# 1-COMPLEMENTED SUBSPACES OF SPACES WITH 1-UNCONDITIONAL BASES

BEATA RANDRIANANTOANINA

Abstract. We prove that if $X$ is a complex strictly monotone sequence space with 1-unconditional basis, $Y \subseteq X$ has no bands isometric to $\ell_2^2$ and $Y$ is the range of norm-one projection from $X$, then $Y$ is a closed linear span a family of mutually disjoint vectors in $X$.

We completely characterize 1-complemented subspaces and norm-one projections in complex spaces $\ell_p(\ell_q)$ for $1 \leq p, q < \infty$.

Finally we give a full description of the subspaces that are spanned by a family of disjointly supported vectors and which are 1-complemented in (real or complex) Orlicz or Lorentz sequence spaces. In particular if an Orlicz or Lorentz space $X$ is not isomorphic to $\ell_p$ for some $1 \leq p < \infty$ then the only subspaces of $X$ which are 1-complemented and disjointly supported are the closed linear spans of block bases with constant coefficients.

## 1. Introduction

Projections and norm one projections have been studied by many authors. The question about the form of a (norm-one) projection and the structure of its range arises naturally not only in geometry of Banach spaces, but also in approximation theory, spectral theory, ergodic theory; see, e.g., the surveys [ChP, D] for more detailed discussions of applications.

The difficulty in studying 1-complemented subspaces of spaces with 1-unconditional bases arises from the following classical fact due to Lindenstrauss [L] (cf. also [LT, Theorem 3.b.1])

**Theorem 1.1.** *Every space $Y$ with a 1-unconditional basis is 1-complemented in some symmetric space $X$.*

Thus it seems hopeless to give any characterization of 1-complemented subspaces of, even symmetric, spaces with 1-unconditional bases.

The only class of spaces where the full characterization of 1-complemented subspaces was available are the classical spaces $\ell_p$ and $c_0$. Namely it is well known that

---

1991 *Mathematics Subject Classification.* 46B20,46B45,41A65.

This research was started while the author participated in the Workshop on Linear Analysis and Probability held at Texas A&M University in College Station, Texas, supported by the NSF Research Group grant MCS DMS-9311902, and completed during the stay at Mathematical Sciences Research Institute in Berkeley, California, supported by the NSF grant DMS-9022140.

every subspace of a Hilbert space is 1-complemented (with the unique orthogonal projection) and in $\ell_p$, for $p \neq 2, \infty$, we have the following result:

**Theorem 1.2** ([An], cf. also [LT, Theorem 2.a.4]). *Let $F \subset \ell_p$, where $1 \leq p < \infty$, $p \neq 2$. Then $F$ is 1-complemented if and only if*

(a) *$F$ is isometric to $\ell_p^{\dim F}$,*

*or*

(b) *$F$ is spanned by a family of mutually disjoint vectors.*

It is clear that Theorem 1.2(a) cannot be extended to other spaces. Namely Lindberg [Lg] showed a class of Orlicz functions $\varphi$ (for necessary definitions see Section 2) so that there exists a 1-complemented subspace $F$ in $\ell_\varphi$ such that $F$ is not even isomorphic to $\ell_\varphi$. Altshuler, Casazza and B. L. Lin [ACL] showed a similar example in the class of Lorentz sequence spaces $\ell_{w,p}$. However, both of these examples were spanned by a family of mutually disjoint vectors; in fact they were closed linear spans of a block basis with constant coefficients. Also the symmetric space $X$ constructed in Theorem 1.1 was such that $Y$ was isometrically isomorhic to a closed linear span of a block basis with constant coefficients.

It is well known that all such spans are 1-complemented in any symmetric space ([LT, Theorem 3.a.4]), so in fact all of those examples satisfy condition $(b)$ of Theorem 1.2.

In this paper we prove that indeed Theorem 1.2(b) can be extended to a large class of 1-complemented subspaces of complex spaces with 1-unconditional basis.

Namely, if $X$ is a complex, strictly monotone sequence space with a 1-unconditional basis, $Y \subset X$ is 1-complemented in $X$, and $Y$ does not contain a band isometric to $\ell_2^2$, then $Y$ is spanned by a family of disjointly supported vectors (see Corollary 3.2). It is clear that our restrictions on $X$ and $Y$ are necessary (see Remark after Corollary 3.2 and examples in Section 4).

The above-mentioned assumption on $Y$ is satisfied, for example, in all spaces $X$ that do not have a 1-complemented subspace isometric to $\ell_2^2$. We discuss it in greater detail in Section 4.

In Theorem 3.1 we also describe the form of general 1-complemented subspaces of complex strictly monotone spaces.

Our method of proof cannot be extended to real sequence spaces. We use in particular the fact that every 1-complemented subspace of a complex space with 1-unconditional basis also has a 1-unconditional basis. The analogous fact is false in real spaces [Le, BFL] (see [R1] for the disscusion in special real spaces).

As a consequence of Theorem 3.1 we obtain a complete characterization of 1-complemented subspaces of complex $\ell_p(\ell_q)$, where $1 < p, q < \infty$ (Theorems 5.1 and 5.2).

Further we study the subspaces that are spanned by disjointly supported vectors and are 1-complemented in $X$. Calvert and Fitzpatrick [CF] showed that if all disjointly supported subspaces are 1-complemented in $X$ then $X$ is isometric to $\ell_p$, for some $p$, $1 \leq p < \infty$, or to $c_0$.

In Section 6 we completely characterize the disjointly supported subspaces that are 1-complemented in Orlicz and Lorentz sequence spaces (Theorems 6.1 and 6.3). In particular, if a Lorentz or Orlicz space $X$ is not isomorphic to $\ell_p$ for some $1 \leq p < \infty$ then the only disjointly supported subspaces that are 1-complemented are those guaranteed by [LT, Theorem 3.a.4], i.e. spanned by a block basis with constant coefficients. The results of Section 6 are valid for both real and complex spaces.

**Acknowledgments .** I wish to express my gratitude to Professors W. B. Johnson and A. Koldobsky for valuable suggestions.

## 2. Preliminaries

In the following we will consider complex Banach spaces $X$ with a normalized 1-unconditional basis $\{e_i\}_{i \in I}$, where $\mathbb{N} \supseteq I = \{1, \dots, \dim X\}$. Our results are valid in both the finite- and infinite-dimensional case.

If $x \in X$ we will write $x = (x_i)_{i \in I}$ if

$$x = \sum_{i \in 1}^{\dim X} x_i e_i \quad \text{and} \quad \operatorname{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}.$$

For $x \in X$ we will denote by $x^*$ (or sometimes by $x^N$) a *norming functional* for $x$, that is, $x^* \in X^*$, $\|x^*\|_{X^*} = 1$ and $x^*(x) = \|x\|_X$.

We say (following [KW], see also [ST]) that an element $x \in X$ is *hermitian* if there exists a hermitian projection $P_x$ from $X$ onto $\operatorname{span}\{x\}$.

Equivalently, $x$ is hermitian if and only if for all $y \in X$, $y^*$ norming for $y$, and $x^*$ norming for $x$ we have

$$x^*(y)y^*(x) \in \mathbb{R}.$$

The set of all hermitian elements is denoted $h(X)$.

Let $\{H_\lambda \ : \lambda \in \Lambda\}$ be the collection of maximal linear subspaces of $h(X)$. Then $\{H_\lambda \ : \lambda \in \Lambda\}$ are called *Hilbert components* of $X$. Kalton and Wood [KW] proved that Hilbert components are well-defined and mutually disjoint.

A Hilbert component $H_\lambda$ is called *nontrivial* if $\dim H_\lambda > 1$.

For the careful analysis and properties of Hilbert components of various spaces we refer to [KW] and to expository papers [F, R2]. Here we just want to recall some properties which will be used in our arguments.

First, recall that if $X$ has 1-unconditional basis $\{e_i\}_{i \in I}$ then each basis element is hermitian. Moreover Kalton and Wood proved the following:

**Theorem 2.1** ([KW, Theorem 6.5]). *Let $X$ be a Banach space with a normalized 1-unconditional basis. Then $x \in X$ is hermitian in $X$ if and only if*

(i) $\|y\|_X = \|y\|_2$ *for all* $y \in X$ *with* $\operatorname{supp} y \subset \operatorname{supp} x$, *and*
(ii) *for all* $y, z \in X$ *with* $\operatorname{supp} y \cup \operatorname{supp} z \subset \operatorname{supp} x$ *and for all* $v \in X$ *with* $\operatorname{supp} v \cap \operatorname{supp} x = \varnothing$ *if* $\|y\|_X = \|z\|_X$ *then* $\|y+v\|_X = \|z+v\|_X$.

For our main result we will need the following two facts.

**Proposition 2.2** ([KW, Lemma 5.2]). *Suppose that $x, y$ are hermitian elements in $X$. Denote by $x^*$ a norming functional for $x$.*

*If $x^*(y) \neq 0$ then* $\operatorname{span}\{x, y\} \subset h(X)$.

**Proposition 2.3** ([F, Lemma 4]). *Suppose that $X$ has 1-unconditional basis $\{e_i\}_{i\in I}$ and let $P : X \longrightarrow X$ be norm one projection with range of $P$ equal to $Y$. Then for all $i \in I$, $Pe_i$ is a hermitian element in $Y$.*

We will also frequently use the following well-known fact:

**Proposition 2.4.** *Let $X$ be a Banach space with a 1-unconditional basis. Suppose that $Y \subset X$ is 1-complemented and a norm-one projection $P : X \to Y$ is given by*

$$P(x) = \sum_i y_i^*(x) y_i$$

*where $Y = \overline{\operatorname{span}}\{y_i\}$ and $y_i^*$ is norming for $y_i$ for all $i$.*

*Then for any $y \in Y$ there exists $y^*$ norming for $y$ and constants $K_i$ so that*

$$y^* = \sum_i K_i y_i^*.$$

Moreover, we have

**Proposition 2.5** (Calvert [C]). *Let $X$ be a strictly convex reflexive Banach space with strictly convex dual $X^*$. Let $J : X \longrightarrow X^*$ be the duality map; $\|Jx\| = \|x\|$, $Jx(x) = \|x\|^2$.*

*Then a closed linear subspace $Y$ of $X$ is 1-complemented in $X$ if and only if $J(Y)$ is a linear subspace of $X^*$.*

Finally we recall a few definitions (see [LT]).

