EXCISION IN BANACH SIMPLICIAL AND CYCLIC COHOMOLOGY

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Abstract. We prove that, for every extension of Banach algebras $0 \to B \to A \to D \to 0$ such that B has a left or right bounded approximate identity, the existence of an associated long exact sequence of Banach simplicial or cyclic cohomology groups is equivalent to the existence of one for homology groups. It follows from the continuous version of a result of Wodzicki that associated long exact sequences exist. In particular, they exist for every extension of C^*-algebras.

1. Introduction

Significant results of A. Connes on non-commutative differential geometry [Co1] have led much research interest to the computation of cyclic (co)homology groups in recent years; see, for example, [Lo] and [Co2] for many references and [ChS], [Gr], [He2], [Wo1] and [Wo2] for the continuous theory of these groups. A promising approach to the calculation of cyclic cohomology groups is to break it down by making use of extensions of Banach algebras; this is a standard device in the study of various properties of Banach algebras. Here we establish (Theorem 3.5) that, for every weakly admissible extension of Banach algebras $0 \to B \to A \to D \to 0$, the existence of an associated long exact sequence of Banach simplicial cohomology groups is equivalent to the existence of one for homology groups. Theorem 3.8 shows that the same result is true for cyclic (co)homology groups when B has a left or right bounded approximate identity. Recall that the existence of long exact sequences of simplicial and cyclic *homology* groups associated with an extension of algebras (as distinct from cohomology and, mainly, in an algebraic context) has been studied by M. Wodzicki [Wo2], [Wo3]; see also [Lo]. Wodzicki remarked that his result could easily be extended to the continuous case under some hypotheses on the extension (see Remark (3) and Corollary 4 [Wo2] and Remark 8.5(2) [Wo3]).

From Theorems 3.5, 3.8 and the continuous version of Wodzicki's result (Theorem 4.1) for homology groups we deduce the existence of long exact sequences of simplicial and cyclic cohomology groups for every extension when B has a left or right bounded

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approximate identity. In particular, this is true for every extension of C^*-algebras. This gives an effective tool for computing Banach simplicial and cyclic cohomology groups (see Proposition 4.4). We apply it to some natural classes of Banach algebras.

Finally, I am grateful to M.Wodzicki for suggestions about the continuous homology of Banach algebras.

2. Definitions and notation

We recall some notation and terminology used in the homological theory of Banach algebras.

Throughout the paper id denotes the identity operator. We denote the projective tensor product of Banach spaces by $\hat{\otimes}$ (see, for example, [Pa]). Note that by $Z^{\hat{\otimes}0}\hat{\otimes}Y$ we mean Y and by $Z^{\hat{\otimes}1}$ we mean Z.

Let A be a Banach algebra, not necessarily unital. We will define the Banach version of the cyclic homology $\mathcal{HC}_n(A)$ (see, for example, [Lo], [Wo2] or [He2]). We denote by $C_n(A),\ n = 0,1,...,$ the $(n+1)$ fold projective tensor power $A^{\hat{\otimes}(n+1)} = A\hat{\otimes}\dots\hat{\otimes}A$ of A; we shall call the elements of this Banach space *n-dimensional chains.* We let $t_{\underline{n}} : C_n(A) \to C_n(A),\ n = 0,1,...,$ denote the operator given by $t_{\underline{n}}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n(a_n \otimes a_0 \otimes \cdots \otimes a_{n-1})$, and we set $t_{\underline{0}} = id$. We let $CC_n(A)$ denote the quotient space of $C_n(A)$ modulo the closure of the linear span of elements of the form $x - t_{\underline{n}}x$ where $n = 0,1,\dots$. Note that, by Proposition 4 [He2], Im $(id_{C_n(A)} - t_{\underline{n}})$ is closed in $C_n(A)$ and so $CC_n(A) = C_n(A)/\mathrm{Im}\ (id_{C_n(A)} - t_{\underline{n}})$. We also set $CC_0(A) = C_0(A) = A$.

