FUNCTIONS WHICH ARE ALMOST MULTIPLIERS OF HILBERT FUNCTION SPACES

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ABSTRACT. We introduce a natural class of functions, the *pseudomultipliers*, associated with a general Hilbert function space, prove an extension theorem which justifies the definition, give numerous examples and establish the nature of the 1-pseudomultipliers of Hilbert spaces of analytic functions under mild hypotheses.

The function $1/z$ on the unit disc $\mathbb{D}$ is almost a multiplier of the Hardy space H^2: it misses by only one dimension. That is, there is a closed subspace of H^2 of codimension 1 which is multiplied by $1/z$ into H^2. The same statement holds for the characteristic function of the point 0. These are two key examples of functions that we call *pseudomultipliers* of Hilbert function spaces. Now *multipliers* of the standard function spaces have been much studied: it is a natural generalization to consider functions which fail to be multipliers by only finitely many dimensions. Moreover, for some familiar function spaces one obtains in this way natural classes of functions. For example, a well-known theorem of Adamyan, Arov and Krein [AAK] on s-numbers of Hankel operators can be interpreted as a description of the pseudomultipliers of H^2 (see Theorem 3.4 below). They are the finite modifications of functions of the form f/p where f is bounded and analytic in the unit disc and p is a polynomial which does not vanish on the unit circle. The pseudomultipliers of the Fock space (see Theorem 3.5 below) are the finite modifications of the proper rational functions. We address the question of what can be said about the pseudomultipliers of other popular function spaces.

One can formulate the definition of a pseudomultiplier of a function space in very great generality. In this paper we introduce the notion for the case of Hilbert function spaces. This still covers a very wide variety of spaces, but we are nevertheless able to make significant assertions about them. A major purpose of the paper is to give the

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"correct" formulation of the notion of a pseudomultiplier. To justify our definition we establish an extension theorem (Theorem 2.1). This is the main result of the paper and shows that apparently weaker variants of our definition give rise to essentially the same objects. Other goals are to illustrate the notion by means of a range of examples and to show that a surprising amount can be said about pseudomultipliers of Hilbert spaces of *analytic* functions under modest hypotheses. Virtually all our results depend heavily on a key technical fact, Lemma 2.5.

Our investigation began as a study of certain interpolation problems. It is known [Ag1, Ag2, Q1] that the classical result of Pick on interpolation by bounded analytic functions [P] can be extended to certain other function spaces, some of them having no connection with analyticity. Could the same be true of the interpolation theorem of Adamyan, Arov and Krein which generalises Pick's theorem? This question was discussed but not resolved in [Q2]. To answer it we had to analyse the "Pick kernel"

$$(\lambda, \mu) \mapsto (1 - \varphi(\lambda)\bar{\varphi}(\mu))k(\lambda, \mu) \tag{0.1}$$

corresponding to a reproducing kernel k and a function φ. Pick's theorem concerns the positivity of this kernel on $\mathbb{D}$ when k is the Szegő kernel, while the result of Adamyan, Arov and Krein tells us what happens when it has m negative squares on $\mathbb{D}$. A study of the same question for other kernels led us to the notion of a pseudomultiplier, which proved to be a remarkably fruitful notion despite its great generality. We think of pseudomultipliers as being something like meromorphic functions (plus point discontinuities), though they are defined on arbitrary sets which need have no differentiable or even topological structure - all information is contained in the kernel. Regarding the original question, we shall show in a future paper that, for the most straightforward formulation, the answer is negative: the Adamyan-Arov-Krein property characterizes the Szegő kernel. However, pseudomultipliers are interesting on their own account.

In this paper a *kernel* on Ω will mean a function

$$k : \Omega \times \Omega \to \mathbb{C},$$

where Ω is a set, satisfying

$$k(\lambda, \mu) = k(\mu, \lambda)^- \quad \text{for all } \lambda, \mu \in \Omega.$$

We say that k is *nonsingular* if, for any $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \Omega$, the $n \times n$ matrix $[k(\lambda_i, \lambda_j)]_{i,j=1}^n$ is nonsingular. We call k a *positive* kernel if every $[k(\lambda_i, \lambda_j)]$ above is positive, and *positive definite* if it is both positive and nonsingular. For any non-negative integer m, we say that a kernel k has *m negative squares* if, for any $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \Omega$, the $n \times n$ matrix $[k(\lambda_i, \lambda_j)]_{i,j=1}^n$ has no more than m negative eigenvalues (counted with multiplicity), while for some choice of $n, \lambda_1, \dots, \lambda_n$, it has exactly m negative eigenvalues.

If k is a positive kernel on a set Ω, there is an associated Hilbert space $H(k)$ of functions on Ω for which k is the reproducing kernel [Ar]. That is, for any $f \in H(k)$ and $\lambda \in \Omega$, we have

$$\langle f, k_\lambda \rangle_{H(k)} = f(\lambda),$$

where $k_\lambda \in H(k)$ is defined by $k_\lambda(\mu) = k(\mu, \lambda)$ for $\mu \in \Omega$. By a *Hilbert space of functions on a set* Ω we shall mean a Hilbert space H whose elements are functions on Ω such that the point evaluations $h \mapsto h(\lambda)$ are continuous linear functionals on H for all $\lambda \in \Omega$. Such a space has a reproducing kernel which is a positive kernel on Ω; it may fail to be nonsingular. If H is a Hilbert space of functions on Ω with kernel k and E is a subset of Ω then the space of restrictions of elements of H to E is a Hilbert space of functions on E, whose kernel is the restriction of k to E. If E is large enough these two Hilbert spaces are naturally isomorphic, but for our purposes they are to be regarded as two distinct Hilbert function spaces. This point will be important when we discuss domains of definition of functions.

We say that a function $\psi : \Omega \to \mathbb{C}$ is a *multiplier* of $H(k)$ if $\psi f \in H(k)$ for every $f \in H(k)$. When ψ is a multiplier of $H(k)$, the mapping $f \mapsto \psi f$ is a linear operator, which we denote by M_ψ. By the closed graph theorem, M_ψ is bounded. The *multiplier norm* of ψ is $\|M_\psi\|$. The set of all multipliers of $H(k)$ is a subalgebra of the algebra $\mathcal{L}(H(k))$ of all bounded linear operators on $H(k)$. It is well-known [BB, S] (and will follow from Theorem 2.1 below) that, for any function $\psi : \Omega \to \mathbb{C}$, the following two statements are equivalent:

(a) ψ is a multiplier of $H(k)$ and $\|M_\psi\| \leq 1$, and
(b) the kernel $(\lambda, \mu) \mapsto (1 - \psi(\lambda)\bar{\psi}(\mu))k(\lambda, \mu)$ is positive on Ω.

If H is a Hilbert space and $u, v \in H$ we denote by $u \otimes v$ the rank 1 operator on H given by $x \mapsto \langle x, v \rangle u$. If $\mathcal{C}$ is a class of functions on a set Ω we say that $E \subset \Omega$ is a *set of uniqueness* for $\mathcal{C}$ if $f, g \in \mathcal{C}$ and $f|E = g|E$ imply $f = g$ (here $f|E$ is the restriction of f to E). If k is a positive definite kernel on Ω then $E \subset \Omega$ is a set of uniqueness for $H(k)$ if and only if $\{k_\lambda : \lambda \in E\}$ spans a dense linear subspace of $H(k)$. For a space H of functions on Ω we shall have occasion to use the class

$$HH + HH \stackrel{\text{def}}{=} \{f_1 f_2 + f_3 f_4 : f_j \in H \text{ for } 1 \leq j \leq 4\}.$$

We denote by $s_j(T)$, for $j \geq 0$, the s-numbers of a bounded linear operator T on a Hilbert space H:

$$s_j(T) = \inf\{\|T - R\| : R \in \mathcal{L}(H), \text{ rank } R \leq j\}.$$

We remark that nonsingularity of a positive kernel k on a set Ω can be expressed in terms of its associated Hilbert space H of functions. It is easy to show that k is nonsingular on Ω if and only if, for every finite sequence $\lambda_1, \ldots, \lambda_n$ of points of Ω, there is a function $h \in H$ such that $h(\lambda_1) = 1$ and $h(\lambda_j) = 0$ for $2 \le j \le n$. Another equivalent condition is that $\{k_\lambda : \lambda \in \Omega\}$ is a linearly independent set in H.

1. Pseudomultipliers of Hilbert Function Spaces

The starting point for the notion of an m-pseudomultiplier of a Hilbert space H of functions on a set Ω is simply a function ψ on Ω such that, for some closed m-codimensional subspace E of H, $\psi E \subset H$. However, as our archetype $1/z$ shows, we need to allow the possibility that ψ is not defined on the whole of Ω, and this gives rise to some delicate issues. We illustrate these issues by examples in order to motivate the formal definition. In the two examples which follow k is the Szegő kernel and Ω is $\mathbb{D}$, so that H is H^2. We consider functions $\psi : D_\psi \subset \Omega \to \mathbb{C}$.

Example 1.1. $D_\psi = \mathbb{D} \setminus \{0\}$ and $\psi(\lambda) = \lambda^{-m}$, where $m \in \mathbb{N}$. Here $z^m H^2$ has codimension m in H^2 and ψ multiplies $z^m H^2$ into H^2. Multiplication by ψ determines a bounded linear operator of norm 1 from $z^m H^2$ into H^2; we say that ψ is an m-pseudomultiplier of H^2 and that $s_m(\psi) = 1$.

Example 1.2. $D_\psi = \mathbb{D}$ and ψ is given by $\psi(0) = 1$ and $\psi(\lambda) = 0$ if $\lambda \neq 0$. Here multiplication by ψ determines the zero operator from the 1-codimensional subspace zH^2 of H^2 into H^2. Thus ψ is a 1-pseudomultiplier and $s_1(\psi) = 0$. However ψ is not a 0-pseudomultiplier.

What should we assume about the domain of an m-pseudomultiplier? It should be in some sense "almost all" of Ω, but in what sense? There are various choices one could make; an important purpose of this paper is to show that all the natural assumptions about the domain lead to essentially the same objects, and these are functions which are defined everywhere on Ω with the exception of at most m points. Accordingly we shall incorporate this requirement into the definition, and in the following section we shall justify our decision by showing that weaker assumptions lead to the same class of functions.

A 1-pseudomultiplier of H^2, then, will be a function defined on the complement of at most one point in $\mathbb{D}$. Consider the function ψ which is identically zero on $\mathbb{D} \setminus \{0\}$. This ψ satisfies $\psi z H^2 = \{0\} \subset H^2$. Should we regard ψ as a pseudomultiplier? And as distinct from the function which is identically zero on all of $\mathbb{D}$? We prefer to regard these two functions as the same, and so will include in our definition a condition to ensure that if a pseudomultiplier is undefined at a point of Ω then it is in a precise sense not definable at that point.

