COVERING NUMBERS AND "LOW M^*-ESTIMATE" FOR QUASI-CONVEX BODIES

A. E. LITVAK, V. D. MILMAN, AND A. PAJOR

ABSTRACT. This article gives estimates on covering numbers and diameters of random proportional sections and projections of symmetric quasi-convex bodies in $\mathbb{R}^n$. These results were known for the convex case and played an essential role in development of the theory. Because duality relations can not be applied in the quasi-convex setting, new ingredients were introduced that give new understanding for the convex case as well.

1. Introduction and notation

Let $|\cdot|$ be a euclidean norm on $\mathbb{R}^n$. Let D be an ellipsoid associated with this norm. Denote $A = \sqrt{\frac{n}{k}} \int\limits_{S^{n-1}} \sqrt{\sum_{i=1}^k x_i^2}\, d\sigma(x)$, where σ is the normalized rotation invariant measure on the euclidean sphere S^{n-1}. Then $A = A(n,k) < 1$ and $A \longrightarrow 1$ as $n, k \longrightarrow \infty$. For any star-body K in $\mathbb{R}^n$ define $M_K = \int\limits_{S^{n-1}} \| x \|\, d\sigma(x)$, where $\| x \|$ is the gauge of K. Let M_K^* be M_{K^0}, where K^0 is the polar of K. For any subsets K_1, K_2 of $\mathbb{R}^n$ denote by $N(K_1, K_2)$ the smallest number N such that there are N points $y_1, ..., y_N$ in K_1 such that

$$K_1 \subset \bigcup_{i=1}^N (y_i + K_2).$$

Recall that a body K is called quasi-convex if there is a constant c such that $K + K \subset cK$, and given a $p \in (0,1]$ a body K is called p-convex if for any $\lambda, \mu > 0$ satisfying $\lambda^p + \mu^p = 1$ and any points $x, y \in K$ the point $\lambda x + \mu y$ belongs to K. Note that for the gauge $\| \cdot \| = \| \cdot \|_K$ associated with the the quasi-convex (p-convex) body K the following inequality holds for any $x, y \in \mathbb{R}^n$

$$\| x + y \| \leq C \max\{\| x \|, \| y \|\} \quad (\| x + y \|^p \leq \| x \|^p + \| y \|^p).$$

In particular every p-convex body K is also quasi-convex one and $K + K \subset 2^{1/p} K$. A more delicate result is that for every quasi-convex body K ($K + K \subset cK$) there exists a q-convex body K_0 such that $K \subset K_0 \subset 2cK$, where $2^{1/q} = 2c$. This is Aoki-Rolewicz theorem ([KPR], [R], see also [K], p.47). In this note by a body we always

Litvak's and Milman's research was partially supported by BSF. Research at MSRI is supported in part by NSF grant DMS-9022140.

mean a centrally-symmetric compact star-body, i.e. a body K satisfying $tK \subset K$ for any $t \in [-1,1]$, however not all results use symmetry. Lemma 2, some analog of Lemma 4 for p-convex body K, and Theorem 3 hold in non-symmetric case also.

Let us remind of the so-called "low M^*-estimate" result.

Theorem 1. *Let $\lambda > 0$ and n be large enough. Let K be a convex body in $\mathbb{R}^n$ and $\| \cdot \|$ be the gauge of K. Then there exists a subspace E of $(\mathbb{R}^n, \| \cdot \|)$ such that $\dim E = [\lambda n]$ and for any $x \in E$ the following inequality holds*

$$\| x \| \geq \frac{f(\lambda)}{M_K^*} |x|$$

for some function $f(\lambda)$, $0 < \lambda < 1$.

Remark. Inequality of this type was first proved in [M1] with very poor dependence on λ and then improved in [M2] to $f(\lambda) = C(1-\lambda)$. It was later shown ([PT]), that one can take $f(\lambda) = C\sqrt{1-\lambda}$ (for different proofs see [M3] and [G]).