We say that a Banach space $X$ with 1-unconditional basis is *strictly monotone* if $\|x+y\| > \|x\|$ for all $x, y \geq 0$ with $y \neq 0$.

An *Orlicz function* $\varphi$ is a convex non-decreasing function $\varphi : [0,\infty) \longrightarrow [0,\infty]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$ or $\infty$. To any Orlicz function $\varphi$ we associate the *Orlicz space* $\ell_\varphi$ of all sequences of scalars $x = (x_i)_i$ such that

$$\sum_{i=1}^{\infty} \varphi\Big(\frac{|x_i|}{\rho}\Big) < \infty \quad \text{for some } \rho > 0,$$

with the norm

$$\|x\|_\varphi = \inf\Big\{\rho > 0 : \sum_{i=1}^{\infty} \varphi\Big(\frac{|x_i|}{\rho}\Big) < 1\Big\}.$$

Let $1 \le p < \infty$ and let $w = \{w_i\}_{i \in I}$, where $I = \mathbb{N}$ or $I = \{1, \dots, d\}$, be a non-increasing sequence such that $w_1 = 1$ and $w_i \ge 0$ for all $i$. The Banach space of all sequences of scalars $x = (x_i)_{i \in I}$ for which

$$\|x\|_{w,p} = \sup_{\sigma \in \mathcal{P}(I)} \Bigg(\sum_{i \in I} |x_{\sigma(i)}|^p w_i\Bigg)^{\frac{1}{p}} < \infty,$$

where $\mathcal{P}(I)$ is the set of all permutations of $I$, is denoted $\ell_{w,p}$ and it is called a *Lorentz sequence space* (another notation frequently used in the literature is $d(w,p)$).

Notice that

- $\ell_\varphi$ is strictly monotone if and only if $\varphi(t) > 0$ for all $t > 0$ and $\varphi(t) < \infty$ for all $t \le 1$.
- $\ell_{w,p}$ is strictly monotone if and only if $w_i > 0$ for all $i \in I$.
- $\ell_p(\ell_q)$ is strictly monotone if and only if $p, q \ne \infty$.

For any $1 \le p, q \le \infty$ we denote be $\ell_p(\ell_q)$ the space of sequences of scalars $x = (x_{ij})_{i \in I, j \in J}$ such that

$$\|x\|_{\ell_p(\ell_q)} = \big\|(\|(x_{ij})_{j \in J}\|_{\ell_q})_{i \in I}\big\|_{\ell_p} < \infty.$$

We follow standard notations as defined in [LT] and this is also where we refer the reader for all undefined terms.

## 3. General form of contractive projections

We are now ready to present our main theorem.

**Theorem 3.1.** *Suppose that $X$ is a complex strictly monotone sequence space with 1-unconditional basis $\{e_i\}$ and $X \ne \ell_2$ and let $P$ be the projection of norm 1 in $X$. Let $\{H_\gamma : \gamma \in \Gamma\}$ be the collection of Hilbert components of $Y = PX$. Then $H_\gamma'$s are disjointly supported as elements of $X$.*

*Proof.* By Proposition 2.3 all $\{Pe_i\}_{i \in I}$ are hermitian elements of $Y$. Let $i, j$ be such that $Pe_i$ and $Pe_j$ are not in the same Hilbert component of $Y$. Assume that there exists $k$ such that $k \in \operatorname{supp} Pe_i \cap \operatorname{supp} Pe_j$. If $Pe_k \ne 0$, then $P^* e_k^*$ is a norming functional for $Pe_k$ in $Y$ and $(P^* e_k^*, Pe_i) = (e_k^*, Pe_i) \ne 0$. Thus, by Proposition 2.2, $Pe_k, Pe_i$ are in the same Hilbert component of $Y$. Similarly $Pe_k, Pe_j$ are in the same Hilbert component of $Y$. But then $Pe_i, Pe_j$ are in the same Hilbert component of $Y$ contrary to our assumption.

Thus $Pe_k = 0$.

Now suppose that $Pe_i = \sum_{l\in S} \alpha_l e_l + \alpha_k e_k$ for some $\alpha_l, \alpha_k \neq 0$, where $S = \operatorname{supp} Pe_i \setminus \{k\}$.

Since $P$ is a projection and $P(e_k) = 0$ we get

$$\sum_{l\in S} \alpha_l e_l + \alpha_k e_k = Pe_i = P(Pe_i) = P\left(\sum_{l\in S} \alpha_l e_l\right) + \alpha_k P(e_k) =$$

$$= P\left(\sum_{l\in S} \alpha_l e_l\right) .$$

Hence $S \neq \varnothing$ and by strict monotonicity of $X$

$$\left\|P\left(\sum_{l\in S} \alpha_l e_l\right)\right\| = \left\|\sum_{l\in S} \alpha_l e_l + \alpha_k e_k\right\| > \left\|\sum_{l\in S} \alpha_l e_l\right\|$$

which contradicts the fact that $\|P\| = 1$.

Thus if $Pe_i, Pe_j$ are not in the same Hilbert component of $Y$ then they are disjoint. □

**Corollary 3.2.** *Suppose that $X$ is a complex strictly monotone sequence space with 1-unconditional basis $\{e_i\}$ and $X \neq \ell_2$ and let $P$ be the projection of norm 1 in $X$.*

*Suppose that $Y = PX \subset X$ has no nontrivial Hilbert components. Then there exist disjointly supported elements $\{y_j\}_{j=1}^m$ ($m = \dim PX \leq \infty$) which span $Y = PX$. Moreover, for all $x \in X$,*

$$Px = \sum_{j=1}^m y_j^*(x) y_j,$$

*where $\{y_j^*\}_{j=1}^m \subset X^*$ satisfy $\|y_j\| = \|y_j^*\| = y_j^*(y_j) = 1$ for all $j$.*

*Proof.* By Proposition 2.3 all $\{Pe_i\}_{i\in I}$ are hermitian elements in $Y$.

By our assumption all Hilbert components of $Y$ are one-dimensional so if $Pe_i, Pe_j$ are linearly independent then they belong to different Hilbert components of $Y$. Therefore, by Theorem 3.1, if $Pe_i, Pe_j$ are linearly independent then they are disjoint and $Y$ can be presented as $\operatorname{span}\{Pe_i : i \in I\}$ where $I$ is a collection of such indices $i, j$ that $Pe_i, Pe_j$ are mutually disjoint.

Then $y_i = Pe_i/\|Pe_i\|$ for all $i \in I$, and for each $x \in X$ we have $Px = \sum_{i\in I} C_i y_i$, where $C_i \in \mathbb{C}$ are uniquely determined by $x$. Clearly $y_i^*(x) \overset{\text{def}}{=} C_i(x)$ satisfies the conclusion of the theorem.

Notice also that $\operatorname{supp} y_i^* = \operatorname{supp} y_i$ for all $i \in I$. □

**Remarks.** 1. Notice that the assumption about $X$ being strictly monotone is important. Indeed, Blatter and Cheney [BCh] (see also [B]) showed examples of 1-complemented hyperplanes in $\ell_\infty^3$ that are not spanned by disjointly supported vectors.

2. Also the assumption about $Y$ cannot be removed. We discuss it in greater detail in the next section.
3. As mentioned in Introduction, Calvert and Fitzpatrick [CF] showed that if every subspace of the form described in Corollary 3.2 is 1-complemented in $X$ then $X$ is isometric to $\ell_p$, for some $p$, $1 \leq p < \infty$, or to $c_0$.

As a consequence of Corollary 3.2 we can express 1-complemented subspaces as an intersection of hyperplanes of special simple form (see [BP] for analogous representation in $\ell_p$).

**Corollary 3.3.** *Let $X$ and $Y$ be as described in Corollary 3.2. Then $Y$ can be presented as intersection of kernels of functionals $f_j$, such that* $\operatorname{card}(\operatorname{supp} f_j) \leq 2$ *for all $j$.*

## 4. 1-complemented copies of $\ell_2^2$

In this section we discuss in what situation it is possible that a space $X$ has a 1-complemented subspace $Y$ with nontrivial Hilbert components. This clearly reduces to the question of characterizing under what conditions $X$ can have a 1-complemented subspace $F$ that is isometric to $\ell_2^2$. The question that arises here is:

> *Is it possible that a space $X$ with only 1-dimensional Hilbert components has a 1-complemented subspace isometric to $\ell_2^2$?*

One quickly realizes that the answer is yes:

**Example 1.** Consider the space $X = \ell_2(\ell_1)$ and for $x = (x_{ij})_{i,j} \in \ell_2(\ell_1)$ let $Px = (x_{i1})_i$. Then $PX$ is isometric to $\ell_2^2$.

Further there exists an orthogonal projection $Q$ of $PX$ onto a span of any collection of orthogonal vectors from $PX = \ell_2$.

In Lemma 5.5 we show that if $\ell_p(\ell_q)$ has a 1-complemented subspace $F$ that is isometric to $\ell_2^2$ then either $q = 2$ and $F$ is contained in a Hilbert component of $\ell_p(\ell_q)$ or $p = 2$ and $F$ is similar to the range of $QP$ in Example 1.

Below we present two more examples: a 1-complemented copy of $\ell_2^2$ in a Lorentz space and in a Orlicz space. We do not know a full characterization of spaces $X$ that have 1-complemented copies of $\ell_2^2$, but we suspect that if $X$ is a Lorentz or Orlicz space then $X$ has to be very similar to the examples presented below.

**Example 2.** Consider Lorentz space $\ell_{w,2}$ with weight $w = (1, 1, w_3, w_4, \dots)$. Then $\operatorname{span}\{e_1, e_2\} \subset \ell_{w,2}$ is isometric to $\ell_2^2$ and clearly it is 1-complemented.

