THE SEMI-COMMUTATOR OF TOEPLITZ OPERATORS ON THE BIDISC

CAIXING GU AND DECHAO ZHENG

ABSTRACT. In this paper we characterize when the semi-commutator $T_f T_g - T_{fg}$ of two Toeplitz operators T_f and T_g on the Hardy space of the bidisc is zero. We also show that there is no nonzero finite rank semi-commutator on the bidisc. Furthermore explicit examples of compact semi-commutators with symbols continuous on the bitorus T^2 are given.

1. INTRODUCTION

Let D be the open unit disk in C. Its boundary is the unit circle T. The bidisc D^2 and the torus T^2 are the subsets of C^2 which are Cartesian products of two copies D and T, respectively. Let $d\sigma(z)$ be the normalized Haar measure on T^2. The Hardy space $H^2(D^2)$ is the closure of the polynomials in $L^2(T^2, d\sigma)$ (or $L^2(T^2)$). Let P be the orthogonal projection from $L^2(T^2)$ onto $H^2(D^2)$. The Toeplitz operator with symbol f in $L^\infty(T^2)$ is defined by $T_f(h) = P(fh)$, for all $h \in H^2(D^2)$ and the Hankel operator with symbol f is defined by $H_f(h) = (I - P)(fh)$, for all $h \in H^2(D^2)$.

Let f and g be two bounded functions on T^2. In this paper we study the the semi-commutator $T_f T_g - T_{fg}$ of two Toeplitz operators T_f and T_g on the bidisc. As in the case of the unit disk, the semi-commutator is connected to the Hankel operators by the following relation.

$$T_f T_g - T_{fg} = -H_{\bar{f}}^* H_g. \tag{1}$$

To motivate the problems to be considered in the paper, we shall recall some classical results for the semi-commutators of Toeplitz operators on the Hardy space $H^2(D)$ of the unit disk. Let f_1 and g_1 be bounded functions on the unit circle T. For two Toeplitz operators T_{f_1} and T_{g_1} on the Hardy space $H^2(D)$ of the unit disk, Brown-Halmos [2] shows that the semi-commutator $T_{f_1} T_{g_1} - T_{f_1 g_1}$ is zero if and only if either $\bar{f}_1$ or g_1 is analytic. In other words, $H_{\bar{f}_1}^* H_{g_1}$ is zero if and only if either $H_{\bar{f}_1}$ or H_{g_1} is zero.

In this paper, we will characterize when the semi-commutator $T_f T_g - T_{fg}$ of two Toeplitz operators T_f and T_g on the bidisc is zero. In particular, we note that unlike

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on the unit disk one can have both $H_{\bar{f}}$ and H_g are not zero, but their product $H^*_{\bar{f}}H_g$ is zero. Furthermore we will see that there is no finite rank semi-commutator on the bidisc. But this is false on the unit disk. Indeed it was shown in [1] that for Toeplitz operators T_{f_1} and T_{g_1} on $H^2(D)$, $T_{f_1}T_{g_1} - T_{f_1g_1}$ is of finite rank if and only if either $\bar{f}_1$ or g_1 is an analytic function plus a rational function.

The main question to be considered here is when the semi-commutator $T_fT_g - T_{fg}$ on the bidisc is compact. This problem is connected with the spectral theory of Toeplitz operators on the bidisc and various applications, see [7], [8] and reference therein.

For Toeplitz operators T_{f_1} and T_{g_1} on the unit disk, this hard problem was solved by the combined efforts of Axler, Chang, and Sarason [1] and Volberg [12]. Their beautiful result is that that $T_{f_1g_1} - T_{f_1}T_{g_1}$ is compact if and only if $H^\infty[\bar{f}_1] \bigcap H^\infty[g_1] \subset H^\infty(D) + C(T)$; here $H^\infty[g_1]$ denotes the closed subalgebra of $L^\infty(T)$ generated by $H^\infty(D)$ and g_1 and $C(T)$ the continuous functions on T.

The function theory on the bidisc is quite different from and much less understood than the function theory on the unit disk [9], [3] and [5]. The proof of the above result on the unit disk relies on some deep results and techniques from function theory on the unit disk which are not available from function theory on the bidisc.

In this paper we content ourselves with some partial results for the compactness of $T_fT_g - T_{fg}$. By carefully analyzing the action of $T_fT_g - T_{fg}$ on the reproducing kernel of $H^2(D^2)$ and exploiting the harmonicity of certain functions, we will get a necessary condition for the compactness of $T_fT_g - T_{fg}$. This shows that for a large class of functions f and g, $T_fT_g - T_{fg}$ is compact if and only if it is zero. For example, if f is a trigonometric polynomial on T^2 and g is an arbitrary bounded function on T^2, then for such f and g, there is no compact semi-commutator $T_fT_g - T_{fg}$. Also as a corollary of this condition we see that there are no compact Hankel operators on bidisc, which was proved by M. Cotlar and C. Sadosky [4] using a completely different method. It is natural to guess that for f and g in $L^\infty(T^2)$, $T_fT_g - T_{fg}$ is compact if and only if $T_fT_g - T_{fg}$ is zero. But it is false.

For the class of bounded functions f and g on T^2 of the form $f(z_1, z_2) = f_1(z_1)f_2(z_2)$ and $g(z_1, z_2) = g_1(z_1)g_2(z_2)$, we will show that $T_fT_g - T_{fg}$ is compact if and only if the following two conditions hold.
(1) $f_1(z_1)g_1(z_1) = 0$ and $f_2(z_2)g_2(z_2) = 0$ on T, and
(2) $\lim_{z\to T^2} \|H_{\overline{f}}k_z\|_2\|H_gk_z\|_2 = 0$,
where k_z denotes the normalized reproducing kernel of $H^2(D^2)$. Here the Littlewood-Paley theory on the bidisc and a certain distribution function inequality in Zheng [13] play a key role. We remark that only one condition similar to Condition (2) above is needed for the compactness of the semi-commutator $T_{f_1}T_{g_1} - T_{f_1g_1}$ on the Hardy space $H^2(D)$ of the unit disk. Indeed it was shown in [13] that $T_{f_1}T_{g_1} - T_{f_1g_1}$ is compact on $H^2(D)$ if and only if $\lim_{z_1\to T} \|H_{\overline{f_1}}k_{z_1}\|_2\|H_{g_1}k_{z_1}\|_2 = 0$, where k_{z_1} denotes the normalized reproducing kernel of $H^2(D)$.

Now we outline the plan of the paper. In Section 2 we study when the semi-commutator of two Toeplitz operators is zero or finite rank. In Section 3 we derive a necessary condition for the compactness of $T_fT_g - T_{fg}$. In Section 4, for f and g of the form $f(z_1, z_2) = f_1(z_1)f_2(z_2)$ and $g(z_1, z_2) = g_1(z_1)g_2(z_2)$, we characterize when $T_fT_g - T_{fg}$ is compact. This result shows that the necessary condition for the compactness of $T_fT_g - T_{fg}$ obtained in Section 3 is not sufficient in general. This result also provides us with explicit examples of compact semi-commutators $T_fT_g - T_{fg}$ with symbols f and g continuous on the bitorus T^2.