By duality this theorem is equivalent to the following theorem.

Theorem 1' *Let $\lambda > 0$ and n be large enough. For any convex body K in $\mathbb{R}^n$ there exists an orthogonal projection P of rank $[\lambda n]$ such that*

$$PD \subset \frac{M_K}{f(\lambda)} PK \ ,$$

where c is an absolute constant.

In this note we will extend both theorems to quasi-convex bodies. Because duality arguments can not be applied to a non-convex body these two theorems become different statements. Also "" M_K^* " should be substituted by an appropriate quantity not involving duality. Note that by avoiding the use of convexity assumption we in fact simplified proof also for a convex case.

2. Main results

The following theorem is an extension of Theorem 1'.

Theorem 2. *Let $\lambda > 0$ and n be large enough ($n > c/(1-\lambda)^2$). For any p-convex body K in $\mathbb{R}^n$ there exists an orthogonal projection P of the rank $[\lambda n]$ such that*

$$PD \subset \frac{A_p M_K}{(1-\lambda)^{1+1/p}} PK \ ,$$

where $A_p = const^{\frac{\ln(2/p)}{p}}$.

The proof of this theorem is based on the next three lemmas. The first one was proved by W.B.Johnson and J.Lindenstrauss in [JL]. The second one was proved in [PT] for convex bodies and is the dual form of Sudakov minoration theorem.

Lemma 1. *There is an absolute constant c such that if $\varepsilon > \sqrt{c/k}$ and $N < e^{\varepsilon^2 k/c}$, then for any set of points $y_1, ..., y_N \in \mathbb{R}^n$ and any orthogonal projection P of rank k*

$$\mu\left(\{U \in O_n \mid \forall j : A(1-\varepsilon)\sqrt{k/n}\ |y_j| \le |PUy_j| \le A(1+\varepsilon)\sqrt{k/n}\ |y_j|\}\right) \ge$$
$$\ge 1 - \sqrt{\pi/2} \cdot e^{-\varepsilon^2 k/c}$$

Lemma 2. *Let K be a body such that $K - K \subset aK$. Then $N(D, tK) \le 2e^{2n(aM_K/t)^2}$.*

M. Talagrand gave a direct simple proof of this lemma for a convex case ([T]). Below, using his idea, we prove more general lemma for p-convex bodies, so we do not prove Lemma 2 now.

Lemma 3. *Let B be a star-body, K be a p-convex set, $r \in (0,1)$, $\{x_i\} \subset rB$ and $B \subset \bigcup(x_i + K)$. Then $B \subset t_r K$, where $t_r = \frac{1}{(1-r^p)^{1/p}}$.*

Proof: Obviously $t_r = \max\{\| x \|_K \mid x \in B\}$. Since $B \subset \bigcup(x_i + K)$, for any point x in B there are points x_0 in rB and y in K such that $x = x_0 + y$. Then by maximality of t_r and p-convexity of K we have $t_r^p \le r^p t_r^p + 1$. That proves the lemma. □

Remark. Somewhat similar argument was used by N. Kalton in dealing with p-convex sets.

Proof of Theorem 2:
Any p-convex body K satisfies $K - K \subset aK$ with $a = 2^{1/p}$. By Lemma 1 and Lemma 2, we obtain for $c_p \approx 2^{1/p}$ that if

$$c_p n \left(\frac{M_K}{t}\right)^2 \le \frac{\varepsilon^2 k}{c}$$

and $\varepsilon > \sqrt{c/k}$, then there exist points $x_1, ..., x_N$ in D and an orthogonal projection P of rank k such that

$$PD \subset \bigcup(Px_i + tPK) \quad \text{and} \quad |Px_i| \le (1+\varepsilon)\sqrt{\frac{k}{n}}|x_i| \quad .$$

