CONJUGATE OPERATORS FOR FINITE MAXIMAL SUBDIAGONAL ALGEBRAS

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal trace τ, and let H^∞ be a finite, maximal, subdiagonal algebra of $\mathcal{M}$. Fundamental theorems on conjugate functions for weak*-Dirichlet algebras are shown to be valid for non-commutative H^∞. In particular the conjugation operator is shown to be a bounded linear map from $L^p(\mathcal{M},\tau)$ into $L^p(\mathcal{M},\tau)$ for $1<p<\infty$, and to be a continuous map from $L^1(\mathcal{M},\tau)$ into $L^{1,\infty}(\mathcal{M},\tau)$. We also obtain that if an operator a is such that $|a|\log^+|a| \in L^1(\mathcal{M},\tau)$ then its conjugate belongs to $L^1(\mathcal{M},\tau)$. Finally, we present some partial extensions of the classical Szegö's theorem to the non-commutative setting.

1. introduction

The theory of conjugate functions has been a strong motivating force behind various aspects of harmonic analysis and abstract analytic function spaces. This theory which was originally developed for functions in the circle group $\mathbb{T}$ has found many generalizations to more abstract settings such as Dirichlet algebras in [5] and weak*-Dirichlet algebras in [11]. Results from this theory have been proven to be very fruitful for studying Banach space properties of the Hardy spaces (and their relatives) associated with the algebra involved (see for instance [2] and [14]).

Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal finite trace τ. Arveson introduced in [1], as non-commutative analogues of weak*-Dirichlet algebras, the notion of finite, maximal subdiagonal algebras of $\mathcal{M}$ (see definition below). Subsequently several authors studied the (non-commutative) H^p-spaces associated with such algebras ([13], [17], [18], [20], [21]). In [17], the notion of harmonic conjugates was introduced for maximal subdiagonal algebras generalizing the notion of conjugate functions for weak*-Dirichlet algebras and it was proved that the operation of conjugation is bounded in $L^2(\mathcal{M},\tau)$.

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The main objective of this paper is to combine the spirit of [11] with that of [17] to get a more constructive definition of conjugate operators for the setting of non-commutative maximal subdiagonal algebras; and to study different properties of conjugations for these non-commutative settings. We prove that most fundamental theorems on conjugate operation on Hardy spaces associated with weak*-Dirichlet (see [5] and [11]) remain valid for Hardy spaces associated with finite subdiagonal algebras. In particular, we show that the conjugation operator is a bounded map from $L^p(\mathcal{M},\tau)$ into $L^p(\mathcal{M},\tau)$ for $1 < p < \infty$, and from $L^1(\mathcal{M},\tau)$ into $L^{1,\infty}(\mathcal{M},\tau)$. We conlude that, as in commutative case, (non-commutative) H^p is a complemented subspace of $L^p(\mathcal{M},\tau)$ for $1 < p < \infty$.

Many results in harmonic analysis can be deduced from the classical Szegö's theorem. This very classical fact, although is valid for the more abstract setting of weak*-Dirichlet algebras, is still unknown for the non-commutative case. The last part of this paper is devoted to various results related to Szegö's theorem.

We refer to [19], [22] and [23] for general information concerning von Neumann algebras as well as basic notions of non-commutative integration, to [6] and [16] for Banach space theory and to [10] and [25] for basic definitions from harmonic analysis.

2. Definitions and preliminary results

Throughout, H will denote a Hilbert space and $\mathcal{M} \subseteq \mathcal{L}(H)$ a von Neumann algebra with a normal, faithful finite trace τ. A closed densely defined operator a in H is said to be *affiliated with* $\mathcal{M}$ if $u^*au = a$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. If a is a densely defined self-adjoint operator on H, and if $a = \int_{-\infty}^{\infty} s\, de_s^a$ is its spectral decomposition, then for any Borel subset $B \subseteq \mathbb{R}$, we denote by $\chi_B(a)$ the corresponding spectral projection $\int_{-\infty}^{\infty} \chi_B(s)\, de_s^a$. A closed densely defined operator on H affiliated with $\mathcal{M}$ is said to be τ*-measurable* if there exists a number $s \geq 0$ such that $\tau(\chi_{(s,\infty)}(|a|)) < \infty$.

The set of all τ-measurable operators will be denoted by $\overline{\mathcal{M}}$. The set $\overline{\mathcal{M}}$ is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint operation [19]. For $x \in \overline{\mathcal{M}}$, the generalized singular value function $\mu(x)$ of x is defined by

$$\mu_t(x) = \inf\{s \geq 0 : \tau(\chi_{(s,\infty)}(|x|)) \leq t\}, \quad \text{for } t \geq 0.$$

The function $t \to \mu_t(x)$ from $(0, \tau(I))$ to $[0,\infty)$ is right continuous, non-increasing and is the inverse of the distribution function $\lambda(x)$, where $\lambda_s(x) = \tau(\chi_{(s,\infty)}(|x|))$, for $s \geq 0$. For a complete study of $\mu(.)$ and $\lambda(.)$ we refer to [9].

Definition 1. *Let E be an order continuous rearrangement invariant (quasi-) Banach function space on $(0,\tau(I))$. We define the symmetric space $E(\mathcal{M},\tau)$ of measurable operators by setting:*

$$E(\mathcal{M},\tau)=\{x\in\overline{\mathcal{M}}\quad;\quad \mu(x)\in E\}\quad \textit{and}$$
$$\|x\|_{E(\mathcal{M},\tau)}=\|\mu(x)\|_E,\ \textit{for } x\in E(\mathcal{M},\tau).$$

It is well known that $E(\mathcal{M},\tau)$ is a Banach space (resp. quasi-Banach space) if E is a Banach space (resp. quasi-Banach space), and that if $E=L^p(0,\tau(I))$, for $0<p<\infty$, then $E(\mathcal{M},\tau)$ coincides with the usual non-commutative L^p-space associated with $(\mathcal{M},\tau)$. We refer to [3], [7] and [24] for more detailed discussions about these spaces. For simplicity we will always assume that the trace τ is normalized.

The following definition isolates the main topic of this paper.

Definition 2. *Let H^∞ be a weak*-closed unital subalgebra of $\mathcal{M}$ and let Φ be a faithful, normal expectation from $\mathcal{M}$ onto the diagonal $D=H^\infty\cap(H^\infty)^*$, where $(H^\infty)^*=\{x^*,\ x\in H^\infty\}$. Then H^∞ is called a finite, maximal, subdiagonal algebra in $\mathcal{M}$ with respect to Φ and τ if:*

1. *$H^\infty+(H^\infty)^*$ is weak*-dense in $\mathcal{M}$;*
2. *$\Phi(ab)=\Phi(a)\Phi(b)$ for all $a,b\in H^\infty$;*
3. *H^∞ is maximal among those subalgebras satisfying (1) and (2);*
4. *$\tau\circ\Phi=\tau$.*

For $0<p<\infty$, the closure of H^∞ in $L^p(\mathcal{M},\tau)$ is denoted by $H^p(\mathcal{M},\tau)$ (or simply H^p) and is called the Hardy space associated with the subdiagonal algebra H^∞. Similarly, the closure of $H_0^\infty=\{x\in H^\infty;\Phi(x)=0\}$ is denoted by H_0^p.

Note that Φ extends to $L^2(\mathcal{M},\tau)$ and this extension is an orthogonal projection from $L^2(\mathcal{M},\tau)$ onto $[D]_2$, the closure of D in $L^2(\mathcal{M},\tau)$. Similarly, since $\|\Phi(x)\|_1\le\|x\|_1$ for every $x\in\mathcal{M}$, the operator Φ extends uniquely to a projection of norm one from $L^1(\mathcal{M},\tau)$ onto $[D]_1$, the closure of D in $L^1(\mathcal{M},\tau)$.

3. Harmonic conjugates

Let $\mathcal{A}=H^\infty+(H^\infty)^*$. Since $\mathcal{A}$ is weak*-dense in $\mathcal{M}$, it is norm dense in $L^p(\mathcal{M},\tau)$, where $1\le p<\infty$.

