LOW M^*-ESTIMATES ON COORDINATE SUBSPACES

A. A. GIANNOPOULOS AND V. D. MILMAN

Abstract. Let K be a symmetric convex body in $\mathbf{R}^n$. It is well-known that for every $\theta \in (0,1)$ there exists a subspace F of $\mathbf{R}^n$ with $\dim F = [(1-\theta)n]$ such that

$$(*) \qquad \mathcal{P}_F(K) \supseteq \frac{c\sqrt{\theta}}{M_K} D_n \cap F,$$

where $\mathcal{P}_F$ denotes the orthogonal projection onto F. Consider a fixed coordinate system in $\mathbf{R}^n$. We study the question whether an analogue of $(*)$ can be obtained when one is restricted to choose F among the coordinate subspaces $\mathbf{R}^\sigma$, $\sigma \subseteq \{1, \dots, n\}$, with $|\sigma| = [(1-\theta)n]$. We prove several "coordinate versions" of $(*)$ in terms of the cotype-2 constant, of the volume ratio and other parameters of K. The basic source of our estimates is an exact coordinate analogue of $(*)$ in the ellipsoidal case. Applications to the computation of the number of lattice points inside a convex body are considered throughout the paper.

1. Introduction

Notation. Our setting is $\mathbf{R}^n$ equipped with an inner product $\langle .,. \rangle$ and the associated Euclidean norm defined by $|x| = \langle x, x\rangle^{1/2}$, $x \in \mathbf{R}^n$. We denote the Euclidean unit ball and the unit sphere by D_n and S^{n-1} respectively, and we write σ for the rotationally invariant probability measure on S^{n-1}.

Let K be a symmetric convex body in $\mathbf{R}^n$. Then, K induces in a natural way a norm $\|.\|_K$ on $\mathbf{R}^n$. In what follows we shall denote by X_K the normed space $(\mathbf{R}^n, \|.\|_K)$. As usual, $K^o = \{y \in \mathbf{R}^n : \langle y, x\rangle \leq 1 \text{ for every } x \in K\}$ is the polar body of K, and $X_{K^o} = (\mathbf{R}^n, \|.\|_{K^o})$ is the dual space of X_K.

Finally, we consider the integral parameters

$$M = M_K = \left(\int_{S^{n-1}} \|x\|_K^2 \, \sigma(dx) \right)^{1/2} \quad , \quad M^* = M_{K^o} = \left(\int_{S^{n-1}} \|x\|_{K^o}^2 \, \sigma(dx) \right)^{1/2},$$

which are up to a constant the mean widths of K^o and K respectively.

Results. The following inequality of the second named author plays an important role in developing a proportional theory of high-dimensional convex bodies:

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Theorem A (*Low M^*-estimate*). *There exists a function $f:(0,1)\to \mathbf{R}^+$ such that for every symmetric convex body K in $\mathbf{R}^n$ and for every $\theta\in(0,1)$ one can find a subspace F of $\mathbf{R}^n$ with* $\dim F=[(1-\theta)n]$ *satisfying*

$$\|x\|_K \geq \frac{f(\theta)}{M_{K^o}}\,|x| \quad , \quad x\in F. \tag{1.1}$$

Theorem A was originally proved in [M1] and a second proof using the isoperimetric inequality on S^{n-1} was given in [M2] where it was shown that (1.1) holds with $f(\theta)\geq c\theta$ for some absolute constant $c>0$ (and with an estimate $f(\theta)\geq \theta+o(1-\theta)$ as $\theta\to 1^-$). This was later improved to $f(\theta)\geq c\sqrt{\theta}$ in [PT], see also [M3] for a different proof with this best possible $\sqrt{\theta}$-dependence. Finally, it was proved in [Go] that one can have

$$f(\theta)\geq \sqrt{\theta}(1+O(\frac{1}{\theta n})). \tag{1.2}$$

Moreover, if we fix some $\theta\in(0,1)$ and consider the Grassmannian manifold $G_{n,k}$ of all k-dimensional subspaces of $\mathbf{R}^n$, where $k=k(\theta,n)=[(1-\theta)n]$, equipped with the Haar probability measure $\nu_{n,k}$, then (1.1) holds true with $f(\theta)\geq c\sqrt{\theta}$ for all subspaces F in a subset $\mathcal{A}_{n,k}$ of $G_{n,k}$ which is of almost full measure $\nu_{n,k}(\mathcal{A}_{n,k})$ as $n\to\infty$.

Interchanging the roles of K and K^o, we may equivalently read Theorem A in the following geometric form:

$$\mathcal{P}_F(K)\supseteq \frac{c\sqrt{\theta}}{M_K}\,D_n\cap F, \tag{1.3}$$

where $\mathcal{P}_F$ denotes the orthogonal projection onto F. In this paper we will follow the tradition and continue calling an inclusion of the type (1.3) a "low M^*-estimate" (for K^o).

Among other applications of (1.3), let us mention the quotient of subspace theorem and the reverse Santaló inequality [M1], [BM].

Let $\{e_1,\dots,e_n\}$ be an arbitrary but fixed orthonormal basis of $\mathbf{R}^n$ with respect to $\langle .,.\rangle$. For a subset $\sigma\subseteq\{1,\dots,n\}$ we naturally define the coordinate subspace $\mathbf{R}^\sigma=\{x\in\mathbf{R}^n:\langle x,e_j\rangle=0 \text{ if } j\notin\sigma\}$. We write D_σ for $D_n\cap\mathbf{R}^\sigma$ and Q_σ for the unit cube $Q_n\cap\mathbf{R}^\sigma=[-1,1]^\sigma$ in $\mathbf{R}^\sigma$.

Our purpose is to discuss "low M^*-estimates" in the form (1.3) when one is restricted to choose F among the coordinate subspaces of $\mathbf{R}^n$ of a certain dimension m proportional to n.

In Section 2 we study the case of an ellipsoid E in $\mathbf{R}^n$. It turns out that for *any* orthonormal basis of $\mathbf{R}^n$ one has results analogous to (1.3) with almost the same $\sqrt{\theta}$-dependence on the parameter θ:

Theorem B (*Coordinate low M^*-estimate for ellipsoids*). *Let E be an ellipsoid in $\mathbf{R}^n$ and $\theta \in (0,1)$. Then, there exists $\sigma \subseteq \{1, \dots, n\}$, $|\sigma| \geq (1-\theta)n$, with*

$$\mathcal{P}_\sigma(E) \supseteq \frac{c\sqrt{\theta}}{\log^{1/2}(\frac{2}{\theta}) M_E} \, D_\sigma,$$

where $\mathcal{P}_\sigma$ denotes the orthogonal projection onto $\mathbf{R}^\sigma$, and $c > 0$ is an absolute constant.

It is perhaps surprising that this type of geometric result about ellipsoids is new and non-trivial. Note that our investigation of these questions was started from a simpler fact of the same nature about a special class of ellipsoids, which was discovered in [Gi].

It can be checked that Theorem B is optimal apart from the logarithmic term (see Remark 2.5). A result of the same type can be proved for an ellipsoid E of smaller but sufficiently large dimension living in an arbitrary subspace F of $\mathbf{R}^n$ (Theorem 2.3). We also consider the corresponding problem for sections (instead of projections) of E with coordinate subspaces (Theorem 2.4).

Simple examples show that one cannot achieve the same strong estimate in full generality: for an arbitrary symmetric convex body K and an arbitrary orthonormal basis in $\mathbf{R}^n$. Consider e.g the case of the unit cube Q_n and the standard basis of $\mathbf{R}^n$: observe that $M_{Q_n} \simeq \sqrt{\log n / n}$, while the radius of the largest Euclidean ball contained in any coordinate projection of Q_n is 1. In Section 3 we give a general low M^*-estimate in terms of the cotype-2 constant C_K of X_K:

Theorem C (*M^*-estimate in terms of C_K*). *For an arbitrary symmetric convex body K in $\mathbf{R}^n$ and for any $\theta \in (0,1)$, one can find $\sigma \subseteq \{1, \dots, n\}$, $|\sigma| \geq (1-\theta)n$, satisfying*

$$\mathcal{P}_\sigma(K) \supseteq \frac{c_1 \theta}{\log^2(\frac{2}{\theta}) h(C_K) M_K} \, D_\sigma,$$

where $h(y) = y \log 2y$, $y \geq 1$, and $c_1 > 0$ is an absolute constant.

Let us note that one can give a simpler argument, based on the isomorphic Sauer-Shelah lemma of S. J. Szarek and M. Talagrand and a factorization theorem of B. Maurey, which results in a weaker estimate of the same type (we sketch it in Remark 3.6). We also obtain results of the same nature in which M_K is replaced by various other "volumic" parameters of K or K^o (see Remark 3.7).

In Section 4 we give a general low M^*-estimate in terms of the volume ratio $vr(K)$ of K:

Theorem D (M^*-*estimate in terms of* $vr(K)$). *Let* K *be a symmetric convex body in* $\mathbf{R}^n$. *For every* $\theta \in (0,1)$, *there exists* $\sigma \subseteq \{1,\dots,n\}$, $|\sigma| \geq (1-\theta)n$, *such that*

$$\mathcal{P}_\sigma(K) \supseteq \frac{1}{[c_2 vr(K)]^{\frac{c_3\log(\frac{2}{\theta})}{\theta}} M_K} D_\sigma,$$

where $c_2, c_3 > 0$ *are absolute constants.*

In Sections 5 and 6 we give some further applications of the low M^*-estimate for ellipsoids. We demonstrate an exact dependence between coordinate sections of an ellipsoid and its polar in the spirit of [M5]. We also apply Theorems 2.2 and 2.4 to questions related to the number of integer or "almost integer" points inside an ellipsoid.

