GENERALIZED BERGMANN METRICS AND INVARIANCE OF PLURIGENERA

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ABSTRACT. An invariant kernel for the pluricanonical system of a projective manifold of general type is introduced. Using this kernel we prove that the Yau volume form on a smooth projective variety has seminegative Ricci curvature. As a biproduct we prove the invariance of plurigenera for smooth projective deformations of manifolds of general type.

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1. Introduction

Let Ω be a bounded domain in C^n and let $\{\phi_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of of L^2 holomorphi n-forms on Ω. Then the Bergmann kernel of Ω is defined by

$$K(z,w) = \sum_{i=1}^{\infty} \phi_i(z)\bar{\phi}_i(w)$$

and the Bergmann Kähler form is defined by

$$\omega = \sqrt{-1}\partial\bar{\partial}\log K(z,z).$$

The Kähler metric associated with ω is called the Bergmann metric of Ω. One of the most important property of the Bergmann metric is the invaiance underthe action of the holomorphic automorphisms of Ω.

In the same way, one can introduce the Bergmann kernel and the Bergmann metric on a complex manifold X such that the space of L^2 canonical forms $H^0_{(2)}(X, \mathcal{O}_X(K_X))$ is very ample. In the case of compact complex manifolds, the class of such manifolds corresponds to the class of complex manifolds whose canonical bundles are very ample. Hence it is a relatively small class.

In this paper, we generalize the Bergmann metric to wider class of manifolds and discuss some applications. Roughly speaking, we introduce a Bergmann kernel function for pluricanonical systems.

Theorem 1. *Let X be a projective manifold of general type. Then for every sufficiently large integer m, there exists a singular Kähler form ω_m on X and a reproducing kernel $K_m(z,w)$ of holomorphic m-ple canonical forms on X with respect to the inner product induced from the $-m$-th powe of the Yau intrisic pseudovolume form on X such that*

(1) *ω_m is invariant under $Aut(X)$,*
(2) *there exists a singular (degenerate) m-ple volume form $K_m(z,z)$ on X such that*

$$\omega_m = \frac{\sqrt{-1}}{m}\partial\bar{\partial}\log K_m(z,z)$$

holds.

Remark 1. *To construct ω_m one can use the inner product defined by the $-m$-th power of the Kobayashi pseudovolume form. Then we have another singular Kähler form. The reason why we use Yau pseudovolume form is mainly the bimeromophic invariance of it. This enables us to use birational geometry instead of biholomorphic geometry. But I do not have any specific example which tells us that these pseudovolume forms are actually different.*

Theorem 2. *Let X be a projective manifold of general type. Let $d\mu$ be the Yau volume form on X. Then the Ricci curvature of $d\mu$ is negative in the sense of current. In fact*

$$d\mu(z) = \limsup_{m\to\infty} K_m(z,z)^{\frac{1}{m}} (z \in X)$$

and

$$-Ric\, d\mu = \lim_{m\to\infty} \omega_m$$

hold.

As an application of this theorem, we prove the following theorem.

Theorem 3. *Let $\pi : X \longrightarrow S$ be a smooth projective family of projective varieties. Assume that every fibre of π is of general type. Then the plurigenera of the fibres are locally constant on S.*

Theorem 3 is a partial answer to the following conjecture.

Conjecture 1. *The plurigenera is invariant under smooth projective deformations.*

There have been several works on this conjecture.

The conjecture is trivial for the family of curves. As for the deformation of surfaces, S. Iitaka([10]) proved the conjecture using classification of compact complex surfaces. In general dimensions an interesting approach was proposed by M. Levine ([18]). He computed the obstruction for the extention of the member of pluricanonical system on a fibre and proved that it always vanishes by using Hodge theory in the case that the member is smooth. On the other hand, N. Nakayama pointed out that if the

minimal model program has been completed (also in the relative case), the conjecture is a direct consequence ([20]). By the completion of minimal model program for the case of 3-folds, J. Kollar and S. Mori proved Conjecture 1 in the case of 3-folds ([13, p.535, Theorem 1.3]). These approaches require some smoothness for a general member of the pluricanonical systems on the fibres.

Our approach is completely different from the above ones. We use the Yau pseudovolume form on the fibre to controle the growth of plurigenera and use the continuity of the pseudovolume form under the deformation. To get the disired result we use the theory of analytic Zariski decompositions and L^2-extention theorem in [21].

The author would like to express his hearty thanks to Professor S. Kobayashi who pointed out the reference [27] and to Professor T. Ohsawa for his frank criticism. He also would like to express his hearty thanks to Professor S. Lu who explained Yau pseudovolume form to him. Finally he would like to express his hearty thanks to MSRI for their hospitality and the stimulating atomosphere.

Remark 2. *Recently T. Mabuchi and I have proved the conjecture in full generality by a different method (in preparation). Hence Theorem 3 has beed generalized. But the proof of Theorem 3 would be of independent interest.*

Part I. Generalized Bergmann Kernels

2. Generalized Bergmann metrics

2.1. Yau pseudovolume form. In [28], S.-T. Yau introduced an intrinsic pseudovolume form on a complex manifold which is very similar to Kobayashi volume form.

Let M be a n-dimensional connected complex manifold. Let Δ^n denote the unit open polydisk in C^n. Let us take a point x on M. Let $f : \Delta^n \longrightarrow M$ be a meromorphic mapping which satisfies the following three conditions:

(1) f is holomorphic on a neighbourhood of O,
(2) $f(O) = x$,
(3) f is nondegenerate at O.

Then we define a pseudovolume form $d\lambda(x)$ at x by

$$d\mu(x) = \inf(f^{-1})^* d\mu_{\Delta^n}$$

where the infimum is taken with respect to all the f which satisfies the above three conditions and $d\mu_{\Delta^n}$ is the Poincaré volume form on Δ^n defined by

$$d\mu_{\Delta^n} = (\prod_{i=1}^{n} \frac{4}{(1- \mid z_i \mid^2)^2})(\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

Let $f : X - \cdots \to Y$ be a meromorphic mapping between complex manifolds. Then we have the inequality:

$$f^* d\mu_Y \leq d\mu_X$$

by the definition of the pseudovolume form. We call this property the volume decreasing property of Yau pseudovolume form. By this property it is clear that Yau pseudovolume form is invariant under bimeromorphic mappings.

It is well known that $d\mu$ is an uppersemicontinuous pseudovolume form on X. It is clear that Yau pseudovolume form is smaller than Kobayashi pseudovolume form (I do not know any example which shows these volume forms are actually different). We call a complex manifold X to be meromorphically measure hyperbolic if for any nonempty open subset of X the measure defined by $d\mu$ is nonzero. It is easy to see that every Kobayashi hyperbolic manifolds are meromorphicall y measure hyperbolic.

Theorem 4. *([28]) Let X be a projective manifold of general type. Then $d\mu$ is nondegenerate on a nonempty Zariski open subset of X. In particular X is meromorphically measure hyperbolic.*

Lemma 1. *Let X be a projective manifold of general type. Then there exists a proper subvariety V of X such that $d\mu$ is continuous on $X - V$.*

Proof. This fact is essentially in [28] and was used in [27] implicitly (see the proof of [27, p. 367, Theorem 8]). The proof goes as follows.

For the first by the definiciton we see that $d\mu$ is uppersemicontinuous. Hence we only need to prove that $d\mu$ is lowersemicontinuous. If we take m sufficiently large, we may assume that $\mid mK_X \mid$ gives a birational rational mapping into a projective space. Let $\sigma_0, \ldots, \sigma_N$ be global holomorphic sections of $mK_X (m >> 1)$ which give a birational rational mapping Φ into P^N and the pullback of the Fubini-Study form

$$\sqrt{-1}\partial\bar{\partial}\log\sum_{i=0}^{N} \mid \sigma_i \mid^2$$

is a strictly positive on X. The existence of $\sigma_0, \ldots, \sigma_N$ follows from Kodaira's lemma (cf.[Appendix][14]). Let V denote the support of the base locus of $\sigma_0, \ldots, \sigma_N$. Let $f_i : \Delta^n \longrightarrow X$ be a sequece of meromorphic mappings such that

(1) f_i is holomorphic near the origin and $\{f_i(O)\}$ converges to a point x on $X - V$,
(2) the Jacobian determinant $\operatorname{Jac} df_{i,O}$ are not zero and converges to a nonzero element.

We note that $f_i^*\sigma_j (0 \leq j \leq N)$ are holomorphic pluricanonica l forms on Δ^n by Hartogs extention theorem. Then by the maximum principle, there exists a constant C independent of i such that

$$\left(\frac{(\sqrt{-1})^{m\frac{n(n-1)}{2}} f_i^*(\sum_{j=0}^{N} \sigma_j \wedge \bar{\sigma}_j)}{d\mu_{\Delta^n}^m}\right) \leq C$$

holds. Here as usual we apply the maximum principle for

$$(\frac{(\sqrt{-1})^{m\frac{n(n-1)}{2}} f_{i,\varepsilon}^{*}(\sum_{j=0}^{N} \sigma_j \wedge \bar{\sigma}_j)}{d\mu_{\Delta^n}^{m}}),$$

where

$$f_{i,\varepsilon}(z) = f_i((1-\varepsilon)z) \quad ((1-\varepsilon)z = ((1-\varepsilon)z_1, \dots, (1-\varepsilon)z_n)) \quad (0 < \varepsilon << 1)$$

and letting ε tend to 0.

Then by the normal family argument (Montel's theorem), we see that there exists a subsequence $\{f_k\}$ of $\{f_i\}$ such that $\{f_k^{*}\sigma_j\}(j = 0, \dots, N)$ converges to a holomorphic m-ple canonical forms on Δ^n compact uniformly. Then $\{\Phi \circ f_k\}$ converges to a meromorphic mapping from Δ^n into P^N. By the biratonality of Φ $\{f_k\}$ of $\{f_i\}$ converges to a meromorphic mapping $f_\infty : \Delta^n \longrightarrow X$ such that

$$df_{\infty,O} = \lim_{k\to\infty} df_{k,O}$$

holds. We note that by the assumption $x \in X - V$ and the nondegeneracy condition of the Jacobian, f_∞ is holomorphic at O. Then by this fact it is easy to see that $d\mu$ is continuous on $X - V$. Q.E.D.

The following corollary is an easy consequence of the above theorem.

Corollary 1. *Let X be a projective manifold of general type. Then there exits a proper subvariety V in X such that for every $x \in X - V$, there exists a meromorphic mapping $f : \Delta^n \longrightarrow X$ which satisfies the following three conditions:*

(1) *f is holomorphic on a neighbourhood of O,*
(2) *$f(O) = x$ holds,*
(3) *$(f^{-1})^{*}(d\mu_{\Delta^n})(x) = d\mu_X(x)$ holds.*

2.2. Construction of the generalized Bergmann kernels. In this subsection we shall prove Theorem 1. Because in some applications, it is more convinient to construct the generalized Bergmann kernel on the universal covering space, we shall consider not only the projective manifolds of general type but also its universal covering spaces. The proof we present here is for the universal coverings. If we restrict ourselves to the case of the manifold itself, the proof is much more straightforward. But it seems to be natural to consider the universal covering in connection withapplications. The following proposition follows from the definition of Yau pseudovolume form.

Proposition 1. *Let M be a complex manifold and let $\pi : \tilde{M} \longrightarrow M$ be an unramified covering. Then $d\mu_{\tilde{M}} = \pi^{*}d\mu_M$ holds. In particular M is meromorphically measure hyperbolic if and only if $\tilde{M}$ is meromorphically measure hyperbolic.*

Let X be a meromorphically measure hyperbolic manifold. We define the subspace $H_m = H_m(X)$ of $H^0(X, \mathcal{O}_X(mK_X))$ by

$$H_m = \{\eta \in H^0(X, \mathcal{O}_X(mK_X)) \mid \ (\sqrt{-1})^{m\frac{n(n-1)}{2}} \int_X \frac{\eta \wedge \bar{\eta}}{d\mu_X^m} d\mu_X < \infty\}.$$

We introduce an inner product on H_m by

$$(\eta, \tau) = (\sqrt{-1})^{m\frac{n(n-1)}{2}} \int_X \frac{\eta \wedge \bar{\tau}}{d\mu_X^m} d\mu_X.$$

The inner product exists by Schwarz inequality.

Lemma 2. *H_m is a Hilbert space.*

Proof. Let $\{\eta_k\}$ be a Cauchy sequence in H_m. Let $f : \Delta^n \longrightarrow X$ be an embedding into a relatively compact subset of X. We shall prove $\{f^*\eta_k\}$ converges. By the volume decreasing property of Yau pseudovolume form, we see that $\{f^*\eta_k\}$ is a Cauchy seqence with respect to the inner product

$$(\eta, \tau) = (\sqrt{-1})^{m\frac{n(n-1)}{2}} \int_X \frac{\eta \wedge \bar{\tau}}{d\mu_{\Delta^n}^m} d\mu_{\Delta^n}.$$

Hence $\{f^*\eta_k\}$ converges to a m-ple canonical form. Since f is arbitrary, $\{\eta_k\}$ converges. Q.E.D.

Let $\{\phi_i\}$ be a complete orthonormal basis for H_m. We set

$$K_m(z, w) = \sum_i \phi_i(z)\bar{\phi}_i(w)$$

and

$$\omega_m = \frac{\sqrt{-1}}{m} \partial\bar{\partial} \log K_m(z, z).$$

$K_m(z, w)$ exists since for every $x \inf X$, $ev_x : H_m \longrightarrow C \simeq \mathcal{O}_X(mK_X)/\mathcal{M}_x$ defined by

$$ev_x(f) = f(x)$$

is a bounded functional. If H_m is very ample, ω_m is a C^∞-Kähler metric on X. In this case we call ω_m the m-ple Bergmann metric of X.

Let X be a projective manifold of general type and let $\pi : \tilde{X} \longrightarrow X$ be an universal covering of X. Let ω_X be a Kähler form on X. Let $H^0_{(2)}(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}))$ be the space of L^2 holomorphic sections of $mK_{\tilde{X}}$ with respect to $\pi^*\omega_X$. Then by the standard L^2-estimates for $\bar{\partial}$ operator, we see that for every large m, $H^0_{(2)}(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}))$ defines

a bimeromorphic mapping on to its image in some infinite dimensional projective space.

Lemma 3. *$H_m(\tilde{X}) = H^0_{(2)}(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}))$ holds.*

Proof. For the first we shall prove that

$$H^0(X, \mathcal{L}^2(mK_X, d\mu_X^{-m})) \simeq H^0(X, \mathcal{O}_X(mK_X))$$

holds for every positive integer m, where $\mathcal{L}^2(mK_X, d\mu_X^{-m})$is the sheaf defined by

$$\mathcal{L}^2(mK_X, d\mu_X^{-m})(U) = \{\sigma \in \Gamma(U, \mathcal{O}_X(mK_X)) \mid \frac{\mid \sigma \mid^2}{d\mu^m} \in L^1_{loc}(U)\}.$$

Let m be a sufficiently large positive integer such that $\mid mK_X \mid$ gives a birational rational mapping from X into a projective space. Let

$$\pi_m : X_m \longrightarrow X$$

be a resolution of Bs $\mid mK_X \mid$ and let

$$\pi_m^* \mid mK_X \mid = \mid P_m \mid + F_m$$

be the decomposition into the free part and the fixed part. Then by the assumption P_m is nef and big. We identify $\mathcal{O}_{X_m}(P_m)$ as a subsheaf of $\mathcal{O}_{X_m}(\pi_m^*(mK_X))$ Then by Kodaira's lemma, we see that there exists an effective Q-divisor E such that $P_m - \varepsilon E$ is an ample Q-divisor for every sufficiently small positive rational number ε. Let $r = r(\varepsilon)$ be a positive integer such that $r(P_m - \varepsilon E)$ is a very ample Cartier divisor. Let $\sigma_0, \ldots, \sigma_N$ be a basis of $H^0(X_m, \mathcal{O}_{X_m}(r(P_m - \varepsilon E)))$. Then for every meromorphic mapping $f : \Delta^n \longrightarrow X$ we have the inequality

$$\frac{(\sqrt{-1})^{rm\frac{n(n-1)}{2}} \sum_{j=0}^N f^*\sigma_j \wedge \bar{\sigma}_j}{d\mu_{\Delta^n}^{rm}} \leq C$$

holds, where C is a positive constant independent of f. Hence we have that

$$d\mu_X \geq \frac{1}{\sqrt[rm]{C}} (\sqrt{-1})^{\frac{n(n-1)}{2}} (\sum_{j=0}^N \sigma_j \wedge \bar{\sigma}_j)^{\frac{1}{rm}}$$

holds on X. This implies that

$$\pi_m^* \mathcal{L}^2(mrK_X, d\mu_X^{-mr})) \supseteq \mathcal{O}_{X_m}(r(P_m - \varepsilon E))$$

holds.