Let $\lambda = k/n$. Denote $r = (1+\varepsilon)\sqrt{\lambda}$. Lemma 3 gives us

$$PD \subset tt_r PK \quad \text{for} \quad t = \frac{\sqrt{cc_p}M_K}{\varepsilon\sqrt{\lambda}} \quad \text{and} \quad \varepsilon^2 > \frac{c}{\lambda n}\ ,\ r < 1 \quad .$$

Choose

$$\varepsilon = \frac{1-\sqrt{\lambda}}{2\sqrt{\lambda}}\ .$$

Then for n large enough we get

$$PD \subset \frac{A_p M_K}{(1-\lambda)^{1+1/p}} PK\ ,$$

for $A_p = const^{\frac{\ln(2/p)}{p}}$. This completes the proof. □

Theorem 2 can be formulated in the global form.

Theorem 2' *Let K be a p-convex body in $\mathbb{R}^n$. Then there is an orthogonal operator U such that*

$$D \subset A_p^{'} M_K (K + UK), \quad \textit{where } A_p^{'} = const^{\frac{\ln(2/p)}{p}} .$$

This theorem can be proved independently, but we show how it follows from Theorem 2.

Proof of Theorem 2': It follows from the proof of Theorem 2 that actually the measure of such projections is large. So we can choose two orthogonal subspaces E_1, E_2 of $\mathbb{R}^n$ such that $\dim E_1 = [n/2]$, $\dim E_2 = [(n+1)/2]$ and

$$P_i D \subset A_p^{''} M_K P_i K ,$$

where P_i is the projection on the space E_i $(i = 1, 2)$. Denote $I = id_{\mathbb{R}^n} = P_1 + P_2$ and $U = P_1 - P_2$. So $P_1 = 1/2(I + U)$ and $P_2 = 1/2(I - U)$. Then U is an orthogonal operator and for any $x \in D$ we have

$$x = P_1 x + P_2 x \subset 1/2\ A_p^{''} M_K (I + U)K + 1/2\ A_p^{''} M_K (I - U)K =$$

$$= A_p^{''} M_K \frac{K + K}{2} + A_p^{''} M_K \frac{UK - UK}{2} = A_p^{'} M_K (K + UK) .$$

That proves Theorem 2'. □

Let us complement Lemma 2 by mentioning how covering number $N(K, tD)$ can be estimated. In convex case this estimate is given by Sudakov inequality, using quantity M^*. More precisely, if K is a convex body, then

$$N(K, tD) \leq 2e^{cn(M_K^*/t)^2} .$$

Of course, using duality for a non-convex setting leads to a weak result, and we suggest below a substitution for quantity M^*.

For two quasi-convex bodies K, B define the following number

$$M(K, B) = \frac{1}{|K|} \int\limits_K \| x \|_B \, dx ,$$

where $|K|$ is volume of K, and $\| x \|_B$ is the gauge of B. Such numbers are considered in [MP1], [MP2] and [BMMP].

Lemma 4. *Let K and B be star-bodies. Assume $B + B \subset aB$. For $\theta > 0$ denote by $c_\theta = c_\theta(K)$ the best possible constant such that*

$$\| x + y \|_K^\theta + \| x - y \|_K^\theta \leq 2 \cdot c_\theta^\theta \left(\| x \|_K^\theta + \| y \|_K^\theta \right) \quad \textit{for any} \quad x, y \in \mathbb{R}^n .$$

Then

$$N(K, tB) \leq 2 c_\theta^n e^{(cn/\theta)(aM(K,B)/t)^\theta} , \quad \textit{where } c \textit{ is an absolute constant.}$$

Note, that in a case $\theta = 1$ which correspond to the general convex case this lemma was announced in [MP2].