Recall that the cotype-2 constant C_K of X_K is the smallest constant $\lambda > 0$ for which

$$\left(\mathrm{Ave}_{\varepsilon_j=\pm 1}\|\sum_{j=1}^m \varepsilon_j x_j\|_K^2\right)^{1/2} \geq \frac{1}{\lambda}\left(\sum_{j=1}^m \|x_j\|_K^2\right)^{1/2}$$

holds for all choices of $m \in \mathbf{N}$ and $\{x_j\}_{j\leq m}$ in X_K. We refer to [MS] and [TJ] for basic facts about type, cotype and p-summing operators which are used below. The letter c will always denote an absolute positive constant, not necessarily the same in all its occurrences. By $|.|$ we denote the cardinality of a finite set, volume of appropriate dimension, and the Euclidean norm (this will cause no confusion).

2. Ellipsoidal case

In this Section we consider the case of an arbitrary ellipsoid E in $\mathbf{R}^n$. There exists a linear isomorphism $T : \mathbf{R}^n \to \mathbf{R}^n$ such that $T(E) = D_n$. It will be convenient for us to write E in the form

$$E = \{x = \sum_{j=1}^n x_j e_j \in \mathbf{R}^n : |\sum_{j=1}^n x_j u_j| \leq 1\}, \tag{2.1}$$

where $u_j = T(e_j)$, $j = 1,\dots,n$. Writing E in this way, we can easily express M_E in terms of the u_j's as follows:

$$M_E = \left(\int_{S^{n-1}} \|x\|_{T^{-1}(D_n)}^2\,\sigma(dx)\right)^{1/2} = \left(\int_{S^{n-1}} |\sum_{j=1}^n x_j u_j|^2\,\sigma(dx)\right)^{1/2} \tag{2.2}$$

$$= \left(\frac{1}{n}\sum_{j=1}^n |u_j|^2\right)^{1/2}.$$

Under the extra assumption that $|u_j| \leq 1,\ j = 1, \dots, n$, an estimate for coordinate projections of E was given in [Gi] in connection with the problems of the Banach-Mazur distance to the cube and the proportional Dvoretzky-Rogers factorization. Its proof combines the structure of the ellipsoid with the well-known Sauer-Shelah lemma and factorization arguments analogous to the ones in [BT, Theorem 1.2]:

Lemma 2.1. *Let $E_\tau = \{x = \sum_{j\in\tau} x_j e_j \in \mathbf{R}^\tau : |\sum_{j\in\tau} x_j u_j| \leq 1\}$, where $u_j \in \mathbf{R}^n$, $j \in \tau$, with $|u_j| \leq 1$. Then, for every $\zeta \in (0,1)$ there exists $\sigma \subseteq \tau,\ |\sigma| \geq (1-\zeta)|\tau|$, such that*

$$\mathcal{P}_\sigma(E_\tau) \supseteq c\sqrt{\zeta}\, D_\sigma,$$

where $c > 0$ is an absolute constant. $\square$

One more step is needed in order to obtain a low M^*-estimate for coordinate subspaces in the ellipsoidal case:

Theorem 2.2. *Let E be an ellipsoid in $\mathbf{R}^n$. For every $\theta \in (0,1)$ there exists a subset σ of $\{1, \dots, n\}$ with $|\sigma| \geq (1-\theta)n$, such that*

$$\mathcal{P}_\sigma(E) \supseteq \frac{c\sqrt{\theta}}{\log^{1/2}(\frac{2}{\theta}) M_E}\, D_\sigma,$$

where $c > 0$ is an absolute constant.

Proof: We write E in the form (2.1) and assume as we may that $M_E = 1$. If $\rho = \{j \leq n : |u_j| \geq \sqrt{2/\theta}\}$, then by (2.2) we have $2|\rho|/\theta \leq \sum_{j\leq n} |u_j|^2 = n$, hence $|\rho| \leq \theta n/2$. Consider the sets of indices:

$$\tau_0 = \{j \leq n : |u_j| \leq 1\},$$
$$\tau_k = \{j \leq n : e^{k-1} < |u_j| \leq e^k\}, \quad k \geq 1.$$

If $k_0 = [\log(\sqrt{2/\theta})] + 1$, we have $|\bigcup_{0\leq k\leq k_0} \tau_k| \geq n - |\rho| \geq (1 - \frac{\theta}{2})n$.

We define $\zeta_k = \frac{\theta n}{2} \frac{e^k/\sqrt{|\tau_k|}}{\sum_k e^k \sqrt{|\tau_k|}}$ for all $k \leq k_0$ with $\tau_k \neq \emptyset$, and consider the set $I = \{k \leq k_0 : \tau_k \neq \emptyset \text{ and } \zeta_k < 1\}$. For each $k \in I$ we can apply Lemma 2.1 for the ellipsoid $E_{\tau_k} = E \cap \mathbf{R}^{\tau_k}$ to find $\sigma_k \subseteq \tau_k$ with $|\sigma_k| \geq (1-\zeta_k)|\tau_k|$ such that

$$\mathcal{P}_{\sigma_k}(E_{\tau_k}) \supseteq c_1 \frac{\sqrt{\zeta_k}}{e^k}\, D_{\sigma_k}, \tag{2.3}$$

where c_1 is the constant from Lemma 2.1. Finally, we set $\sigma = \bigcup_{k\in I} \sigma_k$. Note that the above choice of ζ_k's implies that

$$|\bigcup_{k=0}^{k_0} \tau_k| - |\sigma| \ \leq \ \sum_{k=0}^{k_0} \zeta_k |\tau_k| \ = \ \frac{\theta n}{2},$$

and therefore, $|\sigma| \geq (1-\theta)n$.

Suppose that $w \in D_\sigma$. If we write $w = \sum_{k\in I} w_k$, where $w_k = \mathcal{P}_{\sigma_k}(w)$, then by (2.3),

$$w \in \frac{1}{c_1}\sum_{k\in I}|w_k|\frac{e^k}{\sqrt{\zeta_k}}\,\mathcal{P}_{\sigma_k}(E\cap \mathbf{R}^{\tau_k}) \subseteq \frac{1}{c_1}\left(\sum_{k\in I}|w_k|\frac{e^k}{\sqrt{\zeta_k}}\right)\mathcal{P}_\sigma(E), \tag{2.4}$$

and since $w \in D_\sigma$ was arbitrary, an application of the Cauchy-Schwartz inequality shows that

$$D_\sigma \subseteq \frac{1}{c_1}\left(\sum_{k\in I}\frac{e^{2k}}{\zeta_k}\right)^{1/2}\mathcal{P}_\sigma(E). \tag{2.5}$$

Inserting our ζ_k's in the sum above, we conclude that

$$D_\sigma \subseteq \frac{1}{c_2\sqrt{\theta n}}\left(\sum_{k=0}^{k_0} e^k\sqrt{|\tau_k|}\right)\mathcal{P}_\sigma(E). \tag{2.6}$$

It remains to give an upper bound for the sum $\sum_{k\leq k_0} e^k\sqrt{|\tau_k|}$: to this end, note that for $k=1,\dots,k_0$, we have $|\tau_k|e^{2k-2} \leq \sum_{j\in\tau_k}|u_j|^2 \leq n$ and thus $e^k\sqrt{|\tau_k|}\leq e\sqrt{n}$ for $k=1,\dots,k_0$ which allows a first upper bound of the order of $k_0\sqrt{n}$.

We partition the set of indices $\{0,1,\dots,k_0\}$ setting

$\varphi_0 = \{k\leq k_0 : |\tau_k| \leq \frac{1}{k_0}\frac{n}{e^{2k-2}}\}$,
$\varphi_s = \{k\leq k_0 : \frac{e^{s-1}}{k_0}\frac{n}{e^{2k-2}} < |\tau_k| \leq \frac{e^s}{k_0}\frac{n}{e^{2k-2}}\}, \quad s\geq 1.$

If $s_0 = [\log k_0]+2$, we have $\bigcup_{0\leq s\leq s_0}\varphi_s = \{0,1,\dots,k_0\}$, and for every $s=1,\dots,s_0$ we easily check that

$$|\varphi_s|\frac{e^{s-1}}{k_0}\frac{n}{e^{2k-2}}e^{2k-2} \leq \sum_{k\in\varphi_s}\sum_{j\in\tau_k}|u_j|^2 \leq n,$$

which means that

$$|\varphi_s| \leq \frac{k_0}{e^{s-1}}, \tag{2.7}$$

for all $s\leq s_0$. By the definition of φ_s and by (2.7), we can now estimate the sum in (2.6) as follows:

$$\sum_{k=0}^{k_0} e^k\sqrt{|\tau_k|} = \sum_{s=0}^{s_0}\sum_{k\in\varphi_s} e^k\sqrt{|\tau_k|} \leq \sum_{s=0}^{s_0}|\varphi_s|\frac{e^k e^{s/2}\sqrt{n}}{\sqrt{k_0}e^{k-1}} \tag{2.8}$$

$$\leq \frac{e\sqrt{n}}{\sqrt{k_0}}\sum_{s=0}^{s_0}\frac{k_0}{e^{s-1}}e^{s/2} \leq e^2\Big(\sum_{s=0}^{\infty}e^{-s/2}\Big)\sqrt{n}\sqrt{k_0} \leq c_3\sqrt{k_0}\sqrt{n}.$$

Therefore, (2.6) becomes

$$D_\sigma \subseteq \frac{1}{c_4\sqrt{\theta}}\sqrt{k_0}\mathcal{P}_\sigma(E), \tag{2.9}$$

which completes the proof, since $k_0 \simeq \log(2/\theta)$ and we had assumed that $M_E = 1$. □

We proceed to prove an extension of Theorem 2.2 concerning the case where E is an ellipsoid of dimension $m < n$ living in an arbitrary m-dimensional subspace F of $\mathbf{R}^n$. If m is proportional to n, with m/n sufficiently close to 1, then we still have coordinate projections of E of large dimension containing large Euclidean balls. This result will be useful for our treatment of the general case in Sections 3 and 4:

Theorem 2.3. *Let $\varepsilon \in (0,1)$ and F be a subspace of $\mathbf{R}^n$ with $\dim F = m \geq (1-\varepsilon)n$. Then, for every non-degenerate ellipsoid E in F and for every $\zeta \in [c_1\varepsilon \log(\frac{2}{\varepsilon}), 1)$ there exists $\sigma \subseteq \{1, \dots, n\}$ with $|\sigma| \geq (1-\zeta)n$, such that*

$$\mathcal{P}_\sigma(E) \supseteq \frac{c\sqrt{\zeta}}{2\sqrt{2}\log^{1/2}(\frac{2}{\zeta})M_E}\, D_\sigma,$$

where c is the constant from Theorem 2.2 and $c_1 = \max\{\frac{8}{c^2}, \frac{1}{\log 2}\}$.

