A NOTE ON THE M^*-LIMITING CONVOLUTION BODY

ANTONIS TSOLOMITIS

ABSTRACT. We introduce the mixed convolution bodies of two convex symmetric bodies. We prove that if the boundary of a body K is smooth enough then as δ tends to 1 the δ–M^*–convolution body of K with itself tends to a multiple of the Euclidean ball after proper normalization. On the other hand we show that the δ–M^*–convolution body of the n–dimensional cube is homothetic to the unit ball of ℓ_1^n.

1. INTRODUCTION

Throughout this note K and L denote convex symmetric bodies in $\mathbb{R}^n$. Our notation will be the standard notation that can be found, for example, in [2] and [4]. For $1 \le m \le n$, $V_m(K)$ denotes the m–th mixed volume of K (i.e. mixing m copies of K with $n-m$ copies of the Euclidean ball $\mathcal{B}_n$ of radius one in $\mathbb{R}^n$). Thus if $m = n$ then $V_n(K) = \mathrm{vol}_n(K)$ and if $m = 1$ then $V_1(K) = w(K)$ the mean width of K.

For $0 < \delta < 1$ we define the m–th mixed δ–convolution body of the convex symmetric bodies K and L in $\mathbb{R}^n$:

Definition 1.2. The m–th mixed δ–convolution body of K and L is defined to be the set,

$$C_m(\delta; K, L) = \{x \in \mathbb{R}^n \ : \ V_m\left(K \cap (x + L)\right) \ge \delta V_m(K \cap L)\}.$$

It is a consequence of Brunn–Minkowski inequality for mixed volumes that these bodies are convex.

If we write $h(u)$ for the support function of K in the direction $u \in \mathbb{S}^{n-1}$ then we have,

$$w(K) = 2M_K^* = 2\int_{\mathbb{S}^{n-1}} h(u)d\nu(u), \tag{1.1}$$

where ν is the Lebesgue measure of $\mathbb{R}^n$ restricted on $\mathbb{S}^{n-1}$ and normalized so that $\nu(\mathbb{S}^{n-1}) = 1$. In this note we study the limiting behavior of $C_1(\delta; K, K)$ (which we will abbreviate with $C_1(\delta)$) as δ tends to 1 and K has a C_+^2 boundary. For simplicity we will call $C_1(\delta)$ the "δ–M^*–convolution body of K".

Research at MSRI partially supported by NSF grant DMS-9022140.

We are looking for suitable $\alpha \in \mathbb{R}$ so that the limit

$$\lim_{\delta \to 1^-} \frac{C_1(\delta)}{(1-\delta)^\alpha}$$

exists (convergence in the Hausdorff distance). In this case we call the limiting body "the limiting M^*–convolution body of K".

We prove that for a convex symmetric body K in $\mathbb{R}^n$ with C_+^2 boundary the limiting M^*–convolution body of K is homothetic to the Euclidean ball. We also get a sharp estimate (sharp with respect to the dimension n) of the rate of the convergence of the δ–M^*–convolution body of K to its limit. By C_+^2 we mean that the boundary of K is C^2 and that the principal curvatures of $bd(K)$ at every point are all positive.

We also show that some smoothness condition on the boundary of K is necessary for this result to be true, by proving that the limiting M^*–convolution body of the n–dimensional cube is homothetic to the unit ball of ℓ_1^n.

We want to thank Professor V.D.Milman for his encouragement and his guidance to this research and for suggesting the study of mixed convolution bodies.