2. Zero semi-commutators

Let Z denote the set of all integers, Z_+ the set of all nonnegative integers and Z_- the set of all negative integers. As in [10] we consider multiple Fourier series on the bitorus T^2. The multiple Fourier series on the bitorus T^2 can be viewed as the Fourier transformation on $L^1(T^2)$. For f in $L^1(T^2)$, the Fourier transformation of f on $Z \times Z$ is given by

$$f_m = (\frac{1}{2\pi})^2 \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta_1}, e^{i\theta_1}) e^{i(m,\theta)} d\theta_1 d\theta_2$$

where $m = (m_1, m_2) \in Z \times Z$, $\theta = (\theta_1, \theta_2)$ and $(m, \theta) = m_1\theta_1 + m_2\theta_2$. By Theorem 1.7 in [SW], the Fourier transformation is injective, i.e. If $f \in L^1(T^2)$ and $f_m = 0$ for all $m \in Z \times Z$, then $f \equiv 0$. We recall also that by using multiple Fourier series,

$$L^2(T^2) = \{f : f = \sum_{m \in Z \times Z} f_m e^{i(m,x)}, \sum_{m \in Z \times Z} |f_m|^2 < \infty\}$$

$$H^2(D^2) = \{h : h = \sum_{m \in Z_+ \times Z_+} h_m e^{i(m,x)}, \sum_{m \in Z_+ \times Z_+} |h_m|^2 < \infty\}$$

and

$$Pf = \sum_{m \in Z_+ \times Z_+} f_m e^{i(m,x)} \text{ for } f = \sum_{m \in Z \times Z} f_m e^{i(m,x)} \in L^2(T^2).$$

Theorem 1. *Let f and g be two bounded functions on the torus T^2. The following are equivalent.*
(1) $T_fT_g - T_{fg}$ is a finite rank operator.
(2) $T_fT_g - T_{fg}$ is zero.
(3) For each i ($i = 1, 2$) either $\bar{f}$ or g is analytic in z_i.

Proof. We first show that (3) implies (2). Assume that for each i ($i = 1, 2$) either $\bar{f}$ or g is analytic in z_i. Without loss of generality, assume that $\bar{f}$ is analytic in z_1 for and g is analytic in z_2. Then it is easy to see that

$$(I - P)(gh_1) = \sum_{m=(m_1,m_2) \in Z_- \times Z_+} a_m z^m$$

for all $h_1 \in H^2(D^2)$ and

$$(I-P)(\bar{f}h_2) = \sum_{m=(m_1,m_2)\in Z_+\times Z_-} b_m z^m$$

for all $h_2 \in H^2(D^2)$. Therefore

$$(H_g h_1, H_{\bar{f}} h_2) = 0$$

for all $h_1, h_2 \in H^2(D^2)$. That is, $H_{\bar{f}}^* H_g = 0$. So $T_f T_g - T_{fg} = -H_{\bar{f}}^* H_g = 0$

It is obvious that (2) implies (1). Now we prove that (1) implies (3). That is, assume that $T_f T_g - T_{fg}$ is a finite rank operator, we will show that for each i ($i = 1, 2$) either $\bar{f}$ or g is analytic in z_i. Without loss of generality, we assume that $i = 1$. We write f and g as

$$\bar{f} = \sum_{i=-\infty}^{\infty} f_i(z_2) z_1^i = \sum_{i=-\infty}^{\infty} f_i z_1^i$$

$$g = \sum_{i=-\infty}^{\infty} g_i(z_2) z_1^i = \sum_{i=-\infty}^{\infty} g_i z_1^i$$

Let $\alpha, \beta, k, l \in Z_+$. Then by a straightforward computation we have

$$\begin{aligned}
& (H_f^* H_g z_1^k z_2^\alpha, z_1^l z_2^\beta) \\
= \ & (H_g z_2^\alpha z_1^k, H_f z_2^\beta z_1^l) \\
= \ & \sum_{i\le -(k+1)} (g_i(z_2) z_2^\alpha, f_{-(i+k-l)}(z_2) z_2^\beta) + \sum_{i\ge -k} ((I_2 - P_2) g_i(z_2) z_2^\alpha, f_{-(i+k-l)}(z_2) z_2^\beta)
\end{aligned}$$

where I_2 is the identity on $L^2(T)$ and P_2 is the projection from $L^2(T)$ onto $H^2(D)$. Therefore

$$\begin{aligned}
& (H_g z_1^k z_2^\alpha, H_f z_1^l z_2^\beta) - (H_g z_1^{k+1} z_2^\alpha, H_f z_1^{l+1} z_2^\beta)) \\
= \ & (g_{-(k+1)}(z_2)(z_2)^\alpha, f_{-(l+1)}(z_2)(z_2)^\beta) - ((I_2 - P_2) g_{-(k+1)}(z_2)(z_2)^\alpha, f_{-(l+1)}(z_2)(z_2)^\beta) \\
= \ & (P_2 g_{-(k+1)}(z_2)(z_2)^\alpha, f_{-(l+1)}(z_2)(z_2)^\beta).
\end{aligned}$$

Let S_1 denote the multiplication by z_1 on $H^2(D^2)$, i.e., $S_1 h = z_1 h$ for $h \in H^2(D^2)$. The above relation implies that

$$(S_1^{*l} H_f^* H_g S_1^k - S_1^{*(l+1)} H_f^* H_g S_1^{k+1}) h_2(z_2) = T_{\overline{f_{-(l+1)}}} T_{g_{-(k+1)}} h_2(z_2)$$

for all $h_2 \in H^2(D)$. Therefore, if $H_f^* H_g$ is a finite rank operator on $H^2(D^2)$, then $T_{\overline{f_{-(l+1)}}} T_{g_{-(k+1)}}$ is a finite rank operator on $H^2(D)$. By a result in [2], we have that either $f_{-(l+1)}$ or $g_{-(k+1)} = 0$. Hence either $f_{-(l+1)} = 0$ for all $k \ge 0$ or $g_{-(k+1)} = 0$ for all $l \ge 0$. That is either $\bar{f}$ or g is analytic in z_1. This finishes the proof of the theorem.

3. A necessary condition for compactness

In this section we will give a necessary condition for the compactness of the semi-commutator $T_f T_g - T_{fg}$.

Before going to the main result of this section, we need some notations and definitions. We use z to denote the vector (z_1, z_2) in C^2 of two dimensional complex plane. Since for any z in D^2, the pointwise evaluation of functions in $H^2(D^2)$ at z is a bounded functional, there is a function K_z in $H^2(D^2)$ such that

$$f(z) = (f, K_z)$$

for all f in $H^2(D^2)$. $K_z(w)$ is called the reproducing kernel for $H^2(D^2)$ and sometimes we use K(z,w) to denote $K_z(w)$.