**Example 3.** Consider 4 dimensional Orlicz space $\ell_\varphi$ where

$$\varphi(t) = \begin{cases} t^2 & \text{if } 0 \leq t \leq a \\ (1+a)t - a & \text{if } a \leq t \leq 1 \end{cases}$$

and $\sqrt{2/3} < a < 1$. That is,

$$\|(x_1, \dots, x_4)\|_\varphi = \inf\left\{\lambda : \sum_{i=1}^{4} \varphi\left(\frac{|x_i|}{\lambda}\right) \le 1\right\} .$$

In fact we have $\sum_{i=1}^4 \varphi\left(\frac{|x_i|}{\|x\|_\varphi}\right) = 1$ for all $x \in \ell_\varphi$. Let $F = \ker(e_1^* + e_2^* + e_3^*) = \operatorname{span}\{e_1 - e_2, e_1 - e_3, e_4\}$. then, if $x = (x_1, x_2, x_3, x_4) \in F$ then $\|x\|_\varphi$ is a number such that

$$\sum_{i=1}^{3} \frac{|x_i|^2}{\|x\|_\varphi^2} + \varphi\left(\frac{|x_4|}{\|x\|_\varphi}\right) = 1 \tag{4.1}$$

Indeed, suppose that $x \in F$ and denote $\|x\|_\varphi = \alpha$. Then

$$1 = \sum_{i=1}^{4} \varphi\left(\frac{|x_i|}{\alpha}\right) \ge \sum_{i=1}^{4} \frac{|x_i|^2}{\alpha^2} \tag{4.2}$$

Assume that there is $j$, $1 \le j \le 3$ such that $\frac{|x_j|}{\alpha} > a$. Then, by (4.2)

$$\sum_{i \ne j} \frac{|x_i|^2}{\alpha^2} < 1 - a^2 .$$

Hence

$$\left|\frac{x_1 + x_2 + x_3}{\alpha}\right| \ge \frac{|x_j|}{\alpha} - \sum_{\substack{i=1 \\ i \ne j}}^{3} \frac{|x_i|}{\alpha} \ge a - \sqrt{2}\,\sqrt{1 - a^2} > 0$$

since $a > \sqrt{2/3}$. But this contradicts the fact that $x \in F = \ker(e_1^* + e_2^* + e_3^*)$. Thus $\frac{|x_j|}{\alpha} \le a$ for $j = 1, 2, 3$, so

$$1 = \sum_{i=1}^{4} \varphi\left(\frac{|x_i|}{\alpha}\right) = \sum_{i=1}^{3} \frac{|x_i|^2}{\alpha^2} + \varphi\left(\frac{|x_4|}{\alpha}\right) .$$

Equation (4.1) and Theorem 2.1 immediately imply that $e_1 - e_2$ and $e_1 - e_3$ are hermitian in $F$ and belong to the same Hilbert component of $F$, but clearly $F$ is not isometric to $\ell_2$.

Moreover (4.1) implies that $F$ is 1-complemented in $\ell_\varphi$. Indeed, define $P : \ell_\varphi \to F$ by:

$$P(x_1, x_2, x_3, x_4) = \left(x_1 - \frac{1}{3}(x_1 + x_2 + x_3), x_2 - \frac{1}{3}(x_1 + x_2 + x_3), x_3 - \frac{1}{3}(x_1 + x_2 + x_3), x_4\right).$$

Notice that $Q : \ell_2^3 \to \ell_2^3$ defined by

$$Q(x_1, x_2, x_3) = \left(x_1 - \frac{1}{3}(x_1 + x_2 + x_3), x_2 - \frac{1}{3}(x_1 + x_2 + x_3), x_3 - \frac{1}{3}(x_1 + x_2 + x_3)\right)$$

is the norm one projection on this Hilbert space. Let $x \in \ell_\varphi$. Denote $y = Px$ and $\|Px\|_\varphi = \beta$. Then, by (4.1)

$$\begin{aligned} 1 &= \sum_{i=1}^{3} \frac{|y_i|^2}{\beta^2} + \varphi\left(\frac{|y_4|}{\beta}\right) = \frac{\|(y_1, y_2, y_3)\|_2^2}{\beta^2} + \varphi\left(\frac{|y_4|}{\beta}\right) \\ &= \frac{\|Q(x_1, x_2, x_3)\|_2^2}{\beta^2} + \varphi\left(\frac{|y_4|}{\beta}\right) \le \frac{\|(x_1, x_2, x_3)\|_2^2}{\beta^2} + \varphi\left(\frac{|x_4|}{\beta}\right) \\ &= \frac{|x_1|^2}{\beta^2} + \frac{|x_2|^2}{\beta^2} + \frac{|x_3|^2}{\beta^2} + \varphi\left(\frac{|x_4|}{\beta}\right) \le \\ &\le \sum_{i=1}^{4} \varphi\left(\frac{|x_i|}{\beta}\right) . \end{aligned}$$

Thus $\|x\|_\varphi \geq \beta = \|Px\|_\varphi$, i.e. $\|P\| \leq 1$.

We finish this section with a lemma characterizing 1-complemented subspaces $F$ in $X$ that are isometric to $\ell_2^2$ in terms of norming functionals.

**Lemma 4.1.** *If* $\operatorname{span}\{x, y\} \subset X$ *is* 1*-complemented in* $X$ *and* $\operatorname{span}\{x, y\}$ *is isometric to* $\ell_2^2$*, then there exist* $x^*$ *norming for* $x$ *and* $y^*$ *norming for* $y$ *such that, for all* $a, b \in \mathbb{C}$ *with* $|a|^2 + |b|^2 = 1$*, the functional* $\bar{a}x^* + \bar{b}y^*$ *is norming for* $ax + by$.

*Proof.* Let $P : X \to \operatorname{span}\{x, y\}$ be a norm-one projection. Then there exist $x^*$ norming for $x$ and $y^*$ norming for $y$ such that

$$P(u) = x^*(u)x + y^*(u)y \quad \text{ for all } u \in X$$

and $x^*(y) = y^*(x) = 0$. Moreover, $\operatorname{span}\{x^*, y^*\} \subset X^*$ is isometric to $\ell_2^2$. Thus

$$(\bar{a}x^* + \bar{b}y^*)(ax + by) = a\bar{a}x^*(x) + \bar{b}by^*(y) = |a|^2 + |b|^2 = \|\bar{a}x^* + \bar{b}y^*\|_{x^*}\|ax + by\|_x. \qquad \square$$

## 5. 1-COMPLEMENTED SUBSPACES OF $\ell_p(\ell_q)$

The main results of this section (Theorems 5.1 and 5.2) completely characterize 1-complemented subspaces of the complex space $\ell_p(\ell_q)$, $1 < p, q < \infty$, $p \neq q$.

To formulate the theorem conveniently we will need some notation. For all $x$ in $\ell_p(\ell_q)$ we will write $x = \sum_{i=1}^n x_i$, where $n = \dim \ell_p \leq \infty$, and $x_i \in \ell_q$ for all $i \leq n$. We will also write

$$x = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} e_{ij},$$

where $m = \dim \ell_q \leq \infty$ and $e_{ij}$ are standard basis elements in $\ell_p(\ell_q)$. We will distinguish two types of support of $x$:

- the usual support

$$\operatorname{supp} x = \{(i,j) \subset \{1,\dots,n\} \times \{1,\dots,m\} : x_{ij} \neq 0\},$$

  which we will sometimes call the *scalar support* of $x$, and

- the *vector support*

$$\text{v-supp}\, x = \{i \subset \{1,\dots,n\} : \ell_q \ni x_i \neq \vec{0}\}.$$

Thus we will have notions of disjointness of elements in $\ell_p(\ell_q)$ in the scalar and vector senses.

**Theorem 5.1.** *Let $1 < p, q < \infty$ with $p \neq q, 2$. Consider the complex space $\ell_p(\ell_q)$. Then $Y \subset \ell_p(\ell_q)$ is 1-complemented if and only if there exist $\{v^i\}_{i=1}^{\dim Y} \subset \ell_p(\ell_q)$ so that*

$$Y = \overline{\operatorname{span}}\{v^i\}_{i=1}^{\dim Y},$$

*and for all $i \neq j \leq \dim Y$ one of the following conditions holds:*

(a) $\text{v-supp}\, v^i \cap \text{v-supp}\, v^j = \varnothing$,

*or*

(b) $\text{v-supp}\, v^i = \text{v-supp}\, v^j$, $\|v^i_k\|_q = \|v^j_k\|_q$ *for all* $k \in \text{v-supp}\, v^i$, *and*

  (b1) *if $q \neq 2$ then* $\operatorname{supp} v^i \cap \operatorname{supp} v^j = \varnothing$,

  (b2) *if $q = 2$ then $v^i_k, v^j_k$ are orthogonal for each* $k \in \text{v-supp}\, v^i$.

