \subset G$ be the isotropy subgroup at a reference point $o \in S$, and $\mathfrak{k} \subset \mathfrak{g}$ be corresponding Lie algebras. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ for the Cartan decomposition, where $\mathfrak{m}$ is the real tangent space at o. For the complexifications $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^- + \mathfrak{m}^+$, where $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^- + \mathfrak{m}^+$ is the decomposition of the complexified tangent space into the direct sum of subspaces $\mathfrak{m}^-$ resp. $\mathfrak{m}^+$ of type (0, 1) resp. (1, 0). Let $P \subset G^{\mathbb{C}}$ be the isotropy subgroup at o. We have $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^-$ at the level of Lie algebras. $\mathfrak{m}^+$, $\mathfrak{m}^- \subset \mathfrak{g}^{\mathbb{C}}$ are abelian subalgebras. The corresponding Lie subgroups M^+, $M^- \subset G^{\mathbb{C}}$ are isomorphic to the abelian Lie group $\mathbb{C}^n$. According to the Harish-Chandra Embedding Theorem, the map $M^+ \times K^{\mathbb{C}} \times M^- \to G^{\mathbb{C}}$ defined by $(m^+, k, m^-) \to m^+ \cdot k \cdot m^-$ is an open embedding. Applying to o we have an embedding of the M^+-orbit of o into S. Identifying M^+ with $\mathbb{C}^n$, taking o to be the origin, we regard S as a compactification of $\mathbb{C}^n$ such that translations on $\mathbb{C}^n$ extend to automorphisms of S. Euclidean coordinates on $\mathbb{C}^n \subset S$ will be referred to as Harish-Chandra coordinates.

On $\mathbb{C}^n$, $K^{\mathbb{C}}$ acts as a group of linear transformations, acting faithfully on $T_o(S)$. Denote the corresponding subgroup of $GL(T_o(S))$ by $K_o^{\mathbb{C}}$. For $s \in S$

denote by $P_s \subset G^{\mathbb{C}}$ the isotropy subgroup at s. Consider the homomorphism $\varphi_s : P_s \to GL(T_s(S))$ defined by $\varphi_s(f) = df(s)$. At o, $\mathrm{Ker}(\varphi_o) = M^-$ and $\varphi_o(P_o) = K_o^{\mathbb{C}}$. For $x \in \mathbb{C}^n \subset S$ we use the trivialization of $T(\mathbb{C}^n)$ via Harish-Chandra coordinates to identify $T_x(S)$ with $T_o(S) \cong \mathbb{C}^n$. Since P_s is conjugate to $P_o = P$ via a Euclidean translation in M^+, all $K_x^{\mathbb{C}}$ are identical under the corresponding identification $GL(T_x(S)) \cong GL(n, \mathbb{C})$. Using a finite number of Harish-Chandra coordinate charts, we obtain on S an integrable G-structure with $G = K_o^{\mathbb{C}}$. From now on we will identify $K_o^{\mathbb{C}}$ with $K^{\mathbb{C}}$ and call this the $K^{\mathbb{C}}$-structure on S. For S of rank ≥ 2, $K^{\mathbb{C}} \neq GL(V)$ and we have a non-trivial reduction of the holomorphic tangent bundle.

For details on the Harish-Chandra Embedding Theorem, we refer the reader to Wolf [W].

(1.3) Let S be an irreducible Hermitian symmetric manifold of the compact type and of rank ≥ 2. For the regular family $\pi : X \to \triangle$ as in Theorem 1 we will be considering limits of $K^{\mathbb{C}}$-structures on X_t, $t \neq 0$, as $t \to 0$. In order to have a limit defined everywhere on X_0, we will need the following Hartogs-type extension result. By a holomorphic family $\sigma : M \to \triangle$ of complex manifolds M_t we mean a holomorphic submersion σ with fibers $M_t := \sigma^{-1}(t)$. Denote by $\mathcal{F}_\sigma(M)$ the relative frame bundle where $\mathcal{F}_{\sigma,x}(M) = \mathrm{Isom}(V, T_{\sigma,x}(M))$, $T_{\sigma,x}(M) = \mathrm{Ker}(d\sigma(x))$. For a subgroup $G \subset GL(V)$, by a holomorphic family of G-structures on M we mean a holomorphic G-principal subbundle $\mathcal{G}$ of $\mathcal{F}_\sigma(M)$. The restriction to M_t will be denoted by $\mathcal{G}_t$. We have

Proposition 1. *Let $H \subset GL(V)$ be a reductive algebraic subgroup, $\sigma : M \to \triangle$ be a holomorphic family of n-dimensional complex manifolds M_t. Let $E \subset M_0$ be a complex-analytic subvariety of codimension ≥ 1 and $\mathcal{H} \subset \mathcal{F}_\sigma(M - E)$ be a holomorphic family of H-structures. Then, $\mathcal{H}$ extends holomorphically to M.*

Proof. The problem being local we may assume $M = \triangle^n \times \triangle$ with $\sigma(x, t) = t$. $\mathcal{H} \subset \mathcal{F}_\sigma((\triangle^n \times \triangle) - E)$ is equivalently given by a holomorphic map $f : (\triangle^n \times \triangle) - E \to GL(V)/H$ into the complex homogeneous space $GL(V)/H$, $E \subset \triangle^n \times \triangle$ being of complex codimension ≥ 2. By Matsushima-Morimoto [MM], $GL(V)/H$ is Stein for H reductive. Embedding $GL(V)/H$ as a complex submanifold of some Euclidean space and applying Hartogs' Extension Theorem for holomorphic functions to $(\triangle^n \times \triangle) - E$, we conclude that f and hence $\mathcal{H}$ extend holomorphically to $\triangle^n \times \triangle$, as desired.

(1.4) For the proof of Theorem 1, once a limit holomorphic $K^{\mathbb{C}}$-structure is constructed on the central fiber it will be immediate that the $K^{\mathbb{C}}$-structure is integrable, by the closedness of the integrability condition. To give simple references, we sketch a proof in the special case of G-structures of finite type. To each (holomorphic) G-structure $\mathcal{G} \subset \mathcal{F}(M)$ on a complex manifold M we can associate a prolongation bundle $\mathcal{P}_1 \to \mathcal{G}$, which is a G_1-structure on $\mathcal{G}$ for some $G_1 \subset GL(V \oplus \mathfrak{g})$, $\mathfrak{g}$ denoting the Lie algebra of G (cf. Sternberg [St]). The Lie algebra $\mathfrak{g}_1 \subset \mathfrak{gl}(V \oplus \mathfrak{g})$, called the first prolongation algebra, is constructed from the embedding $\mathfrak{g} \subset \mathfrak{gl}(V)$. Higher prolongation algebras $\mathfrak{g}_{k+1} = (\mathfrak{g}_k)_1$ and prolongation bundles $\mathcal{P}_{k+1} \to \mathcal{P}_k$ can be constructed inductively. The G-structure is said to be of finite type if $\mathfrak{g}_m = 0$ for some m, of type k if k is the first index for which $\mathfrak{g}_k = 0$. $K^{\mathbb{C}}$-structure associated to irreducible Hermitian symmetric spaces S of rank ≥ 2 are of type 2 (cf. Ochiai [Oc]). We have

Proposition 2. *Suppose that we have a holomorphic family $\mathcal{G}$ of G-structures of finite type on a family $\sigma : M \to \triangle$ of complex manifolds M_t. Suppose that $\mathcal{G}_t$ is integrable for all $t \neq 0$. Then, $\mathcal{G}_0$ is also integrable.*

Proof. Under the finite type condition, by going to a prolongation bundle we can assume that $G = \{e\}$ (cf. Sternberg [St, p.338]). Thus, $\mathcal{G}_t$ is a field of frames on M_t, and it is integrable if the frame field corresponds to a coordinate frame field. Since the problem is local, we may assume that $M = \triangle^n \times \triangle$. By assumption, we have holomorphic vector fields $V_1, \ldots, V_n$ on M, so that they define the given frame field and $[V_i, V_j] = 0$ for $t \neq 0$. But then they must satisfy $[V_i, V_j] = 0$ for $t = 0$, too. Hence their integrals give a coordinate system integrating the $\{e\}$-structure $\mathcal{G}_0$.

REMARKS

The integrability of $K^{\mathbb{C}}$-structures holds in a more general setting than Proposition 2. In a forthcoming article, we will show that a $K^{\mathbb{C}}$-structure on a Fano manifold is always integrable.

Finally, to recover Hermitian symmetric spaces from integrable $K^{\mathbb{C}}$-structures, we have

Proposition 3 (Ochiai [Oc]). *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2 with associated $K^{\mathbb{C}}$-structures, $K^{\mathbb{C}} \subset GL(T_o(S))$; $o \in S$. Let M be a compact simply-connected complex manifold with an integrable $K^{\mathbb{C}}$-structure. Then, M is biholomorphic to S.*

§2 Chow spaces and cones of minimal rational curves

(2.1) Let S be an irreducible Hermitian symmetric space of the compact type. The second homology group and hence the Picard group of S are infinite cyclic. Degrees of algebraic curves of S will be measured with respect to an ample line bundle $\mathcal{O}(1)$ which is a generator of the Picard group. An algebraic curve $C \subset S$ is called a minimal rational curve if and only if it is of degree 1. One way of describing minimal rational curves is by means of the first canonical embedding of S into a projective space $\mathbb{P}^N$ (cf. Nakagawa-Takagi [NT]). Complex-analytically this is nothing other than the embedding defined by $\mathcal{O}(1)$, such as the Plücker embedding of a Grassmannian. Identifying S as a subvariety of $\mathbb{P}^N$, the minimal rational curves on S are equivalently rational lines on $\mathbb{P}^N$ lying in S. In particular, they are smooth. One can check the existence of such rational lines using the definition of the first canonical embedding via an embedding of complex Lie groups. The Chow space of all minimal rational curves C on S constitutes a homogeneous space under the action of $G^{\mathbb{C}}$, the identity component of the group $Aut(S)$ of automorphisms of S. For $s \in S$, consider the set $\mathcal{M}_s$ of all minimal rational curves C passing through s and the cone $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ defined as $\{[\alpha] \in \mathbb{P}T_s(S) : T_s(C) = \mathbb{C}\alpha$ for some $[C] \in \mathcal{M}_s\}$. The tangent map $\Phi_s : \mathcal{M}_s \to \mathcal{C}_s$ thus defined is a bijection. We fix a reference point $o \in S$ and call $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ the standard cone associated to S. A remarkable fact about standard cones is that they are themselves Hermitian symmetric spaces of the compact type, of rank ≤ 2, and irreducible except in the case of Grassmannians of rank ≥ 2, where $\mathcal{C}_o$ is a product of two projective spaces. This can be seen from the following table giving a classification of S and a tabulation of their standard cones, together with a description of the inclusion $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective embedding of the Hermitian symmetric manifold $\mathcal{C}_o$. Here in this table alone G will denote a finite simply-connected covering of $Aut_0(S, h)$ and $K \subset G$ the isotropy subgroup. $\mathbb{O}$ will stand for the octonions (Cayley numbers).