Proof: We follow the idea of M.Talagrand of estimating covering numbers in case $K = D$ ([T], see also [BLM] Proposition 4.2). Denote the gauge of K by $\|\cdot\|$ and the gauge of B by $|\cdot|$. Define the measure μ by following

$$d\mu = \frac{1}{A} e^{-\|x\|^\theta} dx \ , \text{ where } \ A \ \text{ chosen such that } \int\limits_{\mathbb{R}^n} d\mu = 1 \ .$$

Let $L = \int\limits_{\mathbb{R}^n} |x| d\mu$. Then $\mu\{|x| \le 2L\} \ge 1/2$. Let $x_1, x_2, ...$ be a maximal set of points in K such that $|x_i - x_j| \ge t$. So the sets $x_i + \frac{t}{a} B$ have mutually disjoint interiors. Let $y_i = \frac{ab}{t} x_i$ for some b. Then, by symmetry of B and convexity of the function e^t, we have

$$\mu\{y_i + bB\} = \frac{1}{A} \int\limits_{bB} e^{-\|x+y_i\|^\theta} dx = \frac{1}{2A} \int\limits_{bB} e^{-\|x+y_i\|^\theta} + e^{-\|x-y_i\|^\theta} dx \ge$$

$$\ge \frac{1}{A} \int\limits_{bB} e^{-\frac{1}{2}(\|x+y_i\|^\theta + \|x-y_i\|^\theta)} dx \ge \frac{1}{A} \int\limits_{bB} e^{-c_\theta^\theta (\|x\|^\theta + \|y_i\|^\theta)} dx =$$

$$= \frac{1}{A} e^{-\|c_\theta y_i\|^\theta} \int\limits_{bB} e^{-\|c_\theta x\|^\theta} dx \ge e^{-(c_\theta ba/t)^\theta} c_\theta^{-n} \mu\{c_\theta bB\}$$

Choose $b = 2L/c_\theta$. Then $\mu\{c_\theta bB\} \ge 1/2$ and, hence,

$$N(K, tB) \le 2 c_\theta^n e^{(2aL/t)^\theta} \ .$$

Now compute L. First, the normalization constant A is equal

$$A = \int\limits_{\mathbb{R}^n} e^{-\|x\|^\theta} dx = \int\limits_{\mathbb{R}^n} \int\limits_{\|x\|}^{\infty} (-e^{-t^\theta})' dt dx = \int\limits_0^\infty \theta t^{\theta-1} e^{-t^\theta} \int\limits_{\|x\| \le t} dx dt =$$

$$= \int\limits_{\|x\| \le 1} dx \int\limits_0^\infty \theta t^{\theta+n-1} e^{-t^\theta} dt = |K| \cdot \Gamma\left(1 + \frac{n}{\theta}\right) \ ,$$

where Γ is the gamma-function. The remaining integral is

$$\int\limits_{\mathbb{R}^n} |x| e^{-\|x\|^\theta} dx = \int\limits_{\mathbb{R}^n} |x| \int\limits_{\|x\|}^{\infty} (-e^{-t^\theta})' dt dx = \int\limits_0^\infty \theta t^{\theta-1} e^{-t^\theta} \int\limits_{\|x\| \le t} |x| dx dt =$$

$$= \int\limits_{\|x\|\le 1} |x|dx \int\limits_0^\infty \theta t^{\theta+n} e^{-t^\theta} dt = |K| \cdot M(K,B) \cdot \Gamma\left(1+\frac{n+1}{\theta}\right) \quad .$$

Using Stirling's formula we get

$$L \approx \left(\frac{n}{\theta}\right)^{1/\theta} M(K,B) \ .$$

That proves the lemma. □

This lemma is an extension of Lemma 2. Indeed, since Euclidean space is a 2-smooth space, then in case $K = D$ being an ellipsoid, we have $c_2(D) = 1$. By direct computation, $M(D,B) = \frac{n}{n+1} M_B$. Thus,

$$N(D, tB) \le 2e^{(cn)(M_B/t)^2} \ .$$

Define the following characteristic of K,

$$\tilde{M}_K = \frac{1}{|K|} \int\limits_K |x|dx \, .$$

By definition, if K is a p-convex body, then for any $x, y \in \mathbb{R}^n$ holds

$$\| x+y \|_K^p + \| x-y \|_K^p \le 2 \cdot (\| x \|_K^p + \| y \|_K^p) \ .$$

So, the last lemma shows that for p-convex body K

$$N(K, tD) \le 2e^{(cn/p)(2\tilde{M}_K/t)^p} \ .$$

The Theorem 3 follows from this estimate by arguments similar of that in [MP].