Let K_{z_1} denote the reproducing kernel of the Hardy space $H^2(D)$ of the unit disk at the point z_1 in D, and k_{z_1} the normalized reproducing kernel of the Hardy space $H^2(D)$ at the point $z_1 \in D$. Namely,

$$K_{z_1} = \frac{1}{(1-\overline{z_1}w_1)}, \quad k_{z_1} = \frac{(1-|z_1|^2)^{1/2}}{(1-\overline{z_1}w_1)}.$$

It is easy to check that the reproducing kernel K_z of the Hardy space $H^2(D^2)$ of the bidisc is the product $K_{z_1}(w_1)K_{z_2}(w_2)$ of the reproducing kernels of $H^2(D)$. So the normalized reproducing kernel k_z of $H^2(D^2)$ is also the product $k_{z_1}(w_1)k_{z_2}(w_2)$ of the normalized reproducing kernels of $H^2(D)$. We observe that k_z weakly converges to zero in $H^2(D^2)$ as z tends to the boundary of D^2. For a function f in the space $L^2(T^2)$, i.e.,

$$f(w) = \sum_{m \in Z\times Z} f_m w^m, \quad w \in T^2,$$

where f_m is a sequence of numbers such that

$$\sum_{m\in Z\times Z} |f_m|^2 < \infty,$$

the bi-harmonic (in short, harmonic) extension $f(z)$ of f is defined via the Possion integral

$$\begin{aligned} f(z) &:= \int_T \int_T f(w) \frac{(1-|z_1|^2)}{|1-\overline{z_1}w_1|^2} \frac{(1-|z_2|^2)}{|1-\overline{z_2}w_2|^2} d\sigma(w_1) d\sigma(w_2) \\ &= (f(w)k_z(w), k_z(w)). \end{aligned}$$

We write the power series expansion of the harmonic extension $f(z)$ of f as follows:

$$\begin{aligned} f(z) &= \sum_{m\in Z\times Z} f_m z^m \\ &:= f_{++}(z) + f_{+-}(z) + f_{-+}(z) + f_{--}(z), \qquad (2) \end{aligned}$$

where

$$f_{++}(z) := \sum_{m\in Z_+\times Z_+} f_m z^m, \quad f_{+-}(z) := \sum_{m\in Z_+\times Z_-} f_m z^m$$

$$f_{-+}(z) := \sum_{m\in Z_-\times Z_+} f_m z^m, \quad f_{--}(z) := \sum_{m\in Z_-\times Z_-} f_m z^m.$$

Also let

$$f_+(z) = f_{++}(z) + f_{+-}(z), \quad f_-(z) = f_{-+}(z) + f_{--}(z), \tag{3}$$

where for example, $m = (m_1, m_2) \in Z_- \times Z_+$ means that $m_1 \in Z_+$ and $m_2 \in Z_-$, z^m the product $\bar{z}_1^{|m_1|} z_2^{m_2}$.

Theorem 2. *Let f and g be two bounded functions on the bitorus T^2. Then $T_f T_g - T_{fg}$ is compact implies that the following two statements hold.*
(1) For all $z_1 \in D$ and all $z_2 \in T$,

$$\frac{\partial f}{\partial z_1}(z_1, z_2)\frac{\partial g}{\partial \bar{z}_1}(z_1, z_2) = 0.$$

(1) For all $z_2 \in D$ and all $z_1 \in T$,

$$\frac{\partial f}{\partial z_2}(z_1, z_2)\frac{\partial g}{\partial \bar{z}_2}(z_1, z_2) = 0.$$

Proof. Without loss of generality, we prove statement (1). The proof of statement (2) is similar. We write f and g as (see above for the notations)

$$f(z) := f_{++}(z) + f_{+-}(z) + f_{-+}(z) + f_{--}(z)$$

$$:= f_+(z) + f_-(z)$$

and similarly

$$g(z) := g_{++}(z) + g_{+-}(z) + g_{-+}(z) + g_{--}(z)$$

$$:= g_+(z) + g_-(z).$$

We compute the action of a Toeplitz operator T_g for g as above on the normalized reproducing kernel $k_z(w)$ as follows:

$$\begin{aligned} & T_g k_z(w_1, w_2) \\ = \; & g_{++}(w_1, w_2)k_z + g_{+-}(w_1, z_2)k_z + g_{-+}(z_1, w_2)k_z + g_{--}(z_1, z_2)k_z \end{aligned} \tag{4}$$

for $z = (z_1, z_2)$ in D^2, where for example

$$f_{+-}(w_1, z_2) = \sum_{(m_1,m_2)\in Z_+\times Z_-} g_m w_1^{m_1} \bar{z}_2^{|m_2|}$$

for all $w_1 \in T$ and $z_2 \in D$. We remark that in fact the above formula (4) holds for any $g \in L^2(T^2)$.

For convenience, we consider $T_{\bar{f}}T_g - T_{\bar{f}g}$ instead of $T_f T_g - T_{fg}$. We write that

$$\begin{aligned} & ((T_{\bar{f}}T_g - T_{\bar{f}g})k_z, k_z) \\ = & -\int_{T^2} \overline{f(w)} g(w) |k_z(w)|^2 d\sigma(w) + (T_g k_z, T_f k_z) \\ := & \; h_1(z) + h(z) \end{aligned} \tag{5}$$

for any z in D^2, where

$$h_1(z) = -\int_{T^2} \overline{f(w)} g(w) |k_z(w)|^2 d\sigma(w),$$

$$h(z) = (T_{\bar{f}}T_g k_z, k_z).$$

Furthermore by (5), we write $h(z)$ as

$$\begin{aligned} h(z) = (T_{\bar{f}}T_g k_z, k_z) &= (T_{\bar{f}}T_g k_z, k_z) \\ := \; & h_{11}(z) + h_{12}(z) + h_{21}(z) + h_{22}(z) \end{aligned}$$

where

$$h_{11}(z) := (T_{g_+} k_z(w), T_{f_+} k_z(w)), \quad h_{12}(z) := (T_{g_+} k_z(w), T_{f_-} k_z(w)), \tag{6}$$

$$h_{21}(z) := (T_{g_-} k_z(w), T_{f_+} k_z(w)), \quad h_{22}(z) := (T_{g_-} k_z(w), T_{f_-} k_z(w)). \tag{7}$$

Let $m = (m_1, m_2) \in Z \times Z$ be fixed. Let $\theta = (\theta_1, \theta_2)$ and

$$U_\theta = diag\{e^{i\theta_1}, e^{i\theta_2}\}.$$

Set

$$H_1(z) := \int_{T^2} h_1(U_\theta z) e^{i(m,\theta)} d\theta$$

$$H_{ij}(z) := \int_{T^2} h_{ij}(U_\theta z) e^{i(m,\theta)} d\theta, \quad i, j = 1, 2$$

for $z \in D^2$, where $h_1(U_\theta z)$ and $h_{ij}(U_\theta z)$ are obtained by replacing z with $U_\theta z$ in the definition of h_1 and h_{ij} respectively. For example, by (4), we have

$$\begin{aligned} H_{12}(z) \; := \; & \int_{T^2} (g_{++}(w_1, w_2) k_{U_\theta z} + g_{+-}(w_1, e^{i\theta_2} z_2) k_{U_\theta z}, \\ & f_{-+}(e^{i\theta_1} z_1, w_2) k_{U_\theta z} + f_{--}(e^{i\theta_1} z_1, e^{i\theta_2} z_2) k_{U_\theta z}) e^{i(m,\theta)} d\theta. \end{aligned} \tag{8}$$

For a fixed $z_1 \in D$ and $u_2 \in T$, let $z_{2\alpha}$ converge to the point u_2 on T, then $z_\alpha = (z_1, z_{2\alpha})$ converges to the boundary point (z_1, u_2) of the bidisc D^2. We claim that $H_1(z)$, $H_{11}(z)$, $H_{12}(z)$, $H_{21}(z)$ and $H_{22}(z)$ are continuous on $\overline{D^2}$; and furthermore if we denote the limits of $H_1(z)$, $H_{11}(z)$, $H_{12}(z)$, $H_{21}(z)$ and $H_{22}(z)$ as z_α converges

to the boundary point (z_1, u_2) by $H_1(z_1, u_2)$, $H_{11}(z_1, u_2)$, $H_{12}(z_1, u_2)$, $H_{21}(z_1, u_2)$ and $H_{22}(z_1, u_2)$, respectively. Then
$H_1(z_1, u_2)$, $H_{11}(z_1, u_2)$, $H_{12}(z_1, u_2)$ and $H_{21}(z_1, u_2)$ are harmonic in z_1 for any $u_2 \in T$. However $H_{22}(z_1, u_2)$ is not necessary harmonic in z_1. We postpone the proof of the claim and first continue with the proof of the theorem.