Table of irreducible Hermitian symmetric spaces S
of the compact type and their standard cones $\mathcal{C}_o$

Type	G	K	$G/K = S$	$\mathcal{C}_o$	Embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2, n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$\mathrm{Spin}(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

It is possible to give an *a-priori* proof of the fact that $\mathcal{C}_o$ is Hermitian symmetric, as follows. Equipping $S = G/K$ with a canonical Kähler-Einstein metric we can induce on $\mathbb{P}T_o(S)$ and hence on $\mathcal{C}_o$ a Kähler metric. In Mok [M2, App.(III.2), Prop.1, pp.246ff] it was proven that holomorphic sectional curvatures are pinched between $\frac{1}{2}$ and 1 (up to normalization). By Ros [R], we know that such Kähler submanifolds of the projective space have to have parallel second fundamental forms and are hence in particular Hermitian symmetric. It is remarkable that the list of standard cones exhausts the set of all such Kähler submanifolds of projective spaces (equipped with the Fubini-Study metric).

(2.2) Let now $\pi : X \to \triangle$, $X_t \cong S$ for $t \neq 0$, be the regular family of compact Kähler manifolds as in Theorem 1. Let $L \to X$ be the holomorphic line bundle such that $L|_{X_t}$ is a positive generator of the Picard group $\mathrm{Pic}(X_t)$ for all $t \in \triangle$. For $t \neq 0$, a degree-1 curve with respect to L is necessarily rational and will be called a minimal rational curve. Let $C \subset X_0$ be a limit of a sequence of minimal rational curves on $\pi^{-1}(\triangle - \{0\})$ as cycles. Since X_0 is Kähler, and C is of degree

1 with respect to L, it must remain reduced and irreducible (and rational). From now on a curve $C \subset X_0$ will be called a minimal rational curve if and only if it is a limit of minimal rational curves on $\pi^{-1}(\triangle - \{0\})$.

A holomorphic vector bundle V over a Riemann sphere $\mathbb{P}^1$ is said to be semipositive if and only if all rank-1 factors in a Grothendieck decomposition of V are of degrees ≥ 0. Let $f : \mathbb{P}^1 \to X_0$ be a generically injective holomorphic map, and write $C = f(\mathbb{P}^1)$. By abuse of language we say that C has semipositive normal bundle in X_0 if and only if $f^*T_{X_0}$ is semipositive on $\mathbb{P}^1$, which is the same as saying that $\mathrm{Graph}(f) \subset \mathbb{P}^1 \times X_0$ has semipositive normal bundle as an embedded Riemann sphere. Over $t \neq 0$ the Chow space of minimal rational curves fills up X_t. It follows that there exists a unique irreducible component $\mathcal{K}$ of the Chow space of minimal rational curves on X_0, such that curves C corresponding to points on $\mathcal{K}$ sweep through X_0. As a consequence, there exists a subvariety $B \subset X_0$ such that for $x \in X_0 - B$ and for any $[C] \in \mathcal{K}$ passing through x, the normal bundle of C in X_0 is semipositive.

(2.3) We are going to construct quotient spaces of complex manifolds under the action of complex Lie groups. This construction is well-known in the algebraic setting (cf. Mori [Mr]). We will simply translate it to the complex-analytic setting. Let G be a complex Lie group and Z be a complex manifold. A holomorphic G-action on Z is by definition a holomorphic map $F : G \times Z \to Z$ such that (i) for any $g \in G$, $g(z) := F(g, z)$ is a biholomorphism of Z; and (ii) $(gh)(z) = g(h(z))$ for any $g, h \in G$. We are interested to endow the set of orbits Z/G with a complex structure. For the formulation, we say that the G-action is free if and only if $g(z) \neq z$ whenever $g \neq e$, the identity element of G. We say that the G-action has closed orbits if and only if $F(G \times \{z\})$ is closed in Z for all $z \in Z$. The following lemma is standard. (cf. Holmann [Ho, Satz 21]).

Lemma 1. *Let Z be a complex manifold, G be a complex Lie group and $F : G \times Z \to Z$ be a free holomorphic action of G on Z with closed orbits. Then, the set of orbits Z/G can be endowed the structure of a complex manifold such that the canonical map $Z \to Z/G$ is a holomorphic submersion, realizing Z as a holomorphic principal G-bundle.*

Consider the set $\mathcal{H}$ of all holomorphic maps $f : \mathbb{P}^1 \to X$, such that $\deg(f^*L) = 1$ and f^*T is semipositive, where $T \to X$ denotes the relative holomorphic tangent bundle of $\pi : X \to \triangle$. From the exact sequence $0 \to f^*T \to f^*T_X \to \mathcal{O} \to 0$ over $\mathbb{P}^1$ and the vanishing of $H^1(\mathbb{P}^1, E)$ for semipositive holomorphic vector bundles E, it follows that $f^*T_X \cong f^*T \oplus \mathcal{O}$ whenever $f^*T \geq 0$, so that $f^*T_X \geq 0$. Conversely if $f^*T_X \geq 0$, $f^*T_X \cong P \oplus \mathcal{O}^\ell$ where P is a positive bundle. Then, $f^*T_X \to \mathcal{O}$ is induced by a projection $\mathcal{O}^\ell \to \mathcal{O}$, and the kernel f^*T_X is isomorphic to $P \oplus \mathcal{O}^{\ell-1}$, hence semipositive. Thus, in the definition of $\mathcal{H}$, it is equivalent to assume $f^*T_X \geq 0$. Furthermore, for $[f] \in \mathcal{H}$, $f^*T_X \cong f^*T \oplus \mathcal{O}$.

The complex Lie group $Aut(\mathbb{P}^1) \cong SL(2, \mathbb{C})/\{\pm I\}$ acts on $\mathcal{H}$ with closed orbits by the action $\varphi[f] = [f \circ \varphi^{-1}]$ for any $[f] \in \mathcal{H}$ and any $\varphi \in Aut(\mathbb{P}^1)$. Since $f : \mathbb{P}^1 \to X_0$ is generically injective the action is also free, so that by Lemma 1 we have on the quotient space $\mathcal{D} = \mathcal{H}/Aut(\mathbb{P}^1)$ the structure of a complex manifold realizing $\mathcal{H} \to \mathcal{H}/Aut(\mathbb{P}^1)$ as a principal $Aut(\mathbb{P}^1)$-bundle. For each $[f] \in \mathcal{H}$, $f(\mathbb{P}^1) = C \subset X_t$ for some $t \in \triangle$, so that there is a canonical holomorphic map $\nu : \mathcal{H} \to \triangle$. The kernel of the differential $d\tau$ at $[f] \in \mathcal{H}$ is given by $\Gamma(\mathbb{P}^1, f^*T) \subset \Gamma(\mathbb{P}^1, f^*T_X)$, of corank 1, so that $\nu : \mathcal{H} \to \triangle$ is a holomorphic submersion. Correspondingly we obtain a holomorphic submersion $\mathcal{D} \to \triangle$. The fiber $\mathcal{D}_t$ over $t \neq 0$ can be identified with the Chow space of minimal rational curves on X_t. Over $t = 0$, $\mathcal{D}_0$ can be identified with the normalization of a Zariski-open set of the irreducible component $\mathcal{K}$ of the Chow space of minimal rational curves on X_0. Let now $Aut_0(\mathbb{P}^1) \subset Aut(\mathbb{P}^1)$ be the isotropy subgroup at $0 \in \mathbb{P}^1$. Then, $\mathcal{U} := \mathcal{H}/Aut_0(\mathbb{P}^1)$ is a complex manifold. As $Aut(\mathbb{P}^1)/Aut_0(\mathbb{P}^1) \cong \mathbb{P}^1$, the canonical map $\tau : \mathcal{U} \to \mathcal{D}$ realizes $\mathcal{U}$ as a holomorphic $\mathbb{P}^1$-bundle over $\mathcal{D}$.

For $x \in X$ let $\mathcal{H}_x \subset \mathcal{H}$ be the subvariety consisting of all $[f] \in \mathcal{H}$ such that $f(0) = x$. We will say that f is marked at x. We have

Proposition 4. *For any $x \in X$, $\mathcal{H}_x \subset \mathcal{H}$ is a complex submanifold. The canonical action of $Aut_0(\mathbb{P}^1)$ on $\mathcal{H}_x$ induces on $\mathcal{M}_x = \mathcal{H}_x/Aut_0(\mathbb{P}^1)$ the structure of a complex manifold. Furthermore, the canonical projection $\rho : \mathcal{U} \mapsto X$ is a holomorphic submersion such that the fiber over x is $\mathcal{M}_x$.*

Proof. For a holomorphic vector bundle E on $\mathbb{P}^1$ we denote by $E(k)$ the twisted bundle $V \otimes \mathcal{O}(k)$. Let $[f] \in \mathcal{H}_x$. Then, the deformation of $[f]$ within $\mathcal{H}_x$ is unobstructed since $H^1(\mathbb{P}^1, f^*T_X(-1)) = 0$, f^*T_X being semipositive. Thus, $\mathcal{H}_x \subset \mathcal{H}$ is a (closed) complex submanifold such that $T_{[f]}(\mathcal{H}_x) = \Gamma(\mathbb{P}^1, f^*T_X(-1)) \subset \Gamma(\mathbb{P}^1, f^*T_X) = T_{[f]}(\mathcal{H})$. $\Gamma(\mathbb{P}^1, f^*T_X(-1))$ is of constant rank $= \deg(f^*K_X^{-1}) = \deg(K_X^{-1})$, where K_X denotes the canonical line bundle over X. Applying Lemma 1 to $\mathcal{H}_x$ together with the canonical action of $Aut_0(\mathbb{P}^1)$, we obtain on $\mathcal{M}_x$ the structure of a complex manifold. The canonical map $\mathcal{M}_x = \mathcal{H}_x/Aut_0(\mathbb{P}^1) \to \mathcal{H}/Aut_0(\mathbb{P}^1) = \mathcal{U}$ identifies $\mathcal{M}_x$ as a complex submanifold. We denote by $\{f\}$ the class in $\mathcal{U}$ defined by $[f]$. Then $T_{\{f\}}(\mathcal{M}_x) \cong \Gamma(\mathbb{P}^1, f^*T_X(-1))/f_*\Gamma(\mathbb{P}^1, T_{\mathbb{P}^1}(-1))$. Here f_* denotes the homomorphism induced by the differential $df : T_{\mathbb{P}^1} \to f^*T_X$. The canonical map $\mathcal{H} \to X$ sending $[f]$ to $f(0)$ induces a holomorphic map $\rho : \mathcal{U} \to X$ such that $\mathrm{Ker}\, d\rho(\{f\}) = \Gamma(\mathbb{P}^1, f^*T_X(-1))/f_*\Gamma(\mathbb{P}^1, T_{\mathbb{P}^1}(-1)) = T_{\{f\}}(\mathcal{M}_x)$, realizing ρ as a holomorphic submersion with fibers $\mathcal{M}_x$, as desired.