Proof: Suppose that an ellipsoid E is given in F. We can find an orthonormal basis $\{w_1, \dots, w_m\}$ of F and $\lambda_1, \dots, \lambda_m > 0$ such that

$$E = \{x \in F : \sum_{j=1}^{m} \frac{\langle x, w_j\rangle^2}{\lambda_j^2} \leq 1\}.$$

We extend to an orthonormal basis $\{w_j\}_{j\leq n}$ of $\mathbf{R}^n$ and consider the ellipsoid

$$E' = \{x \in \mathbf{R}^n : \sum_{j=1}^{m} \frac{\langle x, w_j\rangle^2}{\lambda_j^2} + \sum_{j=m+1}^{n} \frac{\langle x, w_j\rangle^2}{b^2} \leq 1\},$$

where $b = \sqrt{\varepsilon}/M_E$. It is easy to check that

$$M_{E'}^2 = \frac{1}{n}\left[\sum_{j=1}^{m}\frac{1}{\lambda_j^2} + \frac{n-m}{b^2}\right] = \frac{mM_E^2 + (n-m)M_E^2/\varepsilon}{n} \leq 2M_E^2. \tag{2.10}$$

Let $\zeta \in [c_1\varepsilon\log(\frac{2}{\varepsilon}), 1)$. Applying Theorem 2.2 for E' and taking into account (2.10), we find $\sigma \subseteq \{1, \dots, n\}$ with $|\sigma| \geq (1-\zeta)n$ for which

$$\mathcal{P}_\sigma(E') \supseteq \frac{c\sqrt{\zeta}}{\sqrt{2}\log^{1/2}(2/\zeta)M_E}\, D_\sigma. \tag{2.11}$$

Since $\zeta \geq c_1\varepsilon \log(\frac{2}{\varepsilon})$ and the function $\zeta/\log(\frac{2}{\zeta})$ is increasing on (0,1), one can easily check that

$$\frac{c\sqrt{\zeta}}{\sqrt{2}\log^{1/2}(\frac{2}{\zeta})} \geq 2\sqrt{\varepsilon}. \tag{2.12}$$

On the other hand, we clearly have $E' \subseteq E + bD_n$ and hence $\mathcal{P}_\sigma(E') \subseteq \mathcal{P}_\sigma(E) + bD_\sigma$. Combining this with (2.11) and (2.12) we conclude that

$$\frac{c\sqrt{\zeta}}{\sqrt{2}\log^{1/2}(\frac{2}{\zeta})M_E}\, D_\sigma \subseteq \mathcal{P}_\sigma(E) + \frac{1}{2}\frac{c\sqrt{\zeta}}{\sqrt{2}\log^{1/2}(\frac{2}{\zeta})M_E}\, D_\sigma. \tag{2.13}$$

Claim: If A and B are convex symmetric bodies in $\mathbf{R}^\sigma$ and $A \subseteq B + \frac{1}{2}A$, then $A \subseteq 2B$.

[One easily checks that $A \subseteq (1 + \frac{1}{2} + \ldots + \frac{1}{2^k})B + \frac{1}{2^k}A$ and the claim follows by letting $k \to \infty$.]

Our claim and (2.13) imply that

$$\mathcal{P}_\sigma(E) \supseteq \frac{c}{2\sqrt{2}}\frac{\sqrt{\zeta}}{\log^{1/2}(\frac{2}{\zeta})M_E}\, D_\sigma,$$

and the proof of the theorem is complete. $\square$

Our next result concerns coordinate sections of ellipsoids: again, we are interested in finding large balls contained in them. Using a result of [AM] which was recently improved in [T] (in our case each of them works equally well), we can give an essentially optimal answer to this question when the dimension of the coordinate sections is small (of order roughly not exceeding $\sqrt{n}$):

Theorem 2.4. *Let E be an ellipsoid in $\mathbf{R}^n$. For every $m \leq c\sqrt{n}$ we can find a subset σ of $\{1,\ldots,n\}$ of cardinality $|\sigma| = m$, such that*

$$E \cap \mathbf{R}^\sigma \supseteq \frac{c'}{\sqrt{m}M_E}\, D_\sigma.$$

In the statement above, c and c' are absolute positive constants.

Proof: We write E in the form (2.1). As a consequence of (2.2), observe that for every $s \leq n$ the following identity holds:

$$\mathrm{Ave}_{|\tau|=s}\, M^2_{E\cap\mathbf{R}^\tau} = \left[\binom{n-1}{s-1}\Big/\binom{n}{s}\right]\frac{1}{s}\sum_{j=1}^{n}|u_j|^2 = M_E^2, \tag{2.14}$$

where the average is over all $\tau \subseteq \{1,\ldots,n\}$ with $|\tau| = s$. This means in particular that for every $s \leq n$ we can find τ with $|\tau| = s$ for which $M_{E\cap\mathbf{R}^\tau} \leq M_E$.

Assume that $m \leq c\sqrt{n}$ is given, where $c > 0$ is an absolute constant to be chosen. We choose $s = [\frac{m^2}{c^2}]$ and find τ with $|\tau| = s$ and $M_{E\cap\mathbf{R}^\tau} \leq M_E$. Observe that

$$\mathrm{Ave}_{\varepsilon_j=\pm 1} \|\sum_{j\in\varphi} \varepsilon_j e_j\|_E \leq \sqrt{|\tau|} M_{E\cap\mathbf{R}^\tau} \leq \sqrt{|\tau|} M_E.$$

Hence, if c is small enough, the results of [AM] or [T] allow us to find $\varphi \subseteq \tau$ with $|\varphi| = 2m$ such that

$$\|\sum_{j\in\varphi} \varepsilon_j e_j\|_E \leq c_1\sqrt{|\tau|} M_E, \tag{2.15}$$

for all $(\varepsilon_j)_{j\in\varphi} \in \{-1,1\}^\varphi$, where c_1 is a positive absolute constant. In other words, the coordinate section of E by $\mathbf{R}^\varphi$ satisfies

$$E \cap \mathbf{R}^\varphi \supseteq \frac{1}{c_1\sqrt{|\tau|}M_E}\, Q_\varphi. \tag{2.16}$$

This means that the identity operator $id : \ell_\infty^\varphi \to X_E \cap \mathbf{R}^\varphi$ has norm $\|id\| \leq c_1\sqrt{|\tau|}M_E$, and this implies that $\pi_2(id) \leq c_1 K_G \sqrt{|\tau|} M_E$ where K_G is Grothendieck's constant. Applying Pietch's factorization theorem we can find $(\lambda_i)_{i\in\varphi}$, $\sum_{i\in\varphi}\lambda_i^2 = 1$:

$$\|\sum_{i\in\varphi} t_i e_i\|_E \leq c_1 K_G\sqrt{|\tau|} M_E \left(\sum_{i\in\varphi}\lambda_i^2 t_i^2\right)^{1/2} \tag{2.17}$$

for every choice of reals $(t_i)_{i\in\varphi}$. By Markov's inequality, we find $\sigma_1 \subseteq \varphi$, $|\sigma_1| \geq |\varphi|/2 \geq m$, such that $|\lambda_i| \leq \frac{\sqrt{2}}{\sqrt{|\varphi|}}$ for all $i \in \sigma_1$. Then, for any $(t_i)_{i\in\sigma_1}$ we have

$$\|\sum_{i\in\sigma_1} t_i e_i\|_E \leq c_1 K_G \sqrt{|\tau|} M_E \frac{\sqrt{2}}{\sqrt{|\varphi|}} \left(\sum_{i\in\sigma_1} t_i^2\right)^{1/2}. \tag{2.18}$$

The choice of $|\tau|$ and $|\varphi|$ shows that

$$E \cap \mathbf{R}^{\sigma_1} \supseteq \frac{c'}{\sqrt{m}M_E}\, D_{\sigma_1}, \tag{2.19}$$

for some absolute constant $c' > 0$, and we conclude the proof by choosing any $\sigma \subseteq \sigma_1$ of cardinality $|\sigma| = m$. $\square$

Remark 2.5. An iteration of the argument above shows that one can extend the range of m's for which Theorem 2.4 holds to e.g the set $\{1, \dots, [\sqrt{n}]\}$, with some loss in the constant c'. The dependence on m is sharp as it can be seen by the following example: consider the ellipsoid $E = \{(t_j)_{j\leq n} \in \mathbf{R}^n : |\sum t_j u_j|_{n+1} \leq 1\}$, where $u_j = e_j + e_{n+1}$, $j = 1, \dots, n$, and $\{e_j\}_{j\leq n+1}$ is the standard orthonormal basis in $\mathbf{R}^{n+1}$. Given any $\sigma \subseteq \{1, \dots, n\}$ with $|\sigma| = m$, we have that $(\frac{t}{\sqrt{m}}, \dots, \frac{t}{\sqrt{m}}) \in E \cap \mathbf{R}^\sigma$

precisely when $(1+m)t^2 \leq 1$. In particular, we must have $|t| \leq \frac{1}{\sqrt{m}}$. This means that the largest ball contained in $E \cap \mathbf{R}^\sigma$ cannot have radius larger than $\frac{1}{\sqrt{m}}$. On the other hand, observe that $M_E = \sqrt{2}$.