Note that H^∞ and $(H_0^\infty)^*$ are orthogonal in $L^2(\mathcal{M},\tau)$. This fact implies that $L^2(\mathcal{M},\tau)=H^2\oplus(H_0^2)^*$, and hence that $L^2(\mathcal{M},\tau)=H_0^2\oplus(H_0^2)^*\oplus[D]_2$.

Let $a\in\mathcal{A}$. Then a can be written as $a_1+a_2^*+d$ where a_1 and a_2 belong to H_0^∞ and $d\in D$. In fact, $a=b_1+b_2^*$ with $b_1,b_2\in H^\infty$ and set $d=\Phi(b_1)+\Phi(b_2^*)\in D$ and $a_i=b_i-\Phi(b_i)$, for $i=1,2$.

Since H_0^∞ and $(H_0^\infty)^*$ are orthogonal subsets of $L^2(\mathcal{M},\tau)$, this decomposition is unique. For $u = u_1 + u_2^* + d$ in $\mathcal{A}$, we define $\tilde{u} = iu_2^* - iu_1$. Then $\tilde{u} \in \mathcal{M}$ and $u + i\tilde{u} = 2u_1 + d \in H^\infty$. The operator $\tilde{u}$ will be called the **conjugate** of u.

Our main goal is to study the operation that takes $u \in \mathcal{A}$ into $\tilde{u} \in \mathcal{M}$ as linear operator between non-commutative L^p-spaces. In particular we will extend "$\sim$" to $L^p(\mathcal{M},\tau)$ for $1 \leq p < \infty$. It should be noted that if $\mathcal{M}$ is commutative, then the above definition coincides with the the definition of conjugate functions for weak*-Dirichlet algebras studied in [11].

Remark 1. (i) $\Phi(\tilde{u}) = 0$, *for all* $u \in \mathcal{A}$.

(ii) *If* $u = u^*$, *then the uniqueness of the decomposition implies that* $u_1 = u_2$ *and* $d = d^*$. *Therefore if* $u = u^*$ *then* $\tilde{u} = \tilde{u}^*$.

(iii) *For* $u = u_1 + u_2^* + d \in \mathcal{A}$ *and* $\tilde{u} = i(u_2^* - u_1)$, *the above observation implies that* $u_2^* \perp u_1$ *in* $L^2(\mathcal{M},\tau)$, *so*
$$\|\tilde{u}\|_2^2 = \|u_2^* - u_1\|_2^2 = \|u_2^*\|_2^2 + \|u_1\|_2^2,$$
and since $L^2(\mathcal{M},\tau) = H_0^2 \oplus (H_0^2)^* \oplus [D]_2$ *we get,*
$$\|u\|_2^2 = \|u_1\|_2^2 + \|u_2^*\|_2^2 + \|d\|_2^2,$$
which implies that $\|\tilde{u}\|_2 \leq \|u\|_2$.

As a consequence of (iii), we get the following theorem:

Theorem 1. *There is a unique continuous linear map "$\sim$" from* $L^2(\mathcal{M},\tau)$ *into* $L^2(\mathcal{M},\tau)$ *that coincides with "$\sim$" in* $\mathcal{A}$. *This map is of norm 1, and if* $u \in L^2(\mathcal{M},\tau)$ *then* $u + i\tilde{u} \in H^2$.

We remark that Marsalli has recently proved a version of Theorem 1 (see [17] Corollary 10): he showed that the conjugation operator is bounded in $L^2(\mathcal{M},\tau)$ with bound less than or equal to $\sqrt{2}$.

Extension of the operator "$\sim$" to $L^p(\mathcal{M},\tau)$, $1 < p < \infty$.

In this section, we will extend Theorem 1 from $p = 2$ to all p with $1 < p < \infty$.

The following elementary lemma will be used in the sequel; we will include its proof for completeness.

Lemma 1. *Let* $m \in \mathbb{N}$ *and* $a_1, a_2, \dots, a_m \in \overline{\mathcal{M}}$. *If* $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = 1$, *and* $a_j \in L^{p_j}(\mathcal{M},\tau)$ *for each* $j \leq m$, *then*
$$|\tau(a_1 a_2 \ldots a_m)| \leq \Pi_{j=1}^m \|a_j\|_{p_j}.$$

Proof. Recall that, for $a, b \in \overline{\mathcal{M}}$, the operator a is said to be submajorided by b and write $a \prec\prec b$ if
$$\int_0^\alpha \mu_t(a)dt \leq \int_0^\alpha \mu_t(b)dt, \quad \text{for all } \alpha \geq 0.$$

The lemma will be proved inductively on $m \in \mathbb{N}$:

For $m = 2$, it is the usual Hölder's inequality.

Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $b, c \in \overline{\mathcal{M}}$. Then $\|bc\|_r \le \|b\|_p \cdot \|c\|_q$: this is a consequence of the fact [9, Theorem 4.2 (iii)] that $\mu^l_{(.)}(bc) \prec\prec \mu^l_{(.)}(b)\mu^l_{(.)}(c) \quad$ for all $l \in \mathbb{N}$. So

$$\|bc\|_r^r = \int \mu_t^r(bc)dt \le \int \mu_t^r(b)\mu_t^r(c)dt;$$

then apply the usual Hölder's inequality for functions.

Now assume that the lemma is valid for $m = 1, 2, \ldots, k$. Let $a_1, a_2, \ldots, a_k, a_{k+1} \in \overline{\mathcal{M}}$ and $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} + \frac{1}{p_{k+1}} = 1$. Choose q such that $\frac{1}{q} = \frac{1}{p_k} + \frac{1}{p_{k+1}}$. Then

$$\begin{aligned}\tau(a_1 \ldots a_{k-1} \cdot (a_k a_{k+1})) &\le \Pi_{i=1}^{k-1} \|a_j\|_{p_j} \cdot \|a_k a_{k+1}\|_q \\ &\le \Pi_{i=1}^{k+1} \|a_j\|_{p_j}.\end{aligned}$$

The proof is complete. □

Theorem 2. *For each $1 < p < \infty$, there is a unique continuous linear extension of "$\sim$" from $L^p(\mathcal{M}, \tau)$ into $L^p(\mathcal{M}, \tau)$ with the property that $f + i\tilde{f} \in H^p$ for all $f \in L^p(\mathcal{M}, \tau)$. Moreover there is a constant C_p such that*

$$\|\tilde{f}\|_p \le C_p \|f\|_p \quad \textit{for all } f \in L^p(\mathcal{M}, \tau).$$

Proof. Our proof follows Devinatz's argument ([5]) for Dirichlet algebras, but at number of points, certain non-trivial adjustments have to be made to fit the non-commutative setting.

Let $u \in \mathcal{A}$ be nonzero and self-adjoint; $\tilde{u}$ is self-adjoint. Let $g = u + i\tilde{u} \in H^\infty$. Since $u = u^*$, it is of the form $u = a + a^* + d$, where $a \in H_0^\infty$ and $d = d^* \in D$. Recall that $\tilde{u} = i(a^* - a)$ so $g = 2a + d \in H^\infty$. We get that

$$\Phi(g^{2k}) = \Phi((2a + d)^{2k}) = [2\Phi(a) + \Phi(d)]^{2k} = \Phi(d)^{2k}.$$

So $\Phi((u + i\tilde{u})^{2k}) = \Phi(d)^{2k}$ and taking the adjoint, $\Phi((u - i\tilde{u})^{2k}) = \Phi(d)^{2k}$. Adding these two equalities, we get

$$\Phi[(u + i\tilde{u})^{2k} + (u - i\tilde{u})^{2k}] = 2\Phi(d)^{2k}. \tag{3.1}$$

Now we will expand the operators $(u + i\tilde{u})^{2k}$ and $(u - i\tilde{u})^{2k}$. Note that u and $\tilde{u}$ do not necessarily commute.