By (2.2), there exists a subvariety $B \subset X_0$ such that for any $[C] \in \mathcal{K}$ passing through $x \in X_0 - B$, the normal bundle of C in X_0 is semipositive. For such $x \in X_0 - B$, by abuse of language we will call the compact complex manifold $\mathcal{M}_x$ the Chow space of minimal rational curves marked at x. For the projection $\rho : \mathcal{U} \to X$, we will sometimes write $\mathcal{M}$ for $\rho^{-1}(X - B)$. We have

Proposition 5. *The canonical projection $\rho|_{\mathcal{M}} : \mathcal{M} \to X - B$ is a regular family of projective-algebraic manifolds. Furthermore, for $x \in X_t$, $t \neq 0$, the fiber $\mathcal{M}_x$ is biholomorphic to the standard cone $\mathcal{C}_o$ of the model space S.*

Proof. The only thing to prove is the projective-algebraicity of $\mathcal{M}_x$ for $x \in X_0 - B$. Denote by $\mathcal{K}_x \subset \mathcal{K}$ the subvariety of minimal rational curves passing through x. Since the canonical map $\imath : \mathcal{D}_0 \to \mathcal{K}$ is the normalization of a Zariski-open subset of $\mathcal{K}$ and the Chow space $\mathcal{K}$ is projective-algebraic, $\mathcal{D}_0$ is quasi-projective. As $\tau|_{\mathcal{M}_x} : \mathcal{M}_x \to \mathcal{D}_0$ is the normalization onto $\tau(\mathcal{M}_x)$, $\mathcal{M}_x$ is also projective-algebraic, as desired.

(2.4) For the model space S in Theorem 1 consider the first canonical embedding $S \hookrightarrow \mathbb{P}^N$. For any minimal rational curve $C \subset S$, which is a rational line in $\mathbb{P}^N$, $T_S|_C$ is a semipositive bundle which is a holomorphic subbundle of $T_{\mathbb{P}^N}|_C \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{N-1}$. It follows that $T_S|_C$ must be isomorphic to $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ for some nonnegative integers p, q. By considering deformations of marked minimal rational curves, we conclude that p is the dimension of the standard cone $\mathcal{C}_o$ of S.

Let $C \subset X_t$, $t \in \triangle$, be a minimal rational curve, represented by $f : \mathbb{P}^1 \to X$. We say that C is standard if and only if $f^*T_{X_t} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$. Thus C is standard whenever $t \neq 0$. We note that on X_0, any standard minimal rational curve must be immersed, since the only non-trivial homomorphism in $\Gamma(\mathrm{Hom}(T_{\mathbb{P}^1}, f^*T_{X_0}))$ must be everywhere injective, as $T_{\mathbb{P}^1} \cong \mathcal{O}(2)$. However, we do not know *a priori* that standard minimal rational curves are embedded in X_0. For the existence, we have

Proposition 6. *On X_0 a generic minimal rational curve with semipositive normal bundle is standard.*

For the proof of Proposition 6 we need the following lemma of Mori's.

Lemma 2 (Mori's Breaking-up Lemma ([Mr]). *Let Σ be an algebraic surface and $\sigma : \Sigma \to \Gamma$ be a surjective holomorphic map of Σ over an algebraic curve Γ such that the generic fiber is irreducible and rational. Suppose there exist two distinct holomorphic sections Γ_0 and Γ_∞ of Σ over Γ, each of which can be blown down to a point. Then, some fiber of σ must be reducible.*

Proof of Proposition 6. The proof is similar to that of Mok ([Mk2, Lemma (2.4.3), pp.203 ff.]). We argue by contradiction. If Proposition 6 fails, for every $[f] \in \mathcal{H}$, $f(\mathbb{P}^1)$ is not standard. At $[f] \in \mathcal{H}$, $T_{[f]}(\mathcal{H})$ is identified with $\Gamma(\mathbb{P}^1, f^*T_X)$. As all $f(\mathbb{P}^1)$ are non-standard, we can choose a non-empty open subset Ω of $\mathcal{H}$ and a holomorphic vector field η on Ω such that $\eta([f]) \in \Gamma(\mathbb{P}^1, f^*T_X \otimes [0]^{-1} \otimes [\infty]^{-1}) - f_*\Gamma(\mathbb{P}^1, T_{\mathbb{P}^1})$ for all $[f] \in \Omega$. Here for $\zeta \in \mathbb{P}^1$, $[\zeta]$ denotes the positive divisor line bundle defined by ζ. Fix some $[f_0] \in \Omega$ such that $f_0(0) \neq f_0(\infty)$. Write $x = f_0(0)$ and $y = f_0(\infty)$. An integral curve of η in $\mathcal{H}$ containing $[f_0]$ gives a non-trivial one parameter family of minimal rational curves passing through x and y. As the Chow space $\mathcal{K}$ is algebraic, this means that there exists an algebraic family of minimal rational curves C_λ, parametrized by some algebraic curve Γ such that all C_λ passes through the distinct points x and y. Since all C_λ are irreducible this leads to a contradiction to Lemma 2. The proof of Proposition 6 is completed.

§3 Non-deformability of normalized Chow spaces in the case of Grassmannians

(3.1) Making use of the observation that standard cones are themselves Hermitian symmetric spaces of the compact type, we formulate the following proposition furnishing the inductive step in our proof of Theorem 1.

Proposition 7. *Let S' be an irreducible Hermitian symmetric manifold of the compact type of rank ≥ 2 and of complex dimension n. Suppose Theorem 1 has been established for all S of complex dimension $< n$. Then, for a regular family $\pi : X \to \triangle$ of compact complex manifolds such that $X_t \cong S'$ for $t \neq 0$ and for a generic point $x \in X_0$, the normalized Chow space $\mathcal{M}_x$ of marked minimal rational curves at x is biholomorphic to the standard cone $\mathcal{C}'_o$ of S'.*

For S' different from Grassmannians, Proposition 7 follows from [(2.3), Proposition 4] on the smoothness of $\mathcal{M}_x$ for generic $x \in X_0$. For Grassmannians $S' \cong G(p,q)$, the standard cone $\mathcal{C}'_o$ is biholomorphic to $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, and we have to worry about the possibility of a jump of complex structures in deforming the latter manifold. Nonetheless the deformation we encounter is subordinate to a deformation of Grassmannians, and we assert the following proposition which implies Proposition 7.

Proposition 8. *Let $\pi : X \to \triangle$ be a regular family of compact Kähler manifolds and denote by X_t the fiber $\pi^{-1}(t)$ for all $t \in \triangle$. Suppose that for $t \neq 0$, X_t is biholomorphic to the Grassmannian $G(p,q)$, $p, q \geq 2$. Let $s : \triangle \to X$ be a holomorphic section of $\pi : X \to \triangle$ such that the normalized Chow spaces of marked minimal rational curves $\{\mathcal{M}_{s(t)} : t \in \triangle\}$ define a regular family $\tau : \mathcal{M} \to \triangle$ of projective-algebraic manifolds, so that $\mathcal{M}_{s(t)}$ is biholomorphic to $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$ for $t \neq 0$. Then, the central fiber $\mathcal{M}_{s(0)}$ is also biholomorphic to $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$.*

In the example of the deformation of $\mathbb{P}^1 \times \mathbb{P}^1$ to a Hirzebruch surface Σ_a with $a > 0$ even (cf. Siu [S2, pp.97-98], for example), limits of one of the two kinds of direct factors are decomposable. We contend that such a phenomenon cannot occur in the case of Proposition 8. We have

Lemma 3. *Let $\{t_i\}$ be a sequence of points on $\triangle - \{0\}$ converging to 0. Suppose $Z_i \subset \mathcal{M}_{t_i}$ are projective spaces corresponding either to some $\{[a]\} \times \mathbb{P}^\ell$ or some $\mathbb{P}^k \times \{[b]\}$ under a biholomorphism $\mathcal{M}_{t_i} \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, where $1 \leq k \leq p-1$ and $1 \leq \ell \leq q-1$, and the projective spaces $\mathbb{P}^k$ resp. $\mathbb{P}^\ell$ are understood to be projective-linear in $\mathbb{P}^{p-1}$ resp. $\mathbb{P}^{q-1}$. Suppose the sequence $\{Z_i\}$ converges as cycles of $\mathcal{M}$ to some $Z_0 \subset \mathcal{M}_0$. Then, Z_0 is reduced and irreducible.*

Proof. We work first of all on the model $S = G(p,q)$. Identifying $T_o(S)$, $o \in S$, with the tensor product $\mathbb{C}^p \otimes \mathbb{C}^q$ in the standard way the cone $\mathcal{C}_o \subset T_o(S)$ is equivalently $\{[a \otimes b] : a \in \mathbb{C}^p,\ b \in \mathbb{C}^q; a, b \neq 0\}$. Given $a \in \mathbb{C}^p$, the subvariety $\{[a]\} \times \mathbb{P}^{q-1} \hookrightarrow \mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \hookrightarrow \mathbb{P}(T_0(S))$ is the projective-linear subspace $\mathbb{P}(a \otimes \mathbb{C}^q)$. With respect to the Plücker embedding $\nu : S \hookrightarrow \mathbb{P}^N$ minimal rational curves are mapped biholomorphically onto rational lines in $\mathbb{P}^N$. The projective-linear subspace $\mathbb{P}(a \otimes \mathbb{C}^q) \subset \mathcal{C}_o$ then spans a $(q-1)$-dimensional family of rational lines in $\mathbb{P}^N$ sweeping out a q-dimensional projective linear subspace $\cong \mathbb{P}^q \hookrightarrow \mathbb{P}^N$. The same description is valid for $\mathbb{P}(\mathbb{C}^p \otimes b)$.