The same example shows that the estimate in Theorem 2.2 is best possible apart from the $\log^{1/2}(\frac{2}{\theta})$ term. By Lemma 2.1, this logarithmic term can be removed if all the u_j's are of about the same Euclidean norm.

3. General case: estimate in terms of the cotype-2 constant

In this Section we study the general case, that is K is an arbitrary symmetric convex body in $\mathbf{R}^n$, and $\{e_j\}_{j\leq n}$ is a fixed orthonormal basis. We shall make use of the maximal volume ellipsoid E of K and of the better information we have for coordinate projections of ellipsoids. For this purpose we will also need an estimate for the parameters $A_m(K) = \sup\{(|K\cap F|/|E\cap F|)^{1/m} : \dim F = m\}$, $m = 1, \dots, n$.

It was proved in [BM] that the volume ratio $vr(K) = (|K|/|E|)^{1/n}$ of K is bounded by $f(C_K) = cC_K[\log C_K]^4$, with the power of $\log C_K$ improved to 1 in [MiP]. A third proof of the same fact is given in [M4], where it is also shown that $vr(K) \leq ch(C_K)$, where $h(y) = y \log 2y$, $y \geq 1$, and $c > 0$ is an absolute constant. Our first lemma is a modification of the argument presented in [M4] which provides an estimate for $A_m(K), m \leq n$, in terms of C_K:

Lemma 3.1. *Let K be a symmetric convex body in $\mathbf{R}^n$, and E be the maximal volume ellipsoid of K. If F is an m-dimensional subspace of $\mathbf{R}^n$, then*

$$\left(\frac{|K\cap F|}{|E\cap F|}\right)^{1/m} \leq ch(\sqrt{n/m}C_K),$$

where $h(y) = y\log 2y$, $y \geq 1$.

Proof: We may clearly assume that $E = D_n$. The proof will be based on an iteration schema, analogous to the one in [M4].

We set $K_0 = K$, $\alpha_0 = n$, $\beta_0 = n$, and for $j = 1, 2, \dots$ we define:

(i) $\alpha_j = \log \alpha_{j-1} = \log^{(j)} n$, the j-iterated logarithm of n,
(ii) $\beta_j = \alpha_j M_{(K_{j-1}\cap F)^o}$,
(iii) $K_j = K \cap \beta_j D_n$.

Note that for every j the maximal volume ellipsoid of K_j is D_n. Also, $C_{K_j} \leq 2C_K$ and $d(X_{K_j}, \ell_2^n) \leq \beta_j$. By Sudakov's inequality [Su], [P1] the covering number of $K_{j-1}\cap F$ by $\beta_j D_n \cap F$ can be estimated as follows:

$$N(K_{j-1}\cap F, \beta_j D_n \cap F) = N \leq \exp(c_1 m M^2_{(K_{j-1}\cap F)^o}/\beta_j^2) = \exp(c_1 m/\alpha_j^2),$$

and since, by Brunn's theorem, $|K_{j-1}\cap(x+\beta_j D_n\cap F)|\leq |K_{j-1}\cap \beta_j D_n\cap F|,\ x\in F$, we have $|K_{j-1}\cap F|\leq N|K_j\cap F|$ and hence

$$|K_{j-1}\cap F|^{1/m}\leq \exp(\frac{c_1}{\alpha_j^2})\ |K_j\cap F|^{1/m} \tag{3.1}$$

By well-known results of [DMT], [MP], and [P2] we have the string of inequalities

$$M_{(K_j\cap F)^o}\leq c_2\sqrt{\frac{n}{m}}M_{K_j^o}\leq c_3\sqrt{\frac{n}{m}}T_2(X_{K_j^o})\leq c_4\sqrt{\frac{n}{m}}C_{K_j}\log(2d(X_{K_j},\ell_2^n))$$

and therefore

$$M_{(K_j\cap F)^o}\leq 2c_4\sqrt{\frac{n}{m}}C_K\log(2\beta_j).$$

It follows that the sequence $\{\beta_j\}_{j\geq 0}$ satisfies the relation

$$\beta_{j+1}\leq 2c_4\sqrt{\frac{n}{m}}C_K\alpha_j\log(2\beta_j). \tag{3.2}$$

We stop this procedure at the smallest t for which $\alpha_t<6c_4$. Induction and (3.2) show that

$$\beta_t\leq 36c_4^2\sqrt{\frac{n}{m}}C_K\left[\log(\sqrt{\frac{n}{m}}C_K)+6c_4\right]\leq c'h(\sqrt{n/m}C_K). \tag{3.3}$$

By (3.1) we see that

$$|K\cap F|^{1/m}\leq |K_t\cap F|^{1/m}\exp(c_1[\frac{1}{\alpha_1^2}+\ldots+\frac{1}{\alpha_t^2}])\leq c''|K_t\cap F|^{1/m}, \tag{3.4}$$

since $\sum\frac{1}{\alpha_j^2}$ is easily seen to be uniformly bounded. Taking into account (3.3), (3.4) and the Blaschke-Santaló inequality we conclude that

$$\begin{aligned}\left(\frac{|K\cap F|}{|D_n\cap F|}\right)^{1/m} &\leq c''\left(\frac{|D_n\cap F|}{|(K_t\cap F)^o|}\right)^{1/m}\leq c''M_{(K_t\cap F)^o}\\ &\leq 2c_4c''\sqrt{\frac{n}{m}}C_K\log(2c'h(\sqrt{n/m}C_K))\leq ch(\sqrt{n/m}C_K). \qquad \square\end{aligned} \tag{3.5}$$

Simple examples (see Remark 3.3) show that one cannot compare M_K and M_E even if C_K is small: the only estimate one can give is that $M_E\leq\sqrt{n}M_K$, which is a direct consequence of the fact that $K\subseteq\sqrt{n}E$ by John's theorem. However, there exist subspaces F of $\mathbf{R}^n$ of proportional dimension on which we can compare M_K with $M_{E\cap F}$ reasonably well:

Lemma 3.2. *Let E be the maximal volume ellipsoid of K. For every $\varepsilon\in(0,1)$ there exists a subspace F of $\mathbf{R}^n$ with $\dim F=m\geq(1-\varepsilon)n$ such that*

$$M_{E\cap F}\leq\frac{ch(C_K)\log(\frac{2}{\varepsilon})}{\sqrt{\varepsilon}}\ M_K,$$

where $h(y) = y\log 2y$, $y \geq 1$, and $c > 0$ is an absolute constant.

Proof: Let $\{w_1, \dots, w_n\}$ be an orthonormal basis of $\mathbf{R}^n$ and $\lambda_1 \geq \dots \geq \lambda_n > 0$ such that

$$E = \{x \in \mathbf{R}^n : \sum_{j=1}^{n} \frac{\langle x, w_j\rangle^2}{\lambda_j^2} \leq 1\}.$$

For $k = 1, \dots, n$, set $W_k = \mathrm{span}\{w_k, \dots, w_n\}$. By Lemma 3.1 we have

$$\left(\frac{|K \cap W_k|}{|E \cap W_k|}\right)^{\frac{1}{n-k+1}} \leq c_1 h\Big(\sqrt{\frac{n}{n-k+1}}C_K\Big). \tag{3.6}$$

Note that $E \cap W_k \subseteq \lambda_k(D_n \cap W_k)$, and hence

$$\begin{aligned}\left(\frac{|K \cap W_k|}{|E \cap W_k|}\right)^{\frac{1}{n-k+1}} &\geq \frac{1}{\lambda_k}\left(\frac{|K \cap W_k|}{|D_n \cap W_k|}\right)^{\frac{1}{n-k+1}} \\ &\geq \frac{1}{\lambda_k M_{K\cap W_k}} \geq \frac{1}{c_2\lambda_k\sqrt{\frac{n}{n-k+1}}M_K}.\end{aligned} \tag{3.7}$$

Combining (3.6), (3.7) we obtain

$$\frac{1}{\lambda_k} \leq c_1c_2\sqrt{\frac{n}{n-k+1}}\,h\Big(\sqrt{\frac{n}{n-k+1}}C_K\Big)M_K \quad , \quad k = 1, \dots, n. \tag{3.8}$$

Given $\varepsilon \in (0,1)$, let $m = [(1-\varepsilon)n]$ and set $F_m = \mathrm{span}\{w_1, \dots, w_m\}$. By (3.8) we can estimate $M_{E\cap F_m}$ as follows:

$$\begin{aligned} M_{E\cap F_m} &= \left(\frac{1}{m}\sum_{k=1}^{m}\frac{1}{\lambda_k^2}\right)^{1/2} \\ &\leq c_1c_2C_KM_K\left[\sum_{k=1}^{m}\frac{n^2}{m(n-k+1)^2}\log^2\Big(2\sqrt{\frac{n}{n-k+1}}C_K\Big)\right]^{1/2} \\ &\leq c_1c_2C_K\log\Big(2\sqrt{\frac{n}{n-m}}C_K\Big)\sqrt{\frac{n}{n-m}}M_K \leq \frac{ch(C_K)\log(\frac{2}{\varepsilon})}{\sqrt{\varepsilon}}M_K. \qquad \square \end{aligned} \tag{3.9}$$

Remark 3.3. The estimate (3.9) is essentially sharp, even if C_K is small: to see this, consider the class of bodies $K = K(a,b;s) = \{x \in \mathbf{R}^n : \sum_{j\leq s}\frac{|x_j|}{a} + \sum_{j>s}\frac{|x_j|}{b} \leq 1\}$, where a, b are positive parameters and $s \in \{0, 1, \dots, n-1\}$. It is clear that the ellipsoid of maximal volume in K is $E = E(a,b;s) = \{x \in \mathbf{R}^n : \sum_{j\leq s}\frac{|x_j|^2}{a^2} + \sum_{j>s}\frac{|x_j|^2}{b^2} \leq 1\}$. It is also clear that both the cotype-2 constant and the volume ratio of K are uniformly bounded (independently of n, s, a and b).