Now consider a fixed Hilbert space H of functions on a set Ω with nonsingular kernel k. For $D \subset \Omega$ and any function f on Ω we denote by $f|D$ the restriction of f to D, and we denote by $H|D$ the set $\{h|D : h \in H\}$. For $\psi : D_\psi \subset \Omega \to \mathbb{C}$ and $h \in H$ we denote by ψh the pointwise product of ψ and $h|D_\psi$, so that ψh is a function on

D_ψ. For $E \subset H$ we denote by ψE the set $\{\psi h : h \in E\}$, so that ψE is a set of functions on D_ψ. If E is a closed linear subspace of H, if D_ψ is a set of uniqueness and $\psi E \subset H|D_\psi$ we define the linear operator $M_{\psi,E} : E \to H$ by $M_{\psi,E}h = g$ where g is the unique element of H which extends ψh. By the closed graph theorem $M_{\psi,E}$ is continuous. If D_ψ is a set of uniqueness of H we shall say that $\alpha \in \Omega$ is a *singularity* of ψ (relative to H) if there exists $h \in H$ such that $h(\alpha) = 0$ and ψh has an extension to a function $g \in H$ (necessarily unique) such that $g(\alpha) \neq 0$. Of course this can only happen if $\alpha \notin D_\psi$.

Definition 1.3. Let ψ be a complex-valued function on a subset D_ψ of Ω and let $m \in \mathbb{Z}^+$, the set of non-negative integers. We say that ψ is an *m-pseudomultiplier* of H on Ω if

1. D_ψ is a set of uniqueness for H;
2. $\Omega \setminus D_\psi$ consists of singularities of ψ and contains at most m points;
3. there is a closed subspace E of codimension m in H such that $\psi E \subset H|D_\psi$.

For an m-pseudomultiplier ψ of H we define $s_m(\psi)$ to be the infimum of $\|M_{\psi,E}\|$ over all closed m-codimensional subspaces E of H such that $\psi E \subset H|D_\psi$. We say that ψ is a *pseudomultiplier* of H if it is an m-pseudomultiplier of H for some non-negative integer m.

Notes 1. In the case of a finite set Ω no proper subset is a set of uniqueness (recall that we have supposed the kernel k nonsingular), and so pseudomultipliers are everywhere defined on Ω. Every $\psi : \Omega \to \mathbb{C}$ is an m-pseudomultiplier and $s_m(\psi)$ is the mth singular value of the multiplication operator M_ψ.

2. A 0-pseudomultiplier ψ is the same as a multiplier of H, and $s_0(\psi) = s_0(M_\psi)$.

3. If $m \geq 1$, ψ is defined at all but $m-1$ points of Ω and ψ is an m-pseudomultiplier but not an $(m-1)$-pseudomultiplier then there is a *unique* E with the stated properties, and so $s_m(\psi) = \|M_{\psi,E}\|$. For if $E_1,\ E_2$ are distinct and both have all the stated properties then $E_1 + E_2$ is a closed subspace of H of codimension at most $m-1$ and $\psi(E_1 + E_2) \subset H|D_\psi$. That is, ψ is an $(m-1)$-pseudomultiplier.

In particular, if a 1-pseudomultiplier ψ is defined on the whole of Ω then unless it is a multiplier, there is a unique 1-codimensional subspace E of H such that $\psi E \subset H$. However, it can happen for a 1-pseudomultiplier ψ for which $D_\psi \neq \Omega$ that $\psi H \subset H|D_\psi$ and yet ψ cannot be extended to a multiplier of H: see Example 2.2 below.

4. A related definition of $s_m(\psi)$ and an analysis in the case of "Blaschke kernels" k is in [Q2].

5. For ψ as in the Definition, $\Omega \setminus D_\psi$ consists precisely of the singularities of ψ.

The notion of a pseudomultiplier has as much generality as that of a Hilbert function space, and so it is not to be expected that we can give a very precise taxonomy of pseudomultipliers in general. We do want to give a wide range of examples to illustrate the notion; to introduce a little order into these examples we shall take a preliminary step towards a classification of 1-pseudomultipliers. For this purpose we shall make an additional (but mild) assumption on the pseudomultiplier $\psi : D_\psi \to \mathbb{C}$, to wit that D_ψ is a set of uniqueness for the class $HH + HH$ of functions on Ω. A simple distinction is between the ψ for which $D_\psi = \Omega$ and those for which $\Omega \setminus D_\psi$ contains exactly one point. We shall say that ψ has *defect* 0 or 1 according as the first or second of these holds. Consider first the case of defect 1. There exists a unique $\alpha \in \Omega \setminus D_\psi$, and there are $u, g_1 \in H$ such that $u(\alpha) = 0$, $\psi u = g_1|D_\psi$, and $g_1(\alpha) \neq 0$. There is also a closed 1-codimensional subspace E of H such that $\psi E \subset H|D_\psi$. We claim that

$$E = k_\alpha^\perp = \{h \in H : h(\alpha) = 0\}.$$

For suppose that $h \in H$ is such that $\psi h \in H|D_\psi$ – say $\psi h = g_2|D_\psi$, $g_2 \in H$. On D_ψ we have

$$g_1 h - u g_2 = \psi u h - \psi u h = 0,$$

and so, by the hypothesis on $HH + HH$, $g_1 h = u g_2$ on Ω. Since $u(\alpha) = 0$ and $g_1(\alpha) \neq 0$ we have $h(\alpha) = 0$. Thus $E \subset k_\alpha^\perp$. Since E has codimension 1 it must be that $E = k_\alpha^\perp$. We have shown that, subject to the assumption on $HH + HH$, if ψ is a 1-pseudomultiplier of defect 1 and $D_\psi = \Omega \setminus \{\alpha\}$, then $\psi k_\alpha^\perp \subset H|D_\psi$, and moreover $E = k_\alpha^\perp$ is the only closed 1-codimensional subspace of H such that $\psi E \subset H|D_\psi$.

Now suppose that ψ is of defect 0, i.e. ψ is everywhere defined on Ω. We identify three subcases.

1. ψ is a multiplier of H.
2. ψ is not a multiplier and there is a point $\alpha \in \Omega$ such that $\psi k_\alpha^\perp \subset k_\alpha^\perp$.
3. ψ is not a multiplier and there is a closed subspace E of codimension 1 in H such that $\psi E \subset H|D_\psi$ and E has no zeros in Ω.

These three cases do exhaust the 1-pseudomultipliers of defect 0. For if ψ is not a multiplier then, by Note 3 above, there is a unique closed 1-codimensional subspace E of H such that $\psi E \subset H|D_\psi$. If E is of the form $k_\alpha^\perp$ for some $\alpha \in \Omega$ then we must have $\psi k_\alpha^\perp \subset k_\alpha^\perp$ since $\psi(\alpha)$ is defined, and we are in case 2. The remaining possibility is that E is not $k_\alpha^\perp$ for any α, and case 3 applies.

Let us exhibit examples of all these cases. Defect 1 is exemplified by Example 1.1 ($\psi(z) = 1/z$). Defect 0, case 1 is multipliers – constant functions will do. Example 1.2 (the characteristic function of a point) illustrates defect 0, case 2. Defect 0, case 3 occurs in the following.

Example 1.4. Let k be the Szegő kernel restricted to the disc $\Omega = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| < \frac{1}{2}\}$, take $D_\varphi = \Omega$ and $\varphi(\lambda) = \lambda^{-1}$. Then $s_1(\varphi) = 1$. As a Hilbert space H is the usual Hardy space H^2, but as a function space it is a space of functions on Ω. Multiplication by φ acts contractively from $\mathbf{1}^\perp$ into H, but the elements of $\mathbf{1}^\perp$ have no common zero in Ω.

The latter example looks somewhat unnatural, because we took k on the "wrong" domain Ω, but we are working in such generality that such displeasing examples are included. In the next section we shall make further assumptions on k in order to exclude such phenomena, and we shall be able to make stronger assertions about the structure of pseudomultipliers. Let us return to defect 0, case 2: $\psi k_\alpha^\perp \subset k_\alpha^\perp$. Note that ψ is defined at α, but that the value of $s_1(\psi)$ is independent of $\psi(\alpha)$. We could redefine $\psi(\alpha)$ without affecting the fact that ψ is a 1-pseudomultiplier or the value of $s_1(\psi)$. We shall call such a phenomenon an *anomaly* of ψ. Our example 1.2, the characteristic function of the point 0, has an anomaly at 0. In this example it is possible to redefine $\psi(0)$ to obtain a multiplier of H. This is one easy way to generate 1-pseudomultipliers of defect 0, case 2: we will always obtain either such an object or a multiplier if we take a multiplier and change its value at a single point. However, not all instances of defect 0, case 2 arise in this way.

Example 1.5. Let $\Omega = \mathbb{D}$ and let $H = zH^2 \oplus \mathbb{C}g$ where $g(z) = (z-1)^{-1}$ and the norm in $\mathbb{C}g$ is given by $\|\lambda g\| = |\lambda|$. Let $\psi(z) = z$ if $z \neq 0$ and let $\psi(0) = c$. Then $\psi H(0) \subset H(0)$, so that 0 is an anomaly of ψ, but there is no choice of c which makes ψ a multiplier of H. For suppose that $\psi g \in H$. Then

$$\psi g = zf + \lambda g \text{ for some } f \in H^2, \quad \lambda \in \mathbb{C}.$$

On putting $z = 0$ we find that $\lambda = c$, and so

$$f(z) = \frac{\psi(z) - c}{z} g(z) = \frac{z - c}{z(z-1)} \text{ for } z \neq 0.$$

This contradicts $f \in H^2$, whatever the choice of $c \in \mathbb{C}$.

Let us extend the terminology introduced above. For a closed subspace F of H we denote by P_F the orthogonal projection operator from H to F.

Definition 1.6. Let ψ be a m-pseudomultiplier of H. We say that ψ is of *defect* j if the complement of the domain of ψ contains j points. We say that $\alpha \in \Omega$ is an *anomalous point* or an *anomaly* of ψ if $\psi k_\alpha^\perp \subset k_\alpha^\perp$.

Here are further examples of the variety of behaviour of pseudomultipliers.

Example 1.7. A pole which is not a singularity. $H = H^2$, $D_\varphi = \mathbb{D}$ and

$$\varphi(\lambda) = \begin{cases} \lambda^{-m} & \text{if } \lambda \neq 0, \\ c & \text{if } \lambda = 0, \end{cases}$$

where $m \in \mathbb{N}$ and $c \in \mathbb{C}$. This example differs from Example 1.1 in that φ is defined at 0. Here φ multiplies $z^{m+1}H$, but no m-codimensional subspace of H, into H. It is thus an $(m+1)$-pseudomultiplier but not an m-pseudomultiplier. It has defect 0.