From the chains we form the standard homology complex

$$0 \leftarrow C_0(A) \stackrel{d_0}{\leftarrow} \cdots \leftarrow C_n(A) \stackrel{d_n}{\leftarrow} C_{n+1}(A) \leftarrow \dots, \qquad (\mathcal{C}_\sim(A))$$

where the differential d_n is given by the formula

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) =$$

$$\sum_{i=0}^{n}(-1)^i(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1}(a_{n+1}a_0 \otimes \cdots \otimes a_n).$$

It is not difficult to verify that these d_n induce operators $dc_n : CC_{n+1}(A) \to CC_n(A)$ in the respective quotient spaces. Thus we obtain a quotient complex $\mathcal{CC}_\sim(A)$ of the complex $\mathcal{C}_\sim(A)$. The n-dimensional homology of $\mathcal{C}_\sim(A)$, denoted by $\mathcal{H}_n(A)$, is called the *n-dimensional Banach simplicial homology group* of the Banach algebra A. The n-dimensional homology of $\mathcal{CC}_\sim(A)$, denoted by $\mathcal{HC}_n(A)$, is called the *n-dimensional Banach cyclic homology group* of A.

We also consider the complex

$$0 \overset{dr_{-1}}{\leftarrow} C_0(A) \overset{dr_0}{\leftarrow} \cdots \leftarrow C_n(A) \overset{dr_n}{\leftarrow} C_{n+1}(A) \leftarrow \dots, \qquad (\mathcal{CR}_\sim(A))$$

where the differential dr_n is given by the formula

$$dr_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})$$

and we set $dr_{-1}(a_0) = 0$. The n-dimensional homology of $\mathcal{CR}_\sim(A)$, denoted by $\mathcal{HR}_n(A)$, is called the *n-dimensional Banach Bar homology group* of A.

Note that, by definition, $dc_0 = d_0$, so that $\mathcal{HC}_0(A) = \mathcal{H}_0(A) = A/\mathrm{Im}\ d_0$.

Obviously $\mathcal{H}_n(A)$ is just another way of writing the Hochschild homology group $\mathcal{H}_n(A, A)$ (see [He1; II.5.5]).

For a Banach A-bimodule X, we will denote the n-dimensional Banach cohomology group of A with coefficients in X by $\mathcal{H}^n(A, X)$ (see, for example, [Jo1] or [He1]). Recall that a Banach A-bimodule $M = (M_*)^*$, where M_* is a Banach A-bimodule, is called *dual*. Here E^* is the dual Banach space of a Banach space E. A Banach algebra A such that $\mathcal{H}^1(A, M) = \{0\}$ for all dual A-bimodules M is called *amenable*.

The nth cohomology of the dual complex $\mathcal{C}^\sim(A) \overset{def}{=} \mathcal{C}_\sim(A)^*$, denoted by $\mathcal{H}^n(A)$, is called the *n-dimensional Banach simplicial cohomology group* of the Banach algebra A. The nth cohomology of the dual complex $\mathcal{CC}^\sim(A) \overset{def}{=} \mathcal{CC}_\sim(A)^*$, denoted by $\mathcal{HC}^n(A)$, is called the *n-dimensional Banach cyclic cohomology group* of A (see [Co1] or [He2]). The nth cohomology of the dual complex $\mathcal{CR}^\sim(A) \overset{def}{=} \mathcal{CR}_\sim(A)^*$, denoted by $\mathcal{HR}^n(A)$, is called the *n-dimensional Banach Bar cohomology group* of the Banach algebra A.

Note that, by definition, $\mathcal{HC}^0(A) = \mathcal{H}^0(A)$ coincides with the space $A^{tr} = \{f \in A^* : f(ab) = f(ba)$ for all $a, b \in A\}$ of continuous traces on A.

The canonical identification of $(n+1)-$linear functionals on A and $n-$linear operators from A to A^* shows that $\mathcal{H}^n(A)$ is just another way of writing $\mathcal{H}^n(A, A^*)$.