Let now ω be the Kähler form of a Fubini-Study metric on $\mathbb{P}^N$ such that the Kähler class $[\omega]$ is the positive generator of $H^2(\mathbb{P}^N, \mathbb{Z}) \cong \mathbb{Z}$. For any m-dimensional projective linear subspace $\mathbb{P}^m \hookrightarrow \mathbb{P}^N$ we have $\int_{\mathbb{P}^m} \omega^m = 1$. The Plücker embedding $\nu : G(p,q) \to \mathbb{P}^N$ induces an isomorphism $\nu^* : H^2(\mathbb{P}^N, \mathbb{Z}) \to H^2(G(p,q), \mathbb{Z}) \cong \mathbb{Z}$. For $\{[a]\} \times \mathbb{P}^\ell$ resp. $\mathbb{P}^k \times \{[b]\}$ as in the statement of Lemma 3, let $W \subset G(p,q)$ be the $(\ell+1)$ resp. $(k+1)$-dimensional subvariety swept out by the corresponding set of marked minimal rational curves at 0. Write m for the dimension of W (i.e. $k+1$ or $\ell+1$). As the image $\nu(W) \subset \mathbb{P}^N$ is a projective linear subspace we have

$$\int_W (\nu^* \omega)^m = \int_{\nu(W)} \omega^m = 1 \ .$$

Let now $t_i \in \triangle$ and $Z_i \subset \mathcal{M}_{t_i}$ be as in the hypothesis of Lemma 3 and denote by $W_i \subset X_{t_i} \cong G(p,q)$ the corresponding W's. The convergence of Z_i to some $Z_0 \subset \mathcal{M}_0$ as a subvariety implies the convergence of W_i to some subvariety $W_0 \subset X_0$. Since W_i is of degree 1 with respect to the ample line bundle L, W_0 is reduced and irreducible. For $m = 2$, i.e. k resp. $\ell = 1$, it follows that Z_0 (of pure dimension 1) is reduced and irreducible, since any irreducible component of Z_0 necessarily sweeps out a surface. The general case can be deduced from the case of k resp. $\ell = 1$, as follows. Suppose Z_0 is reducible. Pick $p, q \in Z_0$ to be smooth points lying on different irreducible components, so that any subvariety of Z_0 containing both p and q must be reducible. Since $Z_i \subset \mathcal{M}_{t_i}$ converges to Z_0 as cycles there exist pairwise distinct points $p_i, q_i \in Z_i$ such that $p_i \to p$ and $q_i \to q$ on $\mathcal{M}$. Let $C_i \subset Z_i$ be the unique projective line on $Z_i \cong \mathbb{P}^k$ resp. $\mathbb{P}^\ell$ containing p_i and q_i. Then, C_i are of degree 1 with respect to L, so that C_i converges as cycles to a curve C_o on $\mathcal{M}_0$. By the preceding argument for k resp. $\ell = 1$, C_o must be irreducible, contradicting with our choice of $p, q \in Z_0$. We have thus proven that Z_0 is irreducible. It must then also be reduced since W_0 is reduced.

(3.2) It is possible to deduce Proposition 8 from Lemma 3 by an argument applicable to the more general situation of deformations of products of projective-algebraic manifolds. We have chosen instead to present a proof adapted to the special case of projective spaces more in line with methods used in the article, using minimal rational curves and the tangent map. Generic choices of Z_0 as in Lemma 3 will be shown to be immersed projective spaces and the two families of such cycles will be shown to intersect transversally. From this we will deduce Proposition 8.

Pick $t_0 \in \triangle$, $t_0 \neq 0$, and consider a cycle $Z'_{t_0} \subset \mathcal{M}_{t_0}$ corresponding to some $\mathbb{P}^{p-1} \times \{[b]\}$ under a biholomorphism $\mathcal{M}_{t_0} \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$. Let $\mathcal{D}'$ be the irreducible component of the Chow space $\mathcal{D}$ containing the point $[Z'_{t_0}]$ defined by Z'_{t_0}. Similarly choose $Z''_{t_0} \subset \mathcal{M}_{t_0}$ corresponding to some $\{[a]\} \times \mathbb{P}^{q-1}$ and define $\mathcal{D}'' \subset \mathcal{D}$ to be the corresponding irreducible component of $\mathcal{D}$. We have canonical projection maps $\rho' : \mathcal{D}' \to \triangle$ and $\rho'' : \mathcal{D}'' \to \triangle$. Write $\mathcal{D}'_t$ resp. $\mathcal{D}''_t$ for the fibers over $t \in \triangle$. By studying rational curves on Z', $[Z'] \in \mathcal{D}'$; resp. Z'', $[Z''] \in \mathcal{D}''$; we prove

Lemma 4. *There exists a non-empty Zariski-open subset $U = \mathcal{M}_0 - E \subset \mathcal{M}_0$ such that for every $[Z_0'] \in \mathcal{D}_0'$, Z_0' is an immersed projective space of complex dimension $p-1$ provided that $Z_0' \cap U \neq \emptyset$. Furthermore, for every $[Z_0''] \in \mathcal{D}_0''$, Z_0'' is an immersed projective space of complex dimension $q-1$ provided that $Z_0'' \cap U \neq \emptyset$.*

Proof. First of all, we observe that the biholomorphisms $\mathcal{M}_t \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$ for $t \neq 0$ define integrable holomorphic subbundles $T_t', T_t'' \subset T_t := T_{\mathcal{M}_t}$ on $\mathcal{M}_t$ such that the direct factors $Z_t' \subset \mathcal{M}_t$ corresponding to $\mathbb{P}^{p-1} \times \{[b]\}$, resp. $Z_t'' \subset \mathcal{M}_t$ corresponding to $\{[a]\} \times \mathbb{P}^{q-1}$, are integral submanifolds of T_t' resp. T_t''. As the distribution $z \mapsto T_t'(z)$ resp. $z \mapsto T_t''(z)$ on $\mathcal{M}_t$ are defined by holomorphic maps into Grassmannians, they extend meromorphically to $\mathcal{M}$, so that there exists a proper complex-analytic subvariety $E_1 \subset \mathcal{M}_0$ and holomorphic vector subbundles $T', T'' \subset T$ on $\mathcal{M} - E_1$ such that for $t \neq 0$, $T'\big|_{\mathcal{M}_t} = T_t'$, $T''\big|_{\mathcal{M}_t} = T_t''$. We now assume that $[Z_0'] \in \mathcal{D}_0'$ and $y \in Z_0'$ have been so chosen that $y \notin E_1$.

Pick $t_0 \neq 0$ and $Z_{t_0}' \subset \mathcal{M}_{t_0}$ such that $[Z_{t_0}'] \in \mathcal{D}_{t_0}'$. We have $Z_{t_0}' \cong \mathbb{P}^{p-1}$. Let C_{t_0} be a minimal rational curve on $Z_{t_0}' \cong \mathbb{P}^{p-1}$. For the Chow space $\mathcal{D}$ of $\mathcal{M}$, let $\mathcal{E}' \subset \mathcal{D}$ be the irreducible component containing the point $[C_{t_0}]$. From now on, we replace $\mathcal{E}'$ by its normalization and denote by $\mathcal{E}_t'$ the fiber over t of the canonical projection $\mathcal{E}' \mapsto \triangle$. Similar notations will be used for canonically fibered spaces over $\triangle$. From Lemma 3 we know that for every $[C_0] \in \mathcal{E}_0'$, C_0 is an irreducible rational curve, represented by $f_0 : \mathbb{P}^1 \to \mathcal{M}_0$. There exists a unique irreducible component $\mathcal{E}_{01}'$ of $\mathcal{E}_0'$ whose members cover $\mathcal{M}_0$. Members of all other irreducible components cover a proper subvariety $E_2 \subset \mathcal{M}_0$. By Lemma 3 and the Breaking-up Lemma of Mori, we know that for a generic choice of $C_0 \subset \mathcal{M}_0$ with $[C_0] \in \mathcal{E}_{01}'$, represented by $f_0 : \mathbb{P}^1 \to \mathcal{M}_0$, we have

$$f_0^* T_{\mathcal{M}_0} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{p-2} \oplus \mathcal{O}^{q-1} . \qquad (*)$$

In analogy with (2.4) C is said to be standard whenever $t \neq 0$; or $t = 0$ and $(*)$ is valid for $f_0 : \mathbb{P}^1 \to \mathcal{M}_0$ defining C. We have

Claim: For a generic choice of w_0 on $\mathcal{M}_0$, every rational curve $C_0 \subset \mathcal{M}_0$ with $[C_0] \in \mathcal{E}_0'$, $w_0 \in C_0$, is standard.

Proof of Claim. Since $\cup\{C_0 : [C_0] \in \mathcal{E}_{01}'\}$ covers $\mathcal{M}_0$ there exists a proper complex-analytic subvariety $E_3 \subset \mathcal{M}_0$ such that for any $w_0 \in \mathcal{M}_0 - E_3$ and any $[C_0] \in \mathcal{E}_{01}'$ passing through w_0, defined by $f : \mathbb{P}^1 \to \mathcal{M}_0$, we have $f^* T_{\mathcal{M}_0} \geq 0$. The semipositivity of $f^* T_{\mathcal{M}_0}$ implies the smoothness of $\mathcal{E}_{01}'$ at $[C_0]$. Write $E = E_1 \cup E_2 \cup E_3$. Suppose furthermore that $w_0 \notin E$ and C_0 is smooth at w_0. We are going to prove that C_0 is standard. Denote by $\mathcal{U} \subset \mathcal{E}' \times \mathcal{M}$ the subvariety $\{([C], w) \in \mathcal{E}' \times \mathcal{M} : w \in C\}$. For any $t \in \triangle$, $[C_t] \in \mathcal{E}_t'$ and any smooth point $w \in C_t$ we define $\Theta([C_t], w) \in \mathbb{P}T_w(\mathcal{M}_t)$ to be the projectivization of $T_w(C_t)$. We call Θ the tangent map. Pick a smooth neighborhood V of $([C_0], w_0)$ in $\mathcal{U}$ with the following properties: (i) $([C], w) \in V$ implies that $w \notin E$ and w is smooth on C; (ii) the canonical projection $\mu : V \to \mathcal{M}$ has constant rank and has connected fibers. We note that $\Theta([C], w) \in \mathbb{P}T'_w$ for $([C], w) \in V$. This is the case by the definition of T' whenever C lies over $t \neq 0$. It remains true for C lying over 0 by continuity, as $T' \subset T$ is a holomorphic vector subbundle on $\mathcal{M} - E_1$. We note also that both V and $\mathbb{P}T'$ are of complex dimension $2p + q - 3$. Since $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} \mathbb{P}T'$, the tangent map $\Theta : V \to \mathbb{P}T'$ is either an open immersion or it is unramified exactly outside a non-trivial divisor $D \subset V$. As any $[C_t] \in \mathcal{E}'_t$ represents a standard minimal rational curve $C_t \subset \mathcal{M}_t$ whenever $t \neq 0$, the latter possibility can occur only if all C_0, $[C_0] \in \mathcal{E}'_0 \cap V$ are not standard, contradicting the existence of standard rational curves in $\mathcal{E}'_{01}$. We have proven that for $w_0 \in \mathcal{M}_0 - E$, and for any $[C_0] \in \mathcal{E}'_{01}$ containing w_0 as a smooth point, C_0 is standard. The same can be said for any $[C_0] \in \mathcal{E}'_{01}$ passing through w_0. In fact, C_0 must contain a smooth point $z \in \mathcal{M}_0 - E$. Repeating the same argument at z in place of w_0 we conclude that C_0 must be standard. The proof of the claim is completed.