In connection with examples 1.1 and 1.7 we pose the question: is there a natural notion of the multiplicity of a singularity of a pseudomultiplier for a general kernel?

Example 1.8. The simplest of all infinite-dimensional Hilbert function spaces is ℓ^2 as a space of functions on $\mathbb{N}$ in the natural way. Its kernel is $k(\lambda,\mu) = \delta_{\lambda\mu}$, the Kronecker symbol. No proper subset of $\mathbb{N}$ is a set of uniqueness for ℓ^2, and so pseudomultipliers are defined on all of $\mathbb{N}$. In fact the pseudomultiplier algebra of ℓ^2 is ℓ^∞, which is also the multiplier algebra. For suppose there is an unbounded m-pseudomultiplier ψ. We may pick an increasing sequence (n_j) in $\mathbb{N}$ such that $|\psi(n_j)| \geq j$ and a space E of codimension m in ℓ^2 such that $\psi E \subset \ell^2$. Let P_N be the orthogonal projection operator from ℓ^2 to $\operatorname{span}\{k_1, k_2, \dots, k_N\}$. Choose N so large that $P_N E^\perp$ has dimension m. Then, if $f_1, \dots, f_m$ is a basis of $E^\perp$, the vectors $P_N f_1, \dots, P_N f_m$ are linearly independent. For $n > N$ define $x(n)$ to be $1/j$ if $n = n_j$, 0 otherwise. Pick $\xi \in \operatorname{span}\{k_1, \dots, k_N\}$ such that

$$< \xi, P_N f_j >= - \sum_{n=N+1}^{\infty} x(n) \bar{f}_j(n), \text{ for } 1 \leq j \leq m.$$

This is possible on account of the linear independence of the $P_N f_j$. Now set $x(n) = \xi(n)$ for $1 \leq n \leq N$: then $x \perp f_j$ for $1 \leq j \leq m$. Thus $x \in E$ and hence $\psi x \in \ell^2$. However $|\psi x(n_j)| \geq 1$ for infinitely many points n_j, a contradiction. Thus m-pseudomultipliers of ℓ^2 are bounded.

Example 1.9 (The Fock space Φ)**.** This is the space with kernel $k(\lambda,\mu) = e^{\lambda\bar{\mu}}$ on $\mathbb{C}$. It is also known as the Fischer space and the Bargmann–Fock space. It comprises all entire functions f such that

$$\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dA < \infty$$

where dA is area measure. Alternatively it can be described as the space of functions $f(z) = \sum_{n=0}^\infty a_n z^n$ such that $\sum_{n=0}^\infty |a_n|^2 n! < \infty$. An orthonormal basis of Φ is given by the functions $\left(z^n/\sqrt{n!}\right)_{n=0}^\infty$. The interest of Φ here is that it admits no nonconstant multipliers, but it has plenty of nontrivial pseudomultipliers. Indeed, a non-trivial Hilbert space of entire functions cannot have a non-constant multiplier. For suppose that φ is such a multiplier. It is easily shown that φ is entire, and so, by Liouville's theorem, $\varphi(\mathbb{C})$ is unbounded. Since H is non-trivial its kernel k satisfies $k_\lambda \neq 0$ for λ in a dense subset of $\mathbb{C}$. The relation $M_\varphi^* k_\lambda = \bar{\varphi}(\lambda) k_\lambda$ shows that $\varphi(\mathbb{C}) \subset \sigma(M_\varphi)$, contradicting boundedness of M_φ. In particular therefore the only multipliers of the Fock space are the constants, but $\varphi : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with $\varphi(\lambda) = \lambda^{-1}$ is a 1-pseudomultiplier: it multiplies the 1-codimensional subspace $\{h \in H : h(0) = 0\}$ of H into H. We shall discuss the pseudomultipliers of Φ further in Sec. 3 below.

H^2 has the property that if ψ is a multiplier then $s_m(\psi) = s_0(\psi)$ for all $m \in \mathbb{Z}^+$. A simple example of a space which does not have this property is ℓ^2, as a space of functions on $\mathbb{N}$. Here $\psi(n) = 1/n$ defines a multiplier of ℓ^2; clearly $s_m(\psi) = 1/(m+1)$.

One might imagine that a meromorphic function defining a 1-pseudomultiplier would have a single, simple pole. This is true for nonsingular kernels, as will follow from Theorem 2.6 below, but it is not true if we relax for a moment our standing assumption that H has nonsingular kernel, as the following example shows.

Example 1.10. Let $m \geq 2$ and let H be the closure in H^2 of the space of polynomials in λ^m, as a space of functions on $\mathbb{D}$, and let $\varphi(\lambda) = \lambda^{-m}$ on $\mathbb{D} \setminus \{0\}$. Then φ multiplies $\mathbf{1}^\perp$ into H, hence is a 1-pseudomultiplier, but has a pole of multiplicity m at 0.

Examples 1.1 and 1.9 suggest a conjecture: if H is a Hilbert space of analytic functions on a domain Ω with nonsingular kernel, then for any multiplier ψ of H and any $\alpha \in \Omega$ then the function $\lambda \mapsto \psi(\lambda)/(\lambda - \alpha)$ is a 1-pseudomultiplier of H. However, this is not so, even if H is z-invariant.

Example 1.11. Let $\Omega = \mathbb{D}$ and $D_\varphi = \mathbb{D} \setminus \{0\}$, $\varphi(\lambda) = \lambda^{-1}$. Let H be a z-invariant subspace of the Bergman space $L^2_a(\mathbb{D})$ with nonsingular kernel such that zH has codimension 2 in H. Here z denotes the operation of multiplication by the independent variable. It is shown in [H] how such a subspace can be constructed. One can take H to be of the form $I(A) + I(B)$ where A, B are certain zero sequences of $L^2_a(\mathbb{D})$ and the closed subspace $I(A)$ is $\{f \in L^2_a(\mathbb{D}) : f = 0 \text{ on } A\}$. It is shown that A, B can be chosen so that $A \cup B$ is a "sampling sequence" for $L^2_a(\mathbb{D})$, which ensures that $I(A) + I(B)$ is closed. If $\lambda_1, \dots, \lambda_n$ are distinct points of $\mathbb{D}$ one can find $f \in I(A) + I(B)$ such that $f(\lambda_1) \neq 0$; then the function $\prod_{j=2}^n (\lambda - \lambda_j) f(\lambda) \in H$ vanishes at $\lambda_2, \dots, \lambda_n$ but not at λ_1 and so H has nonsingular kernel. Now suppose that φ is a 1-pseudomultiplier of H. By Definition 1.3 there is a subspace E of codimension one in H and a bounded linear operator $L : E \to H$ such that $(Lh)(\lambda) = \lambda^{-1} h(\lambda)$ for all $h \in E$ and all but one $\lambda \in \mathbb{D} \setminus \{0\}$. Thus each $h \in E$ agrees with an element of zH except for at most two points, and so belongs to zH. Thus $E \subset zH$, a contradiction since E has codimension one and zH has codimension two. Thus φ is not a 1-pseudomultiplier.

By a slight modification of the above construction one can find a z-invariant $H \subset L^2_a(\mathbb{D})$ with nonsingular kernel such that $\varphi(\lambda) = \lambda^{-1}$ is not an m-pseudomultiplier for any integer m. It is shown in [HRS] that H can be chosen so that zH has infinite codimension in H. It follows that φ is not a pseudomultiplier.

Most of our natural-looking examples of pseudomultipliers relate to Hilbert spaces of analytic functions, so it is of interest to show that other types of space can also have non-trivial pseudomultipliers.

Example 1.12. A Sobolev space. Let W be the space $\{f \in L^2(0,1) : f' \in L^2(0,1)\}$, with its usual inner product, as a space of functions on $[0,1]$. The characteristic function of a point is a pseudomultiplier of this space, but a much more interesting example is the function $\varphi(t) = \sqrt{t}$. Since W is an algebra with identity and $\varphi \notin W$ it follows that φ is not a multiplier of W, but φ *is* a 1-pseudomultiplier of W. In fact $\varphi k_0^\perp \subset W$. Consider $f \in W$ such that $f(0) = 0$. Then for $t \in (0,1)$

$$|f(t)| = \left| \int_0^t f'(x)dx \right| \leq \sqrt{t}||f||_W,$$

so that $\varphi' f$ is bounded on $(0,1)$. It follows that $\varphi f \in W$. Hence $\varphi k_0^\perp \subset W$ as claimed.

We shall need the following simple fact.

Theorem 1.13. *If φ is a pseudomultiplier with domain D_φ and φ is the restriction of a multiplier of H then $D_\varphi = \Omega$ and φ is a multiplier of H.*

Proof. We have to show that φ cannot have any singularities in Ω. Suppose, on the contrary, that φ is the restriction of a multiplier θ and that there exist $\alpha \in \Omega$ and $h \in H$ such that $h(\alpha) = 0$ and φh has an extension $g \in H$ such that $g(\alpha) \neq 0$. Now $\theta h \in H$ extends φh, and so $\theta h = g$.Thus

$$0 = \theta(\alpha)h(\alpha) = g(\alpha) \neq 0,$$

a contradiction. □

2. Extension of pseudomultipliers and the Pick kernel

In our definition of an m-pseudomultiplier φ of a Hilbert space H of functions on a set Ω the main idea is that φ multiply a closed m-codimensional subspace of H into H. This is natural enough, as is the requirement that the domain D_φ of φ be a set of uniqueness for H; otherwise φ is more appropriately studied via its action on the proper closed subspace of H spanned by k_λ, $\lambda \in D_\varphi$. However, we have also imposed the condition 2 of Definition 1.3, that $\Omega \setminus D_\varphi$ consist of singularities of φ and have at most m points. This is less obviously appropriate. Why make such a strong assumption on D_φ? It would make perfect sense simply to delete condition 2. Let us temporarily call functions satisfying the resulting definition *weak m-pseudomultipliers*. On the face of it these will comprise a larger class than the m-pseudomultipliers. It is a remarkable fact that under a modest hypothesis on D_φ every weak m-pseudomultiplier can be extended with preservation of $s_m(\cdot)$ to a function which is defined on all but at most m points of Ω and which is an m-pseudomultiplier.