We will first show that if $T_{\bar f}T_g - T_{\bar f g}$ is compact, then $H_{22}(z)$ is harmonic in z_1. Replacing z by $U_\theta z$ in (5) yields

$$((T_{\bar f}T_g - T_{\bar f g})k_{U_\theta z}, k_{U_\theta z}) = h_1(U_\theta z) + h(U_\theta z).$$

Multiplying the above equation by $e^{i(m,\theta)}$ and then integrating with respect to θ give that

$$\int_{T^2} ((T_{\bar f}T_g - T_{\bar f g})k_{U_\theta z}, k_{U_\theta z})e^{i(m,\theta)}d\theta$$

$$= H_1(z) + H_{11}(z) + H_{12}(z) + H_{21}(z) + H_{22}(z).$$

Now note that k_z weakly converges to zero as z goes to the boundary of D^2. Therefore if $T_{\bar f}T_g - T_{\bar f g}$ is compact, then

$$\lim_{(z)_\alpha \to (z_1, u_2)} \int_{T^2} ((T_{\bar f}T_g - T_{\bar f g})k_{U_\theta z_\alpha}, k_{U_\theta z_\alpha})e^{i(m,\theta)}d\theta = 0.$$

Hence by (5), we conclude that

$$\lim_{z_\alpha \to (z_1, u_2)} (H_1(z) + H_{11}(z) + H_{12}(z) + H_{21}(z) + H_{22}(z)) = 0$$

for any $z_1 \in D$ and $u_2 \in T$. That is

$$-H_{22}(z_1, u_2) = H_1(z_1, u_2) + H_{11}(z_1, u_2) + H_{12}(z_1, u_2) + H_{21}(z_1, u_2).$$

Since by our claim $H_1(z_1, u_2)$, $H_{11}(z_1, u_2)$, $H_{12}(z_1, u_2)$ and $H_{21}(z_1, u_2)$ are harmonic in z_1 for any $u_2 \in T$, so $H_{22}(z_1, u_2)$ is harmonic in z_1 for any $u_2 \in T$. Thus $\Delta_{z_1} H_{22}(z_1, u_2) = 0$.

On the other hand, if we write

$$f_-(z_1, z_2) = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k f_-}{\partial \bar z_1^k}(0, z_2)\bar z_1^k$$

and

$$g_-(z_1, z_2) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k g_-}{\partial \bar z_1^k}(0, z_2)\bar z_1^k.$$

Let $m = (0, m_2)$. We note that

$$H_{22}(0, u_2) =$$

$$\sum_{k=1}^{\infty} \frac{1}{(k!)^2} \int_T \overline{\frac{\partial^k f_-}{\partial \bar z_1^k}(0, u_2 e^{i\theta_2})} \frac{\partial^k g_-}{\partial \bar z_1^k}(0, u_2 e^{i\theta_2}) e^{im_2\theta_2} d\theta_2 |z_1|^{2k}.$$

Since $H_{22}(z_1, u_2)$ is harmonic in z_1 for any $u_2 \in T$. Thus $\Delta_{z_1} H_{22}(z_1, u_2) = 0$. In particular, $\Delta_{z_1} H_{22}(z_1, u_2) = 0$ for $z_1 = 0$. Hence

$$\int_T \overline{\frac{\partial f_-}{\partial \bar{z}_1}(0, u_2 e^{i\theta_2})} \frac{\partial g_-}{\partial \bar{z}_1}(0, u_2 e^{i\theta_2}) e^{im_2\theta_2} d\theta_2 = 0$$

for all $m_2 \in Z$ and $u_2 \in T$. We observe that both $\frac{\partial f_-}{\partial \bar{z}_1}(0, u_2)$ and $\frac{\partial g_-}{\partial \bar{z}_1}(0, u_2)$ are in $L^2(T)$. So $\overline{\frac{\partial f_-}{\partial \bar{z}_1}(0, u_2)} \frac{\partial g_-}{\partial \bar{z}_1}(0, u_2)$ is in $L^1(T)$ and the Fourier transformation of $\overline{\frac{\partial f_-}{\partial \bar{z}_1}(0, u_2)} \frac{\partial g_-}{\partial \bar{z}_1}(0, u_2)$ on Z is zero. The injection of the Fourier transformation implies that

$$\overline{\frac{\partial f_-}{\partial \bar{z}_1}(0, u_2)} \frac{\partial g_-}{\partial \bar{z}_1}(0, u_2) = 0$$

for all $u_2 \in T$. But this is same as

$$\overline{\frac{\partial f}{\partial \bar{z}_1}(0, u_2)} \frac{\partial g}{\partial \bar{z}_1}(0, u_2) = 0$$

for all $u_2 \in T$.

Next by using the Möbius transform of the bidisc, we will show that statement (1) in the theorem holds. Let $\phi_z(w)$ denote the Möbius transform

$$\phi_z(w) = (\phi_{z_1}(w_1), \phi_{z_2}(w_2))$$

in the bidisc D^2 for each point $z = (z_1, z_2) \in D^2$. For a fixed point $z \in D^2$, we define a unitary operator U_z on $L^2(D^2)$ by

$$U_z h(w) = h \circ \phi_z(w) k_z(w)$$

for all $h \in H^2(D^2)$. It is easy to see that $U_z^* T_f U_z = T_{f \circ \phi_z}$. Thus

$$T_{f \circ \phi_z} T_{g \circ \phi_z} - T_{(f \circ \phi_z)(g \circ \phi_z)} = U_z^* (T_f T_g - T_{fg}) U_z.$$

Therefore if $T_f T_g - T_{fg}$ is compact, then $T_{f \circ \phi_z} T_{g \circ \phi_z} - T_{(f \circ \phi_z)(g \circ \phi_z)}$ is compact.

Replacing f and g respectively by $f \circ \phi_{(z_1, 0, \cdots, 0)}$ and $g \circ \phi_{(z_1, 0, \cdots, 0)}$ in above analysis, we get that

$$\overline{\frac{\partial (f \circ \phi_{(z_1, 0)})}{\partial \bar{z}_1}(0, u_2)} \frac{\partial (g \circ \phi_{(z_1, 0)})}{\partial \bar{z}_1}(0, u_2) = 0$$

for all $z_1 \in D$ and $u_2 \in T$. But

$$\overline{\frac{\partial (f \circ \phi_{(z_1, 0, \cdots, 0)})}{\partial \bar{z}_1}(0, u_2)} = (|z_1|^2 - 1) \overline{\frac{\partial f}{\partial \bar{z}_1}(z_1, u_2)}.$$

Therefore

$$\overline{\frac{\partial f}{\partial \bar{z}_1}(z_1, u_2)}\frac{\partial g}{\partial \bar{z}_1}(z_1, u_2) = 0$$

for $u_2 \in T$ and $z_1 \in D$. This completes the proof of the theorem except we still need to prove our claim.