The structure of 1-complemented subspaces of $\ell_2(\ell_q)$ can be somewhat more complicated as illustrated in Example 1. We have the following:

**Theorem 5.2.** *Let $1 < q < \infty$ with $q \neq 2$. Consider the complex space $\ell_2(\ell_q) = \ell_2^n(\ell_q^m)$, $n, m \in \mathbb{N} \cup \{\infty\}$. Then $Y \subset \ell_2(\ell_q)$ is 1-complemented if and only if there exist $\{v^i\}_{i=1}^{\dim Y} \subset \ell_p(\ell_q)$ so that*

$$Y = \overline{\operatorname{span}}\{v^i\}_{i=1}^{\dim Y},$$

*and for all $i \neq j \leq \dim Y$ one of the conditions $(a)$ or $(b)$ of Theorem 5.1 holds or*

(c) *for each $k \in \text{v-supp}\, v^i \cap \text{v-supp}\, v^j$ there exists a constant $C_k \in \mathbb{C}$ with*

$$v^i_k = C_k v^j_k$$

*and for every map $\sigma : \{1,\dots,n\} \longrightarrow \{1,\dots,m\}$ such that $(k,\sigma(k)) \in \operatorname{supp} v^i \cup \operatorname{supp} v^j$ whenever $k \in \text{v-supp}\, v^i \cup \text{v-supp}\, v^j$, the vectors*

$$s_\sigma(v^i) = \left( \|v^i_k\|_q \frac{v^i_{k\sigma(k)}}{|v^i_{k\sigma(k)}|} \right)_{k \leq n} \in \ell_2, \text{ and } s_\sigma(v^j) = \left( \|v^j_k\|_q \frac{v^j_{k\sigma(k)}}{|v^j_{k\sigma(k)}|} \right)_{k \leq n} \in \ell_2$$

*(with the convention $0/0 = 0$) are orthonormal.*

**Corollary 5.3.** *If $Y \subset \ell_p(\ell_q)$, $1 < p, q < \infty$, $p \neq q$, is 1-complemented in $\ell_p(\ell_q)$ if and only if $Y$ is isometric to $\sum \oplus_p Y_i$, where each $Y_i$ is isometrically isomorphic to $\ell_q^{d_i}$, $d_i = \dim Y_i \in \mathbb{N} \cup \{\infty\}$.*

*Proof of Corollary 5.3.* The "only if" part follows immediately from Theorems 5.1 and 5.2, and the "if" part when $q \neq 2$ is a simple consequence of [Ko, Lemma 6] and Theorems 5.1 and 5.2. When $q = 2$ the "if" part follows from Lemma 5.5. □

For the proof of Theorems 5.1 and 5.2 we use Theorem 3.1 and the following lemmas:

**Lemma 5.4.** *Suppose that $x^1, x^2$, $\|x^1\| = \|x^2\| = 1$, are disjointly supported (in the scalar sense). Then* $\operatorname{span}\{x^1, x^2\}$ *is 1-complemented in $\ell_p(\ell_q)$, with $1 < p, q < \infty$, $p \neq q$, if and only if one of the following conditions holds:*

(a) $\text{v-supp}\, x^1 \cap \text{v-supp}\, x^2 = \varnothing$, *in this case $F$ is isometric to $\ell_p^2$,*

*or*

(b) $\text{v-supp}\, x^1 = \text{v-supp}\, x^2$ *and* $\|x_i^1\|_q = \|x_i^2\|_q$ *for all* $i \in \text{v-supp}\, x^1$, *in this case $F$ is isometric to $\ell_q^2$,*

**Lemma 5.5.** *Suppose that $F \subset \ell_p(\ell_q)$ is isometric to $\ell_2^2$ and is 1-complemented in $\ell_p(\ell_q)$, $1 \leq p, q < \infty$, $p \neq q$. Then*

(a) *$F$ is spanned by disjointly supported vectors (in the scalar sense),*

(b) $p = 2$,

*or*

(c) $q = 2$ *and there exists a surjective isometry $U$ of $\ell_p(\ell_q)$ such that $UF$ is spanned by disjointly supported vectors (in the scalar sense).*

**Remark.** It follows immediately from Lemma 5.4 that if Lemma 5.5($a$) holds then $p = 2$ or $q = 2$.

**Lemma 5.6.** *Let $x, y \in \ell_2^n(\ell_q^m)$, $n, m \in \mathbb{N} \cup \{\infty\}$, $1 \leq q < \infty$, $q \neq 2$, $\|x\| = \|y\| = 1$. Then $F = \operatorname{span}\{x, y\} \subset \ell_2(\ell_q)$ is isometrically isomorphic to $\ell_2^2$ and 1-complemented in $\ell_2(\ell_q)$ if and only if for each* $i \in \text{v-supp}\, x \cap \text{v-supp}\, y$ *there exists a constant $C_i \in \mathbb{C}$ with*

$$x_i = C_i y_i$$

*and for every map $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$ such that $(i, \sigma(i)) \in \operatorname{supp} x \cup \operatorname{supp} y$ whenever* $i \in \text{v-supp}\, x \cup \text{v-supp}\, y$, *the vectors*

$$s_\sigma(x) = \left( \|x_i\|_q \frac{x_{i\sigma(i)}}{|x_{i\sigma(i)}|} \right)_{i \leq n} \in \ell_2, \text{ and } s_\sigma(y) = \left( \|y_i\|_q \frac{y_{i\sigma(i)}}{|y_{i\sigma(i)}|} \right)_{i \leq n} \in \ell_2$$

*(with the convention $0/0 = 0$) are orthonormal.*

*Proof of Theorems 5.1 and 5.2.* Since $\ell_p(\ell_q)$ is complex, if $Y$ is 1-complemented in $\ell_p(\ell_q)$, then $Y$ has a 1-unconditional basis $\{v^i\}_{i=1}^{\dim Y}$.

Consider $v^i, v^j$ for some $i \neq j \leq \dim Y$.

If $v^i, v^j$ belong to different Hilbert components of $Y$, then by Theorem 3.1, $v^i$ and $v^j$ are scalarly disjoint and conditions $(a), (b)$ of Theorem 5.1 follow from Lemma 5.4 (since it is clear that $\operatorname{span}\{v^i, v^j\}$ is 1-complemented in $Y$ and therefore in $\ell_p(\ell_q)$).

If $v^i, v^j$ are in the same Hilbert component of $Y$, then $\operatorname{span}\{v^i, v^j\}$ is isometric to $\ell_2^2$ and when $p \neq 2$ Lemma 5.5 reduces our considerations to the case of disjointly supported vectors (in the scalar sense), where we apply Lemma 5.4.

When $p = 2$ we apply Lemma 5.6. $\square$

*Proof of Lemma 5.4.* Notice that when $z \in \ell_p(\ell_q)$ then the norming functional $z^*$ for $z$ is given by

$$z^* = \frac{1}{\|z\|^{p-1}} \sum_i \|z_i\|_q^{p-q} \sum_j |z_{ij}|^{q-1} \operatorname{sgn}(\overline{z_{ij}}) e_{ij}^*. \tag{5.1}$$

By Proposition 2.5, $\operatorname{span}\{x^1, x^2\}$ is 1-complemented in $\ell_p(\ell_q)$ if and only if for all $a^1, a^2 \in \mathbb{C}$ there exist $K_s$, $(= K_s(a^1, a^2))$, $s = 1, 2$, in $\mathbb{C}$ so that

$$(a^1x^1 + a^2x^2)^* = K_1 x^{1*} + K_2 x^{2*}. \tag{5.2}$$

This is equivalent, by disjointness of $x^1, x^2$ and (5.1), to the existence of constants $K_s$, $(= K_s(a^1, a^2))$, $s = 1, 2$, such that for all $(i, j) \in \operatorname{supp} x^s$:

$$\begin{aligned}\frac{1}{\|a^1x^1 + a^2x^2\|^{p-1}} &\|a^1x_i^1 + a^2x_i^2\|_q^{p-q} |a^s x_{ij}^s|^{q-1} \operatorname{sgn}(\overline{a^s x_{ij}^s}) \\ &= K_s(a^1, a^2) \|x_i^s\|_q^{p-q} |x_{ij}^s|^{q-1} \operatorname{sgn}(\overline{x_{ij}^s}),\end{aligned}$$

which is further equivalent to the existence of constants $C_s$, $(= C_s(a^1, a^2) = K_s(a^1, a^2) \cdot \|a^1x^1 + a^2x^2\|^{p-1})$, $s = 1, 2$, such that for all $i \in \text{v-supp}\, x^s$:

$$\frac{\|a^1x_i^1 + a^2x_i^2\|_q^{p-q}}{\|x_i^s\|_q^{p-q}} = \left(|a^s| + |a^t| \frac{\|x_i^t\|_q^q}{\|x_i^s\|_q^q}\right)^{\frac{p-q}{q}} = C_s(a^1, a^2),$$

where $t \neq s$ and $\{t, s\} = \{1, 2\}$.

Since $p \neq q$ this is equivalent to the conditions that for all $(i, j) \in \operatorname{supp} x^s$:

$$\frac{\|x_i^t\|_q}{\|x_i^s\|_q} = \frac{\|x_j^t\|_q}{\|x_j^s\|_q}.$$

This means that either $\text{v-supp}\, x^1 \cap \text{v-supp}\, x^2 = \varnothing$ or $\text{v-supp}\, x^1 = \text{v-supp}\, x^2$ and $\|x_i^1\|_q = \|x_i^2\|_q$ for all $i \in \text{v-supp}\, x^1$ (since $\|x^1\| = \|x^2\|$).

The fact that $\operatorname{span}\{x^1, x^2\}$ is isometric to $\ell_p^2$ or $\ell_q^2$, resp., follows immediately. $\square$

*Proof of Lemma 5.5.*

First we prove that if Part $(a)$ does not hold then $p=2$ or $q=2$. The proof follows the standard technique of showing that $\ell_2$ can be isometrically embedded in $\ell_p$ only when $p$ is an even integer (see [LV]) and was suggested to me by Alex Koldobsky.

Assume that $F = \operatorname{span}\{x, y\}$, where $\|x\| = \|y\| = 1$ and for all $a, b \in \mathbb{C}$

$$\|ax + by\| = (|a|^2 + |b|^2)^{\frac{1}{2}}.$$

Then

$$(|a|^2 + |b|^2)^{\frac{p}{2}} = \|ax + by\|^p = \sum_{i=1}^{n} \left( \sum_{j=1}^{m} |ax_{ij} + by_{ij}|^q \right)^{\frac{p}{q}}. \tag{5.3}$$

For each $i \le n$ let $S_i = \{j \le m \ : \ (x_{ij}, y_{ij}) \ne (0,0)\} = \{j \le m \ : (i,j) \in \operatorname{supp} x \cup \operatorname{supp} y\}$. We define an equivalence relation $R_i$ on the set $S_i$ by the condition that $(j_1, j_2) \in R_i$ if and only if the pairs $(x_{ij_1}, y_{ij_1})$ and $(x_{ij_2}, y_{ij_2})$ are proportional.