Observe that if D_φ is a set of uniqueness for H then $H|D_\varphi \stackrel{\text{def}}{=} \{h|D_\varphi : h \in H\}$ is a space of functions on D_φ, and $h \mapsto h|D_\varphi$ is a Hilbert space isomorphism of H and $H|D_\varphi$. To say that φ is a weak m-pseudomultiplier of H is the same as saying that φ is an m-pseudomultiplier of $H|D_\varphi$. Our main aim in this section is to show that an m-pseudomultiplier of $H|D_\varphi$ extends with preservation of $s_m(\cdot)$ to an m-pseudomultiplier of H. In fact this is not true in full generality, even for multipliers ($m = 0$), as we show in Example 2.2 below, but the extra hypothesis we need to obtain such an extension holds for nearly all spaces of interest.

Theorem 2.1. *Let H be a Hilbert space of functions on a set Ω with nonsingular kernel k, let $D_\varphi \subset \Omega$ be a set of uniqueness for H and for $HH + HH$. The following are equivalent for a function $\varphi : D_\varphi \to \mathbb{C}$.*

1. *φ is an m-pseudomultiplier of $H|D_\varphi$;*
2. *There exists $t \geq 0$ such that the kernel*

$$(\lambda, \mu) \mapsto \left(t^2 - \varphi(\lambda)\bar{\varphi}(\mu)\right) k(\lambda, \mu) \tag{2.1}$$

has at most m negative squares on D_φ;

3. *There is an m-pseudomultiplier ψ of H which extends φ.*

Moreover, when these conditions hold,

$$s_m(\varphi) = s_m(\psi) = \inf t$$

over all t for which (2.1) has at most m negative squares.

Before we prove the theorem let us look at an example which shows how the conclusions can fail if the hypothesis about $HH + HH$ is omitted.

Example 2.2. Let $\Omega = \mathbb{N}$, let $u = \sqrt{3}(2^{-n})_{n=1}^\infty$, so that u is a unit vector in ℓ^2, and let H be the orthogonal complement of u in ℓ^2. The reproducing kernel of ℓ^2 is $(\lambda, \mu) \mapsto \delta_{\lambda\mu}$ and so the reproducing kernel in H is given by orthogonal projection onto H:

$$k_\lambda(\mu) = [\delta_\lambda - \langle \delta_\lambda, u\rangle u](\mu) = \delta_{\lambda\mu} - \frac{3}{2^{\lambda+\mu}}.$$

This example is simple, but it has some interesting properties. It illustrates pathologies which can occur when pseudomultipliers are studied in excessive generality.

(a) Only the constant sequences are multipliers of H, but there is a rich class of pseudomultipliers. In fact ψ is a pseudomultiplier of $H \Leftrightarrow \psi$ is a 1- pseudomultiplier of $H \Leftrightarrow$ either $\psi \in \ell^\infty$ or ψ is a bounded non-constant function on $\mathbb{N} \setminus \{\alpha\}$ for some $\alpha \in \mathbb{N}$.

(b) For any $\alpha \in \mathbb{N}$ and any non-constant bounded function $\varphi : \mathbb{N} \setminus \{\alpha\} \to \mathbb{C}$, φ is a multiplier of $H|\mathbb{N} \setminus \{\alpha\}$ but φ has no extension to a multiplier of H.

(c) For φ as in (b) and $t = \sup\{|\varphi_n| : n \in \mathbb{N} \setminus \{\alpha\}\}$, the Pick kernel

$$(\lambda, \mu) \mapsto \left(t^2 - \varphi(\lambda)\bar{\varphi}(\mu)\right) k(\lambda, \mu)$$

is positive but φ does not extend to a multiplier of H.

(d) There are functions ψ such that $\psi H \subset H|D_\psi$ but ψ is not a multiplier of H, nor even extendable to a multiplier of H.

Notes. Since $\mathbb{N} \setminus \{\alpha\}$ is a set of uniqueness for H, for any $\alpha \in \mathbb{N}$, (b) shows that the implication $1 \Rightarrow 3$ of Theorem 2.1 fails in general, even when $m = 0$. Likewise (c) shows that $2 \Rightarrow 3$ fails. Here $HH + HH = \ell^1_{\mathbb{N}}$, so that $\mathbb{N} \setminus \{\alpha\}$ is not a set of uniqueness for $HH + HH$.

Let $\varphi = (\varphi_n)_{n=1}^\infty$ be a multiplier of H. Then $\varphi x \in H$, where

$$x = (1, 0, \dots , 0, -2^{n-1}, 0, \dots),$$

the -2^{n-1} being in the nth place. It follows that $\varphi_n = \varphi_1$. Hence all multipliers of H are constant.

Now consider $\psi = (\psi_n)_{n=1}^\infty \in \ell^\infty_{\mathbb{N}}$. It is routine to show that if f is the orthogonal projection of ψu onto $u^\perp$ in ℓ^2 then $\psi f^\perp \subset u^\perp$. Hence ψ is a $1-$pseudomultiplier of H. Thus the set of 1-pseudomultipliers of H contains ℓ^∞; it also contains many other functions. Let $\alpha \in \mathbb{N}$ and let $D = \mathbb{N} \setminus \{\alpha\}$. Then D is a set of uniqueness for H, since if $x \in H$ and we know x_n for $n \neq \alpha$ we can recover x_α from the relation $x \perp u$. Let $\varphi : D \to \mathbb{C}$ be bounded and non-constant. Then φ is a 1-pseudomultiplier of H. Indeed, $\varphi k_\alpha^\perp \subset H$, since $\varphi x \in \ell^2_D$ for any $x \in H$, and every element of ℓ^2_D has a unique extension to an element of H. Moreover, φ has a singularity at α in the sense of Definition 1.3. Since φ is non-constant one can construct $h \in H$ such that $h(\alpha) = 0$ but $(\varphi h)(\alpha) \neq 0$. For example, if $\alpha \notin \{1, 2\}$ and $\varphi(1) \neq \varphi(2)$ we may take $h(1) = \varphi(2)$, $h(2) = -2\varphi(1)$, $h(n) = 0$ for $n > 2$. Hence φ is a 1-pseudomultiplier of H.

We have shown that the 1-pseudomultipliers of H contain ℓ^∞ and all non-constant bounded functions on complements of singleton subsets of $\mathbb{N}$. In fact these are all the pseudomultipliers of H. For let ψ be an m-pseudomultiplier. Suppose ψ is defined everywhere on $\mathbb{N}$ except at a singularity α (the domain of ψ has to be a set of uniqueness, and so can omit at most one point). Pick a closed m-codimensional subspace E of H such that $\psi E \subset H$. Then E has finite codimension as a subspace of ℓ^2 and satisfies $\psi E \subset \ell^2$. It follows that ψ is bounded (see Example 1.8). It is easy to see that if ψ is constant it cannot have a singularity at α. Thus ψ is a bounded non-constant function on $\mathbb{N} \setminus \{\alpha\}$. A similar argument shows that if ψ is everywhere defined on $\mathbb{N}$ then $\psi \in \ell^\infty$. This completes the proof of (a).

(b) Let $D = \mathbb{N} \setminus \{\alpha\}$ for some $\alpha \in \mathbb{N}$. We have $H|D = \ell^2_D$, so that the multipliers of $H|D$ are just the elements of ℓ^∞_D. However, only the constant sequences extend to multipliers of H.

(c) It will be seen below that the implication $1 \Rightarrow 2$ of Theorem 2.1 does not depend on the domain of φ being a set of uniqueness for $HH + HH$, and so φ has the property described.

(d) Let ψ be a non-constant bounded function on $\mathbb{N} \setminus \{\alpha\}$, where $\alpha \in \mathbb{N}$. For any $h \in H$, ψh is an ℓ^2 sequence indexed by $\mathbb{N}\setminus\{\alpha\}$, and it has a (unique) extension to an element of H. Thus $\psi H \subset H|D_\psi$. However, ψ is not a multiplier of H since it is not defined at α. It does not even extend to a multiplier of H since it is non-constant.

Proof of Theorem 2.1 (3) $\Rightarrow$ (1) is immediate from the fact that $h \mapsto h|D_\varphi$ is an isomorphism of Hilbert spaces.

(1) $\Rightarrow$ (2). Let E be a closed m-codimensional subspace of H such that $\varphi E \subset H|D_\varphi$, so that $M \stackrel{\text{def}}{=} M_{\varphi,E}$ is a bounded linear operator from E to H.

For $\lambda \in D_\varphi$ and $h \in E$ we have

$$\begin{aligned}\langle M^* k_\lambda, h\rangle_E &= \langle k_\lambda, Mh\rangle_H = (Mh)(\lambda)^- = \bar\varphi(\lambda)\bar h(\lambda)\\ &= \bar\varphi(\lambda)\langle k_\lambda, h\rangle_H = \langle \bar\varphi(\lambda) P_E k_\lambda, h\rangle_E,\end{aligned}$$

where $P_E : H \to E$ is orthogonal projection. That is,

$$M^* k_\lambda = \bar\varphi(\lambda) P_E k_\lambda \quad \text{for } \lambda \in D_\varphi.$$

For any $t \in \mathbb{R}$ and $\lambda, \mu \in D_\varphi$ we have

$$\begin{aligned}\langle (t^2 - MM^*)k_\mu,\, k_\lambda\rangle &= t^2 k(\lambda,\mu) - \langle M^* k_\mu,\, M^* k_\lambda\rangle\\ &= t^2 k(\lambda,\mu) - \langle \bar\varphi(\mu) P_E k_\mu,\, \bar\varphi(\lambda) P_E k_\lambda\rangle\\ &= t^2 k(\lambda,\mu) - \varphi(\lambda)\bar\varphi(\mu)\langle P_E k_\mu, k_\lambda\rangle\\ &= t^2 k(\lambda,\mu) - \varphi(\lambda)\bar\varphi(\mu)\langle (1 - P_{E^\perp}) k_\mu, k_\lambda\rangle\\ &= (t^2 - \varphi(\lambda)\bar\varphi(\mu))k(\lambda,\mu) + \varphi(\lambda)\bar\varphi(\mu)\langle P_{E^\perp} k_\mu,\, k_\lambda\rangle.\end{aligned}$$

That is,

$$(t^2 - \varphi(\lambda)\bar\varphi(\mu))k(\lambda,\mu) = \langle (t^2 - MM^*)k_\mu,\, k_\lambda\rangle - \varphi(\lambda)\bar\varphi(\mu)\langle P_{E^\perp} k_\mu,\, k_\lambda\rangle.$$

Since $P_{E^\perp}$ has rank m, the kernel $-\varphi(\lambda)\bar\varphi(\mu)\langle P_{E^\perp} k_\mu, k_\lambda\rangle$ has at most m negative squares. If $t \ge \|M\|$ the kernel $\langle (t^2 - MM^*)k_\mu,\, k_\lambda\rangle$ is positive. Thus

$$(t^2 - \varphi(\lambda)\bar\varphi(\mu))k(\lambda,\mu)$$

has at most m negative squares as long as $t \ge \|M\|$, in particular when $t = \|M\| = s_m(\varphi)$.