The vanishing of $\mathcal{HR}^n(A)$ for a Banach algebra A implies the existence of the Connes-Tsygan exact sequence for A. One can find the conditions for this in Theorems 15 and 16 [He2].

The short exact sequence of Banach spaces and continuous linear operators

$$0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0$$

is called *admissible* if there exists a continuous operator $\alpha : W \to Z$ such that $j \circ \alpha = id_W$. Recall that admissibility is equivalent to the existence of a continuous operator
$\beta : Z \to Y$ such that $\beta \circ i = id_Y$. This is also equivalent to the existence of continuous operators $\beta : Z \to Y$ and $\alpha : W \to Z$ such that $\beta \circ i = id_Y$, $j \circ \alpha = id_W$ and $i \circ \beta + \alpha \circ j = id_Z$.

The short exact sequence of Banach spaces and continuous linear operators $0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0$ is called *weakly admissible* if the dual short sequence $0 \to W^* \xrightarrow{j^*} Z^* \xrightarrow{i^*} Y^* \to 0$ is admissible.

Definitions of a (co)chain complex, a morphism of complexes, the homology groups can be found in any text book on homological algebra, for instance MacLane [Ma], Helemskii [He1] (continuous case), Loday [Lo].

A *chain complex* $\mathcal{K}_\sim$ in the category of [Banach] linear spaces is a sequence of [Banach] linear spaces and [continuous] linear operators

$$\cdots \leftarrow K_n \xleftarrow{d_n} K_{n+1} \xleftarrow{d_{n+1}} K_{n+2} \leftarrow \ldots$$

such that $d_n \circ d_{n+1} = 0$ for every n. The *cycles* are the elements of $Z_n = \mathrm{Ker}\ (d_{n-1} : K_n \to K_{n-1})$. The *boundaries* are the elements of $B_n = \mathrm{Im}\ (d_n : K_{n+1} \to K_n)$. The relation $d_{n-1} \circ d_n = 0$ implies $B_n \subset Z_n$. The *homology groups* are defined by $H_n(\mathcal{K}_\sim) = Z_n/B_n$.

A *[continuous] morphism of chain complexes* $\psi_\sim : \mathcal{K}_\sim \to \mathcal{P}_\sim$ in the category of [Banach] linear spaces is a collection of [continuous] linear operators $\psi_n : K_n \to P_n$ such that the following diagram is commutative for any n

$$\begin{array}{ccc} K_n & \xrightarrow{d_{n-1}} & K_{n-1} \\ \downarrow \psi_n & & \downarrow \psi_{n-1} \\ P_n & \xrightarrow{d'_{n-1}} & P_{n-1}. \end{array}$$

Such a morphism obviously induces a map $H_n(\psi_\sim) : H_n(\mathcal{K}_\sim) \to H_n(\mathcal{P}_\sim)$.

A [continuous] morphism of chain complexes $\psi_\sim : \mathcal{K}_\sim \to \mathcal{P}_\sim$ of [Banach] linear spaces is *a [topological] quasi-isomorphism* if $H_n(\psi_\sim) : H_n(\mathcal{K}_\sim) \to H_n(\mathcal{P}_\sim)$ is a [topological] isomorphism for every n.

3. Connections between long exact sequences of homology and cohomology groups

To start with we need the following elementary result from the homological algebra.

3.1 Lemma. *Let*

$$\cdots \to \mathcal{L}_{n-2} \xrightarrow{\zeta_{n-1}} \mathcal{L}_{n-1} \xrightarrow{\zeta_n} \mathcal{L}_n \xrightarrow{\zeta_{n+1}} \mathcal{L}_{n+1} \xrightarrow{\zeta_{n+2}} \mathcal{L}_{n+2} \ldots,$$

be an exact sequence of linear spaces and linear operators. Suppose that ζ_{n-1} and ζ_{n+2} are isomorphisms. Then $\mathcal{L}_n = \{0\}$.