Let $\Lambda_i$ denote the set of equivalence classes for $R_i$ and let $J = \{i \le n \ : \ \operatorname{card}(\Lambda_i) = 1\}$ and $M = \{i \le n \ : \ \operatorname{card}(\Lambda_i) > 1\}$. For each $i \in J$ let $j_i$ be the representative of the equivalence class of $R_i$ (i.e. $j_i \in S_i$) and for $i \in M$ let $\{j_\alpha\}_{\alpha \in \Lambda_i}$ be the set of representatives of each equivalence class. Then there exist positive constants $\{K_i\}_{i \in J}$, $\{C_{i,\alpha}\}_{i \in M, \alpha \in \Lambda_i}$ so that (5.3) can be written as

$$(|a|^2 + |b|^2)^{\frac{p}{2}} = \sum_{i \in J} K_i |ax_{ij_i} + by_{ij_i}|^p + \sum_{i \in M} \left( \sum_{\alpha \in \Lambda_i} C_{i,\alpha} |ax_{ij_\alpha} + by_{ij_\alpha}|^q \right)^{\frac{p}{q}}, \tag{5.4}$$

where the pairs $(x_{ij_\alpha}, y_{ij_\alpha})_{\alpha \in \Lambda_i}$ are mutually linearly independent for each $i \in M$. If there exists $i \in M$ and $\beta \in \Lambda_i$ with $x_{ij_\beta} \cdot y_{ij_\beta} \ne 0$, then there exist $a_0, b_0 \in \mathbb{C}$ with

$$a_0 x_{ij_\beta} + b_0 y_{ij_\beta} = 0$$

and

$$a_0 x_{ij_\alpha} + b_0 y_{ij_\alpha} \ne 0 \quad \text{for all } \alpha \ne \beta, \alpha \in \Lambda_i.$$

Fix $b = b_0$ and differentiate (5.4) with respect to $a$ along the real axis at $a_0$. The left-hand side of (5.4) can be differentiated (in this fashion) infinitely many times. But, if $q$ is not an even integer then the $([q]+1)$-st derivative of $|ax_{ij_\beta} + b_0 y_{ij_\beta}|^q$ does not exist at $a_0$.

Hence, if $q$ is not even, $x_{ij} \cdot y_{ij} = 0$ for all $i \in M, \ j \in S_i$. In particular, this implies that $\operatorname{card}(\Lambda_i) = 2$ for all $i \in M$.

Similarly, if there exists $i \in J$ with $x_{ij_i} \cdot y_{ij_i} \ne 0$, then there exist $a_0, b_0 \in \mathbb{C}$ with $a_0 x_{ij_\beta} + b_0 y_{ij_\beta} = 0$, we fix $b = b_0$ and differentiate (5.4) with respect to $a$ along the real axis at $a_0$. As before, the left-hand side of (5.4) can be differentiated (in this fashion) infinitely many times. But, if $p$ is not an even integer then the $([p]+1)$-st derivative of $|ax_{ij_i} + b_0 y_{ij_i}|^p$ does not exist at $a_0$.

Hence, if $p$ is not even, $x_{ij} \cdot y_{ij} = 0$ for all $i \in J,\ j \in S_i$.

Further, by Lemma 4.1, if $x^*, y^*$ denote norming functionals for $x, y$, respectively, then $\operatorname{span}\{x^*, y^*\}$ is isometric to $\ell_2^2$ and is 1-complemented in $(\ell_p(\ell_q))^* = \ell_{p'}(\ell_{q'})$, where $1/p + 1/p' = 1 = 1/q + 1/q'$. By (5.1) relations analogous to $R_i$ defined using coefficients of $x^*$ and $y^*$ are identical as the original relations $R_i$. Hence, if $q'$ is not even, then $x^*_{ij} \cdot y^*_{ij} = 0$ for all $i \in M,\ j \in S_i$ and if $p'$ is not even, then $x^*_{ij} \cdot y^*_{ij} = 0$ for all $i \in J,\ j \in S_i$.

Thus, by (5.1), if $q$, $q'$, $p$, $p'$ are not all even integers, that is, $q \neq 2$ and $p \neq 2$ then $x$ and $y$ are (scalarly) disjoint. □

We postpone the proof of Part $(c)$, because the proof of Lemma 5.6 is the direct continuation of just presented argument and uses the same notation.

*Proof of Lemma 5.6.*

The "only if" part: By the first part of the proof of Lemma 5.5, since $q \neq 2$, $x_{ij} \cdot y_{ij} = 0$ for all $i \in M,\ j \in S_i$. We will show that $M = \varnothing$.

Since $\operatorname{span}\{x, y\} = \ell_2^2$ we get, by Lemma 4.1, that for all $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$, the functional $\bar{a}x^* + \bar{b}y^*$ is norming for $ax + by$. That is, by (5.1),

$$\sum_i \|ax_i + by_i\|_q^{2-q} \sum_j |ax_{ij} + by_{ij}|^{q-1} \operatorname{sgn}(\overline{ax_{ij} + by_{ij}}) e^*_{ij} =$$
$$\bar{a} \sum_i \|x_i\|_q^{2-q} \sum_j |x_{ij}|^{q-1} \operatorname{sgn}(\overline{x_{ij}}) e^*_{ij} + \bar{b} \sum_i \|y_i\|_q^{2-q} \sum_j |y_{ij}|^{q-1} \operatorname{sgn}(\overline{y_{ij}}) e^*_{ij}.$$

Thus if $i \in M$, by disjointness of $x_i, y_i$, for each $j$ with $(i, j) \in \operatorname{supp} x$ we have

$$\|ax_i + by_i\|_q^{2-q} |ax_{ij}|^{q-1} \operatorname{sgn}(\overline{ax_{ij}}) = \bar{a} \|x_i\|_q^{2-q} |x_{ij}|^{q-1} \operatorname{sgn}(\overline{x_{ij}}).$$

Hence, since $q \neq 2$,

$$\|ax_i + by_i\|_q = |a| \|x_i\|_q = \|ax_i\|_q.$$

Since $x_i$ and $y_i$ are disjoint and $\ell_q$ is strictly monotone we conclude that $y_i = 0$. But this implies that $\operatorname{card}(\Lambda_i) = 1$, which contradicts the fact that $i \in M$. Thus $M = \varnothing$.

Hence $J = \text{v-supp}\, x \cup \text{v-supp}\, y$ and for each $i \in \text{v-supp}\, x \cap \text{v-supp}\, y$ there exists a constant $C_i \in \mathbb{C}$ with

$$x_i = C_i y_i.$$

Now (5.3), (5.4) and definition of $J$ imply that for every map $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$ such that $(i, \sigma(i)) \in \operatorname{supp} x \cup \operatorname{supp} y$ whenever $i \in \text{v-supp}\, x \cup \text{v-supp}\, y$ we have (with the convention $0/0=0$):

$$
\begin{aligned}
|a|^2+|b|^2=\|ax+by\|^2 &= \sum_{i\in J}\left(\sum_{j\in S_i}|ax_{ij}+by_{ij}|^q\right)^{\frac{2}{q}}\\
&=\sum_{i\in J}\left(\sum_{j\in S_i}\left(\frac{|x_{ij}|}{|x_{i\sigma(i)}|}|ax_{i\sigma(i)}+by_{i\sigma(i)}|\right)^q\right)^{\frac{2}{q}}\\
&=\sum_{i\in J}\left(\sum_{j\in S_i}\frac{|x_{ij}|^q}{|x_{i\sigma(i)}|^q}\right)^{\frac{2}{q}}|ax_{i\sigma(i)}+by_{i\sigma(i)}|^2\\
&=\sum_{i\in J}\frac{\|x_i\|_q^2}{|x_{i\sigma(i)}|^2}|ax_{i\sigma(i)}+by_{i\sigma(i)}|^2\\
&=\sum_{i\in J}\left|a\frac{\|x_i\|_q}{|x_{i\sigma(i)}|}x_{i\sigma(i)}+b\frac{\|y_i\|_q}{|y_{i\sigma(i)}|}y_{i\sigma(i)}\right|^2.
\end{aligned}
$$

Thus the vectors

$$
s_\sigma(x)=\left(\|x_i\|_q\frac{x_{i\sigma(i)}}{|x_{i\sigma(i)}|}\right)_{i\le n}\in\ell_2,\ \text{and}\ s_\sigma(y)=\left(\|y_i\|_q\frac{y_{i\sigma(i)}}{|y_{i\sigma(i)}|}\right)_{i\le n}\in\ell_2
$$

(with the convention $0/0=0$) are orthonormal.

The “if” part: It is clear from above calculations that if $x,y$ are of the form described in Lemma 5.6 then $\operatorname{span}\{x,y\}$ is isometrically isomorphic to $\ell_2^2$.

Further, by (5.1) we have for all $a,b\in\mathbb{C}$:

$$
\begin{aligned}
(ax+by)^* &= \frac{1}{(|a|^2+|b|^2)^{\frac12}}\sum_i\|ax_i+by_i\|_q^{2-q}\sum_j|ax_{ij}+by_{ij}|^{q-1}\operatorname{sgn}(\overline{ax_{ij}+by_{ij}})e_{ij}^*\\
&=\frac{1}{(|a|^2+|b|^2)^{\frac12}}\Bigg(\sum_{i\in \text{v-supp}\,x\setminus\text{v-supp}\,y}\|ax_i\|_q^{2-q}\sum_j|ax_{ij}|^{q-1}\operatorname{sgn}(\overline{ax_{ij}})e_{ij}^*\\
&\quad+\sum_{i\in\text{v-supp}\,x\cap\text{v-supp}\,y}\|ax_i+bC_ix_i\|_q^{2-q}\sum_j|ax_{ij}+bC_ix_{ij}|^{q-1}\operatorname{sgn}(\overline{ax_{ij}+bC_ix_{ij}})e_{ij}^*\\
&\quad+\sum_{i\in\text{v-supp}\,y\setminus\text{v-supp}\,x}\|by_i\|_q^{2-q}\sum_j|by_{ij}|^{q-1}\operatorname{sgn}(\overline{by_{ij}})e_{ij}^*\Bigg)
\end{aligned}
$$

$$= \frac{1}{(|a|^2+|b|^2)^{\frac{1}{2}}}\Big(\bar{a}\sum_{i\in \text{v-supp}\, x\setminus \text{v-supp}\, y}\|x_i\|_q^{2-q}\sum_j |x_{ij}|^{q-1}\operatorname{sgn}(\overline{x_{ij}})e^*_{ij}$$
$$+\overline{a+bC_i}\sum_{i\in \text{v-supp}\, x\cap \text{v-supp}\, y}\|x_i\|_q^{2-q}\sum_j |x_{ij}|^{q-1}\operatorname{sgn}(\overline{x_{ij}})e^*_{ij}$$
$$+\bar{b}\sum_{i\in \text{v-supp}\, y\setminus \text{v-supp}\, x}\|y_i\|_q^{2-q}\sum_j |y_{ij}|^{q-1}\operatorname{sgn}(\overline{y_{ij}})e^*_{ij}\Big)$$
$$= \frac{1}{(|a|^2+|b|^2)^{\frac{1}{2}}}(\bar{a}x^*+\bar{b}y^*).$$