(2) $\Longrightarrow$ (3). We begin with a simple observation that explains the introduction of the class $HH + HH$.

Lemma 2.3. *Let $M \in \mathcal{L}(H)$, let $\varphi : D_\varphi \subset \Omega \to \mathbb{C}$ be a function whose domain D_φ is a set of uniqueness of both H and $HH + HH$, and let F be an m-dimensional subspace of H, where $0 \leq m < \infty$. Suppose that, for all $h \in F^\perp$ and $\lambda \in D_\varphi$,*

$$(Mh)(\lambda) = \varphi(\lambda)h(\lambda).$$

Then the subset $\mathcal{S} = \{\lambda : k_\lambda \in F\}$ of Ω contains at most m points, and there is a function $\psi : \Omega \setminus \mathcal{S} \to \mathbb{C}$ such that $\psi(\lambda) = \varphi(\lambda)$ for $\lambda \in D_\varphi \setminus \mathcal{S}$ and

$$(Mh)(\lambda) = \psi(\lambda)h(\lambda) \tag{2.2}$$

for all $\lambda \in \Omega \setminus \mathcal{S}$ and $h \in F^\perp$.

Proof. Since k is nonsingular, the k_λ are linearly independent, and so $\mathcal{S}$ can contain at most m points. Consider any $\lambda \in \Omega \setminus \mathcal{S}$. Pick an $h \in F^\perp$ such that $h(\lambda) \neq 0$ (this is possible, since otherwise $h \in F^\perp$ implies $h \in k_\lambda^\perp$, i.e., $k_\lambda \in F$). Define $\psi(\lambda)$ to be $(Mh)(\lambda)/h(\lambda)$. This definition is independent of the choice of $h \in F^\perp$ such that $h(\lambda) \neq 0$. For suppose h_1, h_2 are two choices. Then for any $\mu \in D_\varphi$ we have

$$(h_1 \cdot Mh_2)(\mu) - (Mh_1 \cdot h_2)(\mu) = h_1(\mu)\varphi(\mu)h_2(\mu) - \varphi(\mu)h_1(\mu)h_2(\mu) = 0.$$

Since D_φ is a set of uniqueness for $HH + HH$ it follows that $h_1 \cdot Mh_2 - Mh_1 \cdot h_2 = 0$, and so

$$\frac{Mh_1(\lambda)}{h_1(\lambda)} = \frac{Mh_2(\lambda)}{h_2(\lambda)}.$$

Thus $\psi : \Omega \setminus \mathcal{S} \to \mathbb{C}$ is well defined.

Now consider $\lambda \in \Omega \setminus \mathcal{S}$ and $g \in F^\perp$. Choose $h \in F^\perp$ such that $h(\lambda) \neq 0$. The same argument as above shows that $h \cdot Mg - Mh \cdot g = 0$. Hence

$$(Mg)(\lambda) = \frac{(Mh)(\lambda)}{h(\lambda)} g(\lambda) = \psi(\lambda)g(\lambda).$$

Thus (2.2) is satisfied. Pick $\lambda \in D_\varphi \setminus \mathcal{S}$ and $h \in F^\perp$ such that $h(\lambda) \neq 0$. Then

$$\varphi(\lambda)h(\lambda) = (Mh)(\lambda) = \psi(\lambda)h(\lambda), \text{ and so } \varphi(\lambda) = \psi(\lambda). \qquad \square$$

Example 1.2 shows that $\mathcal{S}$ *can* contain points of D_φ.

For the remainder of this section denote by $\mathcal{F}$ the set of finite subsets of D_φ. For $F \in \mathcal{F}$, let

$$\mathcal{M}_F = \operatorname{span}\{k_\lambda : \lambda \in F\}$$

and let $\mathcal{R}_\varphi$ be the closed linear span of $\{\bar{\varphi}(\lambda)k_\lambda : \lambda \in D_\varphi\}$.

Define $T_F \in \mathcal{L}(\mathcal{M}_F)$ by

$$T_F k_\lambda = \bar{\varphi}(\lambda)k_\lambda, \quad \text{for } \lambda \in F,$$

and note the following standard calculation. For any $x = \sum_{\lambda \in F} c_\lambda k_\lambda \in \mathcal{M}_F$ we have

$$\|x\|^2 - \|T_F x\|^2 = c^*\big((1 - \varphi(\lambda)\bar{\varphi}(\mu))k(\lambda, \mu)\big)c,$$

where $c = [c_\lambda]_{\lambda \in F}$, regarded as a column vector. It follows that, if φ is an m-pseudomultiplier and $s_m(\varphi) \le 1$, then $1 - T_F^* T_F$ has at most m negative eigenvalues.

Lemma 2.4. *Let φ satisfy condition 2 in Theorem 2.1 and let t_0 be the least value of $t \ge 0$ such that the kernel (2.1) has at most m negative squares on D_φ. There exist functions*

$$a_j : D_\varphi \to \mathbb{C}, \quad \text{for } 1 \le j \le m,$$

and a positive operator B on H such that $\|B\| = t_0^2$, $BH \subset \mathcal{R}_\varphi$, and

$$\langle Bk_\lambda, k_\mu\rangle = \bar{\varphi}(\lambda)\varphi(\mu)\langle k_\lambda, k_\mu\rangle - \sum_{j=1}^{m} \bar{a}_j(\lambda) a_j(\mu) \text{ for all } \lambda, \mu \in D_\varphi. \tag{2.3}$$

Proof. We can assume $t_0 = 1$. For any $F \in \mathcal{F}$, the operator $1 - T_F^* T_F$ has at most m negative eigenvalues. Hence there exists a contraction B_F on $\mathcal{M}_F$ with $0 \le B_F \le 1$ and vectors $u_1^F, \dots, u_m^F \in \mathcal{M}_F$ such that

$$T_F^* T_F = B_F + \sum_{j=1}^{m} u_j^F \otimes u_j^F. \tag{2.4}$$

The u_j^F can be taken to be suitably scaled Schmidt vectors of T_F (some could be zero). Note that (2.4) is equivalent to

$$\bar{\varphi}(\lambda)\varphi(\mu)\langle k_\lambda, k_\mu\rangle = \langle B_F k_\lambda, k_\mu\rangle + \sum_{j=1}^{m} u_j^F(\lambda)^- \, u_j^F(\mu) \tag{2.5}$$

for all $F \in \mathcal{F}$ and $\lambda, \mu \in F$. In particular,

$$\sum_{j=1}^{m} |u_j^F(\lambda)|^2 = |\varphi(\lambda)|^2 \|k_\lambda\|^2 - \langle B_F k_\lambda, k_\lambda\rangle \le |\varphi(\lambda)|^2 \|k_\lambda\|^2 \quad \text{for } \lambda \in F \in \mathcal{F}. \tag{2.6}$$

If $\varphi(\lambda) = 0$ for some λ, then

$$\sum_{j=1}^{m} |u_j^F(\lambda)|^2 = -\langle B_F k_\lambda, k_\lambda\rangle,$$

and we must have $B_F k_\lambda = 0$. Hence $B_F \mathcal{M}_F \subset \mathcal{R}_\varphi$ for all $F \in \mathcal{F}$.

Let $B_F^\sharp = B_F \oplus O_{\mathcal{M}_F^\perp} \in \mathcal{L}(H)$. Then $0 \le B_F^\sharp \le 1$ and $B_F^\sharp H \subset \mathcal{R}_\varphi$ for all $F \in \mathcal{F}$. For each $F \in \mathcal{F}$ let

$$Q_F = \{G \in \mathcal{F} : F \subset G\}.$$

No Q_F is empty, and $Q_{F_1} \cap Q_{F_2} = Q_{F_1 \cup F_2}$. Thus $\{Q_F : F \in \mathcal{F}\}$ is a filter base on $\mathcal{F}$, generating a filter $\mathcal{Q}$. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{F}$ that refines $\mathcal{Q}$. Let α be the mapping from $\mathcal{F}$ to the closed unit ball $\mathcal{B}$ of $\mathcal{L}(H)$ given by $\alpha(F) = B_F^\sharp$. One can think of α as a continuous mapping from $\mathcal{F}$ with the discrete topology to $\mathcal{B}$ with the weak operator topology. $\mathcal{B}$ is compact and so α extends in a unique way to a continuous mapping from the Stone-Čech compactification $\beta\mathcal{F}$ of $\mathcal{F}$ into $\mathcal{B}$. Since $\mathcal{U}$ can be identified with a point of $\beta\mathcal{F}$ it follows that

$$B = \lim_{F\to\mathcal{U}} \alpha(F) = \lim_{F\to\mathcal{U}} B_F^\sharp \tag{2.7}$$

exists in $\mathcal{B}$. Clearly $0 \le B \le 1$ and $BH \subset \mathcal{R}_\varphi$. Recall that $u_j^F(\lambda)$ is defined when $\lambda \in D_\varphi$, $1 \le j \le m$ and $\lambda \in F \in \mathcal{F}$. With the same range of λ and j, extend the definition to all $F \in \mathcal{F}$ by setting $u_j^F(\lambda) = 0$ if $\lambda \notin F$. For fixed j and $\lambda \in D_\varphi$ the complex-valued function $F \mapsto u_j^F(\lambda)$ is bounded on $\mathcal{F}$, by (2.6). Hence

$$a_j(\lambda) \stackrel{\text{def}}{=} \lim_{F\to\mathcal{U}} u_j^F(\lambda)$$

exists for all $\lambda \in D_\varphi$ and $1 \le j \le m$. Again by (2.6) we have, for all $\lambda \in D_\varphi$,

$$\sum_{j=1}^m |a_j(\lambda)|^2 \le |\varphi(\lambda)|^2 \|k_\lambda\|^2. \tag{2.8}$$

We now have to show that we can take limits in (2.5) to obtain (2.3).

Fix $\lambda, \mu \in D_\varphi$ and let $\varepsilon > 0$. By virtue of (2.8) and (2.6) there exists $U \in \mathcal{U}$ such that

$$\left| \sum_j a_j(\lambda)^- a_j(\mu) - \sum_j u_j^F(\lambda)^- u_j^F(\mu) \right| < \frac{\varepsilon}{2} \tag{2.9}$$

whenever $F \in U$. Similarly, there exists $V \in \mathcal{U}$ such that

$$\left| \langle Bk_\lambda - B_F^\sharp k_\lambda,\, k_m \rangle \right| < \frac{\varepsilon}{2}$$

whenever $F \in V$. Pick $F \in U \cap V \cap Q_{\{\lambda\}}$ (this set belongs to $\mathcal{U}$, hence is nonempty). Then $\lambda \in F$, and so $B_F^\sharp k_\lambda = B_F k_\lambda$. Hence

$$|\langle Bk_\lambda - B_F k_\lambda,\, k_\mu \rangle| < \frac{\varepsilon}{2}.$$

Since $F \in U$, inequality (2.9) is also satisfied. Combining these inequalities with (2.5) we obtain

$$\left| \bar\varphi(\lambda)\varphi(\mu)\langle k_\lambda,\, k_\mu\rangle - \langle Bk_\lambda,\, k_\mu\rangle - \sum_j \bar a_j(\lambda) a_j(\mu) \right| < \varepsilon.$$

Since ε was arbitrary, (2.3) is satisfied.