Given $\varepsilon \in (0,1)$, choose $b = a\sqrt{\varepsilon}$, $s = m = (1-\varepsilon)n$. Then, it is easy to check that $M_K \simeq \sqrt{n}\sqrt{\varepsilon}/a$, while $M_{E\cap F_m} \simeq \sqrt{n}/a$.

Also, we can have the ratio M_E/M_K as close to $\sqrt{n}$ as we like: choose, for example, $s = n-1$ and $b = \frac{a}{n-1}$. Then, $M_K \simeq 1/\sqrt{n}b$ while $M_E \simeq 1/b$.

Combining Theorem 2.3 and Lemma 3.2 we prove our M^*-estimate in terms of the cotype-2 constant of X_K:

Theorem 3.4. *Let K be a symmetric convex body in $\mathbf{R}^n$, and $X_K = (\mathbf{R}^n, \|.\|_K)$. For every $\theta \in (0,1)$ there exists $\sigma \subseteq \{1, \dots, n\}$, $|\sigma| \geq (1-\theta)n$, such that*

$$\mathcal{P}_\sigma(K) \supseteq \frac{c\theta}{\log^2(\frac{2}{\theta})h(C_K)M_K}\, D_\sigma,$$

where $h(y) = y\log 2y$, $y \geq 1$, and $c > 0$ is an absolute constant.

Proof: Let E be the maximal volume ellipsoid of K, and set $\varepsilon = \varepsilon(\theta) = \theta/c_2\log(\frac{2}{\theta})$, where $c_2 > 0$ is a constant to be chosen. By Lemma 3.2 we can find a subspace F of $\mathbf{R}^n$ with $\dim F \geq (1-\varepsilon)n$ such that

$$M_{E\cap F} \leq \frac{c_3 h(C_K)\log(\frac{2}{\varepsilon})}{\sqrt{\varepsilon}}\, M_K. \tag{3.10}$$

Observe that if c_2 is large enough, then $\theta \geq c_1\varepsilon\log(\frac{2}{\varepsilon})$ where c_1 is the constant in Theorem 2.3. Thus, we can apply Theorem 2.3 for $E\cap F$ to find $\sigma \subseteq \{1, \dots, n\}$ with $|\sigma| \geq (1-\theta)n$ for which

$$\mathcal{P}_\sigma(E\cap F) \supseteq \frac{c\sqrt{\theta}}{2\sqrt{2}\log^{1/2}(\frac{2}{\theta})M_{E\cap F}}\, D_\sigma. \tag{3.11}$$

Combining (3.10) with (3.11) we finish the proof. $\square$

Remark 3.5. It should be noted that the estimate given by Theorem 3.4 is exact not only when C_K is small (like e.g in the ellipsoidal case), but in the whole range $[1, \sqrt{n}]$ of possible values of C_K i.e even if C_K is extremely large. This can be easily seen if one considers the case of $B_p^n, p > 2$, the unit ball of ℓ_p^n, and the standard coordinate system in $\mathbf{R}^n$. Fix for example $\theta = \frac{1}{2}$. Then, the radius of the largest Euclidean ball inscribed in any $[\frac{n}{2}]$-dimensional coordinate projection of B_p^n is 1, and the well-known estimates for $C_{B_p^n}$ and $M_{B_p^n}$ show that Theorem 3.4 is sharp apart from logarithmic terms. We do not know if the "almost linear" dependence on θ which our method provides is optimal. However, the ellipsoidal case shows that $\sqrt{\theta}$ dependence is the best one might hope for.

Remark 3.6. One can give a weaker estimate, analogous to the one obtained in Theorem 3.4, using the isomorphic Sauer-Shelah lemma of Szarek-Talagrand [ST] and a factorization result of Maurey [Ma] (see also [TJ]). Starting with the body K and the orthonormal basis $\{e_j\}_{j\leq n}$, we have the inequality

$$\mathrm{Ave}_{\varepsilon_j=\pm 1}\|\sum_{j=1}^{n}\varepsilon_j e_j\|_K \leq \sqrt{n}M_K,$$

and therefore, by Markov's inequality we can find $\mathcal{A}\subseteq\{-1,1\}^n$ of cardinality $|\mathcal{A}|\geq 2^{n-1}$ such that $\|\sum\varepsilon_j e_j\|_K\leq 2M_K\sqrt{n}$ whenever $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\in\mathcal{A}$. If we view $\mathcal{A}$ as a set of points in $\mathbf{R}^n$, this means that $\mathcal{A}\subseteq 2M_K\sqrt{n}K$. On the other hand, the isomorphic Sauer-Shelah lemma shows that for some absolute constant $c_1>0$ and for every $\theta\in(0,1)$ there exists $\sigma\subseteq\{1,\dots,n\}$, $|\sigma|\geq(1-\frac{\theta}{2})n$, with $\mathrm{co}(\mathcal{P}_\sigma(\mathcal{A})\supseteq c_1\frac{\theta}{2}Q_\sigma$, and hence

$$\mathcal{P}_\sigma(K)\supseteq\frac{c_1\theta}{4M_K\sqrt{n}}\,Q_\sigma.$$

It follows that if $Y=(\mathbf{R}^{\sigma_1},\|.\|_{K^o})$, then $id:\ell_\infty^{\sigma_1}\to Y^*$ has norm $\|id\|\leq\frac{4M_K\sqrt{n}}{c_1\theta}$, and Maurey's theorem shows that

$$\pi_2(id)\leq c_2\frac{M_K\sqrt{n}}{\theta}\,g(Y^*),$$

where $g(Y^*)=C_{Y^*}\sqrt{1+\log(C_{Y^*})}$. Then, we can apply Pietch's factorization theorem in the context of [BT, Theorem 1.2] to find $\sigma\subseteq\sigma_1$ with $|\sigma|\geq(1-\frac{\theta}{2})|\sigma_1|\geq(1-\theta)n$ for which

$$\left(\sum_{i\in\sigma}t_i^2\right)^{1/2}\leq c_3\frac{M_K g(Y^*)}{\theta^{3/2}}\|\sum_{i\in\sigma}t_ie_i\|_{K^o}$$

is true for all $(t_i)_{i\in\sigma}$. Taking polars in $\mathbf{R}^\sigma$ and using the fact that $C_{Y^*}\leq c_4C_K\|\mathrm{Rad}_{X_K}\|$, we conclude that

$$\mathcal{P}_\sigma(K)\supseteq\frac{c\theta^{3/2}}{f(K)M_K}\,D_\sigma,$$

where $c>0$ is an absolute constant, and $f(K)=C_K\|\mathrm{Rad}_{X_K}\|\sqrt{1+\log(C_K\|\mathrm{Rad}_{X_K}\|)}$.

Remark 3.7. One can modify the proof of Theorem 3.6 to give analogous estimates in which M_K is replaced by other "volumic" parameters of K or K^o.

Consider e.g the sequence of volume numbers of K^o

$$v_s(K^o)=\max\{(|\mathcal{P}_F(K^o)|/|D_n\cap F|)^{1/s}:\ \dim F=s\},\tag{3.12}$$

where $s=1,\dots,n$. As a consequence of the Aleksandrov-Fenchel inequalities, one can easily see that $\{v_s(K^o)\}_{s\leq n}$ is non increasing (see [P1]):

$$v_1(K^o)\geq v_2(K^o)\geq\dots\geq v_n(K^o)=v.rad(K^o).\tag{3.13}$$

Let K be a symmetric convex body in $\mathbf{R}^n$ and let E be the ellipsoid of maximal volume in K as in Lemma 3.4. Using the inverse Santaló inequality in (3.6), (3.7) we get

$$\begin{aligned} \frac{1}{\lambda_k} &\leq \left(\frac{|D_n \cap W_k|}{|K \cap W_k|}\right)^{\frac{1}{n-k+1}} \left(\frac{|K \cap W_k|}{|E \cap W_k|}\right)^{\frac{1}{n-k+1}} \\ &\leq c\left(\frac{|\mathcal{P}_{W_k}(K^o)|}{|D_n \cap W_k|}\right)^{\frac{1}{n-k+1}} c_1 h(\sqrt{\frac{n}{n-k+1}} C_K) \end{aligned} \tag{3.14}$$

for $k = 1, \ldots, n$. By the definition (3.11) of $v_{n-k+1}(K^o)$ this means that

$$\frac{1}{\lambda_k} \leq c_2 h(C_K) v_{n-k+1}(K^o) \sqrt{\frac{n}{n-k+1}} \log(2\sqrt{\frac{n}{n-k+1}}). \tag{3.15}$$