Proof. By the assumption that ζ_{n-1} is an isomorphism, so $\mathrm{Im}\ \zeta_{n-1} = \mathcal{L}_{n-1} = \mathrm{Ker}\ \zeta_n$. Thus $\mathrm{Im}\ \zeta_n = \{0\}$ and, by virtue of the exactness of the sequence, $\mathrm{Ker}\ \zeta_{n+1} = \mathrm{Im}\ \zeta_n = \{0\}$. By the assumption that ζ_{n+2} is an isomorphism, $\mathrm{Ker}\ \zeta_{n+2} = \{0\} = \mathrm{Im}\ \zeta_{n+1}$. Therefore, since $\mathrm{Ker}\ \zeta_{n+1} = \{0\}$ and $\mathrm{Im}\ \zeta_{n+1} = \{0\}$, the result follows. $\square$

3.2 PROPOSITION. *Let $\mathcal{K}_\sim$ and $\mathcal{P}_\sim$ be chain complexes of Banach spaces and continuous linear operators. Suppose there is a continuous morphism of chain complexes $\psi_\sim : \mathcal{K}_\sim \to \mathcal{P}_\sim$ such that, for each n, ψ_n is injective and $\mathrm{Im}\psi_n$ is closed. Then*

(I) *for fixed n, if $H^k(\psi^*_\sim)$ is an isomorphism whenever $n+2 \geq k \geq n-1$, then $H_n(\psi_\sim)$ is a topological isomorphism.*

(II) *for fixed n, if $H_k(\psi_\sim)$ is an isomorphism whenever $n+1 \geq k \geq n-2$, then $H^n(\psi^*_\sim)$ is a topological isomorphism.*

(III) *the following conditions are equivalent:*

(i) *$\psi_\sim$ is a topological quasi-isomorphism of the chain complexes $\mathcal{K}_\sim$ and $\mathcal{P}_\sim$;*

(ii) *$\psi^*_\sim$ is a topological quasi-isomorphism of the cochain, dual complexes $\mathcal{K}^*_\sim$ and $\mathcal{P}^*_\sim$.*

Proof. By the assumption, there is a short exact sequence of complexes

$$0 \to \mathcal{K}_\sim \overset{\psi_\sim}{\to} \mathcal{P}_\sim \overset{\phi_\sim}{\to} \mathcal{L}_\sim \to 0$$

in the category of Banach spaces and continuous linear operators, where $\mathcal{L}_\sim$ is the chain complex $\mathcal{P}_\sim/\mathrm{Im}\psi_\sim$ and $\phi_n : P_n \to P_n/\mathrm{Im}\psi_n$ is the quotient mapping. Hence, by [He1, Theorem 0.5.7], there exists a long exact sequence

$$\cdots \to H_{n+1}(\mathcal{L}_\sim) \overset{\zeta_{n+1}}{\to} H_n(\mathcal{K}_\sim) \overset{H_n(\psi_\sim)}{\to} H_n(\mathcal{P}_\sim) \overset{H_n(\phi_\sim)}{\to} H_n(\mathcal{L}_\sim) \overset{\zeta_n}{\to} H_{n-1}(\mathcal{K}_\sim) \to \dots \tag{3.1}$$

of homology groups, where ζ_n (the connecting morphism), $H_n(\phi_\sim)$ and $H_n(\psi_\sim)$ are continuous linear operators.

We can also consider a short sequence of dual cochain complexes

$$0 \to \mathcal{L}^*_\sim \overset{\phi^*_\sim}{\to} \mathcal{P}^*_\sim \overset{\psi^*_\sim}{\to} \mathcal{K}^*_\sim \to 0$$

in the category of Banach spaces and continuous linear operators. This sequence is exact by [Jo1; Section 1] (see also, [He1; 0.5.2]). Hence, by [He1, Theorem 0.5.7], there exists a long exact sequence

$$\cdots \to H^{n-1}(\mathcal{K}^*_\sim) \overset{\xi_n}{\to} H^n(\mathcal{L}^*_\sim) \overset{H^n(\phi^*_\sim)}{\to} H^n(\mathcal{P}^*_\sim) \overset{H^n(\psi^*_\sim)}{\to} H^n(\mathcal{K}^*_\sim) \overset{\xi_{n+1}}{\to} H^{n+1}(\mathcal{L}^*_\sim) \dots \tag{3.2}$$

of cohomology groups, where ξ_n (the connecting morphism), $H^n(\phi^*_\sim)$ and $H^n(\psi^*_\sim)$ are continuous linear operators.