It remains to show that $\|B\| \ge t_0^2$. By (2.3), for any $t \in \mathbb{R}$ and $\lambda,\ \mu \in D_\varphi$,

$$(t^2 - \varphi(\mu)\bar\varphi(\lambda))k(\mu, \lambda) = \langle (t^2 - B)k_\lambda, k_\mu\rangle - \sum_{j=1}^m a_j(\mu)\bar a_j(\lambda).$$

If $t^2 = \|B\|$ then the first term on the right hand side is positive and the second has m negative squares. Thus $(t^2 - \varphi(\mu)\bar{\varphi}(\lambda))k(\mu,\lambda)$ has m negative squares, and so $t_0^2 \le t^2 = \|B\|$. □

In Example 1.1 (λ^{-m} on $\mathbb{D}$) one can take each B_F to be $1_{\mathcal{M}_F}$, hence $B = 1_H$, and one finds that $a_1(\lambda) = \lambda^{-m}$. In Example 1.2 (the characteristic function of a point) it transpires that $B = 0$ and $a_1 = \varphi$. In both cases a_j fails to belong to H.

Lemma 2.5. *For φ as in Lemma 2.4 there exist A and L in $\mathcal{L}(H)$ such that*

1. *the range and cokernel of both A and L are contained in $\mathcal{R}_\varphi$;*
2. $0 \le A \le 1$ *and* $\operatorname{rank} A \le m$*;*
3. $\|L\| = t_0$ *and, for every $\lambda \in D_\varphi$ and $h \in H$,*

$$(Lh)(\lambda) = \varphi(\lambda)\big(h(\lambda) - (Ah)(\lambda)\big) \tag{2.10}$$

and

$$L^* k_\lambda = (1-A)\bar{\varphi}(\lambda)k_\lambda. \tag{2.11}$$

Proof. The relation (2.3) in Lemma 2.4 can be written

$$\langle \bar{\varphi}(\lambda)k_\lambda,\, \bar{\varphi}(\mu)k_\mu\rangle_H = \langle B^{1/2}k_\lambda \oplus (\bar{a}_1(\lambda),\dots,\bar{a}_m(\lambda)),\, B^{1/2}k_\mu \oplus (\bar{a}_1(\mu)\dots,\bar{a}_m(\mu))\rangle_{H\oplus\mathbb{C}^m}$$

for all $\lambda,\mu \in D_\varphi$. Hence we can define an isometry $V : \mathcal{R}_\varphi \to \mathcal{R}_\varphi \oplus \mathbb{C}^m$ by

$$V\bar{\varphi}(\lambda)k_\lambda = B^{1/2}k_\lambda \oplus (\bar{a}_1(\lambda),\dots,\bar{a}_m(\lambda)) \quad \text{for } \lambda \in D_\varphi. \tag{2.12}$$

The isometry V necessarily has the form

$$V = \begin{bmatrix} C \\ 1\otimes f_1 \\ \vdots \\ 1\otimes f_m \end{bmatrix} \tag{2.13}$$

for some $C : \mathcal{R}_\varphi \to \mathcal{R}_\varphi$ and $f_1,\dots,f_m \in \mathcal{R}_\varphi$. The relation $V^*V = 1$ yields

$$C^*C + f_1\otimes f_1 + \cdots + f_m \otimes f_m = 1.$$

Let

$$A = f_1\otimes f_1 + \cdots + f_m\otimes f_m.$$

Then $0 \le A \le 1$, $AH \subset \mathcal{R}_\varphi$, $\operatorname{rank} A \le m$ and $C^*C = 1 - A$. From (2.12) and (2.13) we have

$$C\bar{\varphi}(\lambda)k_\lambda = B^{1/2}k_\lambda \tag{2.14}$$

$$\bar{\varphi}(\lambda)\bar{f}_j(\lambda) = \bar{a}_j(\lambda)$$

for $\lambda \in D_\varphi$ and $1 \le j \le m$. Extend C to an operator $C^\sharp$ on H by taking the direct sum of C and the zero operator on $\mathcal{R}_\varphi^\perp$, and define

$$L = B^{1/2} C^\sharp : H \to H. \tag{2.15}$$

Clearly the range and cokernel of L are contained in $\mathcal{R}_\varphi$. Since $\|B^{1/2}\| = t_0$ and $\|C\| \le 1$ we have

$$\|L\| \le t_0.$$

On applying C^* to (2.14) we find that, for $\lambda \in D_\varphi$,

$$L^* k_\lambda = (C^\sharp)^* B^{1/2} k_\lambda = C^* B^{1/2} k_\lambda = C^* C \bar\varphi(\lambda) k_\lambda = (1 - A)\bar\varphi(\lambda) k_\lambda.$$

Hence, for any $h \in H$ and $\lambda \in D_\varphi$,

$$\begin{aligned}(Lh)(\lambda) &= \langle Lh, k_\lambda\rangle = \langle h, L^* k_\lambda\rangle = \langle h, (1-A)\bar\varphi(\lambda) k_\lambda\rangle \\ &= \varphi(\lambda)\langle (1-A)h,\, k_\lambda\rangle = \varphi(\lambda)\big(h(\lambda) - (Ah)(\lambda)\big).\end{aligned}$$

It remains to show that $\|L\| \ge t_0$, or equivalently, that the kernel

$$\left(\|L\|^2 - \bar\varphi(\lambda)\varphi(\mu)\right) k(\mu, \lambda)$$

has at most m negative squares. From (2.11) we have, for $\lambda, \mu \in \mathbb{D}_\varphi$,

$$\begin{aligned}\langle L^* k_\lambda, L^* k_\mu\rangle &= \langle (1-A)\bar\varphi(\lambda) k_\lambda, (1-A)\bar\varphi(\mu) k_\mu\rangle \\ &= \bar\varphi(\lambda)\varphi(\mu)\langle (1 - 2A + A^2) k_\lambda, k_\mu\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\left(\|L\|^2 - \bar\varphi(\lambda)\varphi(\mu)\right) k(\mu,\lambda) &= \langle (\|L\|^2 - LL^*) k_\lambda, k_\mu\rangle + \\ &\qquad \bar\varphi(\lambda)\varphi(\mu)\left[\langle (1 - 2A + A^2) k_\lambda, k_\mu\rangle - \langle k_\lambda, k_\mu\rangle\right] \\ &= \text{a positive kernel} - \bar\varphi(\lambda)\varphi(\mu)\langle A(2-A) k_\lambda, k_\mu\rangle.\end{aligned}$$

Since $A(2-A)$ is positive and has rank $\le m$, the right hand side has at most m negative squares, as required. □

We can now prove that (2) $\Rightarrow$ (3) in Theorem 2.1. Suppose that φ satisfies (2), the least value of t being t_0. By Lemma 2.5 there exist A and L in $L(H)$ such that $||L|| = t_0$ and, for all $\lambda \in D_\varphi$ and $h \in H$,

$$Lh(\lambda) = \varphi(\lambda)(h(\lambda) - (Ah)(\lambda)).$$

Moreover A has rank m, so that Ker A has codimension m, and for all $h \in$ Ker A and $\lambda \in D_\varphi$,

$$Lh(\lambda) = \varphi(\lambda) h(\lambda).$$

By Lemma 2.3 there is a subset $\mathcal{S}$ of Ω containing at most m points and a function $\psi : \Omega \setminus \mathcal{S} \to \mathbb{C}$ such that $\psi(\lambda) = \varphi(\lambda)$ for $\lambda \in D_\varphi \setminus \mathcal{S}$ and

$$\psi(\lambda) h(\lambda) = Lh(\lambda) = \varphi(\lambda) h(\lambda) \tag{2.16}$$

for all $h \in \text{Ker } A$ and $\lambda \in \Omega \setminus \mathcal{S}$. Extend ψ to a function on $D_\psi = (\Omega \setminus \mathcal{S}) \cup D_\varphi$ by defining $\psi(\lambda) = \varphi(\lambda)$ for $\lambda \in D_\varphi \cap \mathcal{S}$. Then ψ extends φ and its domain D_ψ is a set of uniqueness for H (since $D_\psi \supset D_\varphi$). There are at most m points in $\Omega \setminus D_\psi \subset \mathcal{S}$, and (2.16) tells us that $\psi\text{Ker } A \subset H$. Thus ψ is an m-pseudomultiplier of H which extends φ. Hence (2) $\Rightarrow$ (3).

Now suppose (1)-(3) hold. From (2.16) we have

$$s_m(\psi) \leq ||L|| = t_0.$$

In the proof of (1) $\Rightarrow$ (2) we showed that the kernel (2.1) has at most m negative squares if $t = s_m(\varphi)$, that is,

$$t_0 \leq s_m(\varphi).$$

If E is an m-codimensional subspace of H such that $\psi E \subset H$ then $\varphi E|D_\varphi \subset H|D_\varphi$ and $||M_{\psi,E}|| = ||M_{\varphi,E}|D_\varphi||$. Taking infima over closed m-codimensional subspaces E such that $\psi E \subset H$ we obtain

$$s_m(\varphi) \leq s_m(\psi).$$

Combining the last three inequalities, we have

$$s_m(\varphi) = t_0 = s_m(\psi).$$

□

We can now derive a simple but informative multiplicity result for pseudomultipliers.