Letting ε tend to 0, by using the above lower estimate for $d\mu_X$ we see that

$$\pi_m^* \mathcal{L}^2(mK_X, d\mu_X^{-m}) \supseteq \mathcal{O}_{X_m}(P_m)$$

holds. Hence for a sufficiently large m we have the desired isomorphism:

$$H^0(X, \mathcal{L}^2(mK_X, d\mu_X^{-m})) \simeq H^0(X, \mathcal{O}_X(mK_X)).$$

By using the ring structure of the canonical ring $R(X, K_X)$, we conclude that the above isomorphim holds for every positive m.

Suppose that there exists an element ϕ in $H^0_{(2)}(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}))$ which is not contained in $\phi \ni H_m$. We set for a positive integer $k > 1$,

$$av(\phi^k) = \sum_{\gamma \in \pi_1(X)} \gamma^* \phi^k.$$

Then as in [26], $av(\phi^k)$ exists and defines an element of $H^0(X, \mathcal{O}_X(mkK_X))$. Then we have for every $x \in X$

$$\mathrm{mult}_x(av(\phi^k)) \geq \mathrm{mult}_x \mathrm{Bs}\,|mkK_X|$$

holds. Let us note that the following simple fact that if a sequence of complex numbers $\{a_\gamma\}$ such that $\sum_\gamma a_\gamma^k$ converges for every integer $k \geq 1$ and

$$\sum_\gamma a_\gamma^k = 0$$

holds for every $k \geq 1$, then $a_\gamma = 0$ for every γ.

Hence we have for every $x \in \tilde{X}$

$$\mathrm{mult}_x(\phi) \geq \lim_{k \to \infty} k^{-1} \mathrm{mult}_{\pi(x)} \mathrm{Bs}\,|mkK_X|$$

holds. This contradicts the assumption that ϕ is not contained in H_m by Proposition 1 and the first half of the proof of this lemma. Q.E.D.

This lemma implies that the kernel K_m exists on $\tilde{X}$.
And $\omega_m = \sqrt{-1}\partial\bar{\partial} \log K_m(z, z)$ defines a singular Kähler form on $\tilde{X}$. We see that by the definition that ω_m is invariant under $\mathrm{Aut}(\tilde{X})$. Hence ω_m definies a singular Kähler form on X.

Corollary 2. *([26]) Let X_1,X_2 be compact unramified quotients of a complex manifold $\tilde{X}$. Then X_1 is of general type if and only if X_2 is of general type.*

Proof. Suppose that X_2 is of general type. Then by Theorem 1, X_1 carries a singular Kähler form ω_m which is the curvature form of a singular hermitian metric of the canonical bundle of X_1. Then by the standard L^2-estimates for $\bar{\partial}$ operator, we see that X_1 is also of general type. Q.E.D.

3. Limit of the Generalized Bergmann kernels

In this section we shall denote $K_m(x, x)$ by $K_m(x)$ for simplicity.

3.1. First properties of Generalized Bergmann kernels.

Lemma 4. *For every $x \in X$*

$$K_m(x) = \max_{\|\phi\|=1} (\phi(z)\bar{\phi}(x))$$

holds.

Proof It is clear that

$$K_m(x) \geq \max_{\|\phi\|=1} (\phi\bar{\phi})(x)$$

holds by the definition of K_m. On the other hand,

$$K_m \leq \max_{\|\phi\|=1} (\phi\bar{\phi})$$

follows from Riesz's theorem. Q.E.D.

Lemma 5.

$$\limsup_{m\to\infty} \int_X K_m^{\frac{1}{m}} \leq \int_X d\mu$$

holds.

Proof. By Hölder inequality we see that

$$\int_X K_m^{\frac{1}{m}} \leq (\int_X \frac{K_m}{d\mu^m} d\mu)^{\frac{1}{m}} \cdot (\int_X d\mu)^{\frac{m-1}{m}}$$

holds. We note that

$$\int_X \frac{K_m}{d\mu^m} d\mu = \dim H^0(X, \mathcal{O}_X(mK_X))$$

holds.

By Riemann-Roch theorem, we see that there exists a constant c such that

$$\dim H^0(X, \mathcal{O}_X(mK_X)) = cm^n + o(m^n)$$

holds. Hence we have that

$$\lim_{m\to\infty} (\int_X \frac{K_m(z,z)}{d\mu^m} d\mu)^{\frac{1}{m}} = 1$$

holds. This implies that

$$\limsup_{m\to\infty} \int_X K_m(z,z)^{\frac{1}{m}} \leq \int_X d\mu$$

holds. Q.E.D.

3.2. Poinwise upper estimate of $d\mu$. Let us consider the function

$$F_m = \frac{d\mu^m}{K_m}$$

on X.

For the first we shall assume that K_X is ample. In this case there exists a point x_0 where F_m takes its minimum since F_m is continuous. We set

$$h_m = (\frac{1}{F_m(z_0)K_m})^{1/m}.$$

Then h_m is a singular hermitian metric on K_X.

Lemma 6. *(*[22, p.103, Lemma 1.2]*) Let X be a smooth projective manifold and let (L,h) be a positive line bundle on X. Let x_0 be a point on X. Let g be the Kähler metric associated with the curvature of h. Choose a normal coordinate $(z_1,\dots,z_n)$ at x_0 such that $x_0=(0,\dots,0)$. We choose a local holomorphic frame e_L of L at x_0 such that the local representation function a is the hermitian metric h has the property*

$$a(x_0)=1, da(x_0)=0.$$

For an n-tuple of integers $(p_1,\dots,p_n)\in Z^n_{\geq 0}$ and an integer $p'>p=p_1+\cdots+p_n$ be a positive integer there exists $m_0>0$ such that for $m>m_0$, there exists a holomorphic section S in $H^0(X,\mathcal{O}_X(mL))$, satisfying

(1)

$$\int_X \parallel S \parallel_{h^m} dV_g = 1,$$

(2)

$$S(z)=\lambda_{(p_1,\cdots,p_n)}(z_1^{p_1}\cdots z_n^{p_n}+O(\mid z\mid^{2p'})e_L^m(1+O(\frac{1}{m^{2p'}})),$$

where

$$\lambda^{-2}_{(p_1,\cdots,p_n)}=\int_{X\ \{\mid z\mid\leq\frac{\log m}{\sqrt{m}}\}} \mid z_1^{p_1}\cdots z_n^{p_n}\mid^2 a^m dV_g.$$

where dV_g be the volume form with respect to g and $\mid z\mid=\sqrt{\sum\mid z_i\mid^2}$.

We call the above S's the peak sections of (L,h) at x_0. Let g be the Kähler metric associated with curv h_m. For a positive integer ℓ we define the number $p(\ell)$ by

$$p(\ell)=\max\{\parallel\sigma\parallel_{h^\ell_m}(x_0)\mid\ \sigma\in H^0(X,\mathcal{O}_X(\ell K_X)),\int_X\parallel\sigma\parallel^2_{h^\ell_m}dV_g=1\},$$

and let σ_ℓ denote the section which attains the maximum. Then by Lemma 6, we see that

$$\lim_{\ell\to\infty}p(\ell)^{1/\ell}=1.$$

Since $h_m \geq d\mu^{-m}$ holds on X by the definition of h_m, by Lemma 6, we see that

$$\int_{X-\{|s|\leq \frac{\log \ell}{\sqrt{\ell}}\}} \| \sigma_\ell \|^2_{d\mu^{-m\ell}} \, dV_g = O(\frac{1}{\ell^2}).$$

Since $d\mu$ is continuous and $h_m(x_0) = d\mu^{-1}(x_0)$ holds , we see that

$$\lim_{\ell\to\infty} (\int_{\{|s|\leq \frac{\log \ell}{\sqrt{\ell}}\}} \| \sigma_\ell \|^2_{d\mu^{-m\ell}} \, dV_g)^{1/\ell} = 1$$

holds. By Lemma 4 this implies that

$$\limsup_{\ell\to\infty} K_\ell(x_0)^{\frac{1}{\ell}} \geq d\mu(x_0)$$

holds.

On the other hand, for a positive integer ℓ we define the number $q(\ell)$ by

$$q(\ell) = \max\{\| \tau \|_{d\mu^{-\ell}}(x_0) \mid \ \tau \in H^0(X, \mathcal{O}_X(\ell K_X)), \int_X \| \sigma \|^2_{d\mu^{-\ell}} \, d\mu = 1\}$$

and let τ_ℓ denote the section which attains the maximum. By Lemma 4, we have

$$q(\ell) = \frac{K_\ell(x_0)}{d\mu^\ell(x_0)}$$

holds. By the inequality

$$d\mu^{-1} \leq h_m,$$

as above we see that L^2 norm of τ_ℓ with respect to $d\mu^{-\ell}$ concentrates around x_0 as ℓ tends to infinity. In fact let $(z_1, \ldots, z_n)$ be a local coordinate around x_0 as in Lemma 6 (with respect to h_m). Let ε be a sufficiently small positive number. Let ρ be a nonnegative C^∞ function of $| z |^2 = \sum_{i=1}^n | z_i |^2$ such that

(1) $\rho \equiv 1$ on $B(O, \varepsilon/2) (= \{z \in X \mid | z | \leq \varepsilon/2\})$,
(2) $| d\rho | \leq 3/\varepsilon$,
(3) $\rho \equiv 0$ on $X - B(0, \varepsilon)$.

If ε sufficiently small, ρ is well defined. Let a be a positive function on $B(x, \varepsilon)$ defined as in Lemma 6. Then there exists a posive constant c such that

$$a \leq (1 - c\sum_{i=1}^n | z_i |^2)$$

holds on $B(O, \varepsilon)$. This follows from the fact that curv h_m is positive at x_0. On the other hand since F_m takes its minimum at x_0, we see that the Taylor expansion of τ_ℓ around x_0 is of the form

$$\tau_\ell = (c_\ell + O(| z |^2)\sigma_m^{\ell/m}$$

holds if $m \mid \ell$, where c_ℓ is a constant depending on ℓ. For the case that m does not divide ℓ we have the similar expansion of the hermitian norm of τ_ℓ with respect to h_m^ℓ.

We set

$$G := \bar{\partial}(\rho\sigma_\ell).$$

Let ψ be a function on X defined by

$$\psi = n\rho\log(\sum_{i=1}^{n} \mid z_i \mid^2)$$

Then by the Taylor expansions of τ_ℓ and a, we see that there exist a positive constant C independent of ℓ and a positive number $0 < \beta < 1$ such that

$$\int_X e^{-\psi} \parallel F \parallel^2_{h^\ell_m} dv \leq C \cdot c_\ell \cdot \beta^\ell.$$

holds, where dv is the pseudovolume form defined by

$$dv = K_m^{\frac{1}{m}}.$$

Then by the standard Hörmander's L^2 estimates for $\bar{\partial}$ opearator , we have that there exists $u \in C^\infty(X, \nu K_X)$ such that

$$\bar{\partial}u = G$$

and

$$\int_X e^{-\psi} \parallel u \parallel^2_{h^\ell_m} dv \leq C_\ell \int_X e^{-\psi} \parallel F \parallel^2_{h^\ell_m} dv \leq C \cdot C_\ell \cdot c_\ell \cdot \beta^\ell$$

holds, where C_ℓ is a positive constant satisfying

$$C_\ell = O(\frac{1}{\ell}).$$

Moreover since $d\mu^{-1} \leq h_m$ we see that

$$\int_X e^{-\psi} \parallel u \parallel^2_{d\mu^{-\ell}} dv \leq C \cdot C_\ell \cdot c_\ell \cdot \beta^\ell$$

holds. By the construction

$$\tilde{\tau}_\ell := \rho\tau_\ell - u$$

is a holomorphic ℓ-ple canonical form on X and

$$\tilde{\tau}_\ell(x_0) = \tau_\ell(x_0)$$

holds. Since $\beta < 1$, we see that

$$\lim_{\ell\to\infty} \frac{\parallel \tau_\ell \parallel}{\parallel \tilde{\tau}_\ell \parallel} = 1$$

holds.

Then it is clear that

$$\limsup_{\ell\to\infty} q(\ell)^{1/\ell} \leq 1$$

holds. Hence we have that

$$\limsup_{\ell\to\infty} K_\ell(x_0)^{\frac{1}{\ell}} = d\mu(x_0)$$

holds. Since m can be arbitrary large we have that

$$d\mu \geq \limsup_{\ell\to\infty} K_\ell^{\frac{1}{\ell}}$$

holds. This implies that the L^2-norm of τ_ℓ concentrates around x_0 as ℓ tends to infinity. This means that as for the asymptotics, we can work only on a small neighbourhood of x_0. By using this fact, we see that

$$\limsup_{\ell\to\infty} q(\ell) \leq 1$$

holds.

Combining the above arguments we see that

$$d\mu(x_0) = \limsup_{\ell\to\infty} K_\ell(x_0)^{1/\ell}$$

holds. Letting m tend to infinity, we see that

$$d\mu \geq \limsup_{\ell\to\infty} K_\ell^{1/\ell}$$

holds on X.

If K_X is not ample, we need to modify the argument as follows. By Kodaira's lemma (cf. [14, Appendix]) there exists an effective Q-divisor E such that $K_X - E$ is ample. We may assume that the support of E is V in Lemma 1. Let r be a positive intger such that rE is a Z-divisor. For every $m \geq r$ we define the modified generalized Bergmann kernel $\tilde{K}_m(z,w)$ by

$$\tilde{K}_m(z,w) = \sum_i \tilde{\phi}_i(z)\bar{\tilde{\phi}}_i(w),$$

where $\{\tilde{\phi}_i\}$ is a orthonormal basis of $H^0(X, \mathcal{O}_X(mK_X - rE))$ with respect to the inner product introduced in 2.2. By Lemma 4 one can show easily that for $x \in X - V$

$$\lim_{m\to\infty} (\frac{\tilde{K}_m(x)}{K_m(x)})^{1/m} = 1$$

holds by looking at the product:

$$H^0(X, \mathcal{O}_X(m_0K_X - rE)) \times H^0(X, \mathcal{O}_X(mK_X)) \to H^0(X, \mathcal{O}_X((m+m_0)K_X - rE)),$$

where m_0 is a sufficiently large positive integer. It is clear that for a sufficiently large m, $d\mu^m/\tilde{K}_m$ is infinity on V and the Ricci form of $\tilde{K}_m$ is strictly negative. Using $\tilde{K}_m$ instead of K_m, we can argue exactly the same way. Letting m tend to infinity, we obtain the following lemma.

Lemma 7.

$$d\mu \geq \limsup_{\ell\to\infty} K_\ell^{\frac{1}{\ell}}$$

holds on X.