(I) Since $H^k(\psi^*_\sim)$ is an isomorphism whenever $n+2 \geq k \geq n-1$, by Lemma 3.1, $H^k(\mathcal{L}^*_\sim) = \{0\}$ whenever $n+2 \geq k \geq n$. Thus, by [Jo1; Section 1], $H_k(\mathcal{L}_\sim) = \{0\}$ for $k = n$ and $k = n+1$, and so $H_n(\psi_\sim)$ is an isomorphism. Hence, by Lemma 7.1.32 [BS], $H_n(\psi_\sim)$ is a topological isomorphism.

(II) Since $H_k(\psi_\sim)$ is an isomorphism whenever $n+1 \geq k \geq n-2$, by Lemma 3.1, $H_k(\mathcal{L}_\sim) = \{0\}$ whenever $n+1 \geq k \geq n-1$. Thus, by [Jo1; Section 1], $H^k(\mathcal{L}^*_\sim) = \{0\}$

for $k = n$ and $k = n+1$, and so $H^n(\psi^*_\sim)$ is an isomorphism. Hence, by Lemma 7.1.32 [BS], $H^n(\psi^*_\sim)$ is a topological isomorphism.

(III) Recall that a complex $\mathcal{L}_\sim$ is exact if and only if its dual complex $\mathcal{L}^*_\sim$ is exact [Jo1; Section 1] (see also [He1; 0.5.2]). Thus the homology groups $H_n(\mathcal{L}_\sim)$ vanish if and only if the cohomology groups $H^n(\mathcal{L}^*_\sim)$ vanish.

Since the sequences (3.1) and (3.2) are exact, the result follows from Lemma 3.1 and Theorem 0.5.10 [He1]. $\square$

The following lemma is widely known.

3.3 Lemma. *Let*

$$0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0 \tag{3.3}$$

be a weakly admissible sequence of Banach spaces and continuous operators. Then, for every Banach space X, the sequence

$$0 \to X \hat{\otimes} Y \stackrel{id_X \otimes i}{\to} X \hat{\otimes} Z \stackrel{id_X \otimes j}{\to} X \hat{\otimes} W \to 0$$

is weakly admissible.

Proof. Since the sequence (3.3) is weakly admissible, there are continuous linear operators $\alpha : Y^* \to Z^*$ and $\beta : Z^* \to W^*$ such that $i^* \circ \alpha = id_{Y^*}$, $\beta \circ j^* = id_{W^*}$ and $\alpha \circ i^* + j^* \circ \beta = id_{Z^*}$.

Further we will use the well-known isomorphism (see, for example, [He1; Theorem 2.2.17])

$$\mathcal{B}(X, Y^*) \to (X \hat{\otimes} Y)^* : \phi \mapsto \Phi_\phi$$

where $\Phi_\phi(x \otimes y) = [\phi(x)](y); x \in X, y \in Y$ and

$$(X \hat{\otimes} Y)^* \to \mathcal{B}(X, Y^*) : f \mapsto \phi_f$$

where $[\phi_f(x)](y) = f(x \otimes y); x \in X, y \in Y$.

We can see that, for $f \in (X \hat{\otimes} Y)^*$, we have $\phi_f \in \mathcal{B}(X, Y^*)$, $\alpha \circ \phi_f \in \mathcal{B}(X, Z^*)$ and $\Phi_{\alpha \circ \phi_f} \in (X \hat{\otimes} Z)^*$. We define $\gamma(f) = \Phi_{\alpha \circ \phi_f}$ for $f \in (X \hat{\otimes} Y)^*$ and $\eta(g) = \Phi_{\beta \circ \phi_g}$ for $g \in (X \hat{\otimes} Z)^*$.