2. The case "$bd(K)$ is a C_+^2 manifold"

In this section we prove the following:

Theorem 2.1. *Let K be a convex symmetric body in $\mathbb{R}^n$ so that $bd(K)$ is a C_+^2 manifold. Then for all $x \in \mathbb{S}^{n-1}$ we have,*

$$\left| \|x\|_{\frac{C_1(\delta)}{1-\delta}} - \frac{c_n}{M_K^*} \right| \le C \frac{c_n}{M_K^*} \left(M_K^* n(1-\delta) \right)^2, \tag{2.1.1}$$

where $c_n = \int_{\mathbb{S}^{n-1}} |\langle x, u\rangle| d\nu(u) \sim 1/\sqrt{n}$ and C is a constant independent of the dimension n. In particular,

$$\lim_{\delta \to 1^-} \frac{C_1(\delta)}{1-\delta} = \frac{M_K^*}{c_n} \mathcal{B}_n.$$

Moreover the estimate (2.1.1) *is sharp with respect to the dimension n*

By "sharp" with respect to the dimension n we mean that there are examples (for instance the n–dimensional Euclidean ball) for which the inequality (2.1.1) holds true if "$\le$" is substituted with "$\ge$" and the constant C is adjusted by a (universal) constant factor.

Before we proceed with the proof we will need to collect some standard notation which can be found in [4]. We write $p : bd(K) \to \mathbb{S}^{n-1}$ for the Gauss map $p(x) = N(x)$ where $N(x)$ denotes the unit normal vector of $bd(K)$ at x. W_x denotes the Weingarten map, that is, the differential of p at the point $x \in bd(K)$. W_u^{-1} is the reverse Weingarten map at $u \in \mathbb{S}^{n-1}$ and the eigenvalues of W_x and W_u^{-1} are respectively the principal curvatures and principal radii of curvature of the manifold $bd(K)$ at $x \in bd(K)$ and $u \in \mathbb{S}^{n-1}$. We write $\|W\|$ and $\|W^{-1}\|$ for the quantities: $\sup_{x \in bd(K)} \|W_x\|$ and $\sup_{u \in \mathbb{S}^{n-1}} \|W_u^{-1}\|$ respectively. These quantities are finite since the manifold $bd(K)$ is assumed to be C_+^2.

For $\lambda \in \mathbb{R}$ and $x \in \mathbb{S}^{n-1}$ we write K_λ for the set $K \cap (\lambda x + K)$. $p_\lambda^{-1} : \mathbb{S}^{n-1} \to bd(K_\lambda)$ is the reverse Gauss map, that is, the affine hyperplane $p_\lambda^{-1}(u) + [u]^\perp$ is tangent to K_λ at $p_\lambda^{-1}(u)$. The normal cone of K_λ at x is denoted by $N(K_\lambda, x)$ and similarly for K. The normal cone is a convex set (see [4]). Finally h_λ will denote the support function of K_λ.

Proof. Without loss of generality we may assume that both the $bd(K)$ and $\mathbb{S}^{n-1}$ are equipped with an atlas whose charts are functions which are Lipschitz, their inverses are Lipschitz and they all have the same Lipschitz constant $c > 0$.

Let $x \in \mathbb{S}^{n-1}$ and $\lambda = \frac{1}{\|x\|_{C_1(\delta)}}$; hence $\lambda x \in bd\,(C_1(\delta))$ and

$$M^*_{K_\lambda} = \delta M^*_K. \tag{2.1.2}$$

We estimate now $M^*_{K_\lambda}$. Let $u \in \mathbb{S}^{n-1}$. We need to compare $h_\lambda(u)$ and $h(u)$. Set $Y_\lambda = bd(K) \cap bd(\lambda x + K)$.

Case 1. $p_\lambda^{-1}(u) \notin Y_\lambda$.

In this case it is easy to see that

$$h_\lambda(u) = h(u) - |\langle \lambda x, u\rangle|.$$

Case 2. $p_\lambda^{-1}(u) \in Y_\lambda$.