We first discuss $H_1(z)$. Let

$$A(z) := \int_{T^2} \overline{f(U_\theta z)} g(U_\theta z) e^{i(m,\theta)} d\theta \tag{9}$$

for $z \in D^2$. Since both f and g are in $L^\infty(T^2)$, we write the harmonic extension of f and g in D^2 as

$$f = \sum_{l \in Z \times Z} f_l z^l \quad \text{and} \quad g = \sum_{j \in Z \times Z} g_j z^j$$

where f_l and g_j are two sequences of numbers satisfying

$$\sum_{l \in Z \times Z} |f_l|^2 < \infty \quad \text{and} \quad \sum_{j \in Z \times Z} |g_j|^2 < \infty.$$

A straightforward computation gives that

$$A(z) = \sum_{j \in Z \times Z} \overline{f_{j+m}} g_j z^{-(j+m)} z^j \tag{10}$$

for $z \in D^2$, where recall that by our convention z_1^{-1} is $\bar{z}_1$, for example. Since

$$\sum_{j \in Z \times Z} |\overline{f_{j+m}} g_j| \le (\sum_{j \in Z \times Z} |f_{j+m}|^2)^{1/2} (\sum_{l \in Z \times Z} |g_j|^2)^{1/2} < \infty,$$

$A(z)$ is continuous on the closure $\overline{D}^2$ of D^2; and indeed $A(z)$ is defined by the series expansion in (10) for all $z \in \overline{D}^2$. But note that here $A(z)$ for $z \in D^2$ is not necessary the harmonic extension of $A(w)$ for $w \in T^2$. By an abuse of notation, we have that (9) holds for all $z \in \overline{D}^2$. It follows from the definition of $H_1(z)$ that for all $z \in D^2$,

$$\begin{aligned} H_1(z) &= -\int_{T^2} \int_{T^2} \overline{f(w)} g(w) |k_{U_\theta z}(w)|^2 d\sigma(w) e^{i(m,\theta)} d\theta \\ &= -\int_{T^2} \int_{T^2} \overline{f(U_\theta w)} g(U_\theta w) e^{i(m,\theta)} d\theta |k_z(w)|^2 d\sigma(w) \\ &= -\int_{T^2} A(w) |k_z(w)|^2 dA(w) \end{aligned}$$

where the second equality is obtained by changing the order of integration and a change of variables from w to $U_\theta w$. That is, $H_1(z)$ in D^2 is the harmonic extension of the continuous function $A(w)$ on T^2. Therefore $H_1(z)$ is continuous on $\overline{D}^2$. Let $\phi_{z_2}(\lambda_2)$ be the Möbius map $\dfrac{z_2-\lambda_2}{1-\overline{z_2}\lambda_2}$. Changing variable $w_2=\phi_{z_2}(\lambda_2)$ in the above integral, we have

$$H_1(z)=-\int_T\int_T A(w_1,\phi_{z_2}(\lambda_2))|k_{z_1}(w_1)|^2dA(w_1)dA(\lambda_2).$$

It follows from Lebesque dominated convergence theorem and the continuity of $A(z)$ on $\overline{D^2}$ that

$$H_1(z,u_2)=\lim_{(z)_\alpha\to(z_1,u_2)}H_{22}(z)=-\int_T A(w_1,u_2)|k_{z_1}(w_1)|^2dA(w_1).$$

That is, $H_1(z,u_2)$ as a function of z_1 on D is the harmonic extension of $A(w_1,u_2)$ as a function of w_1 on T. Therefore $H_1(z,u_2)$ is harmonic in $z_1\in D$ for any $u_2\in T$.

Next we discuss $H_{12}(z)$. Without loss of generality, we consider a term $B(z)$ of H_{12} as in (8).

$$\begin{aligned}B(z)&=\int_{T^2}\int_{T^2}\overline{f_{-+}(e^{i\theta_1}z_1,w_2)}g_{++}(w)|k_{U_\theta z}(w)|^2dA(w)e^{i(m,\theta)}d\theta\\&=\int_{T^2}\int_{T^2}\overline{f_{-+}(U_\theta(z_1,w_2))}g_{++}(U_\theta w)e^{i(m,\theta)}d\theta|k_z(w)|^2dA(w)\end{aligned}$$

where the second equality is obtained by changing the order of integration and a change of variables from w to $U_\theta w$. By integrating with respect to w_1 and noting that $f_{-+}(U_\theta(z_1,w_2))$ is independent of w_1, we get that

$$B(z)=\int_T\int_{T^2}\overline{f_{-+}(U_\theta(z_1,w_2))}g_{++}(U_\theta(z_1,w_2))e^{i(m,\theta)}d\theta|k_{z_2}(w)|^2dA(w_2).$$

Now we define

$$C(z)=\int_{T^2}\overline{f_{-+}(z_1,w_2)}g_{++}(w)e^{i(m,\theta)}d\theta. \tag{11}$$

By the proof of the continuity of $A(z)$, we see that $C(z)$ is continuous on $\overline{D^2}$. By an abuse of notation we see that (11) holds for all $z\in\overline{D^2}$. Indeed we have

$$C(z)=\sum_{j\in Z\times Z}\overline{a_{j+m}}b_jz^{-(j+m)}z^j, \tag{12}$$

for all $z\in\overline{D^2}$, if we write $f_{-+}(z)$ and $g_{++}(z)$ as

$$f_{-+}(z)=\sum_{l\in Z\times Z}a_lz^l\quad\text{and}\quad g_{--}(z)=\sum_{j\in Z\times Z}b_jz^j$$

where, since $f_{-+}(z)$ and $g_{--}(z)$ are in $L^2(T^2)$, a_l and b_j are two sequences of numbers satisfying

$$\sum_{l\in Z\times Z}|a_l|^2<\infty,\quad\text{and}\quad\sum_{j\in Z\times Z}|b_j|^2<\infty.$$

Next note that

$$B(z) = \int_T C(z_1, w_2)|k_{z_2}(w)|^2 dA(w_2).$$

Hence $B(z)$ is continuous on $\overline{D^2}$. Let $\phi_{z_2}(\lambda_2)$ be the Möbius map $\dfrac{z_2 - \lambda_2}{1 - \overline{z_2}\lambda_2}$. We have

$$B(z) = \int_T C(z_1, \phi_{z_2}(\lambda_2)) dA(\lambda_2).$$

It follows from the continuity of $C(z)$ that

$$B(z_1, u_2) = \lim_{(z)_\alpha \to (z_1, u_2)} B(z) = \int_T C(z_1, u_2) dA(\lambda_2) = C(z_1, u_2).$$

To prove that $C(z_1, u_2)$ is harmonic in z_1 for any $u_2 \in T$, we look at the series expansion of $C(z)$ more carefully. We first note that $a_l = 0$ for $l = (l_1, l_2) \in Z \times Z$ such that $l_1 \geq 0$ and $b_j = 0$ except for $j = (j_1, j_2) \in Z_+ \times Z_+$. Therefore in fact $C(z_1, u_2) = 0$ for $m = (m_1, m_2) \in Z_+ \times Z$, and

$$C(z_1, u_2) = z_1^{-m_1} {u_2}^{m_2} \Big(\sum_{0 \leq j_1 < -m_1} \sum_{j_2 \in Z_+} \overline{a_{j+m}} b_j \Big)$$

for $m = (m_1, m_2) \in Z_- \times Z$. This finishes the proof of the claim concerning $H_{21}(z)$.