We can check that $(id_X \otimes i)^* \circ \gamma = id_{(X \hat{\otimes} Y)^*}$, $\eta \circ (id_X \otimes j)^* = id_{(X \hat{\otimes} W)^*}$ and $\gamma \circ (id_X \otimes i)^* + (id_X \otimes j)^* \circ \eta = id_{(X \hat{\otimes} Z)^*}$. The result follows. $\square$

3.4 Lemma. *Let*

$$0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0$$

be a weakly admissible sequence of Banach spaces and continuous operators. Then, for every n, the following conditions are satisfied

(i) *the operator*

$$i^{\hat{\otimes} n} = i \hat{\otimes} \dots \hat{\otimes} i : Y \hat{\otimes} \dots \hat{\otimes} Y \to Z \hat{\otimes} \dots \hat{\otimes} Z$$

is injective and $\mathrm{Im}(i^{\hat{\otimes} n})$ *is closed;*

(ii) $\mathrm{Ker}\, j^{\hat{\otimes} n} = Z^{\hat{\otimes}(n-1)} \hat{\otimes} i(Y) + Z^{\hat{\otimes}(n-2)} \hat{\otimes} i(Y) \hat{\otimes} Z + \dots + i(Y) \hat{\otimes} Z^{\hat{\otimes}(n-1)}$ *and so it is the closure of linear span* $\{z_1 \otimes z_2 \otimes \dots \otimes z_n$*:* at least one z_i belongs *to* $i(Y)\}$.

(iii) *the complex*

$$0 \to Y^{\hat{\otimes} n} \xrightarrow{i^{\hat{\otimes} n}} Z^{\hat{\otimes} n} \xrightarrow{j^{\hat{\otimes} n}} (W)^{\hat{\otimes} n} \to 0$$

is exact at $Y^{\hat{\otimes} n}$ *and* $(W)^{\hat{\otimes} n}$*;*

Proof. (i) We can see that

$$i^{\hat{\otimes} n} = \kappa_n \circ (id_{Z^{\hat{\otimes}(n-1)}} \otimes i) \circ \kappa_{n-1} \circ \cdots \circ (id_{Z \hat{\otimes} Y^{\hat{\otimes}(n-2)}} \otimes i) \circ \kappa_1 \circ (id_{Y^{\hat{\otimes}(n-1)}} \otimes i),$$

where $\kappa_i : Z^{\hat{\otimes}(i-1)} \hat{\otimes} Y^{\hat{\otimes}(n-i)} \hat{\otimes} Z \cong Z^{\hat{\otimes} i} \hat{\otimes} Y^{\hat{\otimes}(n-i)}$, $i = 1, \dots, n$ are given by $\kappa_i(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_n \otimes x_1 \otimes \cdots \otimes x_{n-1}$.

Thus

$$(i^{\hat{\otimes} n})^* = (id_{Y^{\hat{\otimes}(n-1)}} \otimes i)^* \circ (\kappa_1)^* \circ (id_{Z \otimes Y^{\hat{\otimes}(n-2)}} \otimes i)^* \circ \cdots \circ (\kappa_{n-1})^* \circ (id_{Z^{\hat{\otimes}(n-1)}} \otimes i)^* \circ (\kappa_n)^*.$$

By Lemma 3.3, $(i^{\hat{\otimes} n})^*$ is surjective as the composition of surjective operators. Therefore, by [Ed; 8.6.15], $i^{\hat{\otimes} n}$ is injective and $\mathrm{Im}(i^{\hat{\otimes} n})$ is closed in $Z^{\hat{\otimes} n}$.