Theorem 3. *Let $\lambda > 0$ and n be large enough. Let K be a p-convex body in $\mathbb{R}^n$ and $\| \cdot \|$ be the gauge of K. Then there exists subspace E of $(\mathbb{R}^n, \| \cdot \|)$ such that $\dim E = [\lambda n]$ and for any $x \in E$ the following inequality holds*

$$\| x \| \ge \frac{(1-\lambda)^{1/2+1/p}}{a_p \tilde{M}_K} |x| \, ,$$

where a_p depends on p only (more precisely $a_p = const^{\frac{\ln(2/p)}{p}}$).

Proof: By Lemma 4 there are points $x_1, ..., x_N$ in K, such that $N < e^{c_p n (\tilde{M}_K/t)^p}$ and for any $x \in K$ there exists some x_i such that $|x - x_i| < t$. By Lemma 1 there exists an orthogonal projection P on a subspace of dimension δn such that for

$$c_p n \left(\frac{\tilde{M}_K}{t}\right)^p < \frac{\varepsilon^2 \delta n}{c} \quad \text{and} \quad \varepsilon > \sqrt{\frac{c}{\delta n}}$$

we have

$$b|x_i| = (1-\varepsilon) A\sqrt{\delta}|x_i| \le |Px_i| \le (1+\varepsilon) A\sqrt{\delta}|x_i|$$

for every x_i. Let $E = \mathrm{Ker}P$. Then $\dim E = \lambda n$, where $\lambda = 1 - \delta$. Take x in $K \bigcap E$. There is x_i such that $|x - x_i| < t$. Hence

$$|x| \le |x - x_i| + |x_i| \le t + \frac{|Px_i|}{b} = t + \frac{|P(x - x_i)|}{b} \le$$

$$\le t + \frac{|x - x_i|}{b} \le t(1 + \frac{1}{b}) \le \frac{const \cdot t}{(1 - \varepsilon)\sqrt{\delta}}$$

Therefore for n large enough and

$$t = \left(\frac{const \cdot c_p}{\varepsilon^2 \delta}\right)^{1/p} \tilde{M}_K$$

we get

$$\| x \| \ge \frac{const \cdot \varepsilon^2 (1 - \varepsilon) \delta^{1/2 + 1/p}}{c_p^{1/p} \tilde{M}_K} |x| \, .$$

To obtain our result take ε, say, equal to $1/2$. □

As was noted in [MP2] in some cases $\tilde{M}_K << M^*$ and then Theorem 3 gives better estimate than Theorem 1 even for a convex body (in some range of λ). As an example, $K = B(l_1^n)$, $\tilde{M}_K \le const \cdot n^{-1/2}$, but $M_K^* \ge const \cdot n^{-1/2} (\log n)^{1/2}$.

3. Additional remarks

In fact, during the proof of Theorem 2 a more general fact was proved.

Fact. *Let D be an ellipsoid and K be a p-convex body. Let*

$$N(D, K) \le e^{\alpha n} \ .$$

Denote for an integer $1 \le k \le n$ the ratio $\lambda = k/n$. Then for some absolute constant c and

$$\gamma = c\sqrt{\alpha} \ , \quad k \in (\gamma^2 n, (1 - 2\gamma)^2 n)$$

there exists an orthogonal projection P of rank k such that

$$\left(p(1 - \sqrt{\lambda})/2\right)^{1/p} PD \subset PK \ .$$

In terms of entropy numbers this means

$$\frac{\left(p(1 - \sqrt{k/n})/2\right)^{1/p}}{e_k(D, K)} PD \subset PK \ ,$$

where $e_k(D, K) = \inf\{\varepsilon > 0 \mid N(D, \varepsilon K) \le 2^{k-1}\}$.