Inserting this estimate in (3.9) we obtain:

$$M^2_{E\cap F_m} = \frac{1}{m}\sum_{k=1}^{m} \frac{1}{\lambda_k^2} \leq \frac{c_2^2 h^2(C_K)\log^2(\frac{2n}{n-m+1})}{m} \sum_{k=1}^{m} \frac{n}{n-k+1} v^2_{n-k+1}(K^o). \tag{3.16}$$

The monotonicity of volume numbers shows that $v_{n-k+1}(K^o) \leq v_{n-m+1}(K^o),\ k = 1, \ldots, m$, and combining with the fact that

$$\sum_{k=1}^{m} \frac{n}{n-k+1} \leq n \log(\frac{n}{n-m})$$

we arrive at

$$M_{E\cap F_m} \leq \frac{cn}{m} h(C_K) v_{n-m+1}(K^o) \log^{3/2}(\frac{2n}{n-m}). \tag{3.17}$$

Set $m = [(1-\theta)n]$. Then, (3.17) can be rewritten as

$$M_{E\cap F_m} \leq c' h(C_K) v_{[\theta n]}(K^o) \log^{3/2}(\frac{2}{\theta}),$$

and, using Theorem 2.3 exactly as in the proof of Theorem 3.4, we can find $\sigma \subseteq \{1, \ldots, n\}$ with $|\sigma| \geq (1 - c_1\theta\log(\frac{2}{\theta}))n$ for which

$$\mathcal{P}_\sigma(K) \supseteq \frac{c\sqrt{\theta}}{\log^{3/2}(\frac{2}{\theta}) v_{[\theta n]}(K^o) h(C_K)}\, D_\sigma. \tag{3.18}$$

A similar argument shows that for some σ of the same cardinality we have

$$\mathcal{P}_\sigma(K) \supseteq \frac{c\sqrt{\theta}\, w_{[\theta n]}(K)}{\log^{3/2}(\frac{2}{\theta}) h(C_K)}\, D_\sigma, \tag{3.19}$$

where $w_s(K) = \min\{(|K \cap F|/|D_n \cap F|)^{1/s} :\ \dim F = s\},\ s = 1, \ldots, n.$

4. General case: estimate in terms of the volume ratio

In this Section we use the volume ratio $vr(K)$ of K instead of the cotype-2 constant of X_K as a parameter for our low M^*-estimate. Let E be the maximal volume ellipsoid of K. We start with a lemma which estimates the covering number $N(K,E)$ in terms of the volume ratio $vr(K) = (|K|/|E|)^{1/n}$. Our proof is based on Lemma 4.4 from [MS2], actually the argument given there leads to a stronger estimate, but we include a simple proof of what we need here for the sake of completeness. Recall that $N(K,L)$ is the smallest natural number N for which there exist $x_1, \dots, x_N \in \mathbf{R}^n$ with $K \subseteq \bigcup_{i\leq N}(x_i + L)$:

Lemma 4.1. *Let K and L be two symmetric convex bodies in $\mathbf{R}^n$ such that $L \subseteq K$. Then,*

$$N(K,L) \leq c^n \frac{|K|}{|L|},$$

where $c > 0$ is an absolute constant.

Proof: Consider a set N of points in K such that $\|x - x'\|_L \geq 1$ for every $x, x' \in N$, $x \neq x'$, which has the maximal possible cardinality. Observe that the sets $\frac{2}{3}x + \frac{L}{3}$, $x \in N$ have disjoint interiors and, since $L \subseteq K$, they are all contained in K. We easily deduce that

$$|N| \leq 3^n \frac{|K|}{|L|}. \tag{4.1}$$

Finally, it is clear that $K \subseteq \bigcup_{x\in N}(x + L)$, which completes the proof. □

Suppose that K is any symmetric convex body in $\mathbf{R}^n$ and E is the ellipsoid of maximal volume in K. The analogue of Lemma 3.2 in the "volume ratio" formulation is the following:

Lemma 4.2. *Let E be the maximal volume ellipsoid of K. For every $\varepsilon \in (0,1)$ there exists a subspace F of $\mathbf{R}^n$ with $\dim F = m \geq (1-\varepsilon)n$, such that*

$$M_{E\cap F} \leq [c\, vr(K)]^{1/\varepsilon} M_K,$$

where $c > 0$ is an absolute constant.

Proof: As in the proof of Lemma 3.2, let

$$E = \{x \in \mathbf{R}^n : \sum_{j=1}^{n} \frac{\langle x, w_j\rangle^2}{\lambda_j^2} \leq 1\},$$

where $\{w_1, \dots, w_n\}$ is an orthonormal basis of $\mathbf{R}^n$ and $\lambda_1 \geq \dots \geq \lambda_n > 0$. Fix $k \in \{1, \dots, n\}$ and consider the subspace $W_k = \mathrm{span}\{w_k, \dots, w_n\}$. According to

Lemma 4.1, we can find $x_1,\dots,x_N \in K$ such that $N = N(K,E) \leq [c_1 vr(K)]^n$ and $K \subseteq \bigcup (x_i + E)$. Project all the (x_i+E)'s onto W_k. Then,

$$K \cap W_k \subseteq \mathcal{P}_{W_k}(K) \subseteq \bigcup_{j\leq N} \mathcal{P}_{W_k}(x_j+E) = \bigcup_{j\leq N} (\mathcal{P}_{W_k}(x_i) + E\cap W_k), \tag{4.2}$$

and hence, $N(K\cap W_k, E\cap W_k) \leq N(K,E)$. Thus, we can estimate the ratio of the volumes of $K\cap W_k$ and $E \cap W_k$ using (4.2):

$$\left(\frac{|K\cap W_k|}{|E\cap W_k|}\right)^{\frac{1}{n-k+1}} \leq [N(K,E)]^{\frac{1}{n-k+1}} \leq [c_1 vr(K)]^{\frac{n}{n-k+1}}. \tag{4.3}$$

Combining with (3.7) we get

$$\frac{1}{\lambda_k} \leq c_2 \sqrt{\frac{n}{n-k+1}}\, [c_1 vr(K)]^{\frac{n}{n-k+1}} M_K, \qquad k=1,\dots,n. \tag{4.4}$$

We continue as in Lemma 3.2: Given any $\varepsilon \in (0,1)$, we consider the first m for which $m \geq (1-\varepsilon)n$ and set $F_m = \mathrm{span}\{w_1,\dots,w_m\}$. In view of (4.5) we can compare $M_{E\cap F_m}$ with M_K as follows:

$$\begin{aligned} M_{E\cap F_m} &= \left(\frac{1}{m}\sum_{k=1}^m \frac{1}{\lambda_k^2}\right)^{\frac{1}{2}} \qquad (4.5)\\ &\leq c_2 M_K \left(\sum_{k=1}^m \frac{n}{m(n-k+1)}[c_1 vr(K)]^{\frac{2n}{n-k+1}}\right)^{\frac{1}{2}} \\ &\leq M_K [c_3 vr(K)]^{\frac{n}{n-m+1}} \sqrt{\frac{n}{m}} \log^{1/2}(\frac{n}{n-m}), \end{aligned}$$

and the lemma follows with the observation that $\frac{\log(1/\varepsilon)}{1-\varepsilon} \to 1$ as $\varepsilon \to 1^-$. $\square$

Remark 4.3. By well-known results of S.J. Szarek and N. Tomczak-Jaegermann (see [Sz], [STJ]) which were extending previous work of Kashin, if E is the maximal volume ellipsoid of K, then for every $k=1,\dots,n-1$ there exist k-dimensional subspaces F of $\mathbf{R}^n$ for which $E\cap F \subseteq K \cap F \subseteq (c\, vr(K))^{\frac{n}{n-k}} E\cap F$, and this obviously implies that $M_{E\cap F} \leq [c\ vr(K)]^{\frac{n}{n-k}} M_{K\cap F}$. This leads to the same estimate as in Lemma 4.2 above, actually if $E = D_n$ this is true for all subspaces F in a subset $\mathcal{A}$ of $G_{n,k}$ with almost full measure $\nu_{n,k}(\mathcal{A}) > 1 - 2^{-n}$. The argument provided by Lemmata 4.1 and 4.2 gives a concrete example of a subspace on which the weaker "$M_{E\cap F}$ and $M_{K\cap F}$" comparison is true: it can be chosen as the k-dimensional subspace which is coordinate with respect to E and corresponds to the k largest semiaxes of E. If $E = D_n$, then this weak comparison is true for all $F \in G_{n,k}$.

Combining Lemma 4.2 with Theorem 2.3 we prove our volume-ratio result:

Theorem 4.4. *Let K be a symmetric convex body in $\mathbf{R}^n$. For every $\theta \in (0,1)$ there exists $\sigma \subseteq \{1,\dots,n\}$, $|\sigma| \geq (1-\theta)n$, such that*

$$\mathcal{P}_\sigma(K) \supseteq \frac{1}{[c_1 vr(K)]^{\frac{c_2 \log(\frac{2}{\theta})}{\theta}} M_K} D_\sigma,$$

where c_1, c_2 are absolute positive constants.