For $2 \le m \le 2k$, let $\mathcal{S}_m = \{(r_1, r_2, \ldots, r_m) \in \{1, \ldots, 2k-1\}^m; \quad \sum_{j=1}^m r_j = 2k\}$ and set $\mathcal{S} = \cup_{2 \le m \le 2k} \mathcal{S}_m$. For a finite sequence of integers $r = (r_1, r_2, \ldots, r_m)$, we set

$s(r) = \sum_{j=1}^{[m/2]} r_{2j}$. Then

$$(u+i\tilde{u})^{2k} = u^{2k} + (i\tilde{u})^{2k} + \sum_{(r_1,\dots,r_m)\in\mathcal{S}} (u^{r_1}(i\tilde{u})^{r_2}\dots) + ((i\tilde{u})^{r_1}u^{r_2}\dots)$$
$$= u^{2k} + (i)^{2k}\tilde{u}^{2k} + \sum_{(r_1,\dots,r_m)\in\mathcal{S}} (i)^{s(r)}(u^{r_1}\tilde{u}^{r_2}\dots) + (i)^{2k-s(r)}(\tilde{u}^{r_1}u^{r_2}\dots).$$

Similarly,

$$(u-i\tilde{u})^{2k} = u^{2k} + (-i)^{2k}\tilde{u}^{2k} + \sum_{(r_1,\dots,r_m)\in\mathcal{S}} (-i)^{s(r)}(u^{r_1}\tilde{u}^{r_2}\dots) + (-i)^{2k-s(r)}(\tilde{u}^{r_1}u^{r_2}\dots).$$

If $\mathcal{K} = \{r = (r_1, r_2, \dots, r_{2m}) \in \mathcal{S};\ s(r) \in 2\mathbb{N}\}$, then

$$(u+i\tilde{u})^{2k}+(u-i\tilde{u})^{2k} = 2u^{2k}+2(i)^{2k}\tilde{u}^{2k}+2\sum_{r\in\mathcal{K}}(i)^{s(r)}(u^{r_1}\tilde{u}^{r_2}\dots)+(i)^{2k-s(r)}(\tilde{u}^{r_1}u^{r_2}\dots),$$

so from (3.1), we get

$$\Phi(d)^{2k} = \Phi(u^{2k}) + (i)^{2k}\Phi(\tilde{u}^{2k}) + \sum_{r\in\mathcal{K}}(i)^{s(r)}\Phi(u^{r_1}\tilde{u}^{r_2}\dots) + (i)^{2k-s(r)}\Phi(\tilde{u}^{r_1}u^{r_2}\dots).$$

This implies

$$(i)^{2k}\Phi(\tilde{u}^{2k}) = \Phi(d)^{2k} - \Phi(u^{2k}) - \sum_{r\in\mathcal{K}}(i)^{s(r)}\Phi(u^{r_1}\tilde{u}^{r_2}\dots) + i^{2k-s(r)}\Phi(\tilde{u}^{r_1}u^{r_2}\dots).$$

Taking the trace on both sides,

$$|\tau(\tilde{u}^{2k})| \le |\tau(d^{2k})| + |\tau(u^{2k})| + \sum_{r\in\mathcal{K}} |\tau(u^{r_1}\tilde{u}^{r_2}\dots)| + |\tau(\tilde{u}^{r_1}u^{r_2}\dots)|.$$

Applying Lemma 1, with $\frac{1}{p_j} = \frac{r_j}{2k}$, for every $r = (r_1, r_2, \dots, r_m) \in \mathcal{K}$, we get

$$|\tau(\tilde{u}^{2k})| \le |\tau(d^{2k})| + \|u\|_{2k}^{2k} + \sum_{r\in\mathcal{K}}(\|u\|_{2k}^{r_1}\|\tilde{u}\|_{2k}^{r_2}\dots) + (\|\tilde{u}\|_{2k}^{r_1}\|u\|_{2k}^{r_2}\dots).$$

We observe that by the definition of $\mathcal{K}$,

$$\|u\|_{2k}^{2k} + \sum_{r\in\mathcal{K}}(\|u\|_{2k}^{r_1}\|\tilde{u}\|_{2k}^{r_2}\dots) + (\|\tilde{u}\|_{2k}^{r_1}\|u\|_{2k}^{r_2}\dots)$$

is equal to the sum of the terms of the expansion of $(\|u\|_{2k} + \|\tilde{u}\|_{2k})^{2k}$ with $\|\tilde{u}\|_{2k}$ of even exponents between 2 and $2k-2$, i.e.,

$$\|u\|_{2k}^{2k} + \sum_{r\in\mathcal{K}}(\|u\|_{2k}^{r_1}\|\tilde{u}\|_{2k}^{r_2}\dots) + (\|\tilde{u}\|_{2k}^{r_1}\|u\|_{2k}^{r_2}\dots)$$
$$= \binom{2k}{0}\|u\|_{2k}^{2k} + \binom{2k}{2}\|u\|_{2k}^{2k-2}\|\tilde{u}\|_{2k}^{2} + \dots + \binom{2k}{2k-2}\|u\|_{2k}^{2}\|\tilde{u}\|_{2k}^{2k-2}.$$

Since $\tilde{u}$ is self-adjoint, $\tau(\tilde{u}^{2k}) = \|\tilde{u}\|_{2k}^{2k}$ and hence,

$$\|\tilde{u}\|_{2k}^{2k} \leq \|d\|_{2k}^{2k} + \|u\|_{2k}^{2k} + \binom{2k}{2}\|u\|_{2k}^{2}\|\tilde{u}\|_{2k}^{2(k-1)} + \cdots + \binom{2k}{2k-2}\|u\|_{2k}^{2(k-1)}\|\tilde{u}\|_{2k}^{2};$$

and since $\|d\|_{2k}^{2k} \leq \|u\|_{2k}^{2k}$, we have

$$\|\tilde{u}\|_{2k}^{2k} \leq 2\|u\|_{2k}^{2k} + \binom{2k}{2}\|u\|_{2k}^{2}\|\tilde{u}\|_{2k}^{2(k-1)} + \cdots + \binom{2k}{2k-2}\|u\|_{2k}^{2(k-1)}\|u\|_{2k}^{2}.$$

Divide both sides by $\|u\|_{2k}^{2k}$ and set $X_0 = \|\tilde{u}\|_{2k}/\|u\|_{2k}$, we have

$$X_0^{2k} - \binom{2k}{2}X_0^{2(k-1)} - \binom{2k}{4}X_0^{2(k-2)} - \cdots - 2 \leq 0.$$

Hence, X_0 is less than or equal to the largest real root of the polynomial equation

$$X^{2k} - \binom{2k}{2}X^{2(k-1)} - \binom{2k}{4}X^{2(k-2)} - \cdots - 2 = 0.$$

If the largest root is K_{2k}, we have

$$\|\tilde{u}\|_{2k} \leq K_{2k}\|u\|_{2k}.$$

Using Minkowski's inequality, we conclude that for every $f \in \mathcal{A}$ (not necessarily self adjoint), we have

$$\|\tilde{f}\|_{2k} \leq 2K_{2k}\|f\|_{2k}.$$

Since $\mathcal{A}$ is dense in $L^{2k}(\mathcal{M},\tau)$, the inequality above shows that "$\sim$" can be extended as a bounded linear operator from $L^{2k}(\mathcal{M},\tau)$ into $L^{2k}(\mathcal{M},\tau)$, so the theorem is proved for p even.

For the general case, let $2 \leq p < \infty$. Choose k such that $2k \leq p \leq 2k+2$. By [8] (Theorem 2.3), $L^p(\mathcal{M},\tau)$ can be realized as a complex interpolation of the pair $(L^{2k}(\mathcal{M},\tau), L^{2k+2}(\mathcal{M},\tau))$, and we conclude that "$\sim$" is also bounded from $L^p(\mathcal{M},\tau)$ into $L^p(\mathcal{M},\tau)$.