Thus, by Lemma 2.5, $\operatorname{span}\{x,y\}$ is 1-complemented. □

*Proof of Lemma 5.5(c).* Assume that $q=2$. First notice that, if $z\in\ell_p(\ell_2)$ and $\|z\|=1$, then the norming functional $z^*$ for $z$ is given by

$$z^* = \sum_{i=1}^{n}\|z_i\|_2^{p-2}\cdot\sum_{j=1}^{m}\overline{z_{ij}}e_{ij} \tag{5.5}$$

Let $F=\operatorname{span}\{x,y\}$, where $\|x\|=\|y\|=1$ and $\|ax+by\|=(|a|^2+|b|^2)^{1/2}$ for all $a,b\in\mathbb{C}$. Let $U_i:\ell_m^2\to\ell_m^2$ be a surjective isometry such that, for all $i\le n$,

$$U_ix_i=\|x_i\|_2e_1.$$

Define an isometry $U:\ell_p(\ell_2)\to\ell_p(\ell_2)$ by

$$U((z_i)_{i=1}^n)=(U_iz_i)_{i=1}^n.$$

Then $UF=\operatorname{span}\{Ux,Uy\}$ is isometric to $\ell_2^2$ and is 1-complemented in $\ell_p(\ell_2)$.

Thus, by Lemma 4.1, for all $a,b\in\mathbb{C}$ with $|a|^2+|b|^2=1$, we have

$$(aUx+bUy)^*=\bar{a}(Ux)^*+\bar{b}(Uy)^*.$$

Hence, by (5.5) and by the choice of $U$, we get for all $i\le n$ and all $j\ge 2$:

$$\begin{aligned}\|a(Ux)_i+b(Uy)_i\|_2^{p-2}\cdot\left(\bar{a}\cdot 0+\bar{b}\cdot\overline{(Uy)_{ij}}\right) &=\\ \bar{a}\cdot\|(Ux)_i\|_2^{p-2}\cdot 0+\bar{b}\cdot\|(Uy)_i\|_2^{p-2}\overline{(Uy)_{ij}}.\end{aligned} \tag{5.6}$$

Now for each $i$ with $\|(Ux)_i\|_2\neq 0$ we consider two cases: either

1. there exists $j\ge 2$ with $(Uy)_{ij}\neq 0$, or
2. $(Uy)_i=((Uy)_{i1},0,0,\dots,0)$.

In case (1) we have

$$\|a(Ux)_i+b(Uy)_i\|_2^{p-2}=\|(Uy)i\|_2^{p-2}$$

whenever $|a|^2 + |b|^2 = 1$. In particular, since $p \neq 2$ and $\|(Ux)_i\|_2 = \|(Uy)_i\|_2$, we get

$$\left\| a\frac{(Ux)_i}{\|(Ux)_i\|_2} + b\frac{(Uy)_i}{\|(Uy)_i\|_2} \right\|_2 = 1.$$

Thus $(Ux)_i/\|(Ux)_i\|_2$ and $(Uy)_i/\|(Uy)_i\|_2$ form an orthonormal basis for $\ell_2$. Since $(Ux)_i/\|(Ux)_i\| = (1, 0, 0, \dots)$, the vectors $(Ux)_i$ and $(Uy)_i$ are disjoint.

In case (2), i.e. when $(Uy)_i = ((Uy)_{i1}, 0, 0, \dots, 0)$, then by (5.5) and by the form of $U$ we get for all $a, b$ with $|a|^2 + |b|^2 = 1$:

$$\begin{aligned} |a(Ux)_{i1} + b(Uy)_{i1}|^{p-2} \cdot (\bar{a}(Ux)_{i1} + \bar{b}\,\overline{(Uy)_{i1}}) &= \\ \bar{a}(Ux)_{i1}^{p-2}(Ux)_{i1} + \bar{b}|(Uy)_{i1}|^{p-2}\overline{(Uy)_{i1}}. \end{aligned} \tag{5.7}$$

Let $a \in \mathbb{R}_+$ and $b = ce^{i\theta}$, where $c \in \mathbb{R}_+$, $a^2 + c^2 = 1$ and $\theta$ is such that $e^{-i\theta} \cdot \overline{(Uy)_{i1}} = |(Uy)_{i1}|$. Set $\alpha = (Ux)_{i1} > 0$, $\beta = |(Uy)_{i1}| \geq 0$. Then (5.7) becomes

$$(a\alpha + c\beta)^{p-1} = a\alpha^{p-1} + c\beta^{p-1}, \tag{5.8}$$

and this equation holds for all $a, c \geq 0$ with $a^2 + c^2 = 1$.

For any fixed $u, v \geq 0$ define a function $f_{u,v} : [0, 1] \to \mathbb{R}$ by

$$f_{u,v}(a) = au + \sqrt{1 - a^2}v.$$

It is not difficult to check that $f_{u,v}$ attains its maximum on $[0, 1]$ at the point $a_0 = a_0(u, v) = u(u^2 + v^2)^{-1/2}$ and the maximum value of $f_{u,v}$ is equal to $M(u, v) = (u^2 + v^2)^{1/2}$.

Since equation (5.4) can be written as

$$(f_{\alpha,\beta}(a))^{p-1} = f_{\alpha^{p-1},\beta^{p-1}(a)},$$

we have

$$\begin{aligned} a_0(\alpha, \beta) &= a_0(\alpha^{p-1}, \beta^{p-1}), \\ M(\alpha, \beta)^{p-1} &= M(\alpha^{p-1}, \beta^{p-1}). \end{aligned}$$

Thus

$$\left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}\right)^{p-1} = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}.$$

Since $p \neq 2$, we conclude that either $\alpha = 0$ or $\alpha(\alpha^2 + \beta^2)^{-1/2} = 1$, i.e., $\beta = 0$. But $\alpha = (Ux)_{i1} = \|(Ux)_i\|_2$ and $\beta = |(Uy)_{i1}| = \|(Uy)_i\|_2$. Thus $(Ux)_i = 0$ or $(Uy)_i = 0$. Hence $Ux$ and $Uy$ are disjoint. $\square$

**Remark.** Lemmas 5.4-5.6 are all valid (with the presented proofs) both in the complex and real case. We suspect that Theorems 5.1 and 5.2, too, are true in the real case, but our method of proof does not work then.

## 6. 1-complemented disjointly supported subspaces of Orlicz and Lorentz spaces

In this section we fully characterize subspaces of (real or complex) Orlicz and Lorentz spaces that are spanned by disjointly supported elements and 1-complemented.

In particular, it follows from Theorems 6.1 and 6.3 that in "most" Orlicz and Lorentz spaces the only 1-complemented disjointly supported subspaces are those spanned by a block basis with constant coefficients (of some permutation of the original basis).

**Theorem 6.1.** *Let $\ell_\varphi$ be a (real or complex) Orlicz space and let $x, y \in \ell_\varphi$, be disjoint elements such that $\|x\|_\varphi = \|y\|_\varphi = 1$ and $\operatorname{span}\{x, y\}$ is 1-complemented in $\ell_\varphi$.*

*Then one of three possibilities holds:*

(1) $\operatorname{card}(\operatorname{supp} x) < \infty$ *and* $|x_i| = |x_j|$ *for all* $i, j \in \operatorname{supp} x$*; or*
(2) *there exists $p$, $1 \le p \le \infty$, such that $\varphi(t) = Ct^p$ for all $t \le \|x\|_\infty$; or*
(3) *there exists $p$, $1 \le p \le \infty$, and constants $C_1, C_2, \gamma \ge 0$ such that $C_2 t^p \le \varphi(t) \le C_1 t^p$ for all $t \le \|x\|_\infty$ and such that, for all $j \in \operatorname{supp} x$,*
$$|x_j| = \gamma^{k(j)} \cdot \|x\|_\infty$$
*for some $k(j) \in \mathbb{Z}$.*

For the proof of the theorem we will need the following (well-known?) lemma, whose proof is outlined in [Z]. For the convenience of the reader we provide its proof below.