Proof: Let E be the maximal volume ellipsoid of K, and set $\varepsilon = \varepsilon(\theta) = \frac{\theta}{c_2 \log(\frac{2}{\theta})}$, where $c_2 > 0$ is a constant to be chosen. Using Lemma 4.2 we find a subspace F of $\mathbf{R}^n$ with $\dim F \geq (1-\varepsilon)n$ such that

$$M_{E\cap F} \leq [c_4 vr(K)]^{1/\varepsilon} M_K. \tag{4.6}$$

If c_2 is large enough, we easily check that $\theta \geq c_1 \varepsilon \log(\frac{2}{\varepsilon})$ where c_1 is the constant in Theorem 2.3. We can therefore apply Theorem 2.3 for $E \cap F$ to find $\sigma \subseteq \{1,\dots,n\}$ with $|\sigma| \geq (1-\theta)n$, such that

$$\mathcal{P}_\sigma(E\cap F) \supseteq \frac{c\sqrt{\theta}}{2\sqrt{2}\log^{1/2}(\frac{2}{\theta}) M_{E\cap F}} D_\sigma. \tag{4.7}$$

Combining (4.6) with (4.7) we conclude the proof. □

For classes of spaces with uniformly bounded volume ratio, Theorem 4.3 gives an optimal answer as long as, say, $\theta \geq \frac{1}{2}$. The estimate obtained "explodes" if $vr(K)$ is large or if θ is needed to be close to 0.

5. Linear duality relations for coordinate sections of ellipsoids

Let K be a symmetric convex body in $\mathbf{R}^n$. We introduce the integer valued functions $t, t_c : \mathbf{R}^+ \to \mathbf{N}$ defined by

$$t(r) = t(K,r) = \max\{k \leq n : \text{there exists a subspace } E$$
$$\text{with } \dim E = k, \text{such that} \frac{1}{r}|x| \leq \|x\| \text{ for every } x \in E\}$$

and

$$t_c(r) = t_c(K,r) = \max\{k \leq n : \text{there exists a coordinate subspace } E$$
$$\text{with } \dim E = k \text{ such that } \frac{1}{r}|x| \leq \|x\| \text{ for every } x \in E\}.$$

It is easy to see that if K is an ellipsoid in $\mathbf{R}^n$, then $t(K,r) + t(K^o, \frac{1}{r}) \geq n$. In [M5] it is proved that for every body K, for every $r > 0$, and for every $\tau \in (0,1)$, one has a similar duality relation:

$$t(K,r) + t(K^o, \frac{1}{\tau r}) \geq (1-\tau)n - C, \tag{5.1}$$

where $C > 0$ is a universal constant. The proof of this fact is based on the strong form (1.2) of the low M^*-estimate and on the "distance lemma": if $\frac{1}{a}|x| \leq \|x\| \leq b|x|$ for every $x \in \mathbf{R}^n$ and if $(M_K/b)^2 + (M_{K^o}/a)^2 = s > 1$, then $ab \leq \frac{1}{s-1}$.

In this Section we establish a coordinate version of (5.1) in the ellipsoidal case. Our estimate depends on how close the ellipsoid is to being in M-position:

Definition: For a symmetric convex body K in $\mathbf{R}^n$ we denote by λ_K its volume radius: $\lambda_K = (|K|/|D|)^{1/n}$. We also write N_K for $N(K, \lambda_K D)$ and say that K is in *M_δ-position* if $\delta \geq \frac{1}{n}\log N_K$.

Our first lemma provides some simple estimates which show that this position is "stable" under the operations of taking intersection or convex hull with a ball:

Lemma 5.1. *Let K be a symmetric convex body in $\mathbf{R}^n$, and let $r, r_1 > 0$ be given. Define $K_r = K \cap rD$ and $K^{r_1} = \mathrm{co}(K \cup r_1 D)$. Then,*
(i) $N_{K_r} \leq \max\{3^n N_K^2, 9^n N_K\}$.
(ii) $N_{K^{r_1}} \leq 5^n N_K$.

Proof: (i) From the Brunn-Minkowski inequality it easily follows that $|K \cap rD| \geq |K \cap (x + rD)|$, $x \in \mathbf{R}^n$. This implies that $|K| \leq N(K, rD)|K \cap rD|$ or, equivalently,

$$\lambda_K^n \leq N(K, rD)\lambda_{K_r}^n. \tag{5.2}$$

We distinguish two cases:
(1) If $\lambda_K < r$, then $N(K, rD) \leq N_K$ and, by (5.2), $\lambda_K^n \leq N_K \lambda_{K_r}^n$. It follows that

$$N_{K_r} \leq N(K, \lambda_{K_r} D) \leq N_K N(D, \frac{\lambda_{K_r}}{\lambda_K} D) \leq N_K N(D, \frac{1}{N_K} D) \leq 3^n N_K^2.$$

(2) If $\lambda_K > r$, then $N(K, rD) \leq N(K, \lambda_K D)N(D, \frac{r}{\lambda_K}D) \leq N_K 3^n (\frac{\lambda_K}{r})^n$ and hence, by (5.2), $(\frac{r}{\lambda_{K_r}})^n \leq 3^n N_K$. It follows that

$$N_{K_r} \leq N(rD, \lambda_{K_r} D) \leq 3^n (\frac{r}{\lambda_{K_r}})^n \leq 9^n N_K.$$

(ii) We obviously have $\lambda_{K^{r_1}} \geq \max\{\lambda_K, r_1\}$. Also, $K^{r_1} \subseteq K + r_1 D$, which gives

$$N_{K^{r_1}} \leq N(K^{r_1}, 2\lambda_{K^{r_1}} D)N(D, \frac{1}{2}D) \leq 5^n N(K + r_1 D, (\lambda_K + r_1)D) \leq 5^n N_K. \qquad \square$$

For an arbitrary symmetric convex body K, one has in general the information $\lambda_K M_K \geq 1$ as a consequence of the polar coordinates formula for volume. Our next lemma provides an "inverse" inequality in terms of the parameters N_{K^o} and $b = \sup\{\|x\| : x \in S^{n-1}\}$:

Lemma 5.2. *Let K be a symmetric convex body in $\mathbf{R}^n$, and assume that $\|x\| \leq b|x|$ for all $x \in \mathbf{R}^n$. Then,*

$$M_K \leq \frac{c}{\lambda_K} N_{K^o}^{t/n}$$

where $c>0$ is an absolute constant, and $t \leq C(\frac{b}{M_K})^2$.

Proof: Using Theorem 6 from [BLM] (to be more precise, using an argument identical to the one given there and the observation that what is really used is the ratio b/M_K), one can find orthogonal transformations $u_1, \dots, u_t \in O(n)$ such that

$$\frac{M_K}{2} D \subseteq \ T = \frac{1}{t}\sum_{i=1}^{t} u_i(K^o) \ \subseteq 2M_K D, \tag{5.4}$$

with $t \leq C(\frac{b}{M_K})^2$, where $C>0$ is an absolute constant.

On observing that $N(T, \lambda_{K^o}D) \leq [(N(K^o, \lambda_{K^o}D)]^t = N_{K^o}^t$, we can estimate M_K by (5.4) as follows:

$$M_K \leq 2(\frac{|T|}{|D|})^{1/n} \leq 2\lambda_{K^o} N_{K^o}^{t/n}. \tag{5.5}$$

Finally, the Blaschke-Santaló inequality implies that $\lambda_K \lambda_{K^o} \leq 1$, and hence the proof of the Lemma is complete. □

We can now pass to the proof of the main result of this section:

Theorem 5.3. *Let E be an ellipsoid in $\mathbf{R}^n$, and assume that both E and E^o are in M_δ-position. For every $r>0$ and every $\tau \in (0,1)$ we have*

$$t_c(E, r) + t_c(E^o, \frac{u(\tau,\delta)}{r}) \geq (1-\tau)n,$$

where $u(\tau,\delta) = \frac{c\log(\frac{2}{\tau})}{\sqrt{\tau}} e^{\frac{c\delta\log^2(\frac{2}{\tau})}{\tau}}$, and $c>0$ is an absolute constant.

Proof: Let $r>0$ and $\tau \in (0,1)$ be given. Consider the body $E_r = E \cap rD$. Since E_r is $\sqrt{2}$-isomorphic to an ellipsoid, one can easily check that Theorem 2.2 holds for E_r: for every $\theta \in (0,1)$ we can find $\sigma \subseteq \{1, \dots, n\}$ with $|\sigma| \geq (1-\theta)n$ such that $\mathcal{P}_\sigma(E_r^o) \supseteq [g(\theta)/M(E_r^o)]D_\sigma$, where $g(\theta) = c\sqrt{\theta}/2\sqrt{\log(2/\theta)}$ and c is the same constant as in Theorem 2.2.

We distinguish three cases:

Case 1: $\frac{M(E_r^o)}{r} \in [g(\tau), g(1))$.

In this case, consider any $\lambda \in (\tau, 1]$ with $\frac{1}{r}M(E_r^o) < g(\lambda)$. We can find $\sigma_1 \subseteq \{1, \dots, n\}$ with $|\sigma_1| \geq (1-\lambda)n$ such that

$$\mathcal{P}_{\sigma_1}(E_r^o) \supseteq \frac{g(\lambda)}{M(E_r^o)} D_{\sigma_1},$$

and it is easy to check that, for every $x \in \mathbf{R}^{\sigma_1}$, $\max\{\|x\|, \frac{1}{r}|x|\} = \|x\|_{E_r} > \frac{1}{r}|x|$, which means that $\frac{1}{r}|x| \leq \|x\|$, i.e

$$t_c(E, r) \geq (1-\lambda)n. \tag{5.6}$$

Taking the infimum of all λ's for which $\frac{M(E_r^o)}{r} < g(\lambda)$, we conclude that (5.6) also holds for the solution in λ of the equation $M(E_r^o) = rg(\lambda)$.