It is worth to point out that Theorem 2 can be obtained from this results.

We thank E. Gluskin for his remarks on the first draft of this note.

References

[BLM]: Bourgain, J. ; Lindenstrauss, J. ; Milman, V. *Approximation of zonoids by zonotopes.* Acta Math. 162 (1989), no. 1-2, 73–141.

[BMMP]: Bourgain, J. ; Meyer, M. ; Milman, V. ; Pajor, A. *On a geometric inequality.* Geometric aspects of functional analysis (1986/87), 271–282, Lecture Notes in Math., 1317, Springer, Berlin-New York, 1988.

[JL]: Johnson, W. B.; Lindenstrauss, J. *Extensions of Lipschitz mappings into a Hilbert space.* Conference in modern analysis and probability (New Haven, Conn., 1982), 189–206,

[G]: Gordon, Y. *On Milman's inequality and random subspaces which escape through a mesh in* $\mathbb{R}^n$. Geometric aspects of functional analysis (1986/87), 84–106, Lecture Notes in Math., 1317, Springer, Berlin-New York, 1988.

[KPR]: N.J.Kalton, N.T.Peck, J.W.Roberts, *An F-space sampler*, London Mathematical Society Lecture Note Series, 89, Cambridge University Press, Cambridge and New York, 1984.

[K]: König, H. *Eigenvalue Distribution of Compact Operators* , Birkhäuser, 1986.

[M1]: Milman, V. D. Almost Euclidean quotient spaces of subspaces of a finite-dimensional normed space. Proc. Amer. Math. Soc. 94 (1985), no. 3, 445–449.

[M2]: Milman, V. D. *Random subspaces of proportional dimension of finite-dimensional normed spaces: approach through the isoperimetric inequality.* Banach spaces (Columbia, Mo., 1984), 106–115, Lecture Notes in Math., 1166, Springer, Berlin-New York, 1985.

[M3]: Milman, V. D. *A note on a low* M^**-estimate.* Geometry of Banach spaces (Strobl, 1989), 219–229, London Math. Soc. Lecture Note Ser., 158, Cambridge Univ. Press, Cambridge, 1990.

[MP1]: Milman, V. D. ; Pajor, A. *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space.* Geometric aspects of functional analysis (1987–88), 64–104, Lecture Notes in Math., 1376, Springer, Berlin-New York, 1989.

[MP2]: Milman, V. ; Pajor, A. *Cas limites dans des inégalités du type de Khinchine et applications géométriques.* (French) [Limit cases of Khinchin-type inequalities and some geometric applications] C. R. Acad. Sci. Paris Sér. I Math. 308 (1989), no. 4, 91–96.

[PT]: Pajor, A. ; Tomczak-Jaegermann, N. *Subspaces of small codimension of finite-dimensional Banach spaces.* Proc. Amer. Math. Soc. 97 (1986), no. 4, 637–642.

[R]: Rolewicz, S. *Metric linear spaces.* Monografie Matematyczne, Tom. 56. [Mathematical Monographs, Vol. 56] PWN-Polish Scientific Publishers, Warsaw, 1972.

A. E. Litvak, Department of Mathematics, Tel Aviv University, Ramat Aviv, Israel
E-mail address: alexandr@math.tau.ac.il

V. D. Milman, Department of Mathematics, Tel Aviv University, Ramat Aviv, Israel
E-mail address: vitali@math.tau.ac.il

A. Pajor, Universite de Marne-la-Vallée, Equipe de Mathematiques, 2 rue de la Butte Verte, 93166 Noisy-le-Grand Cedex, France
E-mail address: pajor@math.univ-mlv.fr