The proof of the claim concerning $H_{11}(z)$ and $H_{21}(z)$ is similar. So does the proof of the continuity of $H_{22}(z)$. Nevertheless one does not necessarily have that $H_{22}(z_1, u_2)$ is harmonic in z_1. So the claim is established. This completes the proof of the theorem.

We remark that in general the conditions (1) and (2) in Theorem 2 are not sufficient for $T_f T_g - T_{fg}$ to be compact. See Theorem 3 in next section for examples.

Next we use Theorem 2 to derive a result in [4] about compact Hankel operators on the bidisc; see [4] and reference therein for more related results.

Corollary 1. [4] *Let f be a bounded functions on the bitorus T^2. The Hankel operators H_f is compact if and only if it is zero.*

Proof. Note that $-H_f^* H_f = T_{\bar{f}} T_f - T_{\bar{f} f}$. Thus by Theorem 2, H_f is compact implies that for all $z_1 \in D$ and all $z_2 \in T$,

$$\Big|\frac{\partial f}{\partial \bar{z}_1}(z_1, z_2)\Big|^2 = 0$$

and for all $z_2 \in D$ and all $z_1 \in T$,

$$\Big|\frac{\partial f}{\partial \bar{z}_2}(z_1, z_2)\Big|^2 = 0.$$

Therefore f is analytic in D^2. So H_f is zero. The proof is complete.

The following result is an immediate consequence of Theorem 2. See also [11] for related results on the Bergman space of the bidisc.

Corollary 2. *Let f and g be two bounded functions on T^2. If one of the functions f, $\bar{f}$, g, $\bar{g}$ is analytic, then $T_fT_g - T_gT_f$ is zero if and only if $T_fT_g - T_gT_f$ is compact.*

Proof. Without loss of generality, we assume that f is analytic. It is easy to see that in this case $T_fT_g - T_gT_f = T_fT_g - T_{fg}$.

By Theorem 2, if $T_fT_g - T_{fg}$ is compact, then for all $z_1 \in D$ and $z_2 \in T$,

$$\frac{\partial f}{\partial z_1}(z_1, z_2)\frac{\partial g}{\partial \bar{z}_1}(z_1, z_2) = 0,$$

and for all $z_2 \in D$ and all $z_1 \in T$,

$$\frac{\partial f}{\partial z_2}(z_1, z_2)\frac{\partial g}{\partial \bar{z}_2}(z_1, z_2) = 0.$$

Since f is analytic on the unit disk, we conclude that for each i $(i = 1, 2)$ either $\bar{f}$ or g is analytic in z_i. By Theorem 1, this implies that $T_fT_g - T_{fg} = 0$. This completes the proof of the corollary.

We remark that the above corollary is not valid without the assumption that one of the functions f, $\bar{f}$, g, $\bar{g}$ is analytic. An example will be shown in next section.

4. Compact semi-commutators

In this section we will characterize when $T_fT_g - T_{fg}$ is compact for Toeplitz operators T_f and T_g with symbols of the form $f(z_1, z_2) = f_1(z_1)f_2(z_2)$ and $g(z_1, z_2) = g_1(z_1)g_2(z_2)$. This will show that the condition in Theorem 2 is in general not a sufficient condition for $T_fT_g - T_{fg}$ to be compact. Furthermore it provides us with nontrivial compact semi-commutators.

Theorem 3. *Let f_1, f_2, g_1 and g_2 be nonzero bounded functions on the unit circle T. Let f and g be bounded functions on T^2 of the form $f(z_1, z_2) = f_1(z_1)f_2(z_2)$ and $g(z_1, z_2) = g_1(z_1)g_2(z_2)$. Then $T_fT_g - T_{fg}$ is a nonzero compact operator on the Hardy space of the bidisc if and only if the following two conditions hold.*
(1) $f_1(z_1)g_1(z_1) = 0$ and $f_2(z_2)g_2(z_2) = 0$ on T, and
(2) $\lim_{z\to T^2} \|H_{\bar{f}}k_z\|_2\|H_gk_z\|_2 = 0$.

Proof. We first prove the necessity part of the theorem. Assume that $T_fT_g - T_{fg}$ is compact. By Theorem 2, we have that for all $z_1 \in D$ and all $z_2 \in T$,

$$\frac{\partial f_1(z_1)}{\partial z_1}\frac{\partial g_1(z_1)}{\partial \bar{z}_1}f_2(z_2)g_2(z_2) = 0 \tag{13}$$

and for all $z_2 \in D$ and all $z_1 \in T$,

$$\frac{\partial f_2(z_2)}{\partial z_2}\frac{\partial g_2(z_2)}{\partial \bar{z}_2}f_1(z_1)g_1(z_1) = 0. \tag{14}$$

We claim that this implies that Condition (1) holds. If Condition (1) does not hold. Without loss of generality, assume that f_1g_1 is not zero. Then equation (14) gives that

$$\frac{\partial f_2}{\partial z_2}\frac{\partial g_2}{\partial \bar{z}_2} = 0$$

for all $z_2 \in D$. That is either $\bar{f}_2$ or g_2 is analytic in z_2. Say, g_2 is analytic in z_2. Thus g_2f_2 is not zero since otherwise either f_2 or g_2 will be zero. Now equation (13) implies that

$$\frac{\partial f_1}{\partial z_1}\frac{\partial g_1}{\partial \bar{z}_1} = 0.$$

That is either $\bar{f}_1$ or g_1 is analytic in z_1. Say, $\bar{f}_1$ is analytic in z_1. Hence $\bar{f}$ is analytic in z_1 and g is analytic in z_2. By Theorem 1, we get that $T_fT_g - T_{fg}$ is zero. This is a contradiction. Therefore Condition (1) holds.

Next we show that Condition (2) holds. We first need some notations. Let P_1 be the projection defined by

$$P_1(h) = \sum_{m=(m_1,m_2)\in Z_+\times Z} h_m z^m \quad \text{if } h = \sum_{m=(m_1,m_2)\in Z\times Z} h_i(z_2) z^m \in L^2(T^2),$$

and similarly P_2 the projection defined by

$$P_2(h) = \sum_{m=(m_1,m_2)\in Z\times Z_+} h_m z^m \quad \text{if } h = \sum_{m=(m_1,m_2)\in Z\times Z} h_i(z_2) z^m \in L^2(T^2),$$

It is easy to see that $P = P_1P_2 = P_2P_1$. Let $h_n(z_1) \in H^2(D)$ such that h_n converges weakly to zero in $H^2(D)$. For any $h(z_2)$ in $H^2(D)$. Let $H_n(z_1, z_2)$ be $h(z_2)h_n(z_1)$. Note that H_n converges weakly to zero in $H^2(D^2)$ as well. Note also by Condition (1),

$$(T_fT_g - T_{fg})H_n(z_1, z_2) = T_fT_gH_n(z_1, z_2)$$

$$= P_2(f_2P_2g_2h)(z_2)T_{f_1}T_{g_1}h_n(z_1) = -P_2(f_2P_2g_2h)(z_2)H^*_{\overline{f_1}}H_{g_1}h_n(z_1)$$

and $P_2(f_2)P_2(g_2h)(z_2)$ is not zero for some $h \in H^2(D)$. Thus if $T_fT_g - T_{fg}$ is compact, then $\|H^*_{\overline{f_1}}H_{g_1}h_n\|$ converges to zero. Therefore $H^*_{\overline{f_1}}H_{g_1}$ is compact on the Hardy space of the unit disk. The compactness of $H^*_{\overline{f_2}}H_{g_2}$ can be established similarly. It follows from Theorem 2 in [13] that

$$\lim_{z_1\to T} \|H_{\overline{f_1}}k_{z_1}\|_2 \|H_{g_1}k_{z_1}\|_2 = 0,$$

and

$$\lim_{z_2\to T} \|H_{\overline{f_2}}k_{z_2}\|_2 \|H_{g_2}k_{z_2}\|_2 = 0.$$

So

$$\lim_{z_1 \ or z_2\to T} \|H_{\overline{f_1}}k_{z_1}\|_2 \|H_{g_1}k_{z_1}\|_2 \|H_{\overline{f_2}}k_{z_2}|\|_2 \|H_{g_2}k_{z_2}\|_2 = 0.$$