For $1 < p < 2$, from the above case,"$\sim$" is bounded from $L^q(\mathcal{M},\tau)$ into $L^q(\mathcal{M},\tau)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and we claim that as in the commutative case, $(\sim)^* = -(\sim)$.

To see this, let u and v be self adjoint elements of $\mathcal{A}$; we have

$$\Phi((u+i\tilde{u})(v+i\tilde{v})) = \Phi(u+i\tilde{u})\cdot\Phi(v+i\tilde{v}) = \Phi(u)\Phi(v),$$

which implies that

$$\Phi(uv + iu\tilde{v} + i\tilde{u}v - \tilde{u}\tilde{v}) = \Phi(uv - \tilde{u}\tilde{v}) + i\Phi(u\tilde{v} + \tilde{u}v) = \Phi(u)\Phi(v),$$

so

$$\tau(uv - \tilde{u}\tilde{v}) + i\tau(u\tilde{v} + \tilde{u}v) = \tau(\Phi(u)\Phi(v)).$$

Since $\Phi(u)$ and $\Phi(v)$ are self-adjoint, $\tau(\Phi(u)\Phi(v)) \in \mathbb{R}$, and also $\tau(uv - \tilde{u}\tilde{v})$ and $\tau(u\tilde{v} + \tilde{u}v) \in \mathbb{R}$. This implies $\tau(u\tilde{v} + \tilde{u}v) = 0$ and $\tau(u\tilde{v}) = -\tau(\tilde{u}v)$. The proof is complete. □

Extension of the operator "∼" to $L^1(\mathcal{M}, \tau)$.

The following lemma is probably known but we will include its proof for the convenience of the reader.

Lemma 2. *For $u \in \mathcal{M}$, $u \geq 0$, let $f = u + i\tilde{u}$ and $0 < \varepsilon < 1$.*

1. *$I + \varepsilon f$ has bounded inverse with $\|(I + \varepsilon f)^{-1}\| \leq 1$.*
2. *$f_\varepsilon = (\varepsilon I + f)(I + \varepsilon f)^{-1} \in H^\infty$.*
3. *$\mathrm{Re}\,(f_\varepsilon) \geq \varepsilon I$.*
4. *$\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_p = 0 \quad (1 \leq p < \infty)$.*

Proof. (1) Note that f is densely defined and that, for every $x \in D(f)$,

$$\langle (I + \varepsilon f)x,\, x\rangle = \langle (I + \varepsilon u)x,\, x\rangle + i\langle \tilde{u}x,\, x\rangle.$$

Thus $|\langle (I + \varepsilon f)x,\, x\rangle| \geq \|x\|^2$, which implies

$$\|(I + \varepsilon f)x\| \geq \|x\| \quad \text{for all } x \in D(f).$$

So $I + \varepsilon f$ is one-to-one; now for every $y \in R(I + \varepsilon f)$ define $(I + \varepsilon f)^{-1}y$ to be the unique element such that $(I + \varepsilon f)((I + \varepsilon f)^{-1}y) = y$. Then $(I + \varepsilon f)^{-1}$ is linear and, for every $y \in R(I + \varepsilon f)$, we have $\|(I + \varepsilon f)^{-1}y\| \leq \|y\|$. We claim that $R(I + \varepsilon f)$ is dense in H. For this, note that (using similar estimate), $I + \varepsilon f^*$ is one-to-one; if $z \perp R(I + \varepsilon f)$ then $x \to \langle z, (I + \varepsilon f)x\rangle = 0$ is continuous so $z \in D(I + \varepsilon f^*)$ with $\langle (I + \varepsilon f^*)z, x\rangle = 0$ for every $x \in D(I + \varepsilon f)$ so $z = 0$. Hence $(I + \varepsilon f)^{-1}$ can be extended as a bounded operator of norm ≤ 1.

(2) To prove that $f_\varepsilon \in H^\infty$, note that $(I + \varepsilon f)^{-1} \in \mathcal{M}$ with inverse $(I + \varepsilon f) \in H^2$. In particular, the inverse of $(I + \varepsilon f)^{-1}$ lies in $L^2(\mathcal{M}, \tau)$, so from Proposition 1.2 of [18] (see also Proposition 1 of [21]), there exists a unitary operator $a \in \mathcal{M}$ and an operator $b \in H^\infty$ such that $(I + \varepsilon f)^{-1} = ab$. Thus $a^*(I + \varepsilon f)^{-1} = b \in H^\infty$, and since $1 + \varepsilon f \in H^2$, we have $a^* = b(I + \varepsilon f) \in H^2$, so $a^* \in H^2 \cap \mathcal{M}$, which implies $a^* \in H^\infty$. Now a is unitary, that is, $aa^* = a^*a = I$ (i.e $a^* = a^{-1}$), and H^∞ is a Banach algebra, so $a = (a^*)^{-1} \in H^\infty$ implies $(1 + \varepsilon f)^{-1} = ab \in H^\infty$. Hence $f_\varepsilon = (\varepsilon I + f)(I + \varepsilon f)^{-1} \in H^p$ for every $p \geq 1$, so if we can show that $f_\varepsilon \in \mathcal{M}$, the proof is complete. That $f_\varepsilon \in \mathcal{M}$ can be seen as follows:

$$f_\varepsilon = \varepsilon(I + \varepsilon f)^{-1} + f(I + \varepsilon f)^{-1};$$

but $I = (I + \varepsilon f)(I + \varepsilon f)^{-1} = (I + \varepsilon f)^{-1} + \varepsilon f(I + \varepsilon f)^{-1}$ and $(I + \varepsilon f)^{-1} \in \mathcal{M}$, so $\varepsilon f(I + \varepsilon f)^{-1} \in \mathcal{M}$, implying $f(I + \varepsilon f)^{-1} \in \mathcal{M}$.

(3) $\mathrm{Re}\,(f_\varepsilon) = \mathrm{Re}\,((\varepsilon I + f)(I+\varepsilon f)^{-1}) = \mathrm{Re}\,([(\varepsilon I + \varepsilon^2 f) + (1-\varepsilon^2)f)](I+\varepsilon f)^{-1}) = \varepsilon I + (1-\varepsilon^2)\,\mathrm{Re}\,(f(I+\varepsilon f)^{-1})$. Since we assume that $\varepsilon < 1$, it is enough to show that $\mathrm{Re}\,(f(I+\varepsilon f)^{-1}) \geq 0$. For this

$$\begin{aligned}\mathrm{Re}\,(f(I+\varepsilon f)^{-1}) &= \tfrac{1}{2}\left(f(I+\varepsilon f)^{-1} + (I+\varepsilon f^*)^{-1}f^*\right)\\ &= \tfrac{1}{2}(I+\varepsilon f^*)^{-1}\left((I+\varepsilon f^*)f + f^*(I+\varepsilon f)\right)(I+\varepsilon f)^{-1}\\ &= \tfrac{1}{2}(I+\varepsilon f^*)^{-1}\left(2\,\mathrm{Re}\,(f) + 2\varepsilon|f|^2\right)(I+\varepsilon f)^{-1} \geq 0.\end{aligned}$$

(4) We have for every $\varepsilon > 0$,

$$\begin{aligned}f_\varepsilon - f &= (\varepsilon I + f)(I+\varepsilon f)^{-1} - f\\ &= \left((\varepsilon I + f) - f(I+\varepsilon f)\right)(I+\varepsilon f)^{-1}\\ &= \varepsilon(I+f^2)(I+\varepsilon f)^{-1},\end{aligned}$$

so

$$\mu_t(f_\varepsilon - f) \prec\!\prec \varepsilon\mu_t(I+f^2)\mu_t((I+\varepsilon f)^{-1}).$$

Since $\|(I+\varepsilon f)^{-1}\| \leq 1$, we get $\mu_t((I+\varepsilon f)^{-1}) \leq 1$ for every $t > 0$. Also $I + f^2 \in L^p(\mathcal{M},\tau)$ for every $p > 1$, so $\|f_\varepsilon - f\|_p \leq \varepsilon\|I+f^2\|_p \to 0$ (as $\varepsilon \to 0$). The proof is complete. □

Proposition 1. *Let $u \in \mathcal{M}$ with $u \geq 0$, and set $f = u + i\tilde{u}$. There exists a constant K (independent of u) such that, for every $s > 0$,*

$$\tau(\chi_{(s,\infty)}(|f|)) \leq K\frac{\|u\|_1}{s}.$$

Proof. We follow (at least in spirit) the argument of Helson in [10] for the commutative case.