3.3. Lower estimate of the limit of Bergmann kernel. If we have the pointwise inequality

$$\limsup_{m\to\infty} K_m(x,x)^{\frac{1}{m}} \geq d\mu(x) \tag{1}$$

for every $x \in X$, by Lemma 5 , we have the desired equality

$$\limsup_{m\to\infty} K_m(x)^{\frac{1}{m}} = d\mu(x)$$

for every $x \in X$. Hence we shall prove (1). Let m be a sufficiently large positive integer. Let V be a proper subvariety of X as in Lemma 1 and let x be a point on $X - V$. Let $\| \ \|_m$ denote the hermitian norm on mK_X with respect to the hermitian metric K_m^{-1} and let $\| \ \|$ denote the hermitian norm with respect to $d\mu^{-m}$.

Let $\sigma \in H^0(X, \mathcal{O}_X(mK_X))$ be a section of mK_X such that $\| \sigma \|_m = 1$ and $(\sigma, \sigma) = 1$ where $(\ , \ )$ denotes the inner product with respect to the hermitian metric $d\mu^{-m}$ as in the last section. Then by Lemma 4, x is the point where $\| \sigma \|$ takes its maximum. Hence we have that

$$d \| \sigma \| (x) = 0$$

holds. Let $f : \Delta^n \longrightarrow X$ be a meromorphic mapping such that

(1) f is holomorphic on a neighbourhood of O,
(2) $f(O) = x$,
(3) $(f^{-1})^* d\mu_{\Delta^n} = d\mu(x)$.

Such f exists by Corollary 1. If we restrict f^{-1} on a small neighbourhood of x, we may consider f^{-1} as a local coordinate around x. We shall denote the coordinate by $(z_1, \ldots, z_n)$. We note that the inequality

$$d\mu \leq \prod_{i=1}^{n} \frac{4\sqrt{-1} dz_i \wedge d\bar{z}_i}{(1 - | \ z_i \ |^2)^2}$$

holds on a neighbourhood of x by the definition of $d\mu$ and the equality holds at x. Let g_m be the hermitian metric on mK_X on the neighbourhood defined by

$$g_m = (\prod_{i=1}^{n} \frac{4\sqrt{-1} dz_i \wedge d\bar{z}_i}{(1 - | \ z_i \ |^2)^2})^{-m}.$$

Then it is clear that

$$d \| \sigma \|_{g_m} (x) = 0$$

holds. This implies that

$$d(\frac{\prod_{i=1}^n(1-\mid z_i\mid^2)^{2m}\sigma\wedge\bar{\sigma}}{\prod_{i=1}^n(dz_i\wedge d\bar{z}_i)^m})(x)=0$$

holds. Hence if we write $(dz_1\wedge\cdots\wedge dz_n)^m$ as

$$(dz_1\wedge\cdots\wedge dz_n)^m=b(z)\sigma\quad(f\in\mathcal{O}_{X,x})$$

on a neighbourhood of x, we have that

$$b(z_1,\ldots,z_n)=c+\sum_{i,j}c_{ij}z_iz_j+O(\mid z\mid^3)$$

holds, where c, c_{ij} are constants. Let ε be a sufficiently small positive number. Let ρ be a nonnegative C^∞ function of $\mid z\mid^2=\sum_{i=1}^n\mid z_i\mid^2$ such that

(1) $\rho\equiv 1$ on $B(x,\varepsilon/2)(=\{z\in X\mid\mid z\mid\leq\varepsilon/2\})$,
(2) $\mid d\rho\mid\leq 3/\varepsilon$,
(3) $\rho\equiv 0$ on $X-B(0,\varepsilon)$.

If ε sufficiently small, ρ is well defined.

Let a be a positive function on $B(x,\varepsilon)$ defined by

$$a:=\parallel\sigma\parallel_m^2.$$

We note that $\limsup_{m\to\infty}K_m^{-1/m}$ has strictly positive curvature on $X-V$. This fact follows from for a sufficiently large ℓ $\tilde{K}_\ell^{1/\ell}$ defined in the last subsection has a strictly positive curvature on X and satisfies the inequality

$$\tilde{K}_\ell^{1/\ell}\leq C_\ell d\mu$$

on X for some positive constant C_ℓ by the result in the last subsection. Then applying Lemma 6 to the cases $(p_1,\ldots,p_n)=(1,0,\ldots,0),\ldots,(0,\ldots,0,1)$ we see that there exists a posive constant c independent of m such that

$$a\leq(1-c\sum_{i=1}^n\mid z_i\mid^2)^m$$

holds for every sufficiently large m. We set for a positive integer ν

$$F=\bar{\partial}(\rho\frac{(dz_1\wedge\ldots\wedge dz_n)^{\otimes\nu}}{c_\nu}),$$

where c_ν is a positive number defined by

$$c_\nu=4^{-n\nu}\int_{\Delta^n}\prod_{i=1}^n(1-\mid z_i\mid^2)^{\nu-1}d\lambda$$

where $d\lambda$ is the usual Lebesgue measure.

Let ψ be a function on X defined by

$$\psi = n\rho \log(\sum_{i=1}^{n} \mid z_i \mid^2)$$

Then by the Taylor expansions of b and a, we see that there exist a positive constant C independent of ν and a positive number $0 < \alpha < 1$ such that

$$\int_X e^{-\psi} \parallel F \parallel^2_{m,\nu} dv \leq C(\frac{d\mu^m(x)}{K_m(x)})^{\nu/m}\alpha^\nu.$$

holds, where $\parallel \ \parallel_{m,\nu}$ denotes the hermitian norm induced by $K_m^{-\nu/m}$ and dv is the pseudovolume form defined by

$$dv = K_m^{\frac{1}{m}}.$$

Then by the standard Hörmander's L^2 estimates for $\bar{\partial}$ opearator , we have that there exists $u \in C^\infty(X, \nu K_X)$ such that

$$\bar{\partial} u = F$$

and

$$\int_X e^{-\psi} \parallel u \parallel^2_{m,\nu} dv \leq C_\nu \int_X e^{-\psi} \parallel F \parallel^2_{m,\nu} dv \leq C \cdot C_\nu (\frac{d\mu^m(x)}{K_m(x)})^{\nu/m}\alpha^\nu$$

holds, where C_ν is a positive constant satisfying

$$C_\nu = O(\frac{1}{\nu}).$$

By the construction

$$S_{m,\nu} := \rho \frac{(dz_1 \wedge \cdots \wedge dz_n)^{\otimes \nu}}{c_\nu} - u$$

is a holomorphic ν-ple canonical form on X and

$$S_{m,\nu}(x) = \frac{(dz_1 \wedge \cdots \wedge dz_n)^{\otimes \nu}}{c_\nu}(x)$$

holds. We note that

$$\lim_{\nu \to \infty} (\frac{(dz_1 \wedge \cdots \wedge dz_n)^{\otimes \nu}}{c_\nu})^{\frac{1}{\nu}}(O) = d\mu_{\Delta^n}(O)$$

holds. Hence if

$$\frac{d\mu(x)}{\sqrt[m]{K_m(x)}}\alpha < 1$$

holds, by the above estimate and Lemma 7 we see that

$$d\mu(x) \leq \limsup_{m \to \infty} K_m(x)^{\frac{1}{m}}$$

holds.

Let U be the subset of $X - V$ defined by

$$U := \{x \in X - V \mid (\limsup_{m\to\infty} K_m^{\frac{1}{m}})(x) \ = d\mu(x)\}.$$

Then by the proof of Lemma 7, we see that U is nonempty. We note that since $\sqrt{-1}\partial\bar{\partial}\log(\limsup_{m\to\infty} K_m^{\frac{1}{m}})$ is a closed positive current, $\limsup_{m\to\infty} K_m^{\frac{1}{m}}$ is upersemicontinuous on X. Then by Lemma 1 we see that U is closed. On the other hand by the above consideration we see that U is open. Hence we conclude that U is equal to $X - V$. This implies that the equality

$$\limsup_{m\to\infty} K_m^{\frac{1}{m}} = d\mu$$

holds on $X - V$. Since the curvature of both $\limsup_{m\to\infty} K_m^{\frac{1}{m}}$ and $d\mu$ express the first Chern class of X. We see that

$$\limsup_{m\to\infty} K_m^{\frac{1}{m}} = d\mu$$

holds on X.

4. Invariance of Plurigenera

In this section we shall prove Theorem 3.

4.1. Analytic Zariski decomposition. The notion of analytic Zariski decompositon (AZD) was introduced by the author to study big line bundles on an projective variety. It is known such a decomposition exists for any big line bundles on projective manifolds ([25]). For the first we shall recall the definition of singular hermitian metrics.

Definition 1. *Let L be a holomorphic line bundle over a complex manifold M. h is said to be a singular hermitian metric on L, if there exists a C^∞-hermitian metric h_0 on L and a locally L^1-function φ such that*

$$h = e^{-\varphi}h_0$$

holds.

For a singular hermitian line bundle (L, h) as above , we define the curvature $\operatorname{curv} h$ of (L, h) by

$$\operatorname{curv} h = \operatorname{curv} h_0 + \sqrt{-1}\partial\bar{\partial}\varphi.$$

We define the sheaf $\mathcal{L}^2(\mathcal{O}_X(L), h)$ by

$$\mathcal{L}^2(\mathcal{O}_X(L), h)(U) = \{\sigma \in \Gamma(U, \mathcal{O}_X(L)) \mid h(\sigma, \sigma) \in L^1_{loc}(U)\},$$

where U runs open subset of M. We call $\mathcal{L}^2(\mathcal{O}_X(L), h)$ the sheaf of germs of L^2 holomorphic sections of (L, h). By a theorem of Nadel [19], we see that $\mathcal{L}^2(\mathcal{O}_X(L), h)$

is coherent if the curv h is bounded from below by a minus of some Kähler form locally.

We shall define an analogy of Zariski decompositions for line bundles over a compact complex manifold.

Definition 2. *Let (L,h) be a singular hermitian lins bunndle over a compact complex manfold M. We call (L,h) an analytic Zariski decomposition, if it satisfies the following conditions.*

(1) *curv h is a closed positive $(1,1)$-current on M,*
(2) *the natural morphism $H^0(X,\mathcal{O}_X(mL)) \to H^0(X,\mathcal{L}^2(\mathcal{O}_X(mL),h^m))$ is an isomorphim for every positive integer m.*

Theorem 5. *Let X be a smooth projective variety of general type and let $d\mu$ be the Yau pseudovolume form on X. Then $(K_X, d\mu^{-1})$ is an analytic Zariski decomoposition of K_X.*

Proof. By Theorem 2, curv $d\mu^{-1}$ is a closed positive current. By Lemma 3 (and its proof), we have the isomorphism:

$$H^0(X,\mathcal{L}^2(\mathcal{O}_X(mK_X),d\mu^{-m})) \simeq H^0(X,\mathcal{O}_X(mK_X))$$

for every positive m. Q.E.D.

4.2. Growth estimate for the sections.

Definition 3. *Let L be a holomorphic line bundle over a compact complex n-manifold M. Let $gw(L)$ be the number defined by*

$$gw(L) = \limsup_{m\to\infty} m^{-n} \dim H^0(M,\mathcal{O}_M(mL)).$$

We call $gw(L)$ the growth of L.

Let X be a smooth projective variety of general type and let $d\mu$ be the Yau pseudovolume form on X. Let Θ_μ denote the curvature of $d\mu^{-1}$. Then we have the Lebesgue decomposition:

$$\Theta_\mu = \Theta_{\mu,abc} + \Theta_{\mu,sing},$$

where $\Theta_{\mu,abc}$ denotes the absolutely continuous part of Θ_μ and $\Theta_{\mu,sing}$ denotes the singular part of Θ_μ. We shall study the meaning of this decomposition.

Theorem 6.

$$\int_X (\Theta_{\mu,abc})^n = (2\pi)^n n! gw(K_X)$$

holds.

Proof. Let $\pi_m : X_m \longrightarrow X$ be a resolution of the base locus of the complete linear system $|mK_X|$. Let

$$|\pi_m^* mK_X| = |P_m| + F_m$$

be the decomposition of $|\pi_m^* mK_X|$ into the free part and the fixed part. We shall take $\{\pi_{m!}\}$ such that $\pi_{(m+1)!}$ factors throgh $\pi_{m!}$. We shall write $\pi_{(m+1)!}$ as

$$\pi_{(m+1)!} = \xi_{m+1} \circ \pi_{m!}$$

Then it is clear that

$$P_{(m+1)!} - \xi_{m+1}^*(m+1)P_{m!}$$

is effective. Since $P_{(m+1)!}$ and $P_{m!}$ are nef, we see that

$$P_{(m+1)!}^n - (\xi_{m+1}^*(m+1)P_{m!})^n \geq 0$$

holds for every positive integer m. This implies that

$$\lim_{m\to\infty} \frac{P_{m!}^n}{(m!)^n}$$

exists. It is clear that

$$gw(K_X) \geq \frac{1}{(2\pi)^n n!} \lim_{m\to\infty} \frac{P_{m!}^n}{(m!)^n}$$

holds. But by the work of Fujita ([4, p.1,Theorem]) we see that the equality

$$gw(K_X) = \frac{1}{(2\pi)^n n!} \lim_{m\to\infty} \frac{P_{m!}^n}{(m!)^n}$$

holds. On the other hand by the definition of ω_m , we have that

$$\int_X \omega_{m,abc}^n = m^{-n} P_m^n$$

holds. Since

$$\Theta_\mu = \lim_{m\to\infty} \omega_m$$

holds, we have that

$$\frac{1}{(2\pi)^n n!} \int_X (\Theta_{\mu,abc})^n \leq gw(K_X)$$

holds. On the other hand, the Lelong number $n(\Theta_\mu), n(\omega_m)$ of Θ_μ, ω_m respectively satisfy the inequality

$$n(\Theta_\mu, x) \leq n(\omega_m, x) \tag{2}$$

for every $x \in X$ (see the proof of Lemma 3). And we note that the de Rham cohomology class of Θ_μ and ω_m are $2\pi c_1(K_X)$. Let B_m denote the base locus of

mK_X. By the construction the support of $n(\Theta_\mu, x), n(\omega_m, x)$ are contained in B_m. We shall compute the integral

$$I_m = \int_{X-B} (\Theta_\mu^n - \omega_m^n).$$

Then we have

$$I_m = \int_{X-B} (\sqrt{-1}\partial\bar{\partial} \log \frac{d\mu}{K_m^{\frac{1}{m}}})(\Theta_\mu^{n-1} + \Theta_\mu^{n-2}\omega_m + \cdots + \omega_m^{n-1})$$

holds. We note that

$$\Theta_\mu^{n-1} + \Theta_\mu^{n-2}\omega_m + \cdots + \omega_m^{n-1}$$

is an absolutely continuous positive form on $X - B_m$. Then by Stokes theorem and (2), we conclued that I_m is nonnegative. Hence we have

$$\frac{1}{(2\pi)^n n!} \int_X \Theta_{\mu,abc}^n \geq g w_(K_X)$$

holds. This completes the proof of Theorem 6. Q.E.D.

Remark 3. *The other proof will be given (implicitly), in the proof of Theorem 7 below (see also the proof of Lemma 11).*

4.3. Analytic Zariski Decomposition of the family. Let $\pi : X \longrightarrow S$ be a smooth projective family of general type variety. To prove Theorem 3, we may assume that S is the unit open disk Δ in C. For every $t \in S$, we denote the fibre $\pi^{-1}(t)$ by X_t. Let $d\mu_t$ denote the Yau pseudovolume form on X_t and let Θ_t denote the curvature of $(K_{X_t}, d\mu_t^{-1})$.

Lemma 8. *There exists a proper subvariety V which does not contain any fibre of π such that $d\mu_t$ is a continuous relative pseudovolume form on $X - V$.*

Proof. Since $K_{X/S}$ is relativiely big by Kodaira's lemma [14, Appendix] , we have a finite number of global sections $\sigma_0, \ldots \sigma_N$ of $R^0\pi_*\mathcal{O}_X(mK_{X/S})$ which generates an ample subbundle of mK_{X_t} for every $t \in \Delta$. Then we can apply the same proof as that of Lemma 1. This proves the lowersemicontinuity of $d\mu_t$.