Let $y_\lambda = p_\lambda^{-1}(u)$ and $y'_\lambda = y_\lambda - \lambda x \in bd(K)$. The set $N(K_\lambda, y_\lambda) \cap \mathbb{S}^{n-1}$ defines a curve γ which we assume to be parametrized on $[0,1]$ with $\gamma(0) = N(K, y_\lambda)$ and $\gamma(1) = N(K, y'_\lambda)$. We use the inverse of the Gauss map p to map the curve γ to a curve $\tilde\gamma$ on $bd(K)$ by setting $\tilde\gamma = p^{-1}\gamma$. The end points of $\tilde\gamma$ are y_λ (label it with A) and y'_λ (label it with B). Since $u \in \gamma$ we conclude that the point $p^{-1}(u)$ belongs to the curve $\tilde\gamma$ (label this point by Γ). Thus we get:

$$0 \le h(u) - h_\lambda(u) = |\langle \vec{A\Gamma}, u\rangle|.$$

It is not difficult to see that the cosine of the angle of the vectors $\vec{A\Gamma}$ and u is less than the largest principal curvature of $bd(K)$ at Γ times $|\vec{A\Gamma}|$, the length of the vector $\vec{A\Gamma}$. Consequently we can write,

$$0 \le h(u) - h_\lambda(u) \le \|W\| |\vec{A\Gamma}|^2.$$

In addition we have,

$$\begin{aligned} |\vec{A\Gamma}| &\le \text{length}\left(\tilde\gamma|_A^\Gamma\right) \le \text{length}\left(\tilde\gamma|_A^B\right) \\ &= \int_0^1 |d_t\tilde\gamma| dt = \int_0^1 |d_t p^{-1}\gamma| dt \\ &\le \|W^{-1}\| \text{length}(\gamma) \le \frac{2}{\pi}\|W^{-1}\| |p(y_\lambda) - p(y'_\lambda)|, \end{aligned}$$

where $|\,\mathbf{.}\,|$ is the standard Euclidean norm. Without loss of generality we can assume that the points y_λ and y'_λ belong to the same chart at y_λ. Let φ be the chart mapping $\mathbb{R}^{n-1}$ to a neighborhood of y_λ on $bd(K)$ and ψ the chart mapping

$\mathbb{R}^{n-1}$ on a neighborhood of $N(K, y_\lambda)$ in $\mathbb{S}^{n-1}$. We assume, as we may, that the graph of γ is contained in the range of the chart ψ. It is now clear from the above series of inequalities that

$$|\vec{A\Gamma}| \leq c_0 \|W^{-1}\| |\psi^{-1} p\varphi(t) - \psi^{-1} p\varphi(s)|,$$

where t and s are points in $\mathbb{R}^{n-1}$ such that $\varphi(t) = y_\lambda$ and $\varphi(s) = y'_\lambda$ and $c_0 > 0$ is a universal constant. Now the mean value theorem for curves gives,

$$\begin{aligned} |\vec{A\Gamma}| &\leq C\|W^{-1}\| \|\|W\| |t - s| \\ &\leq C\|W^{-1}\| \|\|W\| |y_\lambda - y'_\lambda| \\ &= C\|W^{-1}\| \|\|W\| \lambda, \end{aligned}$$

where C may denote a different constant every time it appears. Thus we have,

$$0 \leq h(u) - h_\lambda(u) \leq C\|W\| \left(\|W^{-1}\| \|\|W\|\right)^2 \lambda^2.$$

Consequently,

$$\int_{\mathbb{S}^{n-1} \setminus p_\lambda(Y_\lambda)} (h(u) - |\langle \lambda x, u \rangle|)\, d\nu(u) + \int_{p_\lambda(Y_\lambda)} \left(h(u) - C\lambda^2\right) d\nu(u)$$

$$\leq M^*_{K_\lambda} = \delta M^*_K \leq$$

$$\int_{\mathbb{S}^{n-1} \setminus p_\lambda(Y_\lambda)} (h(u) - |\langle \lambda x, u \rangle|)\, d\nu(u) + \int_{p_\lambda(Y_\lambda)} h(u) d\nu(u),$$

where C now depends on $\|W\|$ and $\|W^{-1}\|$.