Let u and f be as in the statement of the proposition, and fix $0 < \varepsilon < 1$. Set f_ε as in Lemma 2. For $s \in (0,\infty)$ fixed, consider the following transformation on $\{z; \mathrm{Re}\,(z) \geq 0\}$:

$$A_s(z) = 1 + \frac{z-s}{z+s} \quad \text{for all } z \in \{w, \mathrm{Re}\,(w) \geq 0\}.$$

It can be checked that the part of the plane $\{z; |z| \geq s\}$ is mapped to the half disk $\{w;\ \mathrm{Re}\,(w) \geq 1\}$; this fact is very crusial in the argument of [10] for the commutative case. Although we are unable to verify this fact through functional calculus, one can still recapture its consequences by taking the trace in every step.

Note that $\sigma(f_\varepsilon)$ is a compact subset of $\{z; \mathrm{Re}\,(z) \geq \varepsilon\}$. By the analytic functional calculus for Banach algebras,

$$A_s(f_\varepsilon) = I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1} \in H^\infty$$

and therefore (since A_s is analytic)

$$\Phi(A_s(f_\varepsilon)) = A_s(\Phi(f_\varepsilon)). \tag{3.2}$$

Claim 1. $\Phi(f_\varepsilon) = \Phi(u)_\varepsilon$.

In fact $f_\varepsilon = (\varepsilon I + f)(I+\varepsilon f)^{-1} \in H^\infty$ and $f_\varepsilon \cdot (I+\varepsilon f) = \varepsilon I + f$, so $\Phi(f_\varepsilon)\Phi(I+\varepsilon f) = \Phi(\varepsilon I + f)$. But $\Phi(f) = \Phi(u)$, so we get $\Phi(f_\varepsilon)(I+\varepsilon\Phi(u)) = \varepsilon I + \Phi(u)$, and the claim follows.

Claim 2. $\mathrm{Re}\,(I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1}) \geq 0$.

For this we have

$$\begin{aligned}
\mathrm{Re}\,(I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1}) &= I + \tfrac{1}{2}\left((f_\varepsilon - sI)(f_\varepsilon + sI)^{-1} + (f_\varepsilon^* + sI)^{-1}(f_\varepsilon^* - sI)\right) \\
&= I + \tfrac{1}{2}(f_\varepsilon^* + sI)^{-1}\left((f_\varepsilon^* + sI)(f_\varepsilon - sI) + (f_\varepsilon^* - sI)(f_\varepsilon + sI)\right)(f_\varepsilon + sI)^{-1} \\
&= I + (f_\varepsilon^* + sI)^{-1}\left(|f_\varepsilon|^2 - s^2 I\right)(f_\varepsilon + sI)^{-1} \\
&= (f_\varepsilon^* + sI)^{-1}\left((f_\varepsilon^* + sI)(f_\varepsilon + sI) + |f_\varepsilon|^2 - s^2 I\right)(f_\varepsilon + sI)^{-1} \\
&= 2(f_\varepsilon^* + sI)^{-1}\left(|f_\varepsilon|^2 + s\,\mathrm{Re}\ f_\varepsilon\right)(f_\varepsilon + sI)^{-1},
\end{aligned}$$

and the claim follows from the fact that $\mathrm{Re}\,(f_\varepsilon) \geq \varepsilon I$.

Note that since $\Phi(u)$ is self-adjoint, so are $\Phi(u)_\varepsilon$ and $A_s(\Phi(u)_\varepsilon)$. We conclude from (3.2) that $\tau\,(I + (\Phi(u)_\varepsilon - sI)(\Phi(u)_\varepsilon + sI)^{-1}) \in \mathbb{R}$, and therefore

$$\tau\left(I + \mathrm{Re}\,((f_\varepsilon - sI)(f_\varepsilon + sI)^{-1})\right) = \tau(A_s(\Phi(u)_\varepsilon)). \tag{3.3}$$

To prove the proposition, let $P = \chi_{(s,\infty)}(|f_\varepsilon|)$. The projection P commutes with $|f_\varepsilon|$ and we have

$$\mathrm{Re}\left[I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1}\right] = (f_\varepsilon^* + sI)^{-1}\left[2|f_\varepsilon|^2 + 2s\,\mathrm{Re}\,(f_\varepsilon)\right](f_\varepsilon + sI)^{-1};$$

but since $\mathrm{Re}\,(f_\varepsilon) \geq \varepsilon I$, we get

$$2|f_\varepsilon|^2 + 2s\,\mathrm{Re}\,(f_\varepsilon) \geq 2|f_\varepsilon|^2 + 2s\varepsilon I,$$

and hence

$$\tau\left[\mathrm{Re}\,(I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1})\right] \geq \tau\left[(2|f_\varepsilon|^2 + 2s\varepsilon I)(f_\varepsilon + sI)^{-1}(f_\varepsilon^* + sI)^{-1}\right]. \tag{3.4}$$

Lemma 3. *Let a and b be operators in $\overline{\mathcal{M}}$ with $a \geq 0$, $b \geq 0$, and let P be a projection that commutes with a. Then $\tau(ab) \geq \tau(P(ab)P)$.*

To see this, notice that, since P commutes with a, $PaP \leq a$, so $b^{1/2}PaPb^{1/2} \leq b^{1/2}ab^{1/2}$, implying that $\tau(b^{1/2}PaPb^{1/2}) \leq \tau(b^{1/2}ab^{1/2})$ and

$$\begin{aligned}
\tau(P(ab)P) &= \tau(P(ab)) \\
&= \tau(PaPb) = \tau(b^{1/2}PaPb^{1/2}) \\
&\leq \tau(b^{1/2}ab^{1/2}) = \tau(ab).
\end{aligned}$$

The lemma is proved.

Applying Lemma 3 for $a = 2|f_\varepsilon| + 2s\varepsilon I$ and $b = (f_\varepsilon + sI)^{-1}(f_\varepsilon^* + sI)^{-1}$, we obtain

(3.5)
$$\tau\left[\mathrm{Re}\,(I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1})\right] \geq \tau[(2P|f_\varepsilon|^2 + 2s\varepsilon P)(f_\varepsilon + sI)^{-1}(f_\varepsilon^* + sI)^{-1}].$$

Note that $(f_\varepsilon+sI)^{-1}(f_\varepsilon^*+sI)^{-1} = (|f_\varepsilon|^2 + 2s\,\mathrm{Re}\,(f_\varepsilon) + s^2I)^{-1}$ and $|f_\varepsilon|^2+2s\,\mathrm{Re}\,(f_\varepsilon)+s^2I \leq |f_\varepsilon|^2 + 2s|f_\varepsilon| + s^2I$.

Lemma 4. *Let A, B, C be positive operators such that*
(i) *A^{-1} and B^{-1} exists, and*
(ii) *$A \leq B$.*
Then $\tau(CB^{-1}) \leq \tau(CA^{-1})$.