The uppersemicontinuity of $d\mu_t$ follows from the projectivity of π. Since π is projective, we may assume that the family X is embedded in a projective space P^N by an very ample subsehaf of the relative m-ple canonical bundle for some large m as above. Let $f_0 : \Delta^n - \cdots \to X_0$ be a meromorphic mapping which is holomorphic at O and nondegenerate at O. We want to construct a meromorphic extention $f : \Delta^n \times \Delta(\rho) - \cdots \to X$ such that $f(z,0) = f_0(z)(z \in \Delta^n)$ and $f(z,t) \in X_t(z \in \Delta^n)$, where ρ is a sufficiently small positive number and $\Delta(\varepsilon)$ is the open polydisk of radius ρ. We claim that such an extension exists if we replace $f(z)$ by $f((1-\varepsilon)z)$ for every positive number $\varepsilon < 1$. Let $F_1, \ldots, F_r \in \mathcal{O}_\Delta(\Delta)[X_0, \ldots, X_N]$ be a homogeneous polynomials with coefficients in $\mathcal{O}_\Delta(\Delta)$ such that for every $t \in \Delta$ $F_{i,t} := F_i(t)(1 \leq i \leq r)$ is a

set of defining equations of X_t in P^N, where we have considered F_i as a holomorphic mapping from Δ to $C[X_0, \dots, X_N]$. We set $F := (F_1, \dots, F_r) : C^{N+1} \times \Delta \longrightarrow C^r$. To construct f is equivalent to find set of holomorphic functions $g(z,t) = (g_0, \dots, g_N) : \Delta^n \times \Delta(\varepsilon) \longrightarrow C^N$ such that the equation

$$(*) \qquad F(g(z,t),t) = 0$$

holds with the initial condition

$$g(z,0) = g_0$$

where g_0 is a lifting of f_0 to a holomorphic mapping to C^{N+1}. We shall expand the solution $g(z,t)$ as

$$g(z,t) = \sum_{\nu=0}^{\infty} g_{|\nu}(z) t^{\nu}.$$

Then $g_{|1}(z)$ satisfies the equation:

$$\frac{\partial(F_1, \dots, F_r)}{\partial(X_0, \dots, X_N)}(g_0)^t g_{|1}(z) + \frac{\partial F}{\partial t}(g_0) = 0.$$

This equation for $g_{|1}(z)$ has a holomorphic solution on Δ^n since the polar set of g_0 is of codimension ≥ 2 on Δ^n and the equation $(*)$ is equivalent to a equation with $r = 1$ by a generic projection to P^{n+1}. The equation for $g_{|\nu}$ is of the form:

$$\frac{\partial(F_1, \dots, F_r)}{\partial(X_0, \dots, X_N)}(g_0)^t g_{|\nu} + \text{terms with holomorphic coefficients in } g_{|i} (0 \leq i \leq \nu - 1) = 0.$$

Inductively we can determine holomorphic functions $g_{|\nu}(\nu \geq 1)$. To assure the convergence of the formal solution, we need to construct $g_{|\nu}(\nu \geq 1)$ with some estimate. Let ε be a positive number less than 1. By using the standard argument (see for example [17]), if we replace $f(z)$ by $f((1-\varepsilon)z)$ we can construct a solution $g_{|\nu}$ with the estimate

$$\mid g_{|\nu}(z) \mid \leq C \cdot M^{\nu} \quad (z \in \Delta^n)$$

for some positive constants C and M independent of ν. This assures the convergence of the solution $g(z,t)$ for $\mid t \mid < 1/M$.

The uppersemicontinuity of $d\mu_t$ follows immediately from the existence of this extention. Q.E.D.

Lemma 9. *$\int_{X_t} \Theta^n_{t,abc}$ is a constant function on $t \in S$.*

We set

$$I(t) := \int_{X_t} \Theta^n_{t,abc}.$$

By Theorem 6, we have that

$$I(t) = gw(K_{X_t})$$

holds. By the well known uppersemicontinuity theorem of the cohomology (cf. [16, p.351, Theorem 7.8]), we have that $I(t)$ is constant outside of a countably many points on S and strictly larger on these points.

On the other hand by Lebesgue-Fatous' lemma , for every convergent sequence $\{t_j\}$ in S, we have that

$$I(\lim_{\mathrm{J}\to\infty} t_j) \leq \liminf_{\mathrm{J}\to\infty} I(t_j)$$

holds. This implies that $I(t)$ is constant on S. Q.E.D.

4.4. Comparison with the global growth. Let $\pi : X \longrightarrow S$ be a smooth projective family of varieties of general type over the unit open polydisc in C. To prove Theorem 3 we may assume that this family is embedded into an irrational pencil

$$\varphi : Y \longrightarrow C$$

such that

(1) Y is smooth,
(2) Y is of general type,
(3) the ristriction morphism:

$$H^0(Y, \mathcal{O}_Y(mK_Y)) \to R_0\varpi_*\mathcal{O}_Y(mK_Y)_t$$

is surjective for every $t \in C$.

We may assume the third condition by the semipositivity of the direct image of the relative pluricanonical sheaf ([11, Thorem 1][5]). Let $d\mu$ be the ristriction of the Yau volume form on Y to X and let Θ be the curvature of $(K_X, d\mu^{-1})$. By Theorem 2, we see that Θ is a closed positive current on X. By the construction we see that the restriction $\Theta_{|t} : \Theta \mid_{X_t}$ is well defined for every $t \in S$.

Lemma 10. *For every t, we have the equality*

$$I(t) = \int_X \Theta^n_{t,abc} = \int_{X_t} (\Theta_{|t})^n_{abc}$$

Proof By Lemma 6, $I(t)$ is a constant function on S. On the other hand outside a countably many points on S, the equality

$$I(t) = \int_{X_t} (\Theta_{|t})^n_{abc}$$

holds by the construction of $\Theta_{|t}$ and the upperseimicontinuity theorem. Then by the trivial inequality,

$$I(t) \geq \int_{X_t} (\Theta_{|t})^n_{abc}$$

we completes the proof of Lamma 7. Q.E.D.

We set

$$\mathcal{F}_{|t}(m) := \mathcal{L}^2(mK_{X_t}, d\mu^{-m}_{|X_t}).$$

Lemma 11.

$$\mathcal{F}_{|t}(m) = \mathcal{L}^2(X_t, d\mu_t^{-m})$$

holds.

Proof. To prove this result we shall use the construction of analytic Zariski decompositions via parabolic Monge-Ampère equation in Part 2. We note that we are considering the case of canonical bundles. Then by the construction there exists a Kähler-Einstein form ω_1,ω_2 on a nonempty Zariski open subset U of X_t such that

$$\mathcal{L}^2(X_t, d\mu_t^{-m}) = \mathcal{L}^2(X_t, (\omega_1^n)^{-m})$$

and

$$\mathcal{F}_{|t}(m) = \mathcal{L}^2(X_t, (\omega_1^n)^{-m})$$

holds for every $m \geq 1$. Since ω_1, ω_2 are Kähler-Einstein on U by the construction of these currents (see Lemma 34,35 in Part 2) we have the pointwise inequality,

$$\omega_1^n \geq \omega_2^n$$

holds on U. On the other hand by the proof of Theorem 6, we see that

$$\int_U \omega_1^n = \int_U \Theta^n_{t,abc}$$

and

$$\int_U \omega_2^n = \int_U \Theta^n_{|t,abc}$$

hold. Hence by Lemma 10, we have the equality,

$$\int_U \omega_1^n = \int_U \omega_2^n$$

holds. Combining the pointwise inequality between ω_1^n and ω_2^n, we have that

$$\omega_1 = \omega_2$$

holds. Since $(K_X, (\omega_i^n)^{-1})(i = 1, 2)$ are AZD, we complete the proof of Lemma 11. Q.E.D.

Remark 4. *Lemma 11 is the only part which requires the use of singular Kähler-Einstein metrics. If one succeed in avoiding this very analytic tool, the proof of Theorem 3 will become much easier.*

4.5. Use of L^2-extention theorem.

Lemma 12. *The restriction morphism*

$$H^0(X, \mathcal{O}_X(K_X \otimes \mathcal{L}^2(\mathcal{O}_X(mK_X), d\mu^{-m})) \to H^0(X_t, \mathcal{O}_X(K_{X_t} \otimes \mathcal{F}_{|t}(m)))$$

is surjective for every $t \in S$.

Proof. This directly follows from the L^2-extention theorem of [21]. By Kodaira's lemma, we see that there exists a effective Q-divisor on X, such that $K_X - E$ is relatively ample on X. Then there exists an integer r such that $r(K_X - E)$ is a very ample Cartier divisor on X. Let H be a smooth member of $\mid r(K_X - E) \mid$. Then $X - H - \mathrm{Supp}\, E$ is Stein and moreover K_X is trivial on $X - H - \mathrm{Supp}\, E$. Now we can apply the extension theorem of [21] and completes the proof of Lemma 12. Q.E.D.

Now we shall completes the proof of Theorem 3. By Theorem 5 and Lemma 11, we see that

$$H^0(X_t, \mathcal{O}_X(K_{X_t} \otimes \mathcal{F}_{|t}(m)) \simeq H^0(X_t, \mathcal{O}_{X_t}((m+1)K_{X_t}))$$

holds. By Lemma 12, we completes the proof of Theorem 3.

Part II. Analytic construction of AZD

This part is more or less independent of the last part. We shall prove the following theorem. More general results are in [25].

Theorem 7. *Let* X *be a smooth projective manifold of general type. Then there exists a nonempty Zariski open subset* U *of* X *and a closed positive* $(1,1)$*-current* ω_E *such that*

(1) ω_E *is a Kähler-Einstein form on* U *with constant negative Ricci curvature,*
(2) $(K_X, (\omega_E^n)^{-1})$ *is an analytic Zariski decomposition.*

5. Deformation of Kähler form I

In this section we shall consider Hamilton's equation on a smooth projective variety of general type and determine the maximal existence time for the smooth solution. This section is more or less independent from the other sections. Hence a reader who is familiar with Monge-Ampère equation may skip this section. But the basic method of the estimate of parabolic Monge-Ampère equations is introduced in this section.

5.1. Hamilton's equation. Let X be a smooth projective n-fold of general type. We consider the initial value problem:

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}_\omega - \omega \ \text{ on } X \times [0,T) \tag{3}$$

$$\omega = \omega_0 \ \text{ on } X \times \{0\}, \tag{4}$$

where
ω_0 : a C^∞-Kähler form on X,
T : the maximal existence time for C^∞-solution.

Since

$$\frac{\partial}{\partial t}(d\omega) = -d\omega \ \text{ on } X \times [0,T)$$
$$d\omega_0 = 0 \ \text{ on } X \times \{0\},$$

we have that $d\omega = 0$ on $X \times [0,T)$, i.e., the equation preserves the Kähler condition.

5.2. Reduction to the parabolic Monge-Ampère equation. Let ω denote the de Rham cohomology class of ω in $H^2_{DR}(X,R)$. Since $-(2\pi)^{-1}\mathrm{Ric}_\omega$ is a first Chern form of K_X, we have

$$[\omega] = (1-\exp(-t))2\pi c_1(L) + \exp(-t)[\omega_0]. \tag{5}$$

Let Ω be a C^∞-volume form on X and let

$$\omega_\infty = curv\, \Omega.$$

We set

$$\omega_t = (1-\exp(-t))\omega_\infty + \exp(-t)\omega_0. \tag{6}$$

Since $[\omega] = [\omega_t]$ on $X \times \{t\}$ for every $t \in [0,T)$, there exists a C^∞-function u on $X \times [0,T)$ such that

$$\omega = \omega_t + \sqrt{-1}\partial\bar\partial u. \tag{7}$$

By (3), we have

$$\frac{\partial}{\partial t}(\omega_t + \sqrt{-1}\partial\bar\partial u) = \sqrt{-1}\partial\bar\partial \log(\omega_t + \sqrt{-1}\partial\bar\partial u)^n - (\omega_t + \sqrt{-1}\partial\bar\partial u)$$

Hence

$$\exp(-t)(\omega_\infty - \omega_0) + \sqrt{-1}\partial\bar\partial(\frac{\partial u}{\partial t})$$
$$= \sqrt{-1}\partial\bar\partial \log(\omega_t + \sqrt{-1}\partial\bar\partial u)^n - \sqrt{-1}\partial\bar\partial \log\, \Omega + \exp(-t)(\omega_\infty - \omega_0).$$

Then (3) is equivalent to the initial value problem:

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar\partial u)^n}{\Omega} - u \ \text{ on } X \times [0,T), \tag{8}$$

where

$$u = 0 \quad \text{on } X \times \{0\}.$$

Let

$$A(X) = \{[\eta] \mid \eta : \text{Kähler form on } X\} \subset H^2_{DR}(X, R)$$

be the Kähler cone of X. Since $[\omega]$ moves on the segument connecting $[\omega_0]$ and $[\omega_\infty] = 2\pi c_1(K_X)$, we cannot expect T to be ∞, unless $2\pi c_1(K_X)$ is on the closure of $A(X)$ in $H^2_{DR}(X, R)$. We shall determine T. It is standard to see that $T > 0$ ([8]).

Theorem 8. *If $\omega_0 - \omega_\infty$ is a Kähler form, then T is equal to*

$$T_0 = \sup\{t > 0 \mid [\omega_t] \in A(X)\}.$$

The proof of Theorem 8 is almost parallel to that of [26, p.126, Theorem 3].

5.3. C^0-estimate.

Lemma 13. *If $\omega_0 - \omega_\infty$ is a Kähler form, then there exists a constant C_0 such that*

$$\frac{\partial u}{\partial t} \leq C_0 \exp(-t).$$

Proof.

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \Delta_\omega \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - \exp(-t) tr_\omega(\omega_0 - \omega_\infty)$$

holds by defferentiating (5) by t. By the maximum principle, we have

$$\frac{\partial u}{\partial t} \leq (\max \log \frac{\omega_0^n}{\Omega}) \exp(-t).$$

Q.E.D.

To estimate u from below, we modify (10) as

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial} u)^n}{\omega_t^n} + f_t - u \quad \text{on } X \times [0, T_1), \tag{9}$$

where

$$u = 0 \quad \text{on } X \times \{0\},$$

where

$$f_t = \log \frac{\omega_t^n}{\Omega} \tag{10}$$

and

$$T_1 = \min\{\sup\{t > 0 \mid \omega_t > 0\}, T\}. \tag{11}$$

If $t \in [0, T_1)$, we have

$$\log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega_t^n} = \int_0^1 \frac{d}{ds} \log \frac{(\omega_t + \sqrt{-1}s\partial\bar{\partial}u)^n}{\omega_t^n} ds = \int_0^1 \Delta_s u ds,$$

where Δ_s is the Laplacian with respect to the Kähler form $\omega_t + \sqrt{-1}s\partial\bar{\partial}u$. Then by the minimum principle, (11) and Lemma 13, we have

Lemma 14.

$$u \geq -C_0 \exp(-t) + \min_X f_t \ \ \textit{on } X \times \{t\}, t \in [0, T_1).$$

We note that this estimate is depending on t and C_0 is independent of the choice of Ω.