Rearranging and using c_n for the quantity $\int_{\mathbb{S}^{n-1}} |\langle x, u \rangle| d\nu(u)$ and the fact $\lambda = 1/\|x\|_{C_1(\delta)}$ we get:

$$\left| \|x\|_{\frac{C_1(\delta)}{1-\delta}} - \frac{c_n}{M^*_K} \right| \leq \frac{c_n}{M^*_K} \left(\frac{\int_{p_\lambda(Y_\lambda)} |\langle x, u \rangle| d\nu(u)}{c_n} + C\lambda \frac{\mu\left(p_\lambda(Y_\lambda)\right)}{c_n} \right).$$

We observe now that for $u \in p_\lambda(Y_\lambda)$, $|\langle x, u \rangle| \leq \text{length}(\gamma)/2 \leq \|W\| \lambda$. Using this in the last inequality and the fact that $p_\lambda(Y_\lambda)$ is a "band" around an equator of $\mathbb{S}^{n-1}$ of width at most length$(\gamma)/2$ we get,

$$\begin{aligned} \left| \|x\|_{\frac{C_1(\delta)}{1-\delta}} - \frac{c_n}{M^*_K} \right| &\leq \frac{c_n}{M^*_K} C n \lambda^2 \\ &\leq \frac{c_n}{M^*_K} C n \frac{(1-\delta)^2}{\|x\|^2_{\frac{C_1(\delta)}{1-\delta}}}. \end{aligned}$$

Our final task is to get rid of the norm that appears on the right side of the latter inequality. Set

$$T = \frac{\|x\|_{C_1(\delta)/1-\delta}}{c_n / M^*_K}.$$

We have shown that

$$T^2|T-1| \le C\frac{M_K^*}{c_n}n(1-\delta)^2.$$

If $T \ge 1$ then we can just drop the factor T^2 and we are done. If $T < 1$ we write $T^2|T-1|$ as $(1-(1-T))^2(1-T)$ and we consider the function

$$f(x) = (1-x)^2 x \ : \ (-\infty, \frac{1}{3}) \to \mathbb{R}.$$

This function is strictly increasing thus invertible on its range, that is, f^{-1} is well defined and increasing in $(-\infty, \frac{4}{27})$. Consequently if

$$C\frac{M_K^*}{c_n}n(1-\delta)^2 \le \frac{4}{27}, \tag{2.1.3}$$

we conclude that,

$$\begin{aligned} 0 \le 1 - T &\le f^{-1}\left(C\frac{M_K^*}{c_n}n(1-\delta)^2\right) \\ &\le C\frac{M_K^*}{c_n}n(1-\delta)^2. \end{aligned}$$

The last inequality is true since the derivative of f^{-1} at zero is 1. Observe also that the convergence is "essentially realized" after (2.1.3) is satisfied. □

We now proceed to show that some smoothness conditions on the boundary of K are necessary, by proving that the limiting M^*–convolution body of the n–dimensional cube is homothetic to the unit ball of ℓ_1^n. In fact we show that the δ–M^*–convolution body of the cube is already homothetic to the unit ball of ℓ_1^n.

Example 2.3. *Let $P = [-1,1]^n$. Then for $0 < \delta < 1$ we have,*

$$C_1(P) = \frac{C_1(\delta; P, P)}{1-\delta} = n^{3/2} vol_{n-1}(\mathbb{S}^{n-1}) B_{\ell_1^n}.$$

Proof. Let $x = \sum_{j=1}^n x_j e_j$ where $x_j \ge 0$ for all $j = 1, 2, \ldots, n$ and e_j is the standard basis of $\mathbb{R}^n$. Let $\lambda > 0$ be such that $\lambda x \in bd\,(C_1(\delta))$. Then,

$$P \cap (\lambda x + P) = \{y \in \mathbb{R}^n \ : \ y = \sum_{j=1}^n y_i e_i, -1 + \lambda x_j \le y_j \le 1, j = 1, 2, \ldots, n\}.$$