To prove this lemma, observe that $\tau(CB^{-1}) = \tau(\alpha\beta)$, where $\alpha = C^{1/2}B^{-1}A^{1/2}$ and $\beta = A^{-1/2}C^{-1/2}$. By Hölder's inequality,
$$\begin{aligned}\tau(CB^{-1}) &\leq \tau(|\alpha|^2)^{1/2}\,\tau(|\beta|^2)^{1/2} = \tau(\alpha^*\alpha)^{1/2}\,\tau(\beta^*\beta)^{1/2} = \tau(\alpha\alpha^*)^{1/2}\tau(\beta^*\beta)^{1/2}\\ &= \tau(C^{1/2}B^{-1}A^{1/2}A^{1/2}B^{-1}C^{1/2})^{1/2}\,\tau(C^{1/2}A^{-1/2}A^{-1/2}C^{1/2})^{1/2}\\ &= \tau(C^{1/2}B^{-1}AB^{-1}C^{1/2})^{1/2}\,\tau(CA^{-1})^{1/2}.\end{aligned}$$
But since $A \leq B$, we get $C^{1/2}B^{-1}(A)B^{-1}C^{1/2} \leq C^{1/2}B^{-1}C^{1/2}$, and therefore
$$\tau(CB^{-1}) \leq \tau(CB^{-1})^{1/2}\,\tau(CA^{-1})^{1/2}$$
which shows that $\tau(CB^{-1}) \leq \tau(CA^{-1})$. The proof of Lemma 4 is complete.

Applying Lemma 4 to
$$\begin{aligned}A &= (f_\varepsilon^* + sI)(f_\varepsilon + sI),\\ B &= |f_\varepsilon|^2 + 2s|f_\varepsilon| + s^2I\end{aligned}$$
and
$$C = 2|f_\varepsilon|^2P + 2s\varepsilon P,$$
We obtain from (3.5) that
$$\tau\left[\mathrm{Re}\,(I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1})\right] \geq \tau(CA^{-1}) \geq \tau(CB^{-1})$$
and hence
$$\tau\left[\mathrm{Re}\,(I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1})\right] \geq \tau\left[(2|f_\varepsilon|^2P + 2s\varepsilon P)(|f_\varepsilon|^2 + s|f_\varepsilon| + s^2I)^{-1}\right].$$
Since $|f_\varepsilon|^2P \geq s^2P$, we get that

(3.6)
$$\tau\left[\mathrm{Re}\,(I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1})\right] \geq \tau\left[P(|f_\varepsilon|^2 + 2s\varepsilon I + s^2I)(|f_\varepsilon|^2 + 2s|f_\varepsilon| + s^2I)^{-1}\right].$$

If we denote by $E^{|f_\varepsilon|}$ the spectral decomposition of $|f_\varepsilon|$, then

$$P(|f_\varepsilon|^2 + 2s\varepsilon I + s^2 I)(|f_\varepsilon|^2 + 2s|f_\varepsilon| + s^2 I)^{-1} = \int_s^\infty \frac{t^2 + 2s\varepsilon + s^2}{t^2 + 2st + s^2}\, dE_t^{|f_\varepsilon|}.$$

Let

$$\psi_{\varepsilon,s}(t) = \frac{t^2 + 2s\varepsilon + s^2}{t^2 + 2st + s^2} \quad \text{for } t \in [s, \infty).$$

One can show that $\psi_{\varepsilon,s}$ attains its (unique) minimum at $t_0 = \varepsilon + \sqrt{\varepsilon^2 + \varepsilon s + s^2}$, and therefore that

$$\int_s^\infty \frac{t^2 + 2s\varepsilon + s^2}{t^2 + 2st + s^2}\, dE_t^{|f_\varepsilon|} \geq \psi_{\varepsilon,s}(t_0) P,$$

so we deduce from (3.6) that

$$\tau\left[\mathrm{Re}\,(I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1})\right] \geq \psi_{\varepsilon,s}(t_0)\tau(P).$$

To finish the proof, recall from (3.3) that

$$\tau\left[\mathrm{Re}\,(I + (f_\varepsilon - sI)(f_\varepsilon + sI)^{-1})\right] = \tau\left[I + (\Phi(u)_\varepsilon - sI)(\Phi(u)_\varepsilon + sI)^{-1}\right],$$

so

$$\begin{aligned}\tau(P) &\leq \frac{1}{\psi_{\varepsilon,s}(t_0)}\tau\left[I + (\Phi(u)_\varepsilon - sI)(\Phi(u)_\varepsilon + sI)^{-1}\right] \\ &= \frac{1}{\psi_{\varepsilon,s}(t_0)}\tau\left[2\Phi(u)_\varepsilon(\Phi(u)_\varepsilon + sI)^{-1}\right].\end{aligned}$$

But $(\Phi(u)_\varepsilon + sI)^{-1} = \frac{1}{s}(\frac{\Phi(u)_\varepsilon}{s} + I)^{-1}$ has norm $\leq 1/s$, hence

$$\tau(P) \leq \frac{2}{\psi_{\varepsilon,s}(t_0)}\frac{\|u_\varepsilon\|_1}{s}.$$

Now taking $\varepsilon \to 0$, we get $\|u_\varepsilon\|_1 \to \|u\|_1$, and $\tau(P) \to \tau(\chi_{(s,\infty)}(|f|))$. Note that

$$\psi_{\varepsilon,s}(t_0) = \frac{2\varepsilon^2 + 3\varepsilon s + 2s^2 + 2\varepsilon\sqrt{\varepsilon^2 + \varepsilon s + s^2}}{2\varepsilon^2 + 3\varepsilon s + 2s^2 + 2s(1+\varepsilon)\sqrt{\varepsilon^2 + \varepsilon s + s^2}}$$

so $\lim_{\varepsilon \to 0} \psi_{\varepsilon,s}(t_0) = 1/2$.

Hence $\tau(\chi_{(s,\infty)}(|f|)) \leq 4\,\|u\|_1/s$. The proof is complete. $\square$

Recall that $L^{1,\infty}(\mathcal{M}, \tau) = \{a \in \overline{\mathcal{M}};\ \sup_{t>0} t\mu_t(a) < \infty\}$.

Set $\|a\|_{1,\infty} = \sup_{t>0} t\mu_t(a)$ for $a \in L^{1,\infty}(\mathcal{M}, \tau)$. As in the commutative case, $\|.\|_{1,\infty}$ is equivalent to a quasinorm in $L^{1,\infty}(\mathcal{M}, \tau)$, so there is a fixed constant C such that, for every $a, b \in L^{1,\infty}(\mathcal{M}, \tau)$, we have $\|a + b\|_{1,\infty} \leq C(\|a\|_{1,\infty} + \|b\|_{1,\infty})$.

For $u \in \mathcal{M}$, let $Tu = u + i\tilde{u}$. From Theorem 1, T is linear and Proposition 1 can be restated as follows:

$$\text{For any } u \in \mathcal{M} \text{ with } u \geq 0, \text{ we have } \|Tu\|_{1,\infty} \leq 4\|u\|_1;$$

this implies that for $u \geq 0$,

$$\|\tilde{u}\|_{1,\infty} \leq C(4+1)\|u\|_1 = 5C\|u\|_1.$$

Now suppose that $u \in \mathcal{M}$, $u = u^*$, $u = u_+ - u_-$ and $\tilde{u} = \tilde{u}_+ - \tilde{u}_-$. Then

$$\|\tilde{u}\|_{1,\infty} \leq C(\|\tilde{u}_+\|_{1,\infty} + \|\tilde{u}_-\|_{1,\infty}) \leq 5C^2\|u\|_1.$$

Similarly, if we require only $u \in \mathcal{M}$, we have $u = \mathrm{Re}\,(u) + i\,\mathrm{Im}\,(u)$ and by linearity, $\tilde{u} = \widetilde{\mathrm{Re}(u)} + i\,\widetilde{\mathrm{Im}(u)}$, and as above,

$$\|\tilde{u}\|_{1,\infty} \leq 10C^3\|u\|_1.$$

We are now ready to extend "$\sim$" in $L^1(\mathcal{M},\tau)$: If $u \in L^1(\mathcal{M},\tau)$, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ such that $\|u - u_n\|_1 \to 0$ as $n \to \infty$. Then

$$\|\tilde{u}_n - \tilde{u}_m\|_{1,\infty} \leq 10C^3\|u_n - u_m\|_1,$$

and since $\|u_n - u_m\|_1 \to 0$ as $n, m \to \infty$, the sequence $(\tilde{u}_n)_n$ converges in $L^{1,\infty}(\mathcal{M},\tau)$ to an operator $\tilde{u}$. This defines $\tilde{u}$ for $u \in L^1(\mathcal{M},\tau)$. This definition can be easily checked to be independent of the sequence $(\tilde{u}_n)_n$ and agree with the conjugation operator defined for $p > 1$.