5.4. C^2-estimate. For the next we shall obtain a C^2-estimate of u.

Lemma 15. *(*[29, p.351, (2.22)]*) Let M be a compact Kähler manifold and let ω, $\tilde{\omega}$ be Kähler forms on M. Assume that there exists a C^∞-function φ such that*

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi.$$

We set

$$f = \log \frac{\tilde{\omega}^n}{\omega^n},$$

Then for every positive constant such that

$$C + \inf_{i \neq j} R_{i\bar{i}j\bar{j}} > 1,$$

$$\exp(C\varphi)\tilde{\Delta}(\exp(-C\varphi)(n + \Delta\varphi)) \geq$$

$$(\Delta f - n^2 \inf_{i \neq j} R_{i\bar{i}j\bar{j}}) - Cn(n + \Delta\varphi)$$

$$+(C + \inf_{i \neq j} R_{i\bar{i}j\bar{j}})(n + \Delta\varphi)^{\frac{n}{n-1}} \exp(-\frac{f}{n-1})$$

holds, where
$R_{i\bar{i}j\bar{j}}$: the bisectional curvature of ω,
Δ : the Laplacian with respect to ω.

Applying this lemma to ω_t and $\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}u$, we have:

Lemma 16. *For every $C > 0$ depending only on $t \in [0, T_1)$ such that*

$$C + \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(t) > 1 \quad \text{on } X \times \{0\},$$

$$\exp(Cu)(\Delta_\omega - \frac{\partial}{\partial t})(\exp(-Cu) tr_{\omega_t}\omega) \geq$$

$$-(\Delta_t \log \frac{\omega_t^n}{\Omega} + n^2 \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(t) + n)$$

$$-C(n - \frac{1}{C} - \frac{\partial u}{\partial t}) tr_{\omega_t}\omega - \exp(-t) tr_{\omega_t}((\omega_0 - \omega_\infty) \cdot \sqrt{-1}\partial\bar{\partial}u)$$

$$+(C + \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(t)) \exp(\frac{1}{n-1}(-\frac{\partial u}{\partial t} - u + \log \frac{\omega_t^n}{\Omega}))(tr_{\omega_t}\omega)^{\frac{n}{n-1}}$$

holds, on $X \times \{t\}$) ($t \in [0, T_1)$), where
Δ_t : Laplacian with respect to ω_t,
$R_{i\bar{i}j\bar{j}}(t)$: the bisectional curvature of ω_t
and $tr_{\omega_t}((\omega_0 - \omega_\infty) \cdot \sqrt{-1}\partial\bar{\partial}u)$ denotes the trace (with respect to ω_t) of the product of the endomorphisms $A, B \in End(TX)$ defined by

$$\begin{aligned} \omega_t(A(Z_1) \wedge \bar{Z}_2) &= (\omega_0 - \omega_\infty)(Z_1 \wedge \bar{Z}_2) \\ \omega_t(B(Z_1) \wedge \bar{Z}_2) &= (\sqrt{-1}\partial\bar{\partial}u)(Z_1 \wedge \bar{Z}_2), \end{aligned}$$

where the pair (Z_1, Z_2) runs in $TX \times_X TX$.

Proof. Let

$$f = \log \frac{\omega^n}{\omega_t^n} = \frac{\partial u}{\partial t} + u - \log \frac{\omega_t^n}{\Omega}.$$

Then by Lemma 15, we have

$$\exp(Cu)\Delta_\omega(\exp(-Cu) tr_{\omega_t}\omega)$$

$$\geq (\Delta_t f - n^2 \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(t)) - Cn(n + \Delta_t u)$$

$$+(C + \inf_{i \neq j} R_{i\bar{i}j\bar{j}})(tr_{\omega_t}\omega)^{\frac{n}{n-1}} \exp(-\frac{f}{n-1}).$$

Since

$$\Delta_t f = \Delta_t(\frac{\partial u}{\partial t} + u - \log \frac{\omega_t^n}{\Omega})$$

$$= \Delta_t \frac{\partial u}{\partial t} + tr_{\omega_t}\omega - n - \Delta_t \log \frac{\omega_t^n}{\Omega}$$

and

$$\exp(Cu)\frac{\partial}{\partial t}(\exp(-Cu) tr_{\omega_t}\omega)$$

$$= -C\frac{\partial u}{\partial t} tr_{\omega_t}\omega + tr_{\omega_t}\frac{\partial \omega}{\partial t} - tr_{\omega_t}\frac{\partial \omega_t}{\partial t} \cdot \omega$$

$$= -C\frac{\partial u}{\partial t}tr_{\omega_t}\omega + \Delta_t\frac{\partial u}{\partial t} - \exp(-t)tr_{\omega_t}(\omega_0 - \omega_\infty) + \exp(-t)tr_{\omega_t}(\omega_0 - \omega_\infty)\cdot\omega,$$

we obtain the lemma. Q.E.D.

Let ε be an arbitrary small positive number. We set

$$T_1(\varepsilon) = \min\{\sup\{t > 0 \mid \omega_t > 0\} - \varepsilon, T\}$$

and let C be a positive number such that

$$C + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}(t) > 1$$

for all $t \in [0, T_1(\varepsilon)]$. Then since the function $x\exp(-x)$ is bounded on $[0,\infty)$, by the maximum principle and Lemma 16, we have that if $\exp(-Cu)tr_{\omega_t}\omega$ take its maximum at $(x_0, t_0) \in X \times [0, T_0(\varepsilon)]$, we have

$$tr_{\omega_t}\omega(x_0, t_0) < C_\varepsilon$$

for some $C_\varepsilon > 0$ depending only on ε. Then by the C^0-estimate of u , Lemma 13 and Lemma 14, by the maximum principle for parbolic equations we have that there exists a positive constant $C'_{1,\varepsilon}$ such that

$$tr_{\omega_t}\omega < C'_{1,\varepsilon}.$$

Hence we obtain:

Lemma 17. *There exists a positive constant $C_{1,\varepsilon}$ depending only on $T_1(\varepsilon)$ such that*

$$\| u \|_{C^2(X)} \leq C_{2,\varepsilon}$$

for every $t \in [0, T_1(\varepsilon))$, where $\| \ \|_{C^r(X)}$ is the C^2-norm with repsect to ω_0.

Now by [23], for every $r \geq 2$ there exists a positive constant $C_{r,\varepsilon}$ depending only on $T_1(\varepsilon)$ such that

$$\| u \|_{C^r(X)} \leq C_{r,\varepsilon}.$$

Letting ε tend to 0, we have that

$$T \geq T_1$$

holds. Since $[\omega_{T_0}]$ is on the closure of the Kähler cone $A(X)$, by changing Ω properly, we can make $T_0 - T_1 > 0$ arbitarary small. Hence we conclude that $T = T_0$. This completes the proof of Theorem 9. Q.E.D.

6. Deformation of Kähler form II

In this section we shall construct a Kähler form on a Zariski open subset of X by using a initial value problem similar to (3) in the last section. In this section we use the same notation as in the last section.

6.1. The current ω_E. To state our theorem we need the following definitions.

Definition 4. *Let D be a R-Cartier divisor on a projective variety Y. Then the stable base locus of D is defined by*

$$SBs(D) = \cap_{\nu>0} SuppBs \mid [\nu D] \mid .$$

Definition 5. *Let D be a Cartier divisor on a projective variety Y and let $\Phi_{|D|}; Y - \cdots \to P^{N(\nu)}$ be the rational map associated with $\mid \nu D \mid$. Let $\mu_\nu : Y_\nu \longrightarrow Y$ be a resolution of the base locus of $\mid \nu D \mid$ and let $\tilde{\Phi}_{|\nu D|}; \tilde{X} \longrightarrow P^{N(\nu)}$ be the associated morphism. We set*

$$E(\nu D) = \overline{\mu_\nu(\tilde{E}(\nu D) \cap (Y - Supp\ Bs \mid \nu D \mid))} \ \textit{(Zariski closure)}$$

and call it the exceptional locus of $\mid \nu D \mid$. It is easy to see that $E(\nu D)$ is independent of the choice of the resolution of the base locus μ_ν. We set

$$SE(D) = \cap_{\nu>0} E(\nu D)$$

and call it the stable exceptional locus of D.

We set

$$S = \mathrm{SBs}(L) \cup SE(L).$$

The main result in this section is the following theorem.

Theorem 9. *There exists a d-closed positive $(1,1)$-current ω_E on X such that*

(1) *ω_E is smooth on a nonempty Zariski open subset U of X,*
(2) *$\omega_E = -Ric_{\omega_E}$ holds on U.*
(3) *$[\omega_E] = 2\pi c_1(L)$.*

6.2. Kodaira's Lemma. Let ν be a sufficiently large positive integer such that

(1) $\mid \nu L \mid$ gives a birational rational map from X into a projective space.
(2) Supp Bs$\mid \nu L \mid = \mathrm{SBs}(L)$.

Let $f_\nu : X_\nu \to X$ be a resolution of the base locus of $\mid \nu L \mid$ and let

$$F_\nu = \sum_i b_i^\nu F_i^\nu$$

be the fixed part of $\mid f_\nu^*(\nu L) \mid$. We take f_ν so that F^ν is a divisor with normal crosssings. We set

$$\tilde{b}_i^\nu = b_i^\nu / \nu.$$

Let σ_i^ν be a global holomorphic section of $\mathcal{O}_{X_\nu}(F_i^\nu)$ with divisor F_i^ν. Then there exist hermitian metrics $\| \ \|$ on $\mathcal{O}_{X_\nu}(F_i^\nu)$'s such that

$$\omega_\infty^\nu = f_\nu^* \omega_\infty + \sum \sqrt{-1} \tilde{b}_i^\nu \partial\bar{\partial} \log \| \sigma_i^\nu \|^2$$

is positive on $f_\nu^{-1}(X - S)$, if ν is sufficiently large. We may assume

$$\log \| \sigma_i^\nu \| \leq 0$$

holds for every i. We set

By Kodaira's lemma ([14, Appendix] we have the following lemma

Lemma 18. *There exists an effective Q-divisor*

$$R_\nu = \sum r_j^\nu R_j^\nu$$

on X_ν such that

$$f_\nu^*(K_X) - \sum \tilde{b}_i^\nu F_i^\nu - R^\nu$$

is an ample Q-divisor on X_ν.

We note that εR_ν has the same property as R_ν for $\varepsilon \in [0, 1]$. Let τ_j^ν be a global section of $\mathcal{O}_{X_\nu}(R_j^\nu)$ with divisor R_j^ν. Then there exists hermitian metrics $\| \ \|$ on $\mathcal{O}_{X_\nu}(R_j^\nu)$ such that

$$\omega_\infty^\nu + \sum_j \sqrt{-1} r_j^\nu \partial\bar{\partial} \log \| \tau_j^\nu \|^2$$

is a smooth Kähler form on X_ν and $\| \tau_j^\nu \| \leq 1$ holds on X_ν for all j. We set

$$\delta_\nu = \sum_j \sqrt{-1} r_j^\nu \log \| \tau_j^\nu \|^2 \ .$$

Then for every $\varepsilon \in [0, 1]$,

$$\omega_\infty^\nu + \varepsilon \sqrt{-1} \partial\bar{\partial} \delta^\nu$$

is a smooth Kähler form on X_ν.

We set

$$\xi_\nu = \sum_i \tilde{b}_i^\nu \log \| \sigma_i^\nu \|^2 \ .$$

6.3. Construction of a suitable ample divisor. To construct Kähler-Einstein current on X, we use the Dirichlet problem for parabolic Monge-Ampère equation. Hence we shall construct a strongly pseudoconvex convex exhaustion of a Zariski open subset of X_ν with certain properties. We fix sufficiently large ν hereafter. Let

$$\Phi : X_\nu \longrightarrow P^N$$

be a embedding of X_ν into a projective space. Let

$$\pi_\alpha; X_\nu \longrightarrow P^n (\alpha = 1, \dots m)$$

be generic projections and we set

$$W_\alpha : \text{the ramification divisor of } \pi_\alpha, H_\alpha := \pi_\alpha^*(\{z_0 = 0\}),$$

where $[z_0 : \dots : z_n]$ be the homogeneous coordinate of P^n. For simplicity we shall denote the support of a divisor by the same notation as the one, if without fear of confusion. If m is sufficiently large, we may assume the following conditions:

(1) $\cap_{\alpha=1}^m (W_\alpha + H_\alpha) = \phi$,
(2) $D := (F_\nu + \sum_{\alpha=1}^m (W_\alpha + H_\alpha))_{red}$ is an ample divisor with normal crossings.
(3) D contains $S \cup R_\nu$.
(4) $K_{X_\nu} + D$ is ample.

Then $X_\nu - D$ is strongly pseudoconvex and the following lemma is necessary for our purpose.

Lemma 19. *There exists a positive strongly plurisubharmonic exhaustion function φ of $X_\nu - D$ such that $\omega_\varphi = \sqrt{-1}\partial\bar{\partial}\varphi$ is a complete Kähler form on $X_\nu - D$.*

Proof. Let $D = \sum_k D_k$ be the irreducible decomposition of D and let λ_k be a global holomorphic section of $\mathcal{O}_{X_\nu}(D_k)$ with divisor D_k. Then there exist hermitian metrics $\| \ \|$'s on $\mathcal{O}_{X_\nu}(D_k)$'s such that

$$-\sum_k \sqrt{-1}\partial\bar{\partial} \log \| \lambda_k \|^2$$

is a smooth Kähler form on X_ν. We set for a positive number ι

$$\varphi = -\sum_k \log \| \lambda_k \| - \iota \log\log \frac{1}{\| \lambda_k \|}.$$

Then if we choose ι sufficiently small, then

$$\omega_\varphi = \sqrt{-1}\partial\bar{\partial}\varphi$$

is a complete Kähler form on $X_\nu - D$. Clearly by adding a sufficiently large positive number, we can make the exhuastion φ to be positive on $X_\nu - D$. Q.E.D.

Remark 5. *As one see in Lemma 24 below, ω_φ has a bounded Poincaré growth.*

6.4. The Dirichlet problem. We set for $c > 0$,

$$K_c = \{x \in X_\nu : \varphi(x) \leq c\}.$$

It is easy to see that we may assume that there exists a positive constant c_0 such that the boundary ∂K_c is smooth for every $c \geq c_0$. We fix such c and set

$$K := K_c$$

for simplicity.

In the estimate of u, we try to make the estimate independent of $c \geq c_0$ for the later use.

Since $X_\nu - D$ is canonically biholomorphic to a Zariski open subset of X, we may consider K as a compact subset of X. We consider the following Dirichlet problem for a parabolic Monge-Ampère equation.

$$\begin{cases} \frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u & \text{on } K \times [0,T) \\ u = (1 - e^{-t^4})\xi_\nu & \text{on } \partial K \times [0,T) \quad (10) \\ u = 0 & \text{on } K \times \{0\}, \end{cases}$$

where Ω is a smooth volume form on X,

$$\omega_t = (1 - e^{-t^4})curv\, h + e^{-t^4}\omega_0$$

and T is a maximal existence time for the smooth solution on $\bar{K}$(the closure in the usual topology). We shall assume that

$$\omega_0^n = \Omega$$

holds. It is easy to find such ω_0 and Ω by using the solution of Calabi's conjecture ([29]). We set

$$\omega_\infty = curv\, h.$$

By multiplying a common sufficiently large positive number to ω_0 and Ω, if neceessary, we may assume that

$$\omega_0 + \mathrm{Ric}\,\Omega = \omega_0 - \omega_\infty > 0$$

holds. Please do not confuse u with the one in the last section. We use the same notation for simplicity. For the first we shall show

Theorem 10. *T is infinite and*

$$u_\infty = \lim_{t \to \infty} u$$

exists in C^∞-topology on $\bar{K}$.

6.5. C^0-estimate. We note that by the above choice of ω_0,Ω and the Dirichlet condition, the Dirichlet problem (10) is compatible up to 3-rd order on the corner $\partial K \times \{0\}$.

Hence by the standard implicit function theorem, we see that T is positive. Suppose T is finite. Then if $\lim_{t\to T} u$ exist on K in C^∞-topology, then this is a contradiction. Because again by the implicit function theorem, we can continue the solution a little bit more. Hence to prove Theorem 10 it is sufficient to obtain an estimate of C^k-norm of u on K which is independent of t.

We begin with C^0-estimate.