Theorem 2.6. *Let H be a Hilbert space of functions with nonsingular kernel on a set Ω and let φ be an m-pseudomultiplier of H. For any $\xi \in \mathbb{C}$ such that $|\xi| > s_m(\varphi)$, the equation $\varphi(\lambda) = \xi$ has at most m solutions for $\lambda \in D_\varphi$.*

Proof. Pick L, A as in Lemma 2.5. Suppose there are n points $\lambda_1, \ldots, \lambda_n \in D_\varphi$ which are solutions of the equation, where $n > m$. Since A has rank at most m there exist scalars $c_1, \ldots, c_n$, not all zero, such that $\sum c_j A k_{\lambda_j} = 0$. Let $h = \sum c_j k_{\lambda_j}$. Then $h \neq 0$, and

$$L^* h = \sum c_j L^* k_{\lambda_j} = \sum c_j \bar{\xi}(k_{\lambda_j} - A k_{\lambda_j}) = \bar{\xi} h,$$

and so $\bar{\xi}$ is an eigenvalue of L^*. However

$$||L^*|| = s_m(\varphi) < |\xi|,$$

a contradiction. □

We conclude this section with an aside about the key Lemma 2.5. How should we understand the rank m Hermitian operator A associated with an m-pseudomultiplier

φ? Let us consider the case $m = 1$, and put $A = f \otimes f$ for some $f \in H$. The relation (2.10) becomes

$$(Lh)(\lambda) = \varphi(\lambda)\,(h(\lambda) - \langle h, f\rangle f(\lambda)) \tag{2.17}$$

for $h \in H$, $\lambda \in D_\varphi$. What is the connection between the pseudomultiplier φ and the function f? We propose that f be regarded as a singular vector of the (unbounded) operator M_φ on H corresponding to the singular value $s_0(\varphi) = \infty$. Strictly speaking this does not make sense, but we can give it a meaning as follows.

Consider first the case of a finite kernel k on $\Omega = \{\lambda_1, \ldots, \lambda_n\}$, and define T on H by $Tk_\lambda = \bar{\varphi}(\lambda)k_\lambda$ (so that $T = T_\Omega$ in the notation above). Then T has a maximising vector u such that, for some contraction B on H,

$$T^*T = B + u \otimes u.$$

Here $||u|| = ||T||$, and in the notation of Lemma 2.4, $u = a_1$. By equation (2.14), $\varphi f = u$, and so

$$f = \frac{1}{\varphi}u = (T^*)^{-1}u.$$

Thus $u = T^*f$. Hence f is a maximising vector of T^* and $||f|| = 1$. In other words f is a singular vector of the multiplication operator M_φ corresponding to $s_0(\varphi)$. For finite kernels, then, our statement about f is meaningful and true.

Now consider a nonsingular kernel on a general set Ω and apply the foregoing observation to T_F for each $F \in \mathcal{F}$. Then there is a unit maximising vector $f_F \in \mathcal{M}_F$ and a contraction B_F on $\mathcal{M}_F$ such that

$$T_F^*T_F = B_F + T_F^*f_F \otimes T_F^*f_F.$$

Along the ultrafilter $\mathcal{U}$, f_F tends weakly to a limit, which must equal f. If $||f|| < 1$ then φ is a multiplier of H (put $h = f$ in (2.17)). If $||f|| = 1$ then we have $f_F \to f$ in norm as $F \to \mathcal{U}$. That is, f is the norm limit of unit maximising vectors of the approximating finite rank compressions T_F^*, $F \in \mathcal{U}$ of the multiplication operator M_φ.

3. Pseudomultipliers of analytic kernels

What are the pseudomultipliers of the familiar Hilbert spaces of analytic functions? The examples we began with ($1/z$ and the characteristic function of a point) might lead one to the optimistic hope that the 1-pseudomultipliers are simply the multipliers modified by a removable singularity or a simple pole. We already know that, for a general analytic kernel, this can fail to be true in at least three ways: the domain Ω could be too small (Example 1.4), H could really be a space of functions of z^2 (Example 1.10) or there could be more subtle structural reasons (Example 1.11). Nevertheless, it *is* true for a wide class of kernels, as we show in Theorem 3.3 below.

Let Ω be a domain in $\mathbb{C}$ and let H be a Hilbert space of analytic functions on Ω with reproducing kernel k. Such a kernel will be called an *analytic kernel.* There is one simple observation we can make about pseudomultipliers of such kernels.

Theorem 3.1. *Let k be a nonsingular analytic kernel on a domain Ω and let φ be an m-pseudomultiplier of the associated Hilbert space H of functions on Ω. Let the defect of φ be d. There is a set $\mathcal{S} \subset D_\varphi$, containing at most $m-d$ points, such that φ is analytic at every point of $D_\varphi \setminus \mathcal{S}$. Moreover, each point of $\Omega \setminus D_\varphi$ is a pole of φ and each point of $\mathcal{S}$ is either a removable singularity or a pole of φ.*

Note that it *can* happen that a point of D_φ is a pole of φ: recall Example 1.7.

Proof. By Lemma 2.5 there exist bounded linear operators $A,\ L$ on H such that $0 \leq A \leq 1$, A has rank at most m and, for every $h \in H$ and $\lambda \in D_\varphi$,

$$(Lh)(\lambda) = \varphi(\lambda)\left(h(\lambda) - (Ah)(\lambda)\right).$$

We claim that φ is analytic at any $\beta \in \Omega$ such that $k_\beta \notin AH$. Indeed, for such β, Ker A is not a subset of $k_\beta^\perp$, and so there exists $h \in H$ such that $Ah = 0$ and $h(\beta) \neq 0$. We have $Lh = \varphi h$, and so $\varphi = (Lh)/h$ on any neighbourhood of β on which h does not vanish. Thus φ is analytic at β.

It follows that $k_\beta \in AH$ for the d points of $\Omega \setminus D_\varphi$. Since k is nonsingular and AH has dimension $\leq m$, the set

$$\mathcal{S} \stackrel{\text{def}}{=} \{\beta \in D_\varphi : k_\beta \in AH\}$$

can contain at most $m-d$ points. The preceding paragraph shows that φ is analytic at each point of $D_\varphi \setminus \mathcal{S}$.

Now consider any point $\alpha \in \Omega \setminus D_\varphi$. Then α is a singularity of φ. That is, there exist $u,\ g \in H$ such that $u(\alpha) = 0,\ \varphi u = g|D_\varphi$ and $g(\alpha) \neq 0$. Clearly $u \neq 0$, so that α is an isolated zero of u. Hence we can write $\varphi = g/u$ on a punctured neighbourhood of α, and so φ has a pole at α.

Consider any point $\alpha \in \mathcal{S}$. Since φ is analytic in a punctured neighbourhood of α it must be the case that α is either a removable singularity, a pole or an essential singularity of φ. In the latter case, by the great Picard theorem, φ attains all but one complex values infinitely often, contradicting Theorem 2.6. Hence one of the other two alternatives holds.

□

This is an opportune moment to point out that pseudomultipliers of Hilbert spaces of analytic functions of several complex variables are not interesting objects, at least as we have defined them here. By exactly the argument above, an m-pseudomultiplier φ of such a space is analytic at all but at most m points. Now a singularity of φ in the sense of Definition 1.3 would have to be an isolated pole of φ, and an analytic function of several variables cannot have any such. Thus the m points where analyticity fails are removable singularities, and φ is just an analytic function plus some point discontinuities. However, it is possible the notion of pseudomultiplier could be developed for Hilbert spaces of *vector-valued* functions so as to apply to analytic functions of several variables.

Corollary 3.2. *Let H be a Hilbert space of entire functions having nonsingular kernel on $\mathbb{C}$, and let φ be an m-pseudomultiplier of defect d on H. There exists a rational function ψ of degree at most m which agrees with φ at all except at most $m-d$ points of D_φ.*

Here the degree of ψ is defined to be the sum of the multiplicities of the poles of ψ, including ∞ if applicable. The points at which φ and ψ differ can include poles of ψ.

Proof. By Theorem 3.1 there is a set $\mathcal{S} \subset D_\varphi$ containing at most $m-d$ points, consisting of poles and removable singularities of φ, such that φ is analytic on $D_\varphi \setminus \mathcal{S}$. Let ψ be the function obtained from φ by removing any removable singularities and deleting from the domain of φ any point $\alpha \in \mathcal{S}$ which is a pole of $\varphi|D_\varphi \setminus \{\alpha\}$. Then ψ is meromorphic in the whole complex plane. Again by the great Picard theorem and Theorem 2.6, ψ cannot have an essential singularity at ∞ and hence is rational. Clearly ψ agrees with φ except at points of $\mathcal{S}$. □

In order to get more detailed information about the nature of pseudomultipliers of analytic kernels we consider kernels satisfying the following two axioms:

(AK1): H is invariant under the operation M_z of multiplication by the independent variable;

(AK2): Every bounded linear operator on H which commutes with M_z is a multiplication operator.

Not all spaces of interest have these properties (e.g., the Fock space does not satisfy (AK1)), but many of them do. An easy consequence of (AK1) is

$$M_z^* k_\lambda = \bar{\lambda} k_\lambda \tag{3.1}$$

for all $\lambda \in \Omega$.

Remark. (AK2) holds if (AK1) is true and either H contains the polynomials on Ω as a dense subspace or every eigenvector of M_z^* has the form ck_α for some $c \in \mathbb{C}$ and $\alpha \in \Omega$.

For suppose $TM_z = M_zT$. In the former case it is plain that T is multiplication by $M\mathbf{1}$. In the latter we have, for any $\lambda \in \Omega$,

$$M_z^* T^* k_\lambda = T^* M_z^* k_\lambda = \bar{\lambda} T^* k_\lambda$$

and so T^*k_λ is either 0 or an eigenvector of M_z^* with eigenvalue $\bar\lambda$. Hence, by assumption,

$$T^*k_\lambda = \bar\psi(\lambda)k_\lambda$$

for some $\psi(\lambda) \in \mathbb{C}$. For any $h \in H$ and $\lambda \in \Omega$

$$(Th)(\lambda) = \langle Th, k_\lambda\rangle = \langle h, T^*k_\lambda\rangle = \langle h, \bar\psi(\lambda)k_\lambda\rangle = \psi(\lambda)h(\lambda).$$

Thus ψ is a multiplier and $T = M_\psi$.

Theorem 3.3. *Let H be a Hilbert space of analytic functions on a domain Ω. Suppose that H has nonsingular kernel and that* (AK1) *and* (AK2) *hold. Let $\varphi : D_\varphi \subset \Omega \to \mathbb{C}$ be a 1-pseudomultiplier of H. Then there exists $\alpha \in \mathbb{C}$ and a multiplier θ of H such that D_φ is either Ω or $\Omega \setminus \{\alpha\}$ and*

$$\varphi(\lambda) = \frac{\theta(\lambda)}{\lambda - \alpha} \quad \textit{for } \lambda \in \Omega \setminus \{\alpha\}.$$

We do not assert that $\alpha \in \Omega$ in general. Observe that the conclusion holds when φ is a multiplier or even a function that differs from a multiplier at a single point $\alpha \in \Omega$ (take $\theta(\lambda) = (\lambda - \alpha)\varphi(\lambda)$).