Letting $n \to \infty$ in the inequality $\|\tilde{u}_n\|_{1,\infty} \leq 10C^3\|u_n\|_1$, we obtain the following theorem ($H^{1,\infty}$ denotes the closure of H^∞ in $L^{1,\infty}(\mathcal{M},\tau)$):

Theorem 3. *There is a unique extension of "$\sim$" from $L^1(\mathcal{M},\tau)$ into $L^{1,\infty}(\mathcal{M},\tau)$ with the following property: $u + i\tilde{u} \in H^{1,\infty}$ for all $u \in L^1(\mathcal{M},\tau)$, and there is a constant K such that $\|\tilde{u}\|_{1,\infty} \leq K\|u\|_1$ for all $u \in L^1(\mathcal{M},\tau)$.*

Corollary 1. *For any p with $0 < p < 1$ there exists a constant K_p such that*

$$\|\tilde{u}\|_p \leq K_p\|u\|_1 \quad \textit{for all } u \in L^1(\mathcal{M},\tau).$$

Proof. It is enough to show that such a constant exists for $u \in \mathcal{M}$, $u \geq 0$. Recall that for $u \in \mathcal{M}$, the distribution $\lambda_s(u)$ equals $\tau(\chi_{(s,\infty)}(u))$.

Let $F(s) = 1 - \lambda_s(u) = \tau(\chi_{(0,s)}(u))$. Assume that $\|u\|_1 \leq 1$. From proposition 1,

$$1 - F(s) \leq \frac{4}{s}\|u\|_1 \leq \frac{4}{s}.$$

Note that F is a non-increasing right continuous function and for $p > 0$,

$$\tau(|\tilde{u}|^p) = \int_0^1 \mu_t(|\tilde{u}|)^p dt = \int_0^\infty s^p dF(s) \leq 1 + \int_1^\infty s^p dF(s).$$

If A is a point of continuity for $F(A > 1)$, then

$$\int_1^A s^p dF(s) = [s^p(F(s)-1)]_1^A + p\int_1^A (1-F(s))s^{p-1}ds.$$

Since $1 - F(s) \leq \frac{4}{s}$, we get that both $[s^p(F(s)-1)]_1^A$ and $\int_1^A (1-F(s))s^{p-1}ds$ are bounded for $0 < p < 1$, that is, $\int_0^1 \mu_t(|\tilde{u}|)^p dt$ has bound independent of u. $\square$

The **Riesz projection** $\mathcal{R}$ can now be defined as in the commutative case: for every $a \in L^p(\mathcal{M}, \tau)$, $(1 \leq p \leq \infty)$,

$$\mathcal{R}(a) = \frac{1}{2}(a + i\tilde{a} + \Phi(a)).$$

From Theorem 2, one can easily verify that $\mathcal{R}$ is a bounded projection from $L^p(\mathcal{M}, \tau)$ onto H^p for $1 < p < \infty$. In particular H^p is a complemented subspace of $L^p(\mathcal{M}, \tau)$. For $p = 1$, Theorem 3 shows that $\mathcal{R}$ is bounded from $L^1(\mathcal{M}, \tau)$ into $H^{1,\infty}$.

Our next result gives a sufficient condition on an operator $a \in L^1(\mathcal{M}, \tau)$ so that its conjugate $\tilde{a} \in L^1(\mathcal{M}, \tau)$.

Theorem 4. *There exists a constant K such that for every $a \in \mathcal{M}$,*

$$\|\tilde{a}\|_1 \leq K\tau(|a|\log^+|a|) + K.$$

Proof. Since our proof of Theorem 2 follows the same line of argument as in [5], one can deduce as in [11](Corollary 2h) that there is an absolute constant C such that $\|\tilde{a}\|_p \leq Cpq\|a\|_p$ for all $a \in L^p(\mathcal{M}, \tau)$, $1 < p < \infty$ and $1/p + 1/q = 1$. The conclusion of the theorem can be deduced as a straightforward adjustment of the commutative case in [25](vol .II, p. 119); we will present it here for completeness.

Let $a \in \mathcal{M}$; we will assume first that $a \geq 0$. Let $(e_t)_t$ be the spectral decomposition of a. For each $k \in \mathbb{N}$, let $P_k = \chi_{[2^{k-1}, 2^k)}(a)$ be the spectral projection relative to $[2^{k-1}, 2^k)$. Define $a_k = aP_k$ for $k \geq 1$ and $a_0 = a\chi_{[0,1)}(a)$. Clearly $a = \sum_{k=0}^\infty a_k$ in $L^p(\mathcal{M}, \tau)$ for every $1 < p < \infty$. By linearity, $\tilde{a} = \sum_{k=0}^\infty \tilde{a}_k$. For every $k \in \mathbb{N}$, $\|\tilde{a}_k\|_1 \leq \|\tilde{a}_k\|_p \leq Cp^2(p-1)^{-1}\|a_k\|_p$. Since $a_k \leq 2^k P_k$, we get for $1 < p < 2$,

$$\|\tilde{a}_k\|_1 \leq 4C\frac{1}{p-1}2^k\tau(P_k)^{\frac{1}{p}}.$$

If we set $p = 1 + \frac{1}{k+1}$ and $\epsilon_k = \tau(P_k)$, we have

$$\|\tilde{a}_k\|_1 \leq 4C(k+1)2^k\epsilon_k^{\frac{k+1}{k+2}}.$$

Taking the summation over k,

$$\|\tilde{a}\|_1 \leq \sum_{k=0}^\infty 4C(k+1)2^k\epsilon_k^{\frac{k+1}{k+2}}.$$

We note as in [25] that if $J = \{k \in \mathbb{N};\ \epsilon_k \leq 3^{-k}\}$ then

$$\sum_{k\in J} 4C(k+1)2^k \epsilon_k^{\frac{k+1}{k+2}} \leq \sum_{k=0}^{\infty} 4C(k+1)2^k (3^{-k})^{\frac{k+1}{k+2}} = \alpha < \infty.$$

On the other hand, for $k \in \mathbb{N} \setminus J$, $\epsilon_k^{\frac{k+1}{k+2}} \leq \epsilon_k.3^{\frac{k}{k+2}} \leq \beta\epsilon_k$ where $\beta = \sup_k 3^{\frac{k}{k+2}}$. So we get

$$\begin{aligned}\|\tilde{a}\|_1 &\leq \alpha + 4C\beta \sum_{k=0}^{\infty}(k+1)2^k\epsilon_k \\ &\leq \alpha + 4C\beta(\epsilon_0 + 4\epsilon_1) + 4C\beta\sum_{k\geq 2}(k+1)2^k\epsilon_k.\end{aligned}$$

Since for $k \geq 2$, $k+1 \leq 3(k+1)$, we get

$$\|\tilde{a}\|_1 \leq \alpha + 16C\beta + 24C\beta\sum_{k\geq 2}(k-1)2^{k-1}\epsilon_k.$$

To complete the proof, notice that for $k \geq 2$,

$$\begin{aligned}(k-1)2^{k-1}\epsilon_k &= \int_{2^{k-1}}^{2^k} (k-1)2^{k-1}\ d\tau(e_t) \\ &\leq \int_{2^{k-1}}^{2^k} \frac{t\log t}{\log 2}\ d\tau(e_t).\end{aligned}$$

Hence by setting $K_1 = \max\{\alpha + 16C\beta, 24C\beta/\log 2\}$, we get:

$$\|\tilde{a}\|_1 \leq K_1 + K_1\tau\left(a\log^+(a)\right).$$

Now for a general element $b \in \mathcal{M}$, we decompose b as $b = (b_+^{(1)} - b_-^{(1)}) + i(b_+^{(2)} - b_-^{(2)})$ where $b_+^{(i)}$ and $b_-^{(i)}$ are positive operators for $i = 1, 2$. One can easily verify that $\|\tilde{b}\|_1 \leq K + K\tau(|b|\log^+(|b|))$ where $K = 4K_1$. The proof is complete. $\square$

Remark 2. *From Theorem 4, one can deduce that if* $a \in L^1(\mathcal{M}, \tau)$ *is such that* $|a|\log^+(|a|)$ *belongs to* $L^1(\mathcal{M}, \tau)$ *then* $\tilde{a} \in L^1(\mathcal{M}, \tau)$.