Lemma 20. *There exists a constant C_0^+ such that*

$$\frac{\partial u}{\partial t} \leq C_0^+ e^{-t} \quad \text{on } K \times [0,T)$$

holds.

Proof We set

$$\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}u$$

and

$$\tilde{\Delta} = tr_\omega \sqrt{-1}\partial\bar{\partial}.$$

As in the proof of Lemma 3.1, we have

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \tilde{\Delta}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - 4t^3 e^{-t^4} tr_\omega(\omega_0 - \omega_\infty)$$

holds on $K \times [0,T)$. Since

$$u = (1 - e^{-t^4})\xi_\nu \quad \text{on } \partial K \times [0,T)$$

and $\omega_0 - \omega_\infty$ is a Kähler form on X, by maximal principle

$$\frac{\partial u}{\partial t} \leq C_0^+ e^{-t} \text{ on } K \times [0,T)$$

holds for

$$C_0^+ = \max\{\max_{x\in K} \log \frac{\omega_0^n}{\Omega}(x), \max_{x\in\partial K} \xi_\nu(x)\}.$$

Q.E.D.

We set

$$v = u - (1 - e^{-t^4})\xi_\nu,$$
$$\Omega_\nu = \exp(\xi_\nu)\Omega,$$

and

$$\hat{\omega}_t = \omega_t + (1 - e^{-t^4})\sqrt{-1}\partial\bar{\partial}\xi_\nu.$$

Then v satisfies the equation:

$$\begin{cases} \frac{\partial v}{\partial t} &= \log \frac{(\hat{\omega}_t+\sqrt{-1}\partial\bar{\partial}v)^n}{\Omega_\nu} - v \quad \text{on } K\times[0,T) \\ v &= 0 \qquad\qquad\qquad\qquad \text{on } \partial K\times[0,T) \quad (11) \\ v &= 0 \qquad\qquad\qquad\qquad \text{on } K\times\{0\} \end{cases}$$

We note that $\hat{\omega}_t$ is a Kähler form on $(X_\nu - D)\times[0,\infty]$ and $\hat{\omega}_\infty = \omega_\infty^\nu$. Then since

$$\log \frac{(\hat{\omega}_t+\sqrt{-1}\partial\bar{\partial}v)^n}{\hat{\omega}_t^n} = \int_{a=0}^1 \tilde{\Delta}_a v,$$

where $\tilde{\Delta}_a$ is the Laplacian with respect to the Kähler form

$$\hat{\omega}_t + a\sqrt{-1}\partial\bar{\partial}v,$$

by maximum principle and Lemma 19, we have that

$$v \geq \min\{\min_{x\in K} \log\frac{\hat{\omega}_t^n}{\Omega_\nu}(x) - C_0^+ e^{-t}, 0\} \text{ on } K\times[0,T)$$

holds. Hence we have

Lemma 21.

$$u \geq C_0^- + (1-e^{-t^4})\xi_\nu \ \ \textit{on } K\times[0,T),$$

where

$$C_0^- = \min\{\inf_{(x,t)\in K\times[0,\infty)} \log\frac{\hat{\omega}_t^n}{\Omega_\nu}(x,t), 0\} - C_0$$

We note that C_0^- may depend on K because $\log(\hat{\omega}^n/\Omega_\nu)$ may not be bounded from below on $X_\nu - D$. To obtain the C^0-estimate from below which is independent of K, we shall consider for $\varepsilon\in(0,1]$,

$$v_\varepsilon = u - (1-e^{-t^4})(\xi_\nu + \varepsilon\delta_\nu).$$

Then by the same argument, we have

Lemma 22. *Let $\varepsilon\in(0,1]$. Then there exists a constant $C_0^-(\varepsilon)$ which is independent of K such that*

$$u \geq C_0^-(\varepsilon) + (1-e^{-t^4})(\xi_\nu+\varepsilon\delta_\nu) \ \textit{on } K\times[0,T)$$

The reason why $C_0^-(\varepsilon)$ is independent of K is simply because

$$\hat{\omega}_\infty + \varepsilon\sqrt{-1}\partial\bar{\partial}\delta_\nu$$

on $X_\nu - D$ extends to a smooth Kähler form on X_ν and

$$\exp(\xi_\nu + \varepsilon\delta_\nu) f_\nu^* \Omega$$

is a smooth semipositive (n,n) form on X_ν.

6.6. C^1-estimate on ∂K. Hereafter we estimate derivatives of v basically by using the method in [2]. But our estimates is a little bit more complicated because we are working on a quasi-projective variety which cannot admits a global flat Kähler metric. We set

$$\psi_c = b(\varphi - c),$$

where b is a positive constant. We note that

$$\psi_c = 0 \ \text{ on } \partial K$$

and

$$\psi_c < 0 \quad \text{on } K.$$

Then since ω_φ is a complete Kähler form of Poincaré growth, if we take b sufficiently large

$$\log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\psi_c)^n}{\Omega_\nu} - \psi_c \geq 0 \text{ on } K \times [0,\infty)$$

holds. It is easy to see that we can take b independent of c and t. Then we have

$$\begin{cases} \frac{\partial(v-\psi_c)}{\partial t} & \geq \log \frac{(\hat{\omega}_t+\sqrt{-1}\partial\bar{\partial}v)^n}{(\hat{\omega}_t+\sqrt{-1}\partial\bar{\partial}\psi_c)^n} - (v-\psi_c) \quad \text{on } K \times [0,T) \\ v - \psi_c & = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial K \times [0,T) \\ v - \psi_c & = -\psi_c \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } K \times \{0\} \end{cases}$$

Since

$$\log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}v)^n}{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\psi_c)^n} = \int_0^1 \dot{\Delta}_a(v-\psi_c)da,$$

where $\dot{\Delta}_a$ is the Laplacian with respect to the Kähler form

$$\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\{(1-a)\phi_c + av\}$$

, by the maximum principle, we obtain

$$v \geq \psi_c \quad \text{on } K \times [0,T).$$

On the other hand, trivially

$$\hat{\Delta}_t v \geq -n \ \text{ on } K \times [0,T)$$

holds, where $\hat{\Delta}_t$ is the Laplacian with respect to the Kähler form $\hat{\omega}_t$. Let h be the C^∞-function on $\bar{K} \times [0,T)$ such that

$$\begin{cases} \hat{\Delta}_t h &= -n \quad \text{on } K \times [0,T) \\ h &= 0 \quad \text{on } \partial K \times [0,T). \end{cases}$$

Then by the maximum principle, we have

$$v \leq h \ \text{ on } K \times [0,T).$$

Hence we have

$$\psi_c \leq v \leq h \ \text{ on } K \times [0,T). \tag{14}$$

Now to fix C^k-norms on $X_\nu - D$, we shall construct a complete Kähler-Einstein form on $X_\nu - D$.

We quote the following theorem.

Theorem 11. *(*[12]*) Let $\bar{M}$ be a nonsingular projective manifold and let B be an effective divisor with only simple normal crossings. If $K_{\bar{M}} + B$ is ample, then there exists a unique (up to constant multiple) complete Kähler-Einstein form on $M = \bar{M} - B$ with negative Ricci curvaure.*

By the construction of D, D is a divisor with simple normal crossings and $K_{X_\nu} + D$ is ample. Hence by Theorem 11, there exists a complete Kähler-Einstein form ω_D on $X_\nu - D$ such that

$$\omega_D = -\mathrm{Ric}_{\omega_D}.$$

Then we have

$$\| dv \| \leq \max\{\| dh \|, \| d\psi_c \|\} \ \text{ on } \partial K \times [0,T),$$

where $\| \ \|$ is the pointwise norm with respect to ω_D. To make this estimate independent of K, we need to use special properties of ω_D.

Definition 6. *Let V be an open set in C^n. A holomorphic map from V into a complex manifold M of dimension n is called a* quasi-coordinate map *iff it is of maximal rank everywhere on V. $(V;$ Euclidean coordinate of $C^n)$ is called a* local quasi-coordinate *of M.*

Lemma 23. *(cf.* [12, p.405, Lemma 2 and pp. 406-409] *) There exists a family of local quasi-coordinates $\mathcal{V} = \{(V; v^1, \ldots, v^n)\}$ of $X_\nu - D$ with the following properties.*

1. *$X_\nu - D$ is covered by the images of $(V; v^1, \ldots, v^n)$'s.*
2. *The completment of some open neighbourhood of D is covered by a finite number of $(V, v^1, \ldots, v^n)$'s which are local coordinate in the usual sense.*
3. *Each V, as an open subset of the complex Euclidean space C^n, contains a ball of radius $1/2$.*

(4) *There exists positive constants* c_D *and* $\mathcal{A}_k (k = 0, 1, 2, \dots)$ *independent of* V*'s such that at each* $(V, v^1, \dots v^n)$*, the inequalities:*

$$\frac{1}{c_D}(\delta_{ij}) < (g^D_{i\bar{j}}) < c_D(\delta_{ij}),$$

$$\mid (\partial^{|p|+|q|}/\partial v^p \partial \bar{v}^q) g^D_{i\bar{j}} \mid < \mathcal{A}_{|p|+|q|}, \textit{for any multiindices } p \textit{ and } q$$

hold, where $g^D_{i\bar{j}}$ *denote the components of* ω_D *with respect to* v^i*'s.*

Definition 7. $(\bar{M}, B)$ *be a pair of smooth projective variety of dimension* n *and a divisor with simple normal crossings on it. A complete Kähler metric* ω_M *on* $M = \bar{M} - B$ *is said to have bounded Poincaré growth on* (M, B) *if for any polydisk* $\Delta^n = \{(z_1, \dots, z_n) \in C^n \mid\mid z_i \mid < 1 (1 \leq i \leq n)\}$ *in* $\bar{M}$ *such that*

$$\Delta^n \cap B = \{(z_1, \dots, z_n) \in \Delta^n \mid \ z_1 \cdots z_k = 0\} (k \leq n),$$

$\omega_M \mid \Delta^n$ *is quasi-isometric to*

$$\omega_P = \sum_{i=1}^{k} \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{\mid z_i \mid^2 (\log \mid z_i \mid)^2} + \sum_{i=k+1}^{n} \sqrt{-1} dz_i \wedge d\bar{z}_i$$

on every compact subset of Δ^n *and every covariant derivative of* $\omega_M \mid \Delta^n$ *is bounded on every compact subset of* Δ^n*.*

Then by the construction of ω_D, we have:

Lemma 24. *(*[12, pp.400-409]*)* ω_D *has bounded Poincaré growth. on* (X_ν, B)*.*

Remember the definition of φ in Lemma 19. Then the following lemma is trivial.

Lemma 25. $\varphi^{-1} \parallel d\varphi \parallel$ *is uniformly bounded on* $X_\nu - D$*.*

We note that v satisfies the following differential inequality.

$$\Delta_D v \geq -tr_{\omega_D} \hat{\omega}_t \ \text{ on } K \times [0, T),$$

where Δ_D is the Laplacian with respect to ω_D. Let h_D be the solution of the Dirichlet problem

$$\begin{cases} \Delta_D h_D & = -tr_{\omega_D} \hat{\omega}_t & \text{on } K \times [0, T) \\ h_D & = 0 & \text{on } \partial K \times [0, T) \end{cases}$$

Then by the maximum principle, we have

$$v \leq h_D$$

holds on $K \times [0, T)$. Hence

$$\psi_c \leq v \leq h_D$$

holds on $K \times [0, T)$ by (12). Hence by the maximum principle, we have

$$\parallel dv \parallel \leq \max\{\parallel d\psi_c \parallel, \parallel dh_D \parallel\} \text{on } \partial K \times [0, T).$$

By the standard boundary estimate for the second order linear elliptic equations (cf. [6]), we see that h_D is smooth on $\bar{K}$. By using the standard elliptic estimate and Lemma 23, it is easy to obtain an estimate for $\| dh_D \|$ on K. But in this case, since $\omega_\varphi = \sqrt{-1}\partial\bar{\partial}\varphi$ is a complete Kähler form of Poincaré growth, we can find a negative constant b' independent of c and t such that

$$b'\Delta_D(\varphi - c) \leq -tr_{\omega_D}\hat{\omega}_t$$

holds. Then by the maximum principle, we see that

$$h_D \leq b'(\varphi - c) \quad \text{on } K$$

holds.

Then since b and b' are independent of c and t, by Lemma 25, we have:

Lemma 26. *There exists a positive constant C_1' independent of $c \geq c_0$ such that*

$$\| dv \| \leq C_1' c \quad \text{on } \partial K \times [0, T)$$

where $\| \ \|$ is the norm with respect to the Kähler form ω_D.

Remark 6. *As you have seen above, in the proof of Lemma 26, the use of h_D is not unnecessary. We can use a $b'\psi_c$ instead of h_D from the first. The reason why we have used h_D here is that the method can be applicable more general situations.*

6.7. C^1-estimate on K. Let $\pi_\alpha : X_\nu \longrightarrow P^n$ be the generic projection constructed in 6.3. And let

$$Z = \mathrm{Re}(\sum_i \beta_i \frac{\partial}{\partial(z_i/z_0)}),$$

where $(\beta_1, \ldots, \beta_n) \in C^n - \{O\}$. Then

$$\theta = \pi_\alpha^*(Z)$$

is a holomorphic differential operator on $X_\nu - D$ which is meromorphic on X_ν. By operating θ to (11), we have

$$\begin{cases} \frac{\partial(\theta v)}{\partial t} & = \tilde{\Delta}(\theta v) - \theta v + \theta \log \frac{\hat{\omega}^n}{\Omega_\nu} & \text{on } K \times [0, T) \\ \theta v & = \theta v & \text{on } \partial K \times [0, T) \\ \theta v & = 0 & \text{on } \partial K \times \{0\}, \end{cases}$$

where $\tilde{\Delta}$ is the Laplacian with repect to

$$\omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}v.$$

Then by the maximam principle and Lemma 25, we get

$$\| \theta v \| \leq C_1(\theta, K),$$

where

$$C_1(\theta, K) = \max\{\sup_{\partial K \times [0,T)} \| \theta v \|, \sup_{K \times [0,T)} \| \theta \log \frac{\hat{\omega}^n}{\Omega_\nu} \|\}.$$

Then since $\| d \log(\hat{\omega}^n/\Omega_\nu) \|$ is bounded on $X_\nu - D$, if we take m sufficiently large and $\pi_\alpha (1 \leq \alpha \leq m)$ properly, we get :

Lemma 27. *There exists a positive constant $C_1(K)$ which depends on $c \geq c_0$ such that*

$$\| dv \| \leq C_1(K) \quad \text{on } K \times [0,T)$$

holds.

The estimate is getting worse if the point goes far from the boundary because θ has a pole along D.

We set

$$K_c(\varepsilon) = K_c - K_{c-\varepsilon}.$$

Then by the above argument and the construction of D in 6.3, we obtain the following estimate.

Lemma 28. *There exists positive constants C_1 and A_1 independent of $c \geq c_0$ such that*

$$\| dv \| \leq C_1 c \ \text{ on } K(e^{-A_1 c}) \times [0,T).$$

Remark 7. *This idea is inspired by the idea in* [4].

6.8. C^2-estimate on ∂K. In this subsection, we follow the argument in [2, pp. 218-223] and prove:

Lemma 29. *There exists a positive constant C_2' independent of $c \geq c_0$ such that*

$$\| \sqrt{-1}\partial\bar{\partial} v \| \leq \exp(C_2' c) \ \text{ on } \partial K \times [0,T)$$

Let P be a point on ∂K. Choose a coordinates $z_1, \ldots, z_n$ with origin at P such that

(1) $dg^D_{i\bar{j}}(P) = 0$ and $g^D_{i\bar{j}}(P) = \delta_{ij}$.