Proof. By Lemma 2.5 there exist $L \in \mathcal{L}(H)$ such that $\|L\| = s_1(\varphi)$ and $f \in H$ such that $\|f\| \le 1$ and, for all $\lambda \in D_\varphi$,

$$L^*k_\lambda = \bar\varphi(\lambda)k_\lambda - \bar\varphi(\lambda)\bar f(\lambda)f$$

(we have put $A = f \otimes f$). From this equation and (3.1) it follows that

$$(M_z^*L^* - L^*M_z^*)k_\lambda = \bar\varphi(\lambda)\bar f(\lambda)(\bar\lambda f - M_z^* f) \tag{3.2}$$

for all $\lambda \in D_\varphi$. Hence

$$\operatorname{Range}(M_z^*L^* - L^*M_z^*) \subset \operatorname{span}\{f, M_z^* f\}.$$

We consider the three possible dimensions of this span.

Suppose $f = 0$: then by 2.10, for $\lambda \in D_\varphi$ and $h \in H$,

$$(Lh)(\lambda) = \varphi(\lambda)h(\lambda).$$

By Lemma 2.3 and Theorem 1.13 $D_\varphi = \Omega$ and φ is a multiplier of H. (Incidentally, we have in this case

$$s_0(\varphi) = ||M_\varphi|| = ||L|| = s_1(\varphi),$$

so that $s_0(\varphi)$ is a multiple singular value.)

Secondly, suppose f and $M_z^* f$ span a space of dimension one: say $M_z^* f = \bar\alpha f$ for some $\alpha \in \mathbb{C}$. In equation (2.10) replace h by $(M_z - \alpha)h$ to obtain, for all $\lambda \in D_\varphi$ and

$h \in H$,

$$\begin{aligned}(L(M_z-\alpha)h)(\lambda) &= \varphi(\lambda)\left((\lambda-\alpha)h(\lambda)-\langle (M_z-\alpha)h, f\rangle f(\lambda)\right)\\ &= \varphi(\lambda)\left((\lambda-\alpha)h(\lambda)-\langle h, M_z^* f-\bar\alpha f\rangle f(\lambda)\right)\\ &= (\lambda-\alpha)\varphi(\lambda)h(\lambda).\end{aligned}$$

Since $L(M_z-\alpha)$ is bounded, Lemma 2.3 tells us that there is a multiplier θ of H such that

$$\theta(\lambda) = (\lambda-\alpha)\varphi(\lambda) \text{ for all } \lambda \in D_\varphi. \tag{3.3}$$

We claim that φ can have no singularity in Ω other than α. For suppose $\beta \neq \alpha$ is a singularity. Then there exists $h \in H$ such that $h(\beta)=0$ and φh has an extension $u \in H$ such that $u(\beta)\neq 0$. Now $(M_z-\alpha)u \in H$ is an extension of the function $\lambda \mapsto (\lambda-\alpha)(\varphi h)(\lambda)$ on D_φ, that is, of $\theta h|D_\varphi$. Since $\theta h \in H$, it follows that $\theta h = (M_z-\alpha)u$. Evaluating both sides at β we have

$$0=\theta(\beta)h(\beta)=(\beta-\alpha)u(\beta)\neq 0,$$

a contradiction. Thus β cannot be a singularity, and so D_φ is either Ω or $\Omega\setminus\{\alpha\}$. It follows that $D_\varphi\setminus\{\alpha\}=\Omega\setminus\{\alpha\}$, and, from (3.3),

$$\varphi(\lambda)=\frac{\theta(\lambda)}{\lambda-\alpha} \quad \text{for } \lambda\in\Omega\setminus\{\alpha\}$$

as required.

The remaining possibility is that $\operatorname{span}\{f, M_z^* f\}$ has dimension 2. Then, by (3.2), there exist $u, v \in H$ such that

$$M_z^* L^* - L^* M_z^* = f\otimes u - M_z^* f \otimes v. \tag{3.4}$$

On applying both sides to k_λ we find, for $\lambda \in D_\varphi$,

$$\bar\varphi(\lambda)\bar f(\lambda)(\bar\lambda f - M_z^* f) = \bar u(\lambda) f - \bar v(\lambda) M_z^* f.$$

By the linear independence of f and $M_z^* f$,

$$\bar\lambda\bar\varphi(\lambda)\bar f(\lambda)=\bar u(\lambda), \quad \bar\varphi(\lambda)\bar f(\lambda)=\bar v(\lambda)$$

on D_φ. It follows that $u-M_z v$ vanishes on D_φ, hence is zero. Thus (3.4) becomes

$$M_z^* L^* - L^* M_z^* = (f\otimes v)M_z^* - M_z^*(f\otimes v).$$

Consequently $L+v\otimes f$ commutes with M_z. Thus there exists a multiplier θ of H such that $L+v\otimes f = M_\theta$. Then, for $\lambda\in D_\varphi$,

$$\bar\theta(\lambda)k_\lambda = M_\theta^* k_\lambda = L^* k_\lambda + (f\otimes v)k_\lambda = \bar\varphi(\lambda)k_\lambda - \bar\varphi(\lambda)\bar f(\lambda) f + \bar v(\lambda) f = \bar\varphi(\lambda)k_\lambda.$$

Thus φ is the restriction of a multiplier. By Theorem 1.13, φ *is* a multiplier. □

It would be of interest to identify the pseudomultipliers and the corresponding quantities $s_m(\cdot)$ for favourite function spaces. As we have mentioned, in the case of H^2, a well known theorem of Adamyan, Arov and Krein provides a complete description. To present it we introduce some terminology. Suppose that $\varphi,\ \psi$ are functions with domains $D_\varphi,\ D_\psi$ contained in a set Ω. We shall say that ψ is a *finite modification* of φ if $D_\psi \supset D_\varphi,\ D_\psi \setminus D_\varphi$ is finite and $\varphi,\ \psi$ agree at all but finitely many points of D_φ. We recall that $H^\infty_{(\ell)}$ denotes the set of functions of the form $\varphi = f/p$ where $f \in H^\infty$ and p is a polynomial with at most ℓ zeros in $\mathbb{D}$, counting multiplicities, and none of unit modulus; we take D_φ to be the complement of the set of poles of φ in $\mathbb{D}$ (thus we suppose that all removable singularities have been removed).

Theorem 3.4. *A function $\varphi : D_\varphi \subset \Omega \to \mathbb{C}$ is a pseudomultiplier of H^2 if and only if there is a function $\psi \in H^\infty_{(\ell)}$, for some non-negative integer ℓ, such that φ is a finite modification of ψ. Moreover, if φ is an m-pseudomultiplier of H^2, then $s_m(\varphi) = \|\varphi\|_{L^\infty(\mathbb{T})}$.*

This is essentially the main result of [AAK], where however it is expressed in a rather different context. A careful derivation of this form of the theorem from the original one can be found in [Q2, Sec. 7].

We shall conclude with a description of the pseudomultipliers of the Fock space (see Example 1.9). Recall that a rational function is said to be *proper* if it has a finite limit at infinity.

Theorem 3.5. *The pseudomultipliers of the Fock space Φ are precisely the finite modifications of the proper rational functions.*

Proof. Let $\mathcal{E}$ be the space of all entire functions. We claim that if p is a non-zero polynomial, f is an entire function and $pf \in \Phi$ then $f \in \Phi$. Indeed, if p is non-constant, $\{z : |p(z)| < 1\}$ is a bounded subset of $\mathbb{C}$, and so

$$\begin{aligned}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA &= \int_{|p(z)|<1} + \int_{|p(z)|>1} |f(z)|^2 e^{-|z|^2} dA \\ &\le \int_{|p(z)|<1} |f(z)|^2 e^{-|z|^2} dA + \int_{|p(z)|>1} |p(z)f(z)|^2 e^{-|z|^2} dA < \infty.\end{aligned}$$

Let φ be a proper rational function of degree n with denominator p. We show that φ is a pseudomultiplier of Φ. Let $E = \Phi \cap p\mathcal{E}$. Then E has codimension n in Φ: it is given by the vanishing of n linear functionals, which are evaluations of f and some of its derivatives at zeros of p. Write φ as a sum of partial fractions in which each term is either constant or is a constant divided by a divisor of p. The claim above shows that each such term multiplies E into Φ and hence $\varphi E \subset \Phi$. It is clear that each pole of φ

is a singularity in the sense of Definition 1.3, and hence φ is an n-pseudomultiplier, whose domain D_φ is the complement of the set of poles of φ. Now consider the function ψ obtained by changing the value of φ at points $\alpha_1, \dots, \alpha_r \in D_\varphi$ and giving it a value at poles $\beta_1, \dots, \beta_s$ of φ. Then ψ has domain $D_\psi = D_\varphi \cup \{\beta_1, \dots, \beta_s\}$. If β_j has multiplicity n_j then ψ multiplies the closed $(n+r+s)$-codimensional subspace

$$\{f \in \Phi \cap p\mathcal{E} : f(\alpha_j) = 0,\ 1 \le j \le r,\ f^{(n_i)}(\beta_i) = 0,\ 1 \le i \le s\}$$

of Φ into $\Phi | D_\psi$. Points outside the domain of ψ are poles of φ and they remain singularities of ψ. Thus ψ is an $(n+r+s)$-pseudomultiplier of Φ. That is, any finite modification of a proper rational function is a pseudomultiplier of Φ.

Conversely, consider an m-pseudomultiplier ψ of Φ. Let ψ have defect d. By Corollary 3.2 there exists a rational function φ which differs from ψ at no more than $m-d$ points. We have to show that φ is proper.

The function z is not a pseudomultiplier of Φ. For suppose $s_m(z) = t < \infty$. Then by Theorem 2.1 the kernel $(t^2 - \lambda\bar{\mu})e^{\lambda\bar{\mu}}$ has at most m negative squares. However,

$$(t^2 - w)e^w = t^2 + \sum_{n=1}^{\infty} \left(\frac{t^2}{n} - 1\right) \frac{w^n}{(n-1)!},$$

which has infinitely many negative coefficients, a contradiction. It follows that $z \mapsto z - a$ is not a pseudomultiplier, for any $a \in \mathbb{C}$. In fact no non-constant polynomial is a pseudomultiplier of Φ. For suppose that p is such a polynomial and $pE \subset \Phi$ for some closed finite-codimensional space $E \subset \Phi$. Factorize p as $p(z) = (z-a)q(z)$ for some $a \in \mathbb{C}$ and polynomial q. By the claim above, if $(z-a)qE \subset \Phi$ then $(z-a)E \subset \Phi$, contradicting the fact that $z - a$ is not a pseudomultiplier. Hence p is not a pseudomultiplier.

Return to the rational pseudomultiplier φ of Φ and expand it in partial fractions:

$$\varphi = p + \sum u_j$$

where p is a polynomial and each u_j is a constant divided by a power of a linear function. We showed above that each u_j is a pseudomultiplier, and so $\varphi - \sum u_j$ is a pseudomultiplier. That is, p is a pseudomultiplier. It follows that p is constant, and hence that φ is proper. $\square$

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