Now, choose $\mu \in (0,1)$ such that $(1-\lambda)+(1-\mu) = 1-\tau$, and $r_1 > 0$ satisfying $M((E_r)^{r_1})r_1 < g(\mu)$ (this is always possible since the left hand side is decreasing in r_1 and tends to zero as $r_1 \to \infty$). Since $(E_r)^{r_1}$ is 2-isomorphic to an ellipsoid, we can find $\sigma_2 \subseteq \{1,\dots,n\}$, $|\sigma_2| \geq (1-\mu)n$, with

$$\mathcal{P}_{\sigma_2}((E_r)^{r_1}) \supseteq \frac{g(\mu)}{M((E_r)^{r_1})} D_{\sigma_2},$$

thus $\max\{r_1|x|, \|x\|_{E_r^o}\} = \|x\|_{[(E_r)^{r_1}]^o} \geq \frac{g(\mu)}{M((E_r)^{r_1})}|x| > r_1|x|$, i.e $\|x\|_{E^o} \geq \|x\|_{E_r^o} > r_1|x|$ on $\mathbf{R}^{\sigma_2}$, which means that

$$t_c(E^o, \frac{1}{r_1}) \geq (1-\mu)n. \tag{5.7}$$

Again, we may take r_1 to be the solution of the equation $M((E_r)^{r_1})r_1 = g(\mu)$ in r_1.

Combining (5.6) with (5.7) we obtain

$$t_c(E,r) + t_c(E^o, \frac{1}{r_1}) \geq (1-\tau)n, \tag{5.8}$$

and it remains to compare r with r_1. Let us write W for the body $(E_r)^{r_1}$. By the way W has been constructed, it is easily checked that the following are satisfied:

(i) $M(W)r_1 = g(\mu)$ and $M(W^o) \geq M(E_r^o) = rg(\lambda)$.

(ii)$\|x\|_W \leq \frac{1}{r_1}|x|$ and $\|x\|_{W^o} \leq r|x|$, $x \in \mathbf{R}^n$.

(iii) $N_W^{1/n} \leq c_1 N_E^{c_2/n}$ and $N_{W^o}^{1/n} \leq c_1 N_{E^o}^{c_2/n}$, where $c_1, c_2 > 0$ are absolute constants. This is a simple consequence of Lemma 5.1, since both W and W^o are formed from E and E^o with two successive operations of taking intersection and convex hull with balls.

We simply write

$$\frac{r}{r_1} = \frac{r}{M(W^o)}\,\frac{1}{r_1 M(W)}\, M(W)M(W^o)$$

and making use of (i)-(iii) and of Lemma 5.2 we arrive at

$$\frac{r}{r_1} \leq \frac{c}{g(\lambda)g(\mu)} N_{E^o}^{C/ng^2(\mu)} N_E^{C/ng^2(\lambda)}. \tag{5.9}$$

Note that, at some point, we also used the fact that $\lambda_E \lambda_{E^o} \simeq 1$. Finally, assuming that both E and E^o are in M_δ-position, we rewrite (5.9) as follows:

$$\frac{r}{r_1} \leq \frac{c}{g(\lambda)g(\mu)} e^{C\delta/g^2(\lambda)g^2(\mu)}. \tag{5.10}$$

We have $\lambda+\mu=1+\tau$ and with this condition we can easily check that $\frac{1}{g(\lambda)g(\mu)} \leq \frac{c\log(\frac{2}{\tau})}{\sqrt{\tau}}$, which completes the proof in this case.

Case 2: $\frac{M(E_r^o)}{r} \geq g(1)$.

We choose $r_1>0$ such that $M((E_r)^{r_1})r_1 = g(\tau)$ and as above we conclude that $t_c(E^o, \frac{1}{r_1}) \geq (1-\tau)n$. The estimate for r/r_1 is done exactly in the same way, the only difference being that now $r/M(E_r^o) \leq 1/g(1)$.

Case 3: $\frac{M((E_r)^o)}{r} < g(\tau)$.

This is the simplest case since we already have $t_c(E,r) \geq (1-\tau)n$. □

6. Integer points inside an ellipsoid: some remarks

Consider an arbitrary ellipsoid E in $\mathbf{R}^n$. Write E in the form (2.1), so that $\sum_{j\leq n}|u_j|^2 = nM_E^2$. Without loss of generality we may assume that the $|u_j|$'s are arranged in the increasing order, therefore a simple application of Markov's inequality shows that

$$|u_j| \leq \sqrt{\frac{n}{n-j+1}}M_E \quad , \quad j=1,\dots,n. \tag{6.1}$$

Recall that the j-th successive minimum $\lambda_j(E)$ of E is defined by $\lambda_j(E)=\min\{\lambda > 0 : \dim(\mathrm{span}(\lambda E\cap \mathbf{Z}^n)) \geq j\}$. Inequality (6.1) gives an estimate on the successive minima of E in terms of M_E:

Fact I: *Let E be an ellipsoid in $\mathbf{R}^n$. Then, $\lambda_j(E) \leq \sqrt{\frac{n}{n-j+1}}M_E$, $j=1,\dots,n$. In particular, if $M_E \leq 1$ then E contains an integer point different from the origin.*

Note that if $M_E>1$ then E may contain no integer points other than the origin. Consider for example a ball of radius $r=\frac{1}{M_E}$.

Let us concentrate on the case $M_E<1$. If $M_E < |D_n|^{1/n}/2$, then we obviously have $|E|>2^n$ and Minkowski's theorem with its relatives start giving estimates on the cardinality of the set of integer points in E. We are interested in the range $|D_n|^{1/n}/2 < M_E < 1$. From Fact I we know that E contains non-trivial integer points, and using M_E as a parameter we try to estimate the number of them. Theorem 2.4 can be useful in this direction:

Let D_m be the m-dimensional Euclidean unit ball, and define $d(t,m)=|tD_m\cap \mathbf{Z}^m|$ be the cardinality of the set of integer points in tD_m. A simple lower bound for $d(t,m)$ can be given by counting the points with coordinates $0,\pm 1$ in tD_m:

$$d(t,m) \geq \sum_{k=0}^{[t^2]}\binom{n}{k}2^k \geq \binom{n}{[t^2]}2^{[t^2]}. \tag{6.2}$$

By Theorem 2.4, for every $m \leq c_1\sqrt{n}$ we can find $\sigma \subseteq \{1, \dots, n\}$ with $|\sigma| = m$ and $E \cap \mathbf{R}^\sigma \supseteq \frac{c_2}{\sqrt{m}M_E} D_\sigma$, where $c_1, c_2 > 0$ are absolute constants. Assuming that $M_E < c_2$ and using (6.2) we have some non-trivial information: It is clear that

$$|E \cap \mathbf{Z}^n| \geq \max_m \{|E \cap \mathbf{Z}^\sigma| : |\sigma| = m \leq c_1\sqrt{n}\} \tag{6.3}$$

Thus, we have:

Fact II: *Let E be an ellipsoid in $\mathbf{R}^n$ with $M_E < c_2 < 1$. Then,*

$$|E \cap \mathbf{Z}^n| \geq \max_m \{d(\frac{c_2}{\sqrt{m}M_E}, m) : m \leq c_1\sqrt{n}\}$$

$$\geq \max_m \{\binom{n}{[c_2^2/mM_E^2]} 2^{[c_2^2/mM_E^2]} : m \leq c_1\sqrt{n}\}.$$

The question of computation of the number of integer points inside an ellipsoid (or, more generally, inside a symmetric convex body) in $\mathbf{R}^n$ was relaxed in several directions in [M6]. One of the questions stated asks for "almost integer" points inside E in the following precise sense: for a given $\theta \in (0, 1)$, find a projection of E onto some coordinate subspace $\mathbf{R}^\sigma$ with $|\sigma| \geq (1-\theta)n$, which contains as many as possible integer points. Then, E itself will contain many points with $[(1-\theta)n]$ coordinates which are distinct $[(1-\theta)n]$-dimensional integers.

Our low M^*-estimate for ellipsoids provides an answer to this question in terms of M_E. We know that there exists $\sigma \subseteq \{1, \dots, n\}$, $|\sigma| = [(1-\theta)n]$, such that

$$\mathcal{P}_\sigma(E) \supseteq \frac{c\sqrt{\theta}}{\sqrt{\log(\frac{2}{\theta})}M_E} \; D_\sigma.$$

This, and (6.2), lead to the following:

Fact III: *Let E be an ellipsoid in $\mathbf{R}^n$. For every $\theta \in (0, 1)$ there exists $\sigma \subseteq \{1, \dots, n\}$ with $|\sigma| = [(1-\theta)n]$ for which*

$$|\mathcal{P}_\sigma(E) \cap \mathbf{Z}^\sigma| \geq d(\frac{c\sqrt{\theta}}{\sqrt{\log(\frac{2}{\theta})}M_E}, [(1-\theta)n])$$

$$\geq \binom{[(1-\theta)n]}{[\frac{c^2\theta}{\log(\frac{2}{\theta})M_E^2}]} 2^{[\frac{c^2\theta}{\log(\frac{2}{\theta})M_E^2}]}.$$

Clearly, the results in Sections 3 and 4 give analogous estimates for an arbitrary symmetric convex body.

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A. A. Giannopoulos, Department of Mathematics, University of Crete, Iraklion, Crete, Greece

E-mail address: deligia@talos.cc.uch.gr

V. D. Milman, Department of Mathematics, Tel Aviv University, Tel Aviv, Israel

E-mail address: vitali@math.tau.ac.il

Current address: (both authors) Mathematical Sciences Research Institute, 1000 Centennial Road, Berkeley, CA 94720