(2) There exists a positive number $\hat{b}$ such that

$$r = \hat{b}(\varphi - c)$$

satisfies $r_{z_\alpha}(0) = 0$ for $\alpha < n$, $r_{y_n}(0) = 0, r_{x_n} = -1$, where

$$z_\alpha = x_\alpha + \sqrt{-1} y_\alpha$$

and

$$r_{z_\alpha} = \frac{\partial r}{\partial z_\alpha}$$

and so on.

We set $s_1 = x_1, s_2 = y_1, \dots, s_{2n-3} = x_{n-1}, s_{2n-2} = y_{n-1}, s_{2n-1} = y_n = s, s' = (s_1, \dots, s_{2n-1})$. By $\partial\bar{\partial}$-Poincaré lemma, we choose a smooth function ϕ defined on a open neighbourhood U of P such that

$$\hat{\omega}_t = \sqrt{-1}\partial\bar{\partial}\phi$$

holds on U. Let g be a function defined by

$$g = \phi + v.$$

It is clear that to estimate $\sqrt{-1}\partial\bar{\partial}v(P)$ is equivalent to $\sqrt{-1}\partial\bar{\partial}g(P)$ because $\hat{\omega}_t$ is uniformly bounded with respect to ω_D on $X_\nu - D$ by a constant independent of t. . Moreover by Lemma 24, the convariant derivatives of $\hat{\omega}_t$ of any order with respect to ω_D is uniformly bounded with respect to the norm defined by ω_D on $X_\nu - D$ by a constant independent of t. Then by Lemma 23 , we may assume that U contains a ball of radius $1/2c_D$ with center P and any derivatives of ϕ of a fixed order with respect to $(z_1, \dots, z_n)$ is bounded by a constant independent of $c \geq c_0$, if we allow $(U, z_1, \dots, z_n)$ to be a quasi-coordinate. Since the estimate is completely local, this does not cause any trouble in our estimate in this subsection. Hence the C^2-estimate of v on ∂K is reduced completely to the C^2-estimate of g on ∂K.

Sublemma 1. *There exists a positive constant $\dot{C}_2$ independent of $c \geq c_0$ such that*

$$\mid g_{s_i s_j}(0) \mid \leq \dot{C}_2 \quad i, j \leq 2n-1.$$

For r near 0 we may represent g as

$$g = \phi + \sigma r.$$

Then

$$g_{x_n}(0) = \phi_{x_n} - \sigma(0),$$

so that by Lemma 26, $\sigma(0) \leq C_1' c$. Hence

$$g_{s_i s_j}(0) = \phi_{s_i s_j} + \sigma(0) r_{s_i s_j}$$

holds. We note that $r_{s_i s_j} = O(1/c)$ because of the normalization. Hence we get the sublemma.

Sublemma 2. *There exists a positive constant $\ddot{C}_2$ independent of $c \geq c_0$ such that*

$$\mid g_{s_i x_n}(0) \mid \leq \exp(\ddot{C}_2 c)$$

holds.

The proof of Sublemma 2 is a little bit technical.

Writing the Taylor expansion of r up to second order we obtain:

$$r = \mathrm{Re}(-z_n + \sum a_{ij} z_i z_j) + \sum b_{i\bar{j}} z_i \bar{z}_j + O(\mid z \mid^3).$$

Introducing new coordinates of the form

$$\begin{aligned} z'_n &= z_n - \sum a_{ij} z_i z_j, \\ z'_k &= z_k \text{ for } k \leq n-1, \end{aligned}$$

we can write

$$r = -\mathrm{Re}(z'_n) + \sum c_{i\bar{j}} z'_i \bar{z}'_j + O(\mid z \mid^3). \tag{15}$$

It is clear that $(c_{i\bar{j}})$ is positive definite.

We define T_i in a neighbourhood of 0 by

$$T_i = \frac{\partial}{\partial s_i} - \frac{r_{s_i}}{r_{x_n}} \frac{\partial}{\partial x_n}, \text{for } i = 1, \ldots, 2n-1;$$

then $T_i r = 0$ nad we have $T_i(g - \phi) = 0$ on $r = 0$.

We show that for suitable $\varepsilon > 0$, in the region

$$S_\varepsilon = \{x \in U \mid r(x) \leq 0, x_n \leq \varepsilon\},$$

where U is a neighbourhood of the origin, we set

$$w = \pm T_i(g - \phi) + (g_s - \phi_s)^2 - A x_n + B \mid z \mid^2$$

We claim :

(a) For B sufficiently large, $\tilde{L} w \geq 0$;

(b)On ∂S_ε, if A is sufficiently large, $w \leq 0$ holds.

To prove (a) set

$$a = -r_{s_i} / r_{x_n}$$

and consider (we use summation convention)

$$\tilde{L}(T_i g) = T_i \log \Psi(z, g(z)) + g^{p\bar{q}} a_p g_{x_n \bar{q}} + g^{p\bar{q}} a_{\bar{q}} g_{x_n, p} + g^{p\bar{q}} a_{p\bar{q}} g_{x_n}, \tag{16}$$

where

$$\tilde{L} = \tilde{\Delta} - \frac{\partial}{\partial t},$$

and

$$\Psi(z, g(z)) = \frac{\Omega_\nu}{(\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n} \exp(-g\phi).$$

Obeserve that $g^{p\bar{q}}g_{n\bar{q}} = \delta_n^p$ and that

$$\frac{\partial}{\partial x_n} = 2\frac{\partial}{\partial z_n} + \sqrt{-1}\frac{\partial}{\partial s}$$

so that

$$g_{x_n\bar{q}} = 2g_{n\bar{q}} + \sqrt{-1}g_{s\bar{q}}.$$

Thus the second term on the right-hand side of (16) is of the form

$$a_n + g^{p\bar{q}}a_p g_{t\bar{q}} = O(1 + (\sum g^{i\bar{i}})^{1/2}(g^{p\bar{q}}g_{ps}g_{\bar{q}s})^{1/2}).$$

A similar estimate holds for the third term on the right of (16) while the forth term is $O(\sum g^{i\bar{i}})$. Thus by Lemma 19 and the arithmetic-geometric mean inequality, we have

$$\pm\tilde{L}T_i g \leq -C\Psi^{-1/n} - g^{p\bar{q}}g_{ps}g_{\bar{q}s}.$$

Further

$$\begin{aligned}\tilde{L}(g_s - \phi_s)^2 &= 2g^{p\bar{q}}g_{ps}g_{\bar{q}s} + 2(g_s - \phi_s)(\partial_s \log\Psi - \tilde{L}\phi_s)\\ &\geq 2g^{p\bar{q}}g_{ps}g_{\bar{q}s} - C\Psi^{-1/n}\end{aligned}$$

holds on S_ε by the C^1-estimate (Lemma 28) and the arithmetic-geometric mean inequality (if we take ε sufficiently small). Hence we find

$$\begin{aligned}\tilde{L}w &\geq B\sum g^{i\bar{i}} - C\Psi^{-1/n}\\ &\geq 0 \quad \text{on } S_\varepsilon \times [0,T),\end{aligned}$$

if B is sufficiently large. B depends on the C^1-estimate of v on $S_\varepsilon \times [0,T)$ and which is uniform with the weight φ by Lemma 27. This completes the proof of (a).

To prove (b), consider first $\partial S_\varepsilon \cap \partial K$. Here we write $x_n = \rho(s_1, \ldots, s_{2n-1})$ and from (13), we deduce that

$$\rho(t) = \sum_{ij<2n} b_{i\bar{j}}s_i s_j + O(\mid s' \mid^3) \tag{17}$$

with $(b_{i\bar{j}})$ positive definite. Thus on ∂K near O we have

$$x_n \geq a \mid z \mid^2,$$

where a is uniformly bounded from below by a positive constant times $1/c$ where by the construction of K. Also,

$$g(s, \rho(s)) = \phi(s, \rho(s)),$$

so that

$$\mid g_s - \phi_s \mid^2 \leq C \mid s \mid^2 \leq C\rho.$$

Taking A large we obtain (b). By Lemma 28. if we take a positive constant $\ddot{C}_2$ sufficiently large, we may assume that A is bounded from above by a constant times

$\exp(\ddot{C}_2 c)$ for some positive constant $\ddot{C}_2$ independent of $c \geq c_0$. By the maximum principle and (a),(b),

$$w \leq 0 \qquad \text{on } S_\varepsilon$$

holds.

In view of the maximum principle, we have

$$\mid (T_i g)_{x_n}(0) \mid \leq A.$$

This completes the proof of Sublemma 2.

Using, still the special coordinate above we see that to finish our proof of Lemma 29 , we have only to establish the estimate

$$\mid g_{x_n x_n}(O) \mid \leq \exp(\acute{C}_2 c) \tag{18}$$

for some constant $\acute{C}_2$ independent of $c \geq c_0$. By the previous estimates:

$$\begin{aligned} \mid g_{s_i s_j}(O) \mid &\leq \dot{C}_2 \quad (1 \leq i, j \leq 2n-1), \\ \mid g_{s_i x_n} \mid &\leq \exp(\ddot{C}_2 c) \quad (1 \leq i \leq 2n-1), \end{aligned}$$

it suffices to prove

$$\mid g_{n\bar{n}}(O) \mid \leq \exp(C_2' c)$$

for some $\hat{C}_2$ independent of $c \geq c_0$. We may solve the equation

$$\det(g_{i\bar{j}})(O) = \frac{\Omega_\nu}{\omega_D^n}(O)$$

for $g_{n\bar{n}}(O)$. Then since there exists a positive constant $C^\flat$ such that

$$\exp(-C^\flat c) \leq \frac{\omega_D^n}{\Omega_\nu} \leq \exp(C^\flat c)$$

holds on $X_\nu - D$, we see that (16) follows from (17) provided we know the following sublemma.

Sublemma 3. *(*[2, pp. 221-223]*) The $(n-1)$ by $(n-1)$ matrix*

$$(g_{z_\alpha \bar{z}_\beta}(O))_{\alpha,\beta<n} \geq C_3 (\frac{\Omega_\nu}{\omega_D^n})^{\frac{1}{n}} \frac{1}{c} I$$

for some C_3; here I is the $(n-1)$ by $(n-1)$ identity matrix. C_3 is independent of $c \geq c_0$

The proof of this sublemma is very technical.

After subtraction of a linear function we may assume that $\phi_{s_j}(O) = 0, j \leq 2n-1$. To prove Sublemma 3, it suffices to prove

$$\sum_{\alpha,\beta<n} \gamma_\alpha \bar{\gamma}_\beta g_{z_\alpha \bar{z}_\beta}(O) \geq C_3 \mid \gamma \mid^2$$

which we shall do for $\gamma = (1, 0, \ldots, 0)$. We shall show that

$$g_{1\bar{1}}(O) \geq C_4, \tag{19}$$

where C_4 is a positive constant. Let $\tilde{g} = g - \lambda x_n$ with λ so chosen that

$$(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2})\tilde{g}(s_1, \ldots, s_{2n-1}, \rho(s_1, \ldots, s_{2n-1})) = 0 \quad \text{at } O,$$

i.e.

$$0 = g_{1\bar{1}}(O) + \tilde{g}_{x_n}(O)\rho_{1\bar{1}}(O) = g_{1\bar{1}}(O) + (g_{x_n}(O) - \lambda)\rho_{1\bar{1}}(O). \tag{20}$$

Using the fact that any real homogeneous cubic polynomial in (s_1, s_2) admits the unique decomposition

$$\mathrm{Re}(\alpha(s_1 + \sqrt{-1}s_2)^3 + \beta(s_1 + \sqrt{-1}s_2)(s_1 + \sqrt{-1}s_2)^2),$$

we find on expanding $\tilde{g} \mid_{\partial K \cap U}$ in a Taylor series, in $s_1, \ldots, s_{2n-1}$,

$$\begin{aligned} \tilde{g} \mid_{\partial K \cap U} = \mathrm{Re} \textstyle\sum_2^{n-1} a_j z_1 \bar{z}_j + \mathrm{Re}(a z_1 s) \quad & + \mathrm{Re}(p(z_1, \ldots, z_{n-1}) + \beta z_1 \mid z_1 \mid^2) \\ & + O(s_3^2 + \ldots + s_{2n-1}^2), \end{aligned}$$

where p is a holomorphic cubic polynomial.

With the aid of (13), we may replace the term $\beta z_1 \mid z_1 \mid^2$ to $(\rho_{z_1\bar{z}_1}(O))^{-1}\beta z_1 x_n$, if we change the a_j, a and p. Thus by changing the a_j, a and p appropriately we may obtain the inequality:

$$\tilde{g} \mid_{\partial K \cap U} \leq \mathrm{Re}\ p(z) + \mathrm{Re} \sum_2^n a_j z_1 \bar{z}_j + C \sum_{j=2}^n \mid z_j \mid^2 .$$

Let $\check{g} = \tilde{g} - \mathrm{Re}\, p(z)$ and observe that $\check{g}$ satisfies

$$\det(\check{g}_{j\bar{k}}) = \det(\tilde{g}_{j\bar{k}}) = \Psi(z, g(z)).$$

Recall that $\Psi(z, g(z)) \geq \delta > 0$ on a neighbourhood of O, where δ depends on $\Omega_\nu / \omega_D^n(O)$ and the C^1-estimate of v on the neighbourhood. With ε small we see that in the set S_ε, we have $\Psi(z, g(z)) \geq \delta$. Let

$$h = -\delta_0 x_n + \delta_1 \mid z \mid^2 + \frac{1}{B} \sum_2^n \mid a_j z_1 + B z_j \mid^2 .$$

We wish to show that with the suitable choice of $\delta_0, \delta_1, B > 0$ we have $h \geq \check{g}$ on ∂S_ε. First observe that if B is sufficiently large and δ_0 so small that $-\delta_0 x_n + \delta_1 \mid z \mid^2 \geq 0$

on $\partial S_\varepsilon \cap \partial K$ (the dependence of δ_0 and δ_1 is controlled by the Levi form of ∂K). By Lemma 28, if we take B sufficiently large, we have

$$\check{g} \leq h \quad \text{on } \partial S_\varepsilon.$$

The function h is plurisubharmonic and the lowest eigenvalues of the complex Hessian $(h_{i\bar{j}})$ are bounded independently by δ_1 while the other eigenvalues are bounded independently of δ_1.

Hence choosing δ_1 equal to small const. times $\delta^{1/n}$

$$\det(h_{i\bar{j}}) \leq \delta \text{ in } S_\varepsilon$$

holds. By the mximum principle

$$\check{g} \leq h \quad \text{on } S_\varepsilon$$

holds. Hence by the maximum principle

$$\check{g}_{x_n}(O) \leq h_{x_n}(O) = -\delta_0.$$

The desired inequality follows from (19). This completes the proof of Lemma 29.

6.9. C^2-estimate on K. Using the C^2-estimate on ∂K, we shall obtain a C^2-estimate inside K. The method here is the same as in [26].

Let H be a smooth function on $X_\nu - D$ defined by

$$H = \exp(\delta_\nu)(\prod_k \parallel \lambda_k \parallel \prod_k (\log \frac{1}{\parallel \lambda_k \parallel})^{-1})^\varepsilon$$

where δ_ν is the one in 4.2 and $\parallel \lambda_k \parallel$'s are the ones in 4.6 and ε is a sufficiently small positive number such that

$$\omega_H = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial} \log H$$

is a complete Kähler form on $(X_\nu - D) \times [0, \infty]$ which is quasi-isometric to ω_D on $X_\nu - D$, i.e., there exists a positive constant $C(D, H) > 1$ such that

$$\frac{1}{C(D,H)}\omega_H \leq \omega_D \leq C(D,H)\omega_H$$

holds on $X_\nu - D$. We note that ω_H have bounded Poincaré growth so that the bisectional curvature of ω_H is bounded between two constants uniformly on $(X_\nu - D) \times [0, \infty]$.