**Lemma 6.2.** *Let $a > 0$ and suppose that $\varphi : [0, a] \to \mathbb{R}$ is an increasing differentiable function with $\varphi(0) = 0$. Suppose that there exist $a < \alpha, \beta < 1$ so that, for all $u \le a$,*
$$\varphi(\alpha u) = \beta\varphi(u). \tag{6.1}$$
*Then there exist $p > 0$ and $C_1, C_2 > 0$ such that, for all $u \le a$,*
$$C_2 u^p \le \varphi(u) \le C_1 u^p.$$
*Moreover, if $\varphi(u) \not\equiv C \cdot u^p$, there exists $\gamma > 0$ such that* (6.1) *is satisfied (with the corresponding $\beta$) if and only if $\alpha = \gamma^k$ for some $k \in \mathbb{Z}$.*

*Proof of Theorem 6.1.* Let $z \in \ell_\varphi$. By [GH] the norming functional $z^*$ of $z$ is given by:
$$z_i^* = \frac{1}{C}\operatorname{sgn}(\overline{z_i})\varphi'\Big(\frac{|z_i|}{\|z\|_\varphi}\Big),$$
where $C$ is a constant depending on $z$. By Proposition 2.4, for all $b \in \mathbb{C}$ there exist constants $K_1, K_2$ such that
$$(x + by)^* = K_1 x^* + K_2 y^*.$$

Since $x$ and $y$ are disjoint, there exists a constant $K = K(b)$ so that for all $i \in \operatorname{supp} x$

$$\operatorname{sgn}(\overline{x_i})\varphi'\left(\frac{|x_i|}{\|x+by\|_\varphi}\right) = K \cdot \operatorname{sgn}(\overline{x_i})\varphi'(|x_i|).$$

Now for all $,0 < t < 1$ there exists $b \in \mathbb{C}$ so that $\|x+by\|_\varphi = t^{-1}$. Thus, for all $0 < t \le 1$, there exists $C_t > 0$ so that for all $i \in \operatorname{supp} x$

$$\varphi'(|x_i| \cdot t) = C_t \varphi'(|x_i|).$$

Hence for all $i, j \in \operatorname{supp} x$ and for all $t \le 1$

$$\frac{\varphi'(|x_i| \cdot t)}{\varphi'(|x_i|)} = \frac{\varphi'(|x_j| \cdot t)}{\varphi'(|x_j|)}.$$

Set

$$\beta = \frac{\varphi'(|x_i|)}{\varphi'(|x_j|)}, \quad u = |x_j| \cdot t, \quad \alpha = \frac{|x_i|}{|x_j|}.$$

In this notation we have

$$\varphi'(\alpha u) = \beta \varphi'(u)$$

for all $u$ such that $0 \le u \le |x_j|$. Thus

$$\varphi(\alpha u) = \beta \alpha \varphi(u)$$

for all $u$ such that $0 \le u \le |x_j|$.

Let $j \in \operatorname{supp} x$ be such that $|x_j| = \|x\|_\infty$. If there exists $i \in \operatorname{supp} x$ with $|x_i| \ne |x_j| = \|x\|_\infty$ then, by Lemma 6.2, condition (2) or (3) holds. $\square$

*Proof of Lemma 6.2.* Let $p = \log_\alpha(\beta)$. Let $m_0 \in \mathbb{Z}$ be the smallest integer with $\alpha^{m_0} \le a$.

If $\alpha^{m_0} < u \le a$ we have

$$\varphi(u) \le \varphi(a) \le \frac{\varphi(a)}{\alpha^{m_0 p}} \cdot u^p$$

and

$$\varphi(u) \ge \varphi(\alpha^{m_0}) \ge \frac{\varphi(\alpha^{m_0})}{a^p} \cdot u^p.$$

If $\alpha^{m+1} < u \le \alpha^m$ for some $m \le m_0$ we have

$$\varphi(u) \le \varphi(\alpha^m) = \beta\varphi(\alpha^{m-1}) = \cdots = \beta^{m-m_0}\varphi(\alpha^{m_0}) = \frac{\varphi(\alpha^{m_0})}{\beta^{m_0+1}}(\alpha^p)^{m+1} \le \frac{\varphi(\alpha^{m_0})}{\beta^{m_0+1}} \cdot u^p$$

and

$$\varphi(u) \ge \varphi(\alpha^{m+1}) = \beta\varphi(\alpha^m) = \cdots = \beta^{m+1-m_0}\varphi(\alpha^{m_0}) = \frac{\varphi(\alpha^{m_0})}{\beta^{m_0-1}} \cdot (\varphi^p)^m \ge \frac{\varphi(\alpha^{m_0})}{\beta^{m_0-1}} u^p.$$

Set $C_1 = \max\{\varphi(a)/\alpha^{m_0 p},\ \varphi(\alpha^{m_0})/\beta^{m+1}\}$ and $C_2 = \min\{\varphi(\alpha^{m_0})/a^p,\ \varphi(\alpha^{m_0})/\beta^{m_0-1}\}$. Then

$$C_2 u^p \le \varphi(u) \le C_1 u^p$$

for all $u$ with $0 \leq u \leq a$.

Further define a function $h_\varphi : (-\infty, \ln a] \to \mathbb{R}$ by

$$h_\varphi(t) = \frac{d}{ds}(\ln(\varphi(e^s)))\big|_{s=t}.$$

Then, by (6.1), $h_\varphi(t + \ln(\alpha)) = h_\varphi(t)$ for all $t \leq \ln a$. Thus, since $\alpha \neq 1$, either

- $h_\varphi$ is constant, that is, there exists a constant $K$ so that $\varphi(u) = K \cdot u^p$ for all $u \leq a$, or
- $h_\varphi$ is periodic, that is, there exists $w$, with minimal $|w|$, so that $h_\varphi(t+w) = h_\varphi(t)$ for all $t \leq \ln a$.

Thus there exists $\gamma > 0$ (namely $\gamma = e^w$) and $k \in \mathbb{Z}$ such that $\alpha = \gamma^k$. □

Our next theorem describes disjointly supported 1-complemented subspaces of (real or complex) Lorentz sequence spaces.

**Theorem 6.3.** *Let $\ell_{w,p}$, with $1 < p < \infty$, be a real or complex Lorentz sequence space. Suppose that $\{x_i\}_{i\in I}$ are mutually disjoint elements of $\ell_{w,p}$ such that* $\mathrm{card}(I) \geq 2$ *and $F = \overline{\mathrm{span}}\{x_i\}_{i\in I}$ is 1-complemented in $\ell_{w,p}$. Suppose, moreover, that $w_\nu \neq 0$ for all $\nu \leq \Sigma \stackrel{\text{def}}{=} \sum_{i\in I} \mathrm{card}(\mathrm{supp}\, x_i) \quad (\leq \infty)$.*

*Then*

(*a*) *$w_\nu = 1$ for all $\nu \leq \Sigma$,*

*or*

(*b*) *$|x_{il}| = |x_{ik}|$ for all $i \in I$ and all $k, l \in \mathrm{supp}\, x_i$.*

*Proof.* With each element $z \in \ell_{w,p}$ we associate a decreasing sequence of positive numbers $(\tilde{z}_i)_{i=1}^{l(z)}$ and "level sets" $A_i(z)$ defined inductively as follows:

$$\begin{aligned} \tilde{z}_1 &= \|z\|_\infty, & A_1(z) &= \{j \in \mathbb{N} : |z_j| = \tilde{z}_1\}, \\ \tilde{z}_2 &= \max\{|z_j| : j \in \mathbb{N} \setminus A_1(z)\}, & A_2(z) &= \{j \in \mathbb{N} : |z_j| = \tilde{z}_2\}, \end{aligned}$$

and so on. Note that $l(z)$ is the largest integer such that $\tilde{z}_{l(z)} > 0$ and $\mathrm{supp}\, z = \bigcup_{i=1}^{l(z)} A_i(z)$.

For $i \leq l(z)$ introduce also

$$s_0(z) = 0, \quad s_i(z) = \sum_{j=1}^{i} \mathrm{card}(A_i(z)),$$

$$L_i(z) = \{s_{i-1}(z) + 1, \ldots, s_i(z)\} \subset \mathbb{N},$$

and let $\delta_i : A_i(z) \to L_i(z)$ be a bijection.

Finally, for any set $A \subset \mathbb{N}$ denote by $\mathcal{P}(A)$ the set of all permutations of $A$.

In this notation we can easily describe norming functionals $z^N$ for $z$. Namely, for each $j$ with $1 \le j \le l(z)$, there exists a family of coefficients $\{\lambda_\sigma\}_{\sigma\in\mathcal{P}(L_j(z))}$ such that $\lambda_\sigma \ge 0$, $\quad \sum_{\sigma\in\mathcal{P}(L_j)} \lambda_\sigma = 1$ and

$$\big(|z_k^N|\big)_{k\in A_j(z)} = \left(\frac{\tilde{z}_j}{\|z\|}\right)^{p-1} \sum_{\sigma\in\mathcal{P}(L_j(z))} \lambda_\sigma \big(w_{\sigma(\delta_j(k))}\big)_{k\in A_j(z)}.$$

In particular, we can compute the $\ell_1$-norm of $z^N$ restricted to a level set $A_j(z)$

$$\left\|\big(|z_k^N|\big)_{k\in A_j(z)}\right\|_{\ell_1} = \left(\frac{\tilde{z}_j}{\|z\|}\right)^{p-1} \sum_{n\in L_j(z)} w_n. \tag{6.2}$$

Notice that the left hand side of (6.2) does not depend on the choice of the norming functional $z^N$ for $z$.

Now assume that $l(x_1) > 1$. We will show that $w_\nu = 1$ for all natural numbers $\nu \le \Sigma$, where $\Sigma \overset{\text{def}}{=} \sum_{i\in I} \operatorname{card}(\operatorname{supp} x_i)$ as defined above.

If $\nu \le \Sigma$, there exists $n \le l_1(x_1)$ such that

$$\nu - s_n(x_1) \le \Sigma - \operatorname{card}(\operatorname{supp} x_1).$$

Further, there exist $\mu \in \mathbb{N}$ and $\{j_i\}_{i=2}^{\mu} \subset \mathbb{N}$ such that $j_i \le l(x_i)$ for $i = 2, \dots, \mu$ and

$$\sum_{i=2}^{\mu} s_{j_i}(x_i) =: M \ge \nu - s_N(x_1).$$

Choose $\{a_i\}_{i=1}^{\mu} \subset \mathbb{R}_+$ such that, for all $i$ with $2 \le i \le \mu$,

$$a_1(\widetilde{x_1})_1 > a_i(\widetilde{x_i})_{j_i} > a_1(\widetilde{x_1})_2$$

and $a_1(\widetilde{x_1})_n > a_i(\widetilde{x_i})_j$ for all $j > j_i$.