Let now $w_0 \in \mathcal{M}_0 - E$ and $[Z'_0] \in \mathcal{D}'_0$ be such that $w_0 \in Z'_0$. The holomorphic subbundle $T'_0 \subset T_0$ is defined at w_0. For any $[C_0] \in \mathcal{E}'_0$ represented by $f_0 : \mathbb{P}^1 \hookrightarrow \mathcal{M}_0$, $f_0(0) = w_0$, we know that f_0 is an immersion at 0, so that $df_0(0)$ sends $T_0(\mathbb{P}^1)$ to a complex line L_{C_0} in T'_{w_0}. Consider the normalized Chow space $\mathcal{C}^*_{w_0}$ of rational curves marked at w_0 and denote by $\mathcal{C}'_{w_0}$ the irreducible component of $\mathcal{C}^*_{w_0}$ containing $[C_0]$. Then $\mathcal{C}'_{w_0}$ is a $(p-2)$-dimensional complex manifold such that the tangent map $\Theta : \mathcal{C}'_{w_0} \to \mathbb{P}T'_{w_0} \cong \mathbb{P}^{p-2}$ is a local biholomorphism and thus a biholomorphism. The subvariety $S = \cup\{C_0 : [C_0] \in \mathcal{C}'_{w_0}\} \subset \mathcal{M}_0$ must have a tangent space $= \cup\{L_{C_0} : [C_0] \in \mathcal{C}'_{w_0}\} = T'_{w_0}$. For the proof of Lemma 4 we may assume that $p \geq 3$, as the case $p = 2$ is already implied by the Claim. The family of rational curves parametrized by $\mathcal{C}'_{w_0}$ gives rise to a $\mathbb{P}^1$-bundle B over $\mathcal{C}'_{w_0} \cong \mathbb{P}^{p-2}$ with a distinguished section Σ corresponding to the base point w_0. The fact that Θ is a biholomorphism implies that the normal bundle of $\Sigma \cong \mathbb{P}^{p-2}$ in B is isomorphic to $\mathcal{O}(-1)$. For $p \geq 3$ one can then blow down Σ to a smooth point b to obtain $\check{B} \cong \mathbb{P}^{p-1}$. The canonical map $B \mapsto \mathcal{M}_0$ induces a holomorphic map $\nu : \mathbb{P}^{p-1} \cong \check{B} \to \mathcal{M}_0$. Since each $[C_0] \in \mathcal{C}'_{w_0}$ represents a standard rational curve, $\nu : \mathbb{P}^{p-1} \to \mathcal{M}_0$ is an immersion outside $b \in \check{B}$. Since $\nu(\mathbb{P}^{p-1}) = S$ is smooth at w_0 and ν is an immersion outside of a single point 0, of codimension $p - 1 \geq 2$, ν must be a local biholomorphism at 0. The same argument applies *verbatim* to $\mathcal{D}''$ in place of $\mathcal{D}'$, proving Lemma 4.

In the proof of Lemma 4 the integrable holomorphic subbundles T', $T'' \subset T$ were defined on $\mathcal{M} - E$. For $x \in \mathcal{M} - E$ denote by $Z'(x)$ (resp. $Z''(x)$) the unique leaf of the integrable holomorphic distribution $x \mapsto T'_x$ (resp. T''_x) passing through x. For $t \neq 0$ and $x \in \mathcal{M}_t \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, $Z'(x) \cap Z''(x) = \{x\}$, the intersection being transversal at x. For the proof of Proposition 8 we need the following lemma extending this property across generic points of $\mathcal{M}_0$.

Lemma 5. *In the notations of Lemma 4, for any $x \in \mathcal{M}_0 - E$, $Z'(x)$ and $Z''(x)$ are smooth at x. Furthermore, $Z'(x) \cap Z''(x) = \{x\}$, the intersection being transversal at x.*

Proof. Consider a holomorphic family of embeddings $\nu'_t : \mathbb{P}^{p-1} \to \mathcal{M}_t$, $t \in \triangle$, with $\nu'_0 = \nu'$; $\nu'(\mathbb{P}^{p-1}) = Z'(x)$ and similarly a holomorphic family of embeddings $\nu''_t : \mathbb{P}^{q-1} \to \mathcal{M}_t$, $t \in \triangle$, with $\nu''_0 = \nu''$, $\nu''(\mathbb{P}^{q-1}) = Z''(x)$. We know that for $t \neq 0$

$$\nu'_t(\mathbb{P}^{p-1}) \cap \nu''_t(\mathbb{P}^{q-1}) = \{x_t\}$$

for some $x_t \in \mathcal{M}_t$, such that $x_t \mapsto x$. Write $[\mathbb{P}^{p-1}]$ resp. $[\mathbb{P}^{q-1}]$ for the fundamental class of $\mathbb{P}^{p-1}$ resp. $\mathbb{P}^{q-1}$. We have

$$(\nu'_t)_*[\mathbb{P}^{p-1}] \cdot (\nu''_t)_*[\mathbb{P}^{q-1}] = 1$$

for $t \neq 0$ and hence also for $t = 0$. Suppose $\nu'_0(\mathbb{P}^{p-1}) \cap \nu''_0(\mathbb{P}^{q-1})$ is zero-dimensional. Then, $Z'(x)$ and $Z''(x)$ must be smooth at x, x is the unique point of intersection and the intersection must be transversal, as desired. It remains therefore to rule out the existence of a positive-dimensional irreducible component S in $\nu'_0(\mathbb{P}^{p-1}) \cap \nu''_0(\mathbb{P}^{q-1})$. Given any such S of complex dimension s with fundamental class $[S]$,

$$0 \neq [S] \in (\nu'_0)_* H_{2s}(\mathbb{P}^{p-1}, \mathbb{Z}) \cap (\nu''_0)_* H_{2s}(\mathbb{P}^{q-1}, \mathbb{Z}) \ . \tag{1}$$

Define $S'_0 = (\nu'_0)^{-1}(S) \subset \mathbb{P}^{p-1}$, $S''_0 = (\nu''_0)^{-1}(S) \subset \mathbb{P}^{q-1}$. Since $\nu'_0 : \mathbb{P}^{p-1} \to \mathcal{M}_0$ and $\nu''_0 : \mathbb{P}^{q-1} \to \mathcal{M}_0$ are immersions, S'_0 and S''_0 are of pure dimension s. Define $S'_t = \nu'_t(S'_0)$, $S''_t = \nu'_t(S''_0)$. On the one hand,

$$(\nu'_t)_* H_{2s}(\mathbb{P}^{p-1}, \mathbb{Q}) \cap (\nu''_t)_* H_{2s}(\mathbb{P}^{p-1}, \mathbb{Q}) = \{0\} \ . \tag{2}$$

On the other hand, we have $[S'_t] \to k'[S]$; $[S''_t] \to k''[S]$ as homology classes for some positive integer k' and k''. This leads plainly to a contradiction between (1) and (2). The proof of Lemma 5 is completed.

We are now ready to give a proof of Proposition 8.

Proof of Proposition 8. Recall that $\tau : \mathcal{M} \to \triangle$ is a regular family such that $\mathcal{M}_t = \tau^{-1}(t)$ is biholomorphic to $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$ for $t \neq 0$. Denote by T the relative holomorphic tangent bundle of $\tau : \mathcal{M} \to \triangle$. By Lemma 5, there exists a proper complex-analytic subvariety $E \subset \mathcal{M}_0$, holomorphic vector subbundles T' and T'' of $T\big|_{\mathcal{M}-E}$, such that $T'_x \cap T''_x = \{0\}$, $T_x = T'_x \oplus T''_x$ for any $x \in \mathcal{M} - E$; and such that for $x \in \mathcal{M}_t$, $t \neq 0$, the direct sum decomposition $T_x = T'_x \oplus T''_x$ arises from the product decomposition $\mathcal{M}_t \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$. Given a complex vector space W of complex dimension N, positive integers N', N'' such that $N = N' + N''$, the set of direct sum decompositions $W = W' \oplus W''$, with W' resp. $W'' \subset W$ of complex dimensions N' resp. N'', is parametrized by $\mathcal{S} = GL(N, \mathbb{C})/GL(N', \mathbb{C}) \times GL(N'', \mathbb{C})$. Since $G = GL(N', \mathbb{C}) \times GL(N'', \mathbb{C})$ is reductive and $E \subset \mathcal{M}$ is a subvariety of codimension ≥ 2, we can apply the extension result of [(1.3), Proposition 1] to extend the G-structure; i.e., equivalently the direct sum decompositions $W = W' \oplus W''$. As a consequence, we obtain on all of $\mathcal{M}$ a decomposition $T = T' \oplus T''$ as a direct sum of holomorphic vector subbundles.

For $t \neq 0$, the holomorphic distributions on $\mathcal{M}_t$ defined by $x \mapsto T'_x$, T''_x are integrable with closed leaves. The same is true at $t = 0$ by taking limits as $t \mapsto 0$. On $\mathcal{M}_0$, the holomorphic distribution $x \mapsto T'_x$ (resp. $x \mapsto T''_x$) defines a regular family of compact complex manifolds such that the generic fiber is biholomorphic to $\mathbb{P}^{p-1}$ (resp. $\mathbb{P}^{q-1}$), by Lemma 4. $Z'(x) \cap Z''(x) = \{x\}$ remains valid by considering intersection numbers (as in the proof of Lemma 4). From this it follows that $\mathcal{M}_0 \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, as desired.