(ii) We can see that

$$j^{\hat{\otimes} n} = \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n$$

where $\psi_1 = j \otimes id_{W^{\hat{\otimes}(n-1)}}$, $\psi_i = id_{Z^{\hat{\otimes}(i-1)}} \otimes j \otimes id_{W^{\hat{\otimes}(n-i)}}$ for $1 < i < n$ and $\psi_n = id_{Z^{\hat{\otimes}(n-1)}} \otimes j$. By Lemma 3.3, we have $\mathrm{Ker}\ \psi_i = \mathrm{Im}\ id_{Z^{\hat{\otimes}(i-1)}} \otimes i \otimes id_{W^{\hat{\otimes}(n-i)}} = Z^{\hat{\otimes}(i-1)} \hat{\otimes} i(Y) \hat{\otimes} W^{\hat{\otimes}(n-i)}$ and, for $F_i = Z^{\hat{\otimes}(i-1)} \hat{\otimes} i(Y) \hat{\otimes} Z^{\hat{\otimes}(n-i)}$, we obtain

$$(\psi_{i+1} \circ \cdots \circ \psi_{n-1} \circ \psi_n)(F_i) = \mathrm{Ker}\ \psi_i.$$

Thus we have

$$\mathrm{Ker}\ \psi_n = Z^{\hat{\otimes}(n-1)} \hat{\otimes} i(Y);$$
$$\mathrm{Ker}\ (\psi_{n-1} \circ \psi_n) = \psi_n^{-1}(\mathrm{Ker}\ \psi_{n-1}) =$$
$$\mathrm{Ker}\ \psi_n + F_{n-1} = Z^{\hat{\otimes}(n-1)} \hat{\otimes} i(Y) + Z^{\hat{\otimes}(n-2)} \hat{\otimes} i(Y) \hat{\otimes} Z;$$
$$\mathrm{Ker}\ (\psi_{n-2} \circ \psi_{n-1} \circ \psi_n) = (\psi_{n-1} \circ \psi_n)^{-1}(\mathrm{Ker}\ \psi_{n-2}) =$$
$$\mathrm{Ker}\ (\psi_{n-1} \circ \psi_n) + F_{n-2} =$$
$$Z^{\hat{\otimes}(n-1)} \hat{\otimes} i(Y) + Z^{\hat{\otimes}(n-2)} \hat{\otimes} i(Y) \hat{\otimes} Z + Z^{\hat{\otimes}(n-3)} \hat{\otimes} i(Y) \hat{\otimes} Z^{\hat{\otimes} 2};$$
$$\dots\dots$$
$$\mathrm{Ker}\ (\psi_1 \circ \dots \psi_{n-1} \circ \psi_n) = (\psi_2 \circ \dots \psi_{n-1} \circ \psi_n)^{-1}(\mathrm{Ker}\ (\psi_1) =$$
$$\mathrm{Ker}\ (\psi_2 \circ \dots \psi_{n-1} \circ \psi_n) + F_1 =$$
$$Z^{\hat{\otimes}(n-1)} \hat{\otimes} i(Y) + Z^{\hat{\otimes}(n-2)} \hat{\otimes} i(Y) \hat{\otimes} Z + \cdots + i(Y) \hat{\otimes} Z^{\hat{\otimes}(n-1)}.$$

(iii) It follows from (i) and the fact that the projective tensor product of surjective operators is also surjective. □

3.5 Theorem. *Let*

$$0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0 \tag{3.4}$$

be a weakly admissible extension of Banach algebras. Then

(I) *there exists an associated long exact sequence of Banach simplicial homology groups*

$$\cdots \to \mathcal{H}_n(B) \to \mathcal{H}_n(A) \to \mathcal{H}_n(D) \to \mathcal{H}_{n-1}(B) \to \cdots \to \mathcal{H}_0(D) \to 0 \tag{3.5}$$

if and only if there exists an associated long exact sequence of Banach simplicial cohomology groups

$$0 \to \mathcal{H}^0(D) \to \cdots \to \mathcal{H}^{n-1}(B) \to \mathcal{H