On the other hand,

$$\|H_{\overline{f}}k_z\|_2\|H_gk_z\|_2 = \|H_{\overline{f_1}}k_{z_1}\|_2\|H_{g_1}k_{z_1}\|_2\|H_{\overline{f_2}}k_{z_2}\|_2\|H_{g_2}k_{z_2}\|_2.$$

Hence

$$\lim_{z\to T^2}\|H_{\overline{f}}k_z\|_2\|H_gk_z\|_2 = 0$$

as it is equivalent to

$$\lim_{z_1\to T}\|H_{\overline{f_1}}k_{z_1}\|_2\|H_{g_1}k_{z_1}\|_2 = 0,$$

and

$$\lim_{z_2\to T}\|H_{\overline{f_2}}k_{z_2}\|_2\|H_{g_2}k_{z_2}\|_2 = 0.$$

We now turn to the proof of the sufficient part of the theorem. By Condition (1), we have $T_fT_g - T_{fg} = T_fT_g$. Let ϕ and ψ be in $H^2(D^2)$. We observe that

$$T_fT_g\phi = P_1P_2(f_1f_2P_1P_2(g_1g_2\phi))$$

$$= P_1f_1P_1g_1P_2(f_2P_2(g_2\phi))$$

$$= P_1f_1(I-P_1)g_1P_2(f_2(I-P_2)(g_2\phi))$$

$$= P_1f_1P_2(f_2(I-P_1)g_1(I-P_2)(g_2\phi)),$$

where the third equality follows from Condition (1). Therefore

$$< T_fT_g\phi,\psi >=< P_1f_1P_2(f_2(I-P_1)g_1(I-P_2)g_2\phi),\psi >$$

$$=< (I-P_1)g_1(I-P_2)g_2\phi, \overline{f_1f_2}\psi >$$

$$=< (I-P_1)g_1(I-P_2)g_2\phi, (I-P_1)\overline{f_1}(I-P_2)\overline{f_2}\psi > .$$

It is easy to see that $(I-P_1)g_1(I-P_2)g_2\phi$ is co-analytic in z_1 and z_2. So

$$\bigtriangledown_1(I-P_1)g_1(I-P_2)g_2\phi = \frac{\partial}{\partial\overline{z_1}}(I-P_1)g_1(I-P_2)g_2\phi.$$

Since

$$\int_{\partial D}(I-P_1)g_1(I-P_2)g_2\phi(z_1,z_2)d\sigma(z_1) = 0$$

and

$$\int_{\partial D}\frac{\partial}{\partial\overline{z_1}}[(I-P_1)g_1(I-P_2)g_2\phi(z_1,z_2)]d\sigma(z_2) = 0,$$

using the Littlewood-Paley formula, we have

$$
\begin{aligned}
&< (I-P_1)g_1(I-P_2)g_2\phi, (I-P_1)\overline{f_1}(I-P_2)\overline{f_2}\psi > \\
&\quad = \int_D\int_D < \nabla_1 \nabla_2 (I-P_1)g_1(I-P_2)g_2\phi, \nabla_1 \nabla_2 (I-P_1)\overline{f_1}(I-P_2)\overline{f_2}\psi > \times \\
&\qquad\qquad \log\frac{1}{|z_1|}\log\frac{1}{|z_2|}dA(z_1)dA(z_2) \\
&\quad = \int_D\int_D < \nabla_1(I-P_1)g_1 \nabla_2 (I-P_2)g_2\phi, \nabla_1(I-P_1)\overline{f_1} \nabla_2 (I-P_2)\overline{f_2}\psi > \times \\
&\qquad\qquad \log\frac{1}{|z_1|}\log\frac{1}{|z_2|}dA(z_1)dA(z_2) \\
&\quad = I_1 + I_2 + I_3
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 = \int_{rD}\int_{rD} &< \nabla_1 \ \nabla_2 (I-P_1)g_1(I-P_2)g_2\phi, \nabla_1 \ \nabla_2 (I-P_1)\overline{f_1}(I-P_2)\overline{f_2}\psi > \\
&\times \log\frac{1}{|z_1|}\log\frac{1}{|z_2|}dA(z_1)dA(z_2),
\end{aligned}
$$

$$
\begin{aligned}
I_2 = \int_{D/rD}\int_{D} &< \nabla_1 \ \nabla_2 (I-P_1)g_1(I-P_2)g_2\phi, \nabla_1 \ \nabla_2 (I-P_1)\overline{f_1}(I-P_2)\,\overline{f_2}\psi > \\
&\times \log\frac{1}{|z_1|}\log\frac{1}{|z_2|}dA(z_1)dA(z_2),
\end{aligned}
$$

$$
\begin{aligned}
I_3 = \int_{rD}\int_{D/rD} &< \nabla_1 \nabla_2 (I-P_1)g_1(I-P_2)g_2\phi, \nabla_1 \nabla_2 (I-P_1)\overline{f_1}(I-P_2)\overline{f_2}\psi > \\
&\times \log\frac{1}{|z_1|}\log\frac{1}{|z_2|}dA(z_1)dA(z_2).
\end{aligned}
$$

It is easy to see that there is a compact operator K_r such that

$$I_1 =< K_r\phi, \psi > .$$

To deal with I_2 and I_3, we use the distribution function inequality established in [13]. As in the proof of Theorem 7 in [13], we obtain that

$$
\begin{aligned}
&|\int_{D/rD} < \nabla_1(I-P_1)g_1 \nabla_2 (I-P_2)g_2\phi, \nabla_1(I-P_1)\overline{f_1} \nabla_2 (I-P_2)\overline{f_2}\psi > \log\frac{1}{|z_1|}dA(z_1)| \\
&\qquad \le C \sup_{|z_1|>r} \|H_{\overline{f_1}}k_{z_1}\|\|H_{g_1}k_{z_1}\|\| \nabla_2 (I-P_2)g_2\phi\|_{z_1}\| \nabla_2 (I-P_2)\overline{f_2}\psi\|_{z_1},
\end{aligned}
$$

where for $h \in L^2(T^2)$ and z_2 fixed, $\|h\|_{z_1}$ denotes the L^2 norm of $h(z_1, z_2)$ as a function of z_1. Similarly we also have

$$\begin{aligned}&|\int_{D/rD} < \nabla_1 (I-P_1)g_1 \nabla_2 (I-P_2)g_2\phi, \nabla_1(I-P_1)\overline{f_1} \nabla_2 (I-P_2)\overline{f_2}\psi > \log\frac{1}{|z_2|}\, dA(z_2)| \\ &\qquad \le C \sup_{|z_2|>r} \| \|H_{\overline{f_2}}k_{z_2}\|\|H_{g_2}k_{z_2}\|\| \nabla_1 (I-P_1)g_1\phi\|_{z_2}\| \nabla_1 (I-P_1)\overline{f_1}\psi\|_{z_2}.\end{aligned}$$