Remarks

For the Steinness of $\mathcal{S} = GL(N, \mathbb{C})/GL(N', \mathbb{C}) \times GL(N'', \mathbb{C})$, in place of applying Matsushima-Morimoto [MM], we can use the explicit description of $\mathcal{S}$, as the complement of a *hypersurface* D in a product $G' \times G''$ of two Grassmannians, $G' = G(N', N'')$, $G'' = G(N'', N')$, where D is of bidegree (a, b), $a, b > 0$, arising from taking determinants of a set of N vectors of $W \cong \mathbb{C}^N$. $\mathcal{S} = (G' \times G'') - D$ implies that $\mathcal{S}$ is in fact affine-algebraic. In particular, $\mathcal{S}$ is a Stein manifold.

§4 The linear span of tangents to minimal rational curves in Fano manifolds

(4.1) For the proof of Theorems 1 and 1′ we are going to consider distributions arising from cones of minimal rational curves in the central fiber X_0. The key question is integrability. In this section we will formulate results on integrability and non-integrability in the broader context of Fano manifolds.

Let M be a Fano manifold. By Mori [Mr], M is uniruled. Let E be the set of all irreducible components $\mathcal{P}$ of the Chow space of M such that generic points of $\mathcal{P}$ correspond to rational curves with semipositive normal bundles. Fix a positive line bundle L on M and define $\delta_L(\mathcal{P})$ as the degree of members of $\mathcal{P}$ with respect to L. From the proof of [(2.4), Proposition 6], minimizing $\delta_L(\mathcal{P})$ among $\mathcal{P} \in E$ we obtain $\mathcal{K} \in E$ with the following property: A generic point of $\mathcal{K}$ corresponds to an immersed rational curve C given by $f : \mathbb{P}^1 \to M$ such that $(*) f^*T_M \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ for some $p, q \geq 0$. (We note that the arguments in [(2.4), Proposition 6] apply because we can choose points $x, y \in M$ such that every rational curve passing through x resp. y has semipositive normal bundle.) There may be several possible choices of $\mathcal{K}$. From now on we fix one such $\mathcal{K}$. In analogy with (2.4) we will call C a standard minimal rational curve if and only if (implicitly) $[C] \in \mathcal{K}$ and the splitting condition $(*)$ is satisfied.

Let $\mathcal{R} \subset \mathcal{K}$ be the Zarishi-dense subset corresponding to standard minimal rational curves. Consider the universal family maps $\psi : \mathcal{U} \to \mathcal{R}$, $\varphi : \mathcal{U} \to M$. For a given point $x \in M$, let $\mathcal{R}_x = \psi(\varphi^{-1}(x))$ be the set of standard minimal rational curves containing x. From now on to streamline the discussion we will make the following assumption, which is valid in the situation of the present article:

(Assumption) For a generic point $x \in M$, $\mathcal{R}_x$ is irreducible.

The tangent map $\Theta_x : \mathcal{R}_x \to \mathbb{P}T_x(M)$, which sends $r \in \mathcal{R}_x$ to the tangent vector of $\varphi(\psi^{-1}(r))$ at x, is a holomorphic map. Let $\mathcal{K}_x$ be the closure of $\mathcal{R}_x$.

Then Θ_x induces a dominant rational map $\mathcal{K}_x \to \mathcal{C}_x$, where $\mathcal{C}_x$ is the closure of the image $\Theta_x(\mathcal{R}_x)$. Consider the holomorphic distribution defined on a Zariski-dense subset of M by the linear span of $\mathcal{C}_x$ in $\mathbb{P}T_x(M)$. We can extend this distribution to a distribution W on M outside a subvariety $S(W)$ of codimension > 1. Of course, W is the trivial distribution of the whole tangent bundle, if $\mathcal{C}_x$ is linearly non-degenerate in $\mathbb{P}T_x(M)$. We will sometimes refer to W as a meromorphic distribution on M and say that W is integrable to mean that it is integrable on $M - S(W)$.

At a generic point $x \in M$ let $W_x \subset T_x(M)$ be the subspace defining W. By definition, $\mathcal{C}_x \subset \mathbb{P}W_x$ is linearly non-degenerate. A line in $\mathbb{P}W_x$ corresponds to a 2-plane in W, defining a point in $\mathbb{P}\Lambda^2 W_x$. Consider the tangential lines to $\mathcal{C}_x$ at smooth points of $\mathcal{C}_x$. The closure of the corresponding points on $\mathbb{P}\Lambda^2 W_x$ defines the variety of tangential lines $\mathcal{T}_x \subset \mathbb{P}\Lambda^2 W_x$.

(4.2) We develop now a sufficient condition for the integrability of the meromorphic distribution W. We have

Proposition 9. *Suppose the variety of tangent lines $\mathcal{T}_x \subset \mathbb{P}\Lambda^2 W_x$ is linearly non-degenerate at a generic point $x \in M$. Then, W is integrable.*

As W is defined algebro-geometrically using minimal rational curves we do not have a straight-forward way to verify the Frobenius condition on W. Instead, we will first of all reduce the verification to finding at generic points $x \in X_0$ a set of integral surfaces Σ such that $\Lambda^2 T_x(\Sigma)$ generates $\Lambda^2 W_x$. Then, we will show that such integral surfaces Σ can be constructed as pencils of rational curves.

We start with the following general statement on the Frobenius condition.

Lemma 6. *Let $\Omega \subset \mathbb{C}^n$ be a domain and $D \subset T_\Omega$ be a holomorphic distribution on Ω. Then, D is integrable if and only if the following holds.*

- $(*)$ *Given $x \in \Omega$, there exist D-valued holomorphic vector fields α_j, β_j defined on a neighborhood of x, $1 \le j \le N$ for some positive integer N, such that*
  - (i) $[\alpha_j, \beta_j](x) \in D_x$,
  - (ii) $\{\alpha_j(x) \wedge \beta_j(x)\}$ *spans* $\Lambda^2 D_x$.

Proof. The "only if" part is obvious. For the "if" part by Frobenius we need to prove that $[D, D] \subset D$. Fix $x \in \Omega$. Let A, B be holomorphic vector fields with values in D on a neighborhood U of x, and f, g be holomorphic functions on U. Then,

$$[fA, gB] = fg[A, B] + fA(g)B - gB(f)A \equiv fg[A, B] \bmod D \ .$$

It follows that the Lie bracket defines at x a skew-symmetric bilinear form on D_x with values in the quotient space T_x/D_x. The hypotheses (i) and (ii) then imply that this bilinear form vanishes for every $x \in \Omega$. In other words $[D, D] \subset D$ and D is integrable, as desired.

For the proof of Proposition 9 the crucial point is to verify the hypothesis of Lemma 6 for W on some Zariski-dense subset of M using the deformation theory of rational curves. For this purpose we have

Proposition 10. *Let $x \in M$ be a generic point and $C \subset M$ be a standard minimal rational curve passing through x, represented by $f : \mathbb{P}^1 \to M$ such that $f(0) = x$. Write*

$$f^*T\big|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$$

for a decomposition of f^*T into a direct sum of line bundles. Write $df(T_0(\mathbb{P}^1)) = \mathbb{C}\alpha$ and denote by $P_\alpha \subset T_x$ the vector subspace corresponding to $(\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)_0$. Let $\xi \in P_\alpha$ be a vector such that α and ξ are linearly independent. Then, there exists a smooth complex-analytic surface Σ on some neighborhood Ω of x such that

$$T_x(\Sigma) \cong \mathbb{C}\alpha + \mathbb{C}\xi$$

and such that at every point $y \in \Omega$, $T_y(\Sigma) \subset W_y$.

Proof. To simplify notations we assume that C is embedded. Choose a point $y \in C \cap \Omega$, $y \neq x$. Let $s \in \Gamma(C, T_X)$ be a holomorphic section such that $s(x) = \xi$ and $s(y) = 0$. This is possible because $\xi \in P_\alpha$. Let now $\{C_\zeta : \zeta \in \triangle\}$ be a family of smooth standard minimal rational curves given by $f_\zeta : \mathbb{P}^1 \to X_0$ such that $f_0 \equiv f$ and such that $f_\zeta(\infty) = y$ for all $\zeta \in \triangle$. Then, for any $\zeta_0 \in \triangle$ consider the section $s = \frac{\partial f_\zeta}{\partial \zeta}\Big|_{\zeta=\zeta_0} \in \Gamma(\mathbb{P}^1, f_{\zeta_0}^* T_X \otimes \mathcal{O}(-1))$. We have $s(\infty) = 0$. Identify s as a section in $\Gamma(C_{\zeta_0}; T_X)$. For any $u \in C_{\zeta_0}$ with $T_u(C_{\zeta_0}) = \mathbb{C}\alpha'$ we must have $s(u) \in P_{\alpha'}$ as $s(y) = 0$. Let now Σ be the image of $\{(z;\zeta) : |z| < 1; |\zeta| < \varepsilon\}$ under the map $F : \mathbb{P}^1 \times \triangle \to X_0$ defined by $F(z,\zeta) = f_\zeta(z)$. We may assume that $\Sigma \subset \Omega$. Since $\alpha \wedge \xi \neq 0$, for ε sufficiently small Σ is a locally closed complex-analytic surface containing x such that for any point $u \in \Sigma$, $u = F(z, \zeta_0)$,

$$T_u(\Sigma) = \mathbb{C}\alpha' + \mathbb{C}s(u) \subset P_{\alpha'} \subset W_u$$

for s corresponding to $\frac{\partial f}{\partial \zeta}\Big|_{\zeta=\zeta_0}$. The proof of Proposition 10 is completed.

We proceed with the proof of Proposition 9.

Proof of Proposition 9. At a generic point of M, W_x is defined and is spanned by vectors α tangent to minimal rational curves. By Proposition 10, given such an α and $\xi \in P_\alpha$, $\alpha \wedge \xi \neq 0$, there exists an integral surface Σ of W in a neighborhood of x, such that $T_x(\Sigma) = \mathbb{C}\alpha \oplus \mathbb{C}\xi$. Thus there exist W-valued holomorphic vector fields $\tilde{\alpha}$, $\tilde{\xi}$ on a neighborhood of x such that $\tilde{\alpha}(x) = \alpha$; $\tilde{\xi}(x) = \xi$ and $[\tilde{\alpha}, \tilde{\xi}](x) = 0$. By assumption the linear span of such $\alpha \wedge \xi$ at x is $\Lambda^2 W$, so that the hypothesis of Lemma 6 is satisifed on a Zariski-dense subset of $M - \mathcal{S}(W)$. It follows that W is integrable, as desired.