The vertices of $P_\lambda = P \cap (\lambda x + P)$ are the points $\sum_{j=1}^n \alpha_j e_j$ where α_j is either 1 or $-1 + \lambda x$ for all j. Without loss of generality we can assume that $-1 + \lambda x_j < 0$ for all the indices j. Put $\operatorname{sign} \alpha_j = \alpha_j/|\alpha_j|$ when $\alpha_j \neq 0$ and $\operatorname{sign} 0 = 0$. Fix a sequence of α_j's so that the point $v = \sum_{j=1}^n \alpha_j e_j$ is a vertex of P_λ. Clearly,

$$N\,(P_\lambda, v) = N\left(P, \sum_{j=1}^n (\operatorname{sign} \alpha_j) e_j\right).$$

If $u \in \mathbb{S}^{n-1} \cap N(P_\lambda, v)$ then,

$$h_\lambda(u) = h(u) - \left| \langle \sum_{j=1}^{n} (\alpha_j - \operatorname{sign} \alpha_j) e_j, u \rangle \right|.$$

If $\operatorname{sign} \alpha_j = 1$ then $\alpha_j - \operatorname{sign} \alpha_j = 0$ otherwise $\alpha_j - \operatorname{sign} \alpha_j = \lambda x$.

Let $\mathcal{A} \subseteq \{1, 2, \dots, n\}$. Consider the "$\mathcal{A}$–orthant":

$$\mathcal{O}_{\mathcal{A}} = \{y \in \mathbb{R}^n : \langle y, e_j \rangle < 0, \text{ if } j \in \mathcal{A} \text{ and } \langle y, e_j \rangle \geq 0 \text{ if } j \notin \mathcal{A}\}.$$

Then $\mathcal{O}_{\mathcal{A}} = N\left(P, \sum_{j=1}^{n} (\operatorname{sign} \alpha_j) e_j\right)$ if and only if $\operatorname{sign} \alpha_j = 1$ exactly for every $j \notin \mathcal{A}$. Hence,

$$h_\lambda(u) = h(u) - \left| \langle \sum_{j \in \mathcal{A}} \lambda x_j e_j, u \rangle \right|,$$

for all $u \in \mathcal{O}_{\mathcal{A}} \cap \mathbb{S}^{n-1}$. Hence using the facts $M^*_{P_\lambda} = \delta M^*_P$ and $\lambda = 1/\|x\|_{C_1(\delta)}$ we get,

$$\|x\|_{\frac{C_1(\delta)}{1-\delta}} = -\frac{1}{M^*_P} \sum_{\mathcal{A} \subseteq \{1,2,\dots,n\}} \sum_{j \in \mathcal{A}} x_j \int_{\mathcal{O}_{\mathcal{A}} \cap \mathbb{S}^{n-1}} \langle e_j, u \rangle d\nu(u),$$

which gives the result since

$$\int_{\mathcal{O}_{\mathcal{A}} \cap \mathbb{S}^{n-1}} \langle e_j, u \rangle d\nu(u) = \frac{1}{2^{n-1}} \int_{\mathbb{S}^{n-1}} |\langle e_1, u \rangle| d\nu(u).$$

□

References

1. Kiener, K., *Extremalität von Ellipsoiden und die Faltungsungleichung von Sobolev*, Arch. Math. **46** (1986), 162–168.
2. Milman, V. and Schechtmann, G., *Asymptotic theory of finite dimensional normed spaces*, vol. 1200, Springer Lecture Notes, 1986.
3. Schmuckenschläger, M., *The distribution function of the convolution square of a convex symmetric body in* $\mathbb{R}^n$, Israel Journal of Mathematics **78** (1992), 309–334.
4. Schneider, R., *Convex bodies: The Brunn–Minkowski theory*, vol. 44, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1993.
5. Tsolomitis, A., *Convolution bodies and their limiting behavior*, Duke J. Math. (submitted).

Department of Mathematics, 231 W.18th Avenue, Columbus, OH 43210

Current address: Mathematical Sciences Research Institute, 1000 Centennial Road, Berkeley, CA 94720

E-mail address: atsol@math.mps.ohio-state.edu