So

$$\begin{aligned}|I_2| \quad &\le C \sup_{|z_1|>r} \| H_{\overline{f_1}}k_{z_1}\|\|H_{g_1}k_{z_1}\| \times \\ &\int_D \| \nabla_2 (I-P_2)g_2\phi\|_{z_1}\| \nabla_2 (I-P_2)\overline{f_2}\psi\|_{z_1} \log\frac{1}{|z_2|}\, dA(z_2).\end{aligned}$$

By the Holder inequality,

$$\begin{aligned}|I_2| \quad &\le C \sup_{|z_1|>r} \|H_{\overline{f_1}}k_{z_1}\|\|H_{g_1}k_{z_1}\| \times \\ &[\int_D\int_{\partial D} | \nabla_2 (I-P_2)g_2\phi(z_1,z_2)|^2 \log\frac{1}{|z_2|} d\sigma(z_1)dA(z_2)]^{1/2} \times \\ &[\int_D\int_{\partial D} | \nabla_2 (I-P_2)\overline{f_2}\psi(z_1,z_2)|^2 \log\frac{1}{|z_2|} d\sigma(z_1)dA(z_2)]^{1/2}.\end{aligned}$$

By the Littlewood-Paley theorem and the fact that g_2 and f_2 are bounded,

$$|I_2| \le C \sup_{|z_1|>r} \|H_{\overline{f_1}}k_{z_1}\|\|H_{g_1}k_{z_1}\|\|\phi\|\|\psi\|.$$

Similarly we can also get the estimate for I_3,

$$|I_3| \le C \sup_{|z_2|>r} \|H_{\overline{f_2}}k_{z_2}\|\|H_{g_2}k_{z_2}\|\|\phi\|\|\psi\|.$$

In the proof of the necessary part of the theorem we have shown that the second condition is equivalent to

$$\lim_{z_1\to T} \|H_{\overline{f_1}}k_{z_1}\|_2\|H_{g_1}k_{z_1}\|_2 = 0,$$

and

$$\lim_{z_2\to T} \|H_{\overline{f_2}k_{z_2}}\|_2\|H_{g_2}k_{z_2}\|_2 = 0.$$

Hence we conclude that

$$\lim_{r\to 1} \|T_fT_g - T_{fg} - K_r\| = 0.$$

Therefore $T_fT_g - T_{fg}$ is compact. This completes the proof of the theorem.

We now give an example to show that the assumption in Corollary 2 is necessary. Namely, there exist bounded functions f and g such that $T_fT_g - T_gT_f$ is compact, but $T_fT_g - T_gT_f$ is not zero.

Example. Let $f_1(z_1)$ and $g_1(z_1)$ be nonzero continuous functions on the unit circle satisfying the conditions that $f_1(z_1)g_1(z_1) = 0$ and the set $\{z_1 : f_1(z_1) = 0\} \cap \{z_1 : g_1(z_1) = 0\}$ as a subset of T has positive Lebesgue measure. Let $f_2(z_2)$ and $g_2(z_2)$ be nonzero continuous functions on the unit circle such that $f_2(z_2)g_2(z_2) = 0$.

First we show that $T_fT_g - T_gT_f$ is not zero. Note that

$$(T_fT_g - T_gT_f)h_1(z_1) = P_2(f_2)P_2(g_2)(T_{f_1}T_{g_1} - T_{g_1}T_{f_1})h_1(z_1)$$

for all $h_1 \in H^2(D)$. It is easy to see that the assumption on f_2 and g_2 implies that $P_2(f_2)P_2(g_2)$ is not zero. Therefore it suffices to show that $T_{f_1}T_{g_1} - T_{g_1}T_{f_1}$ is not zero. Let S_1 denote the multiplication by z_1 on $H^2(D)$. We observe that $I - S_1S_1^*$ is the rank one operator $e_0 \otimes e_0$, where $e_0 = 1 \in H^2(D)$. Note that

$$\begin{aligned}
& T_{f_1}T_{g_1} - T_{g_1}T_{f_1} - S_1^*(T_{f_1}T_{g_1} - T_{g_1}T_{f_1})S_1 \\
= \; & T_{f_1}T_{g_1} - S_1^*T_{f_1}T_{g_1}S_1 - (T_{g_1}T_{f_1} - S_1^*T_{g_1}T_{f_1}S_1) \\
= \; & S_1^*T_{f_1}S_1S_1^*T_{g_1}S_1 - S_1^*T_{f_1}T_{g_1}S_1 - (S_1^*T_{g_1}S_1S_1^*T_{f_1}S_1 - S_1^*T_{g_1}T_{f_1}S_1) \\
= \; & -S_1^*T_{f_1}(I - S_1S_1^*)T_{g_1}S_1 + S_1^*T_{g_1}(I - S_1S_1^*)T_{f_1}S_1 \\
= \; & -S_1^*T_{f_1}e_0 \otimes S_1^*T_{g_1}^*e_0 + S_1^*T_{g_1}e_0 \otimes S_1^*T_{f_1}^*,
\end{aligned}$$

where the second equality follows from the fact that for Toeplitz operator T_{f_1}, $S_1^*T_{f_1}S_1 = T_{f_1}$. Thus if $T_{f_1}T_{g_1} - T_{g_1}T_{f_1} = 0$, then there exists a contant c such that

$$S_1^*T_{f_1}e_0 = cS_1^*T_{g_1}e_0.$$

That is, there exists a constant d such that $P_1(f_1 - cg_1) = d$. Hence if we denote the function $\overline{f_1} - \overline{cg_1}$ by h, then h is analytic. But by our assumption, $h(z_1) = 0$ on a subset of T with positive Lebesgue measure. Therefore $h_1 = 0$ on T, i.e., $f_1 = cg_1$. This contradicts to the conditions that both f_1 and g_1 are nonzero and $f_1g_1 = 0$ on T. So $T_{f_1}T_{g_1} - T_{g_1}T_{f_1}$ is not zero.

Next we show that $T_fT_g - T_gT_f$ is compact. Since $f_1(z_1)$, $g_1(z_1)$, $f_2(z_2)$ and $g_2(z_2)$ are continuous, so $H_{\overline{f_1}}H_{g_1}$ and $H_{\overline{f_2}}H_{g_2}$ are compact. Hence

$$\lim_{z_1 \to T} \|H_{\overline{f_1}}k_{z_1}\|_2 \|H_{g_1}k_{z_1}\|_2 = 0,$$

and

$$\lim_{z_2 \to T} \|H_{\overline{f_2}k_{z_2}}\|_2 \|H_{g_2}k_{z_2}\|_2 = 0,$$

or equivalently Condition (2) in Theorem 3 holds. Therefore $T_fT_g - T_{fg}$ and $T_gT_f - T_{gf}$ are compact. Note that

$$T_fT_g - T_gT_f = (T_fT_g - T_{fg}) - (T_gT_f - T_{gf}).$$

Hence $T_fT_g - T_gT_f$ is compact.

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Caixing Gu, Mathematics Department, University of California Irvine, Irvine, CA 72717

E-mail address: cgu@math.uci.edu

Dechao Zheng, Mathematics Department, Michigan State University, East Lansing, MI 48824

E-mail address: dechao@math.msu.edu