(4.3) We proceed to examine consequences of integrability of the meromorphic distribution W defined by rational curves. When cones $\mathcal{C}_x \subset \mathbb{P}T_x$ are linearly degenerate, i.e., when W is non-trivial, we will show that $b_2(M) \geq 2$. To start with, we have

Proposition 11. *Suppose the meromorphic distribution W is integrable. Then every leaf $\mathcal{W}$ on $M - \mathcal{S}(W)$ is closed and its topological closure $\overline{\mathcal{W}}$ is a complex-analytic subvariety.*

Proof. Let $x \in M$ be a generic point. It is enough to show that the leaf through x can be compactified to a subvariety.

Let $\psi : \mathcal{U} \to \mathcal{R}$ and $\varphi : \mathcal{U} \to M$ be the universal family. We know that there exists a subvariety $Z \subset M$ so that $\mathcal{R}_x$ and $\mathcal{C}_x$ are irreducible for $x \in M - Z$.

Consider an irreducible subvariety $A \subset M$ which is not contained in $\mathcal{S}(W) \cup Z$ and whose generic part is contained in a single leaf of W. We will define a new irreducible variety $\mathcal{V}(A)$ with the properties: (i) $\mathcal{V}(A)$ is not contained in $\mathcal{S}(W) \cup Z$ and (ii) the generic part of $\mathcal{V}(A)$ is contained in the leaf of W containing the generic part of A.

The variety $\psi(\varphi^{-1}(A))$ may be reducible. But there is an irreducible component $\mathcal{R}(A)$ which contains the irreducible $\mathcal{R}_a$ for a generic $a \in A$. Let $\mathcal{V}(A)$ be the closure of $\varphi(\psi^{-1}(\mathcal{R}(A)))$. Then $\mathcal{V}(A)$ is also irreducible and it contains A. From this, Property (i) is immediate. To check Property (ii), note that generic points of any curve C with $[C] \in \mathcal{R}$ which is not contained in $\mathcal{S}(W)$, is contained in a single leaf of W. Curves corresponding to generic points of $\mathcal{R}(A)$ intersect $A - \mathcal{S}(W)$. Hence generic points of such curves are contained in the leaf of W containing A. But such curves are dense in $\mathcal{V}(A)$. It follows that generic points of $\mathcal{V}(A)$ are contained in the leaf of W containing the generic part of A.

Now given a generic point $x \in M$, consider the irreducible varieties $\mathcal{V}^i(x)$ defined inductively by $\mathcal{V}^1(x) = \mathcal{V}(x)$ and $\mathcal{V}^{i+1}(x) = \mathcal{V}(\mathcal{V}^i(x))$. Since $\mathcal{V}^i(x) \subset \mathcal{V}^{i+1}(x)$, and they are irreducible, we get $\mathcal{V}^n(x) = \mathcal{V}^{n+1}(x)$, where n is the dimension of M. This implies that for a generic smooth point $y \in \mathcal{V}^n(x)$, a curve corresponding to a generic point of $\mathcal{R}_y$ is contained in $\mathcal{V}^n(x)$. Hence $\mathcal{C}_y \subset \mathbb{P}T_y(\mathcal{V}^n(x))$, which implies that $W_y \subset T_y(\mathcal{V}^n(x))$. But the generic part of $\mathcal{V}^n(x)$ is contained in the leaf of W containing x, which implies $T_y(\mathcal{V}^n(x)) \subset W_y$. It follows that at a generic point, $\mathcal{V}^n(x)$ is the leaf of the foliation W. Thus the leaf of W containing x can be compactified to the subvariety $\mathcal{V}^n(x)$.

Proposition 12. *Suppose the meromorphic distribution W is integrable. Let $C \subset M$ be a standard minimal rational curve corresponding to a sufficiently generic point of $\mathcal{R}$. Let $\mathcal{W}_C$ be the closure of the leaf of W containing C and let $\mathcal{W}_1$ be the closure of a leaf distinct from $\mathcal{W}_C$. Then, C is disjoint from $\mathcal{S}(W)$ and $\mathcal{W}_1$.*

Proof. Clearly, $\mathcal{W}_C \cap \mathcal{W}_1$ is contained in the singular locus $\mathcal{S}(W)$. So it is enough to show that a deformation of a standard minimal rational curve C is disjoint from $\mathcal{S}(W)$. This follows from a general result that a rational curve with semipositive normal bundle can be deformed to avoid any codimension-2 set (e.g. [Ko], Proposition 5.2.8). We will give a proof for the reader's convenience.

For notational simplicity we assume that C is embedded. Let $[\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ be the decomposition of the normal bundle N_C of C in M. Choose a point P on the curve away from $\mathcal{S}(W)$. Choose sections σ_i, $1 \leq i \leq p$, of the normal bundle corresponding to independent sections of $[\mathcal{O}(1)]^p$ vanishing at P and sections σ_i, $p+1 \leq i \leq n-1$ which generate the trivial factor $\mathcal{O}^q$ of the normal bunle. These sections σ_i, $1 \leq i \leq n-1$ are pointwise linearly independent outside P. Consider an $(n-1)$-dimensional deformation of the curve corresponding to the linear span of σ_i's inside $H^0(C, N_C)$, the tangent space of deformations of C. Suppose all members of this $(n-1)$-dimensional family intersects $\mathcal{S}(W)$. Since $\mathcal{S}(W)$ has codimension ≥ 2 in M, this means that we have a 1-dimensional subfamily intersecting $\mathcal{S}(W)$ at a fixed point $Q \in \mathcal{S}(W)$. In particular, in the linear span of σ_i's, there exists a section of the normal bundle vanishing at $Q \neq P$. But this is impossible because σ_i's are pointwise independent outside P. Thus, some member of this deformation must be disjoint from $\mathcal{S}(W)$. Proposition 12 follows.

We are now ready to deduce a topological obstruction to the integrability of W when cones are linearly degenerate.

Proposition 13. *Suppose $b_2(M) = 1$ and the distribution W is non-trivial, i.e. $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ is linearly degenerate for a generic $x \in M$. Then, W cannot be integrable.*

Proof. Suppose W is integrable. Let $\mathcal{D}$ be an irreducible subvariety in the Chow space of M whose generic point corresponds to the closure of a generic leaf of W. A generic hypersurface in $\mathcal{D}$ corresponds to a hypersurface H in M which is invariant under W, i.e. it is the union of leaves of W. Let $\mathcal{W}_1$ be the closure of a leaf of W. Then either $\mathcal{W}_1 \subset H$ or $\mathcal{W}_1 \cap H \subset \mathcal{S}(W)$. From the previous proposition, a generic standard minimal rational curve is disjoint from H. But from $b_2(M) = 1$, H is ample, a contradiction.

Remarks

With simple modifications the discussion in §4 applies even if we drop the assumption that generic members of $\mathcal{K}$ represent standard minimal rational curves.

§5 Cones in the central fiber

(5.1) Returning to the situation of Theorem 1, we proceed to examine the meromorphic distribution W on the central fiber X_0 defined at generic points by cones $\mathcal{C}_x \subset \mathbb{P}T_x(X_0)$. To start with, consider the irreducible Hermitian symmetric space S with a base point $o \in S$. Let $\mathcal{C}_o \subset \mathbb{P}T_o$ be the standard cone. As in §4, let $\mathcal{T}_o \subset \mathbb{P}\Lambda^2 T_o$ be the variety of tangential lines to $\mathcal{C}_o$. We denote by $\tilde{\mathcal{C}_o} \subset T_o - \{0\}$ (resp. $\tilde{\mathcal{T}_o} \subset \Lambda^2 T_o - \{0\}$) the preimage of $\mathcal{C}_o$ (resp. $\mathcal{T}_o$) under the canonical projection. We are going to prove

Proposition 14. *$\mathcal{T}_o \subset \mathbb{P}\Lambda^2 T_o$ is linearly non-degenerate.*

For the proof we consider the principal bundle $\mathcal{G} \subset \mathcal{F}(S)$ defining the standard $K^{\mathbb{C}}$-structure on S. It is based on a faithful irreducible representation $\mu : K^{\mathbb{C}} \subset GL(V)$. The cone $\mathcal{C}_o$ can be identified with the $K^{\mathbb{C}}$-orbit of projectivizations of highest weight vectors in $\mathbb{P}V$, via any choice of an element of $\mathcal{G} \subset \mathrm{Isom}(V, T_0)$.

We recall Grothendieck's classification of G-principal bundles over $\mathbb{P}^1$, where G is any connected complex reductive Lie group. Choose a maximal algebraic torus $H \subset G$. Let $\mathcal{O}(1)^*$ be the $\mathbb{C}^*$-principal bundle on $\mathbb{P}^1$, which is just the complement of the zero section of $\mathcal{O}(1)$.

Proposition 15 (Grothendieck [Gro]). *Let $\mathcal{G}$ be a principal G-bundle on $\mathbb{P}^1$. Then, there exists an algebraic one-parameter subgroup $\rho : \mathbb{C}^* \to H$ such that $\mathcal{G}$ is equivalent to the G-bundle associated to $\mathcal{O}(1)^*$ via the action ρ. Furthermore, let $\mathcal{V}$ be a vector bundle associated to $\mathcal{G}$ via a representation $\mu : G \to GL(V)$. Then $\mathcal{V}$ splits as the direct sum of line bundles $\mathcal{O}(<\mu_i, \rho>)$, where $\mu_i : H \to \mathbb{C}^*$ are the weights of μ and $<\mu_i, \rho>$ denotes the integral exponent of the homomorphism $\mu_i \circ \rho : \mathbb{C}^* \to \mathbb{C}^*$.*

Proof of Proposition 14. We apply Grothendieck's result to the case $G = K^{\mathbb{C}}$. For any $\alpha \in \mathcal{C}_x$, we have a minimal rational curve C passing through $o \in S$, tangential to α with $T(S)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ for some positive integers p, q. Restricting $\mathcal{G}$ to C, we obtain a $K^{\mathbb{C}}$-principal bundle on C. From Grothendieck's classification above, we have a one-parameter subgroup $\rho : \mathbb{C}^* \to K^{\mathbb{C}}$ defining $\mathcal{G}|_C$. From the splitting type of $T(S)|_C$, $\rho(\mathbb{C}^*)$ acts on V with three exponents $2, 1, 0$ and gives rise to a decomposition $V = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$. This action preserves the cone $\mathcal{C}_o \subset \mathbb{P}V$.