4. Remarks on Szegö's theorem

Szegö's theorem plays a very important role in theory of weak*-Dirichlet algebras. It is still unknown if it has an extension in the context of subdiagonal algebras. In this section, we discuss different forms of possible extensions of Szegö's theorem.

Proposition 2. *If* a *and* a^{-1} *belong to* H^∞, *then*

$$\inf\left\{\tau(|a^*|^2|I - f|^2);\ f \in H_0^\infty\right\} = \tau(|\Phi(a)|^2).$$

Proof. Let $b = \Phi(a)$ and $p = I - ba^{-1}$. We will equip $\mathcal{M}$ with the following scalar product by setting for every $x, y \in \mathcal{M}$,

$$\langle x, y\rangle = \tau(|a^*|^2 y^* x).$$

One can easily verify, since a is invertible, that $\mathcal{M}$ with $\langle .\, , .\rangle$ is a pre-Hilbertian. We denote the completion of this space by $L^2(\mathcal{M}, |a^*|^2)$. Let B be the closure of H_0^∞ in the space $L^2(\mathcal{M}, |a^*|^2)$. We claim that p is the projection of I into B.

First we will show that $p \in H_0^\infty$ (and thus $p \in B$): clearly $p \in H^\infty$; also $\Phi(p) = I - b\Phi(a^{-1})$ and since both a and a^{-1} belong to H^∞, $I = \Phi(aa^{-1}) = \Phi(a).\Phi(a^{-1})$ so $\Phi(a^{-1}) = b^{-1}$ which implies that $\Phi(p) = 0$.

To prove the claim it is enough to check that $I - p = ba^{-1} \perp H_0^\infty$ for $\langle .\, , .\rangle$: for $f \in H_0^\infty$, we have

$$\begin{aligned}\langle f, I - p\rangle &= \tau(|a^*|^2 (ba^{-1})^* f) \\ &= \tau(aa^*((a^{-1})^* b^*) f) \\ &= \tau(ab^* f)\end{aligned}$$

and since $b \in D$, $ab^* \in H^\infty$ so $\tau(ab^* f) = 0$.

To complete the proof of the theorem, note that $\text{dist}(I, B) = ||I - p||_{L^2(|a^*|^2)}$ so $\inf\left\{||I - f||_{L^2(|a^*|^2)};\ f \in B\right\} = ||I - P||_{L^2(|a^*|^2)}$ i.e

$$\inf\left\{\tau(|a^*|^2|I - f|^2);\ f \in H_0^\infty\right\} = \tau(|a^*|^2|ba^{-1}|^2) = \tau(|b|^2).$$

The proof is complete. $\square$

Remark 3. *Using similar argument as above with $p = I - a^{-1}b$ and the Hilbert space defined by $\langle x, y\rangle = \tau(|a|^2 xy^*)$, one can deduce the following identity:*

$$\inf\left\{\tau(|a|^2|I - f^*|^2);\ f \in H_0^\infty\right\} = \tau(|\Phi(a)|^2).$$

We remark also that if $v \in \mathcal{M}$ is such that $v = v^*$ then there exists $a \in H^\infty$ with $a^{-1} \in H^\infty$ satisfying $|a^*|^2 = e^v$; in fact for such v, $e^{v/2}$ is invertible in $\mathcal{M}$. Apply [21] (Proposition 1) to get operators $a \in H^\infty$ and u unitary in $\mathcal{M}$ such that $au = e^{v/2}$. Since u and $e^{v/2}$ are invertible, it is clear that a is invertible. The fact that $|a^*|^2 = e^v$ is immediate. Similarly, an operator $b \in H^\infty$ with $b^{-1} \in H^\infty$ can also be choosen so that $|b|^2 = e^v$.

Proposition 3. *Let $h \in \mathcal{M}$, $h \geq 0$ and $\log(h)$ exists then*

$$\exp(\tau(\log(h))) \geq \inf\left\{\tau(he^{\text{Re}\, f});\ f \in \mathcal{M}, \tau(f) = 0\right\}.$$

The proof of Proposition 3 is based on the following simple extension of the usual Jensen's inequality:

Lemma 5. *Let $h \in \mathcal{M}$, $h \geq 0$ then $\tau(\log(h)) \leq \log(\tau(h))$.*

Proof. Let $\varphi : [0,\infty) \to \mathbb{R}$ defined by $\varphi(x) = \log(x+1)$. The function φ is continuous, increasing and $\varphi(0) = 0$. We get from [9](Corollary 2.8) and the usual Jensen's inequality for functions that

$$\begin{aligned}\tau(\log(h+I)) &= \int_0^1 \mu_t(\log(h)+I)\ dt \\ &= \int_0^1 \log(\mu_t(h)+1)\ dt \\ &\le \log\left(\int_0^1 \mu_t(h)\ dt + 1\right) \\ &= \log\left(\tau(h+I)\right).\end{aligned}$$

So we have $\tau(\log(h+I)) \le \log(\tau(h+I))$. Fix $\varepsilon > 0$. Applying the same inequality for h/ε, we get

$$\begin{aligned}\tau(\log(h/\varepsilon + I) &\le \log\left(\tau(h/\varepsilon)+1\right) \\ \tau(\log(h+\varepsilon I) - (\log\varepsilon)I) &\le \log\left(\tau(h+\varepsilon I)\right) - \log\varepsilon \\ \tau(\log(h+\varepsilon I)) &\le \log\left(\tau(h)+\varepsilon\right).\end{aligned}$$

By letting $\varepsilon \to 0$, the desired inequality follows. The lemma is proved. $\square$

Proof of Proposition 3. Let $g \in \mathcal{M}$ such that $g = g^*$, g commutes with h and $\tau(g) = 0$. Applying Lemma 5 to the operator he^g, we have $\exp\left(\log(h)\right) \le \tau(he^g)$ and therefore,

$$\exp\left(\tau(\log(h))\right) \le \inf\left\{\tau(he^g);\ g = g^*, g \text{ commutes with } h, \tau(g) = 0\right\}.$$

Let $\lambda = \tau(\log(h))$ and $g = \lambda I - \log(h)$. Clearly $g = g^*$, g commutes with h and $\tau(g) = 0$ and it is easy to check that $\tau(he^g) = \exp\left(\tau(\log(h))\right)$ so the inequality above is in fact an equality i.e

$$\exp\left(\tau(\log(h))\right) = \inf\left\{\tau(he^g);\ g = g^*, g \text{ commutes with } h, \tau(g) = 0\right\}.$$

This implies that

$$\exp\left(\tau(\log(h))\right) \ge \inf\left\{\tau(he^g);\ g = g^*, \tau(g) = 0\right\}.$$

The proof is complete. $\square$

The above proposition leads to the following question: If $h \in H^\infty$, is it true that $\exp(\tau(\log|h|)) \ge |\tau(h)|$? This inequality is known as Jensen's inequality for H^∞.

Remark 4. *In* [4]*, characterization of real functions in $L^1(\mathbb{T})$ that have rearrangement in $\operatorname{Re} H_0^1(\mathbb{T})$ were given (see also* [12] *for another proof). The same characterization was shown to be true for the weak*-Dirichlet algebra setting in* [15]*. It would be interesting to know if such characterization holds for the non-commutative case.*

We note that the proofs given in [12] *and* [15] *made use of Szegö's theorem in a very crusial way.*

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