We set

$$v_H = v - \log H = u - (1 - e^{-t^4})\xi_\nu - \log H,$$

$$\Omega_H = H \cdot \Omega_\nu.$$

Then v_H satisfies the equation

$$\frac{\partial v_H}{\partial t} = \log \frac{(\omega_H + \sqrt{-1}\partial\bar{\partial} v_H)^n}{\Omega_H} - v_H \quad \text{on } K \times [0,T).$$

By Lemma 20 and Lemma 22 v_H satisfies the C^0-estimate:

Lemma 30. *For every sufficiently small positive number ε*

$$\begin{aligned} v_H &\geq C_0^-(\varepsilon) - \log H + (1 - e^{-t^4})\varepsilon\delta_\nu \\ v_H &\leq C_0^+(1 - e^{-t}) - (1 - e^{-t^4})\xi_\nu - \log H \end{aligned}$$

holds on $K \times [0,T)$, where $C_0^+, C_0^-(\varepsilon)$ are constants in Lemma 20 and 22 respectively.

We have the following lemma.

Lemma 31. *(*[26, Lemma 3.2]*)*

$$H^{-C} e^{Cv} (\tilde{\Delta} - \frac{\partial}{\partial t})(e^{-Cv} H^C tr_{\omega_H}\omega) \geq$$

$$(-\Delta_H \log \frac{\omega_H^n}{\Omega_H} - n^2 \inf_{i \neq j} R^H_{i\bar{i}j\bar{j}} - n) + C(n - \frac{1}{C} - \frac{\partial v_H}{\partial t}) tr_{\omega_H}\omega -$$

$$e^{-t} tr_{\omega_H}((\omega_0 - \omega_\infty)\cdot\omega) + (C + \inf_{i\neq j} R^H_{i\bar{i}j\bar{j}}) \exp(\frac{1}{n-1}(-\frac{\partial v_H}{\partial t} - v_H + \log \frac{\omega_H^n}{\Omega})) (tr_{\omega_H}\omega)^{\frac{n}{n-1}},$$

holds on $K \times [0,T)$, where $tr_{\omega_H}((\omega_0 - \omega_\infty)\cdot\omega)$ is defined as in Lemma 3.4, $\inf_{i\neq j} R^H_{i\bar{i}j\bar{j}}$ denotes the infimum of the bisectional curvarue of ω_H on $(X_\nu - D) \times [0,\infty]$ and C is a positive constant such that

$$C + \inf_{i\neq j} R^H_{i\bar{i}j\bar{j}} > 1$$

holds.

The proof of this lemma is the same as one of Lemma 3.2 in [26]. Hence we omit it.

Lemma 32. *If we take C sufficiently large, then there exists a positive constant C_2 independent of $c \geq c_0$ such that*

$$H^C tr_{\omega_D}\omega \leq C_2 \ \text{ on } K \times [0,T)$$

holds.

Proof. By the definition H has zero of order at least $r_j^\nu/2$ along R_j^ν (cf. 6.2). Then by Lemma 29, if we take C sufficiently large, there exists a constant $\tilde{C}_2$ independent of $c \geq c_0$ such that

$$H^C tr_{\omega_D}\omega \leq \tilde{C}_2 \text{ on } \partial K \times [0,T).$$

holds. Suppose $e^{-Cv}H^C tr_{\omega_H}\omega$ takes its maximum at $P_0 \in K \times \{t_0\}(t_0 \in [0,T))$ then by Lemma 31, we have

$$(tr_{\omega_H}\omega)(P_0) \leq \hat{C}_2$$

for a positive constant $\hat{C}_2$ independent of $c \geq c_0$ and C(if it is sufficiently large). Hence in this case we have

$$(H^C tr_{\omega_H}\omega)(P) \leq H^C(P_0)\exp(-C(v(P_0)-v(P)))\hat{C}_2 \quad \text{on } K \times [0,T)$$

holds. By Lemma 30 (since in Lemma 30, we can take ε arbitrarily small), $H\exp(-v) = \exp(-v_H)$ is uniformly bounded from above on $X_\nu - D$. Hence if we change $\hat{C}_2$, if necessary, by the maximum principle for parabolic equations, we may assume that

$$(H^C tr_{\omega_H}\omega)(P) \leq \exp(Cv(P))\hat{C}_2$$

holds on $K \times [0,T)$. We note that ω_D and ω_H are quasi-isometric on $X_\nu - D$. Hence by Lemma 22, if we take C sufficiently large, there exists a positive constant C_2 independent of $c \geq c_0$ such that

$$H^C tr_{\omega_D}\omega \leq C_2 \text{ on } K \times [0,T)$$

holds. Q.E.D.

By [23], the higher order interior estimate on $K \times [0,T)$ follows. As for the boundary estimate of u, we just need to follow the argument in [2]. This completes the proof of Theorem 10.

6.10. Costruction of ω_E. Let u_∞ be as in Theorem 10. Then by the construction

$$\omega_K = \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty$$

is a Kähler-Einstein form on $\bar{K} = \bar{K}_c$. We may assume that $c > 1$. Let us take the exahustion $\{K_{lc}\}_{l=1}^\infty$ of $X_\nu - D$ and let

$$\omega_l^\nu = \omega_{K_{lc}}.$$

Then by Lemma 20 and Lemma 32 and the regularity theorem in [23], we have the following lemma.

Lemma 33. *There exists a subsequence of $\{\omega_l^\nu\}_{l=1}^\infty$ which converges uniformly on every compact subset of $X_\nu - D$ in C^∞-topology to a Kähler-Einstein form ω^ν on $X_\nu - D$.*

Although ω^{ν} is a Kähler-Einstein form, it is not enough good for our purpose.

Let us consider the linear system $\mid m!\nu L \mid$ and construct $\xi_{m!\nu}$ as before. We denote $\xi_{m!\nu}$ by $\xi^{(m)}$ for simplicity. Since $X_{m!\nu} - F^{m!\nu}$ are all biholomorphic to $X - \mathrm{SBs}(L)$, we may consider $\{\xi^{(m)}\}$ as a family of functions on $X_{\nu} - D$. Let us denote $X_{m!\nu}$ by $X^{(m)}$ for simplicity and define $X_{(1)} = X$.

Lemma 34. *We can construct* $\{\xi^{(m)}\}$ *so that*

$$\xi^{(1)} \leq \xi^{(2)} \leq \ldots \leq \xi^{(m)} \leq \xi^{(m+1)} \leq \ldots$$

holds on $X_{\nu} - D$.

Proof. Let $\mathcal{I}_{\mu}$ denote the ideal sheaf of the base scheme $\mathrm{Bs} \mid \mu L \mid$. Then clearly

$$\mathcal{I}_{\mu_1+\mu_2} \hookrightarrow \mathcal{I}_{\mu_1} \otimes_{\mathcal{O}_X} \mathcal{I}_{\mu_2}$$

holds. Hence inductively we can construct a morphism

$$\mu_m : X_{(m)} \longrightarrow X_{(m-1)} \ \ (m \geq 2)$$

such that

(1) $f_{(m)} := \mu_m \circ \cdots \circ \mu_1 : X_{(m)} \longrightarrow X_{(0)} (= X)$ is a resolution of $\mathrm{Bs} \mid m!\nu L \mid$.
(2) The fixed part of $\mid f_{(m)}^{*}(m!L) \mid$ is a divisor with normal crossings on $X_{(m)}$.

An explicit construction of $\{\xi^{(m)}\}_{m=1}^{\infty}$ is as follows. Let $V^{(1)} = \{\eta_i^{(1)}\}_{i=0}^{N(1)}$ be a basis of $H^0(X, \mathcal{O}_X(\nu L))$. By induction for each $m \geq 1$, we can construct a finite subset

$$V^{(m)} = \{\eta_1^{(m)}, \ldots, \eta_{N(m)}^{(m)}\}$$

in $H^0(X, \mathcal{O}_X(m!\nu L)$ with the following properties.

(1) $V^{(m)}$ spans $H^0(X, \mathcal{O}_X(m!\nu L))$.
(2) $V^{(m)}$ contains all the elements of the form:

$$(\eta_{i_1}^{(m-1)})^{\otimes a_1} \otimes \cdots \otimes (\eta_{i_m}^{(m-1)})^{\otimes a_m},$$

$$\sum_{i=1}^{m} a_i = m, a_i \geq 0, 0 \leq i_1 < \ldots < i_m \leq N(m-1).$$

Now we set

$$\xi^{(m)} = \frac{1}{m!\nu} \log\Big(\sum_{i=0}^{N(m)} \frac{(\sqrt{-1})^{m!\nu n^2} \eta_i^{(m)} \wedge \bar{\eta}_i^{(m)}}{\Omega^{m!\nu}} \Big).$$

We may consider $\xi^{(m)}$ as a function on $X_{\nu} - D$. Then by the construction

$$\xi^{(1)} \leq \xi^{(2)} \leq \ldots \leq \xi^{(m)} \leq \ldots$$

holds and

$$\omega_{\infty} + \sqrt{-1}\partial\bar{\partial}\xi^{(m)}$$

is a smooth semipositive form on $X_{(m)}$ and positive on $X_\nu - D$. This completes the proof of the lemma. Q.E.D.

Now we consider the following Dirichlet problem.

$$\begin{cases} \frac{\partial u^{(m)}}{\partial t} &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar\partial u^{(m)})^n}{\Omega} - u^{(m)} \quad \text{on } K \times [0, T_m) \\ u^{(m)} &= (1 - e^{-t^4})\xi^{(m)} \quad \text{on } \partial K \times [0, T_m) \\ u^{(m)} &= 0 \quad \text{on } K \times \{0\}, \end{cases}$$

where T_m is the maximal existence time for smooth solution on $\bar K$.

Lemma 35. *The followings are true.*

(1) *T_m is infinite and $u_\infty^{(m)} = \lim_{t\to\infty} u^{(m)}$ exists in C^∞-topology on $\bar K$.*
(2) *$\omega_1^{(m)} := \omega_\infty + \sqrt{-1}\partial\bar\partial u_\infty^{(m)}$ is a Kähler form on K.*
(3) *If we define a sequence of Kähler forms $\{\omega_l^{(m)}\}_{l=1}^\infty$ in the same manner as the definition of $\{\omega_l^\nu\}_{l=1}^\infty$ above, then there exists a subsequence of $\{\omega_l^{(m)}\}$ which converges uniformly on every compact subset of $X_\nu - D$ uniformly in C^∞-topology.*

Proof. The only difference between the above equation and the equation (10) is that $\nabla^k \xi^{(m)} (k \geq 1)$ is bounded with respect to ω_D with weights different from before. It is easy to find such weights. In fact, there exists a positive constant $C(m,k)$ depending only on m and k such that $H^{C(m,k)} \parallel \nabla^k \xi^{(m)} \parallel$ is bounded by a positive constant on $X_\nu - D$. Hence the previous argument is valid with some weight with respect to H. Hence the first assertion is trivial.

Then by replacing K to K_{lc}, we get a sequence of Kähler-Einstein form $\{\omega_l^{(m)}\}_{l=1}^\infty$ which are defined on K_{lc} respectively $(\omega_l^{(1)} = \omega_l^\nu)$.

We would like to find a subsequence of $\{\omega_l^{(m)}\}_{l=1}^\infty$ which converges in C^∞-topology on every compact subset of $X_\nu - D$.

For the first, replacing ξ_ν by $\xi^{(m)}$, completely analogous estimate as Lemma 22 holds for $u^{(m)}$ with the perturbation $\delta^{(m)}$ completely analogous to δ_ν. As for the C^2-estimate, by the proof of Lemma 35, Lemma 32 also holds for ω_l^m if we replace C and C_2 to appropriate constants independent of l. The rest of the proof is the same as in the proof of Lemma 33. Q.E.D.

Taking subsequence, if necessary, we obtain $\omega^{(m)}$ as before, where $\omega^{(0)} = \omega^\nu$.

We shall consider $\omega^{(m)}$ as a d-closed positive $(1,1)$-current on X by

$$\omega^{(m)} := \omega_\infty + \sqrt{-1}\partial\bar{\partial}\log\frac{(\omega^{(m)})^n}{\Omega},$$

where $\partial\bar{\partial}$ is taken in the sense of a current. This definition is well defined by the C^0-estimate (Lemma 22) and clearly

$$[\omega^{(m)}] = 2\pi c_1(L)$$

holds. Now we want to show that

Proposition 2.

$$\omega_E = \lim_{m\to\infty}\omega^{(m)}$$

exists in the sense of a d-closed positive $(1,1)$-current on X_ν.

Lemma 36. *There exists a positive constant C_3 independent of m such that*

$$(\omega^{(m)})^n \leq C_3\omega_D^n \quad \text{on } X_\nu - D$$

holds.

Proof. Since ω_D is a *complete* Kähler-Einstein form on $X_\nu - D$, by applying the maximum principle to the function

$$\log\frac{(\omega^{(m)})^n}{\omega_D^n},$$

we obtain the lemma. Q.E.D.

Hence $\{(\omega^{(m)})^n\}$ is uniformly bounded from above. For the next we shall show:

Lemma 37. *For every $m \geq 1$, we have that*

$$(\omega^{(m)})^n \leq (\omega^{(m+1)})^n \quad \text{on } X_\nu - D$$

holds.

Proof. $u^{(m+1)} - u^{(m)}$ satisfies the following equations:

$$\begin{cases} \frac{\partial(u^{(m+1)}-u^{(m)})}{\partial t} = \log\frac{(\omega_t+\sqrt{-1}\partial\bar{\partial}u^{(m+1)})^n}{(\omega_t+\sqrt{-1}\partial\bar{\partial}u^{(m)})^n} - (u^{(m+1)} - u^{(m)}) & \text{on } K\times[0,\infty) \\ u^{(m+1)} - u^{(m)} = (1-e^{-t^4})(\xi^{(m+1)} - \xi^{(m)}) & \text{on } \partial K\times[0,\infty) \\ u^{(m+1)} - u^{(m)} = 0 & \text{on } K\times\{0\}. \end{cases}$$

Since

$$\log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u^{(m+1)})^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}u^{(m)})^n} = \int_0^1 \Delta_a^{(m,m+1)}(u^{(m+1)} - u^{(m)})da,$$

where $\Delta_a^{(m,m+1)}(a \in [0,1])$ is the Laplacian with respect to the Kähler form

$$\omega_t + \sqrt{-1}\partial\bar{\partial}\{(1-a)u^{(m)} + au^{(m+1)}\},$$

we see that this equation is of parabolic type. We note that $\xi^{(m+1)} \geq \xi^{(m)}$ on $X_\nu - D$ by the construction. Then by the maximum principle we obtain

$$u^{(m)} \leq u^{(m+1)} \text{on } K \times [0,\infty).$$

We set

$$u_\infty^{(m)} = \lim_{t\to\infty} u^{(m)}.$$

Then we have

$$\omega_1^{(m)} = \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty^{(m)}$$

and

$$(\omega_1^{(m)})^n = \exp(u_\infty^{(m)})\Omega$$

on K. Hence we see that

$$(\omega_1^{(m)})^n \leq (\omega_1^{(m+1)})^n$$

holds on K. By replacing K to K_{lc} and repeating the same argument, we see that

$$(\omega_l^{(m)})^n \leq (\omega_l^{(m+1)})^n$$

holds on K_{lc}. By letting l tend to infinifty, we completes the proof of the lemma. Q.E.D.

Hence $\{(\omega^{(m)})^n\}_{m=1}^\infty$ is monotone increasing and bounded from above uniformly on every compact subset of $X_\nu - D$

$$\omega_E^n := \lim_{m\to\infty}(\omega^{(m)})^n$$

exists.

We shall define a d-closed positive $(1,1)$ current ω_E on X by

$$\omega_E = \sqrt{-1}\partial\bar{\partial}\log\omega_E^n.$$

ω_E is well defined by the C^0-estimates in the last subsection. Then by the construction it is clear that $[\omega_E] = 2\pi c_1(K_X)$ and $(K_X, (\omega_E^n)^{-1})$ is an AZD. This completes the proof of Theorem 7.

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