Taking the inverse of ρ and multiplying by a scalar representation, we get a $\mathbb{C}^*$-representation on V preserving $\mathcal{C}_o$ which fixes $\mathbb{C}\alpha$, acts as t on $\mathcal{H}_\alpha$ and acts as t^2 on $\mathcal{N}_\alpha$, $t \in \mathbb{C}^*$. Choose a generic point $\alpha + \xi + \zeta$ on $\tilde{\mathcal{C}}_o$, with $\xi \in \mathcal{H}_\alpha$ and $\zeta \in \mathcal{N}_\alpha$. The orbit of the $\mathbb{C}^*$-action is $\alpha + t\xi + t^2\zeta$. At $t = t_0$, we further consider the curve $\alpha + e^s t_0 \xi + e^{2s} t_0^2 \zeta$, $s \in \mathbb{C}$. Taking derivative with respect to s at $s = 0$, we get the tangent vector $t_0\xi + 2t_0^2\zeta$ to $\tilde{\mathcal{C}}_o$ at the point $\alpha + t_0\xi + t_0^2\zeta$. The corresponding element of $\tilde{\mathcal{T}}_o$ is

$$(\alpha + t_0\xi + t_0^2\zeta) \wedge (t_0\xi + 2t_0^2\zeta) = t_0\alpha \wedge \xi + 2t_0^2\alpha \wedge \zeta + t_0^3\xi \wedge \zeta \ .$$

Thus, the linear span of $\mathcal{T}_o$ contains $\alpha \wedge \xi, \alpha \wedge \zeta$, and $\xi \wedge \zeta$ for any generic $\alpha \in \mathcal{C}_o$ and $\alpha + \xi + \zeta \in \mathcal{C}_o$. As we vary ξ on $\mathcal{H}_\alpha$, the corresponding vectors $\zeta \in \mathcal{N}_\alpha$ span $\mathcal{N}_\alpha$. Otherwise, $\mathcal{C}_o$ will be contained in the linear subspace $\mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}'_\alpha$ for some proper subspace $\mathcal{N}'_\alpha \subset \mathcal{N}_\alpha$, contradicting the linear non-degeneracy of $\mathcal{C}_o$. It follows that $\mathcal{T}_o$ is linearly non-degenerate.

Remarks

The above proposition can be proved by checking case by case using representation theory. Let $\mathfrak{l} = [\mathfrak{k}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}]$ be the maximal semisimple subalgebra. The representation of $\mathfrak{l}$ on T_o can be identified from the table in Section 2. In Table 5 of Onishchik-Vinberg [OV], the irreducible decomposition of the second exterior power of these representations can be found. We will list them below. The notations are as in [OV], Table 1 (pp. 292-294) and Table 5 (pp. 299-305). π_i denotes the i-th fundamental weight of the simple Lie algebra and $R(\lambda)$ denotes the irreducible representation with the highest weight λ. For $\mathfrak{sl}(p) \times \mathfrak{sl}(q)$, $R(\lambda) \otimes R(\mu)$ denotes the tensor product of the $\mathfrak{sl}(p)$-representation $R(\lambda)$ and the $\mathfrak{sl}(q)$-representation $R(\mu)$.

Type	l	T_o	$\Lambda^2 T_o$
I	$\mathfrak{sl}(p) \times \mathfrak{sl}(q)$	$R(\pi_1) \otimes R(\pi_1)$	$R(\pi_2) \otimes R(2\pi_1) \oplus R(2\pi_1) \otimes R(\pi_2)$
II	$\mathfrak{sl}(n)$	$R(\pi_2)$	$R(\pi_1 + \pi_3)$
III	$\mathfrak{sl}(n)$	$R(2\pi_1)$	$R(2\pi_1 + \pi_2)$
IV	$\mathfrak{so}(n)$	$R(\pi_1)$	$R(\pi_2)$
V	$\mathfrak{so}(10)$	$R(\pi_5)$	$R(\pi_3)$
VI	$\mathfrak{e}_6$	$R(\pi_1)$	$R(\pi_2)$

With the exception of Type I, $\Lambda^2 T_o$ is irreducible. Proposition 14 is obvious in these cases, because the linear span of $\mathcal{T}_o$ is an invariant subspace.

For Type I, we can check it in the following way. Let $v \in T_o$ be the highest weight vector with weight $\pi_1 \otimes \pi_1$. There are two weights maximal among the remaining weights, $\pi_1 \otimes (\pi_2 - \pi_1)$ and $(\pi_2 - \pi_1) \otimes \pi_2$. Corresponding weight vectors are $l \cdot v$ and $l' \cdot v$ respectively, where l (resp. l') denotes an eigenvector of $\mathfrak{sl}(p)$ (resp. $\mathfrak{sl}(q)$) corresponding to the root α_1. Hence the highest weight vectors of $\Lambda^2 T_o$ are $v \wedge (l \cdot v)$ and $v \wedge (l' \cdot v)$. But $v \wedge (l \cdot v)$ and $v \wedge (l' \cdot v)$ correspond to tangential lines to the cone at v. Hence both highest weight vectors are in $\mathcal{T}_o$, as desired.

To study the meromorphic distribution W on X_0 we prove

Proposition 16. *Let $W_o \subset T_o$ be a subspace and $\mathbb{P}T_o \to \mathbb{P}W_o$ be a projection defined by choosing a complementary subspace. Suppose the strict image of $\mathcal{C}_o$ is linearly non-degenerate in $\mathbb{P}W_o$. Then, the tangential lines to the smooth part of the strict image generates $\mathbb{P}\Lambda^2 W_o$.*

Proof. The projection $T_o \to W_o$ induces a surjective map $\Lambda^2 T_o \to \Lambda^2 W_o$. Proposition 16 is thus a direct consequence of Proposition 14.

(5.2) We are ready to finalize the proof of Theorem 1, for which we may assume that S is of rank ≥ 2. We apply results of §4 to $M = X_0$. The family $\mathcal{K}$ is chosen as the family $\mathcal{K}$ of (2.2). Then, the irreducibility assumption made in §4 holds. By [(2.4), Proposition 6] a generic member of $\mathcal{K}$ represents a standard minimal rational curve. W_x, $\mathcal{C}_x$ and $\mathcal{T}_x$ are defined for a generic point $x \in X_0$. Let $\sigma : \Delta \to X$ be a section of π, so that $\sigma(0) = x$. From §3, we know that the family $\mathcal{M}_{\sigma(t)}$ is a trivial family of standard cones $\mathcal{C}_o$. We have the family of rational maps $\Theta_{\sigma(t)} : \mathcal{M}_{\sigma(t)} \to \mathbb{P}T_{\sigma(t)}(X_t)$. Thus, $\Theta_x : \mathcal{M}_x \cong \mathcal{C}_o \to \mathbb{P}T_x(X_0)$ is a (generically finite) rational map, defined by a subsystem of the complete linear system defining the standard embedding of $\mathcal{C}_o$. By applying [(5.1), Proposition 16], $\mathcal{T}_x$ is linearly non-degenerate in $\mathbb{P}\Lambda^2 W_x$. By [(4.2), Proposition 9], W is integrable. Since $b_2(X_0) = 1$, by [(4.3), Proposition 13] W must be the trivial distribution T_{X_0}. Thus, $\mathcal{C}_x \subset \mathbb{P}T_x(X_0)$ is linearly non-degenerate at a generic point x and is the standard embedding given by the complete linear system. By the Hartogs' extension result [(1.2), Proposition 3] there exists on X a holomorphic bundle of cones $\mathcal{C}_x \subset \mathbb{P}T_x$, defined everywhere on X, such that the inclusion is isomorphic to the embedding of the standard cone $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ on the model space S. Thus, we have obtained a holomorphic $K^{\mathbb{C}}$-structure on X_0, which is integrable by the closedness of the integrability condition (cf. [(1.4), Proposition 2]). As X_0 is simply-connected, it follows from the result of Ochiai [(1.4), Proposition 3] that X_0 is biholomorphic to S. The proof of Theorem 1 is completed.

Finally, for Theorem 1′ we note that ρ is projective because $\mathrm{Pic}(S) \cong \mathbb{Z}$. Then, Theorem 1′ follows from Theorem 1 by slicing by algebraic curves Z_0 on the base space Z, noting that if the fiber X_y is biregular to S at some point y on the curve Z_0, then X_z is biregular to S for all $z \in Z_0$ except for at worst a finite number of base points z, since S is infinitesimally rigid.

For completeness we explain here why the apparently stronger Theorem 1 is *a priori* equivalent to Theorem 1′. It suffices to show that modulo at worst a lifting by a ramified covering $\triangle \to \triangle$, the regular family $\pi : X \to \triangle$ can be completed to a regular family $\pi' : X' \to \mathbb{P}^1$ over the Riemann sphere $\mathbb{P}^1$ such that the total space X' is projective-algebraic. The extension can in fact be obtained by gluing the regular family $\pi : X \to \triangle$ with the trivial family on $D = \mathbb{P}^1 - \overline{\triangle(\frac{1}{2})}$, provided that the regular family $\pi : X \to \triangle$ is holomorphically trivial on the annulus $A = \triangle \cap D = \{z : \frac{1}{2} < |z| < 1\}$. The structure group of $\pi|_A : X|_A \to A$, $X|_A = \pi^{-1}(A)$, is the group $Aut(S)$ of automorphisms of S. The identity component $Aut_0(S)$ is a complex Lie group and $Aut(S)/Aut_0(S)$ is finite. Thus, lifting by at worst a finite ramified covering $\varphi : \triangle \to \triangle$ defined by $\varphi(z) = z^p$, we may assume that the structure group is the connected complex Lie group $Aut_0(S)$. It then follows that $\pi|_A : X|_A \to A$ is topologically trivial and hence holomorphically trivial by the Oka-Grauert Principle (Grauert [G], Cartan [C]) for holomorphic principal bundles, allowing us to define an extension $\pi' : X' \to \mathbb{P}^1$ by gluing. Finally we observe that X' is projective-algebraic. Let Λ be the determinant bundle of the relative tangent bundle T on X'. As all fibers are Kähler, $\Lambda|_{X'_t}$ is ample for any $t \in \mathbb{P}^1$. Let H be the hyperplane section line bundle on $\mathbb{P}^1$. Then, $\Lambda \otimes (\pi'^* H)^k$ is ample for k sufficiently large, as desired.

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