To shorten the notation, set $x = \sum_{i=1}^{\mu} a_1 x_1$. Then

$$a_1(\widetilde{x_1})_1 = \tilde{x}_1 \quad \text{and} \quad A_1(x) = A_1(x_1). \tag{6.3}$$

Moreover, there exists $k \in \mathbb{N}$ with $2 < k \le 1 + \sum_{i=2}^{\mu} j_i$ such that, for all $\alpha$ satisfying $2 \le \alpha \le n$,

$$a_1(\widetilde{x_1})_\alpha = \tilde{x}_{k+(\alpha-2)}, \quad A_{k+(\alpha-2)}(x) = A_\alpha(x_1), \tag{6.4}$$

and

$$s_{k-1}(x) = s_1(x_1) + M.$$

Thus

$$L_1(x) = L_1(x_1) = \{1, \dots, s_1(x_1)\}, \tag{6.5}$$

and, for all $\alpha$ with $2 \le \alpha \le n$,

$$L_{k+(\alpha-2)}(x) = M + L_\alpha(x_1) = \{M + s_{\alpha-1}(x_1) + 1, \dots, M + s_\alpha(x_1)\}. \tag{6.6}$$

By Proposition 2.4 there exist norming functionals $x^N$ for $x$ and $x_i^N$ for $x_i$, and constants $K_i$, where $i = 1, \dots, \mu$, such that

$$x^N = \sum_{i=1}^{\mu} K_i x_i^N.$$

Thus, by (6.2) and (6.3),

$$\left(\frac{\tilde{x}_1}{\|x\|}\right)^{p-1} \sum_{j \in L_1(x)} w_j = K_1 \cdot \left(\frac{a_1(\widetilde{x_1})_1}{\|x_1\|}\right)^{p-1} \sum_{j \in L_1(x_1)} w_j.$$

Hence, by (6.5),

$$K_1 \cdot \left(\frac{\|x\|}{\|x_1\|}\right)^{p-1} = 1. \tag{6.7}$$

Moreover, by (6.2) and (6.4), for all $\alpha$ with $2 \leq \alpha \leq n$ we get:

$$\left(\frac{\tilde{x}_{k+(\alpha-2)}}{\|x\|}\right)^{p-1} \sum_{j \in L_{k+(\alpha-2)}(x)} w_j = K_1 \cdot \left(\frac{a_1(\widetilde{x_1})_\alpha}{\|x_1\|}\right)^{p-1} \sum_{j \in L_\alpha(x_1)} w_j.$$

Hence, by (6.7) and (6.6),

$$\sum_{j=s_{\alpha-1}(x_1)+1}^{s_\alpha(x_1)} w_{j+M} = \sum_{j=s_{\alpha-1}(x_1)+1}^{s_\alpha(x_1)} w_j.$$

Since $\{w_j\}$ is a decreasing sequence of numbers we immediately conclude that, for all $\alpha$ with $2 \leq \alpha \leq n$,

$$w_{s_{\alpha-1}(x_1)+1} = w_{s_\alpha(x_1)+M}.$$

Since $M \geq 1$ we get

$$w_{s_1(x_1)+1} = w_{s_2(x_1)+M} = w_{s_2(x_1)+1} = w_{s_3(x_1)+M} = \cdots = w_{s_n(x_1)+M}. \tag{6.8}$$

Finally, choose $\{b_i\}_{i=1}^{\mu} \subset \mathbb{R}_+$ in such a way that, for all $i$ with $2 \leq i \leq \mu$,

$$b_i(\widetilde{x_i})_{j_i} > b_1(\widetilde{x_1})_1.$$

Now set $y = \sum_{i=1}^{\mu} b_i x_i$. Then there exists $t \in \mathbb{N}$, with $1 \leq t \leq 1 + \sum_{i=2}^{\mu} j_i$, such that for all $\alpha$ with $1 \leq \alpha \leq n$ we have

$$b_1(\widetilde{x_1})_\alpha = \tilde{y}_{t+(\alpha-1)} \quad ; A_\alpha(x_1) = A_{t+(\alpha-1)}(y). \tag{6.9}$$

Similarly, as before,

$$s_{t+(\alpha-1)}(y) = s_\alpha(x_1) + M$$

and

$$L_{t+(\alpha-1)}(y) = M + L_\alpha(x_1) = \{M + s_{\alpha-1}(x_1) + 1, \dots, M + s_\alpha(x_1)\}. \tag{6.10}$$

Again, by Proposition 2.4 there exist norming functionals $y^N$ for $y$ and $x_i^N$ for $x_i$, and constants $K_i'$, where $i = 1, \ldots, \mu$, such that

$$y^N = \sum_{i=1}^{\mu} K_i' x_i^N.$$

Thus, by (6.2) and (6.9) we get, for all $\alpha$ with $1 \le \alpha \le n$,

$$\left(\frac{\tilde{y}_{t+(\alpha-1)}}{\|y\|}\right)^{p-1} \cdot \sum_{j\in L_{t+(\alpha-1)}(y)} w_j = K_1' \left(\frac{(\widetilde{x}_1)_\alpha}{\|x\|}\right)^{p-1} \cdot \sum_{j\in L_\alpha(x_1)} w_j.$$

Hence, by (6.10),

$$\left(\frac{1}{\|y\|}\right)^{p-1} \cdot \sum_{j=s_{\alpha-1}(x_1)+1}^{s_\alpha(x_1)} w_{j+M} = K_1' \cdot \left(\frac{1}{\|x\|}\right)^{p-1} \sum_{j=s_{\alpha-1}(x_1)+1}^{s_\alpha(x_1)} w_j. \tag{6.11}$$

If $\alpha = 2$, by (6.8) we conclude that

$$\left(\frac{1}{\|y\|}\right)^{p-1} = K_1' \cdot \left(\frac{1}{\|x\|}\right)^{p-1}.$$

Thus, when $\alpha = 1$, (6.11) becomes

$$\sum_{j=1}^{s_1(x_1)} w_{j+M} = \sum_{j=1}^{s_1(x_1)} w_j.$$

Thus

$$w_1 = w_{s_1(x_1)+M}$$

and since $M \ge 1$, by (6.8) we get

$$w_1 = w_{s_n(x_1)+M}.$$

Since $\nu \le s_n(x_1) + M$ we conclude that

$$1 = w_1 = w_\nu,$$

which ends the proof. □

## References

[ACL] Z. Altshuler, P.G. Casazza, and B.L. Lin. On symmetric basic sequences in Lorentz sequence spaces. *Israel J. Math.*, 15:140–155, 1973.

[An] T. Ando. Contractive projections in $L_p$–spaces. *Pacific J. Math.*, 17:391–405, 1966.

[B] M. Baronti. Norm–one projections onto subspaces of $\ell^\infty$. *Arch. Math.*, 51:242–246, 1988.

[BP] M. Baronti and P. L. Papini. Norm-one projections onto subspaces of $\ell_p$. *Annali di Mat. Pura ed Appl.*, 152:53–61, 1988.

[BCh] J. Blatter and E. W. Cheney. Minimal projections on hyperplanes in sequence spaces. *Annali di Mat. Pura ed Appl.*, 101:215–227, 1974.

[BFL] Y. Benyamini, P. Flinn, and D.R. Lewis. A space without 1-unconditional basis which is 1-complemented in a space with a 1-unconditional basis. *Longhorn Notes, The University of Texas, Texas Funct. Anal. Seminar*, pages 145–149, 1983-84.

[C] B. Calvert. Convergence sets in reflexive Banach spaces. *Proc. Amer. Math. Soc.*, 47:423–428, 1975.

[CF] B. Calvert and S. Fitzpatrick. Characterizing $\ell_p$ and $c_0$ by projections onto hyperplanes. *Boll. Un. Math. Ital.*, (VI) 5:405–410, 1986.

[ChP] E. W. Cheney and K. H. Price. Minimal projections. In A. Talbot, editor, *Approximation Theory, Proceedings of a symposium held at Lancaster, July 1969*, pages 261 – 289. Academic Press, 1970.

[D] I. Doust. Contractive projections on Lebesgue–Bochner spaces. In *Function spaces (Edwardsville, IL, 1994)*, volume 172 of *Lecture Notes in Pure and Appl. Math.*, pages 101–109. Dekker, New York, 1995.

[F] P. Flinn. On a theorem of N.J. Kalton and G.V. Wood concerning 1-complemented subspaces of spaces having an orthonormal basis. *Longhorn Notes, The University of Texas, Texas Functional Analysis Seminar*, pages 135–144, 1983-84.

[GH] R. Grząślewicz and H. Hudzik. Smooth points of Orlicz spaces equipped with Luxemburg norm. *Math. Nachr.*, 155:31–45, 1992.

[KW] N. J. Kalton and G. V. Wood. Orthonormal systems in Banach spaces and their applications. *Math. Proc. Camb. Phil. Soc.*, 79:493–510, 1976.

[Ko] A. L. Koldobskiĭ. Isometries of $L_p(X; L_q)$ and equimeasurability. *Indiana Univ. Math. J.*, 40:677–705, 1991.

[Le] D.R. Lewis. Ellipsoids defined by Banach ideal norms. *Matematika*, 26:18–29, 1979.

[Lg] K.J. Lindberg. On subspaces of Orlicz sequence spaces. *Studia Math.*, 45:119–146, 1973.

[L] J. Lindenstrauss. A remark on symmetric bases. *Israel J. Math.*, 13:317–320, 1972.

[LT] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces, Vol. 1, Sequence spaces.* Springer–Verlag, Berlin–Heidelberg–New York, 1978.

[LV] Y.I. Lyubich and L.N. Vaserstein. Isometric embeddings between classical Banach spaces, cubature formulas, and spherical designs. *Geometriae Dedicata*, 47:327–362, 1993.

[R1] H. P. Rosenthal. Functional Hilbertian sums. *Pacific J. Math.*, 124:417–467, 1986.

[R2] H. P. Rosenthal. On one-complemented subspaces of complex Banach spaces with one-unconditional basis, according to Kalton and Wood. *Israel Seminar on Geometrical Aspects of Functional Analysis*, 1983-4. No IX.

[ST] H. Schneider and R.E.L. Turner. Matrices Hermitian for an absolute norm. *Linear and Multilinear Algebra*, 1:9–31, 1973.

[Z] M. G. Zaidenberg. Groups of isometries of Orlicz spaces. *Soviet Math. Dokl.*, 17:432–436, 1976.



Department of Mathematics, The University of Texas at Austin, Austin, TX 78712

*Current address*: Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720

*E-mail address*: `brandri@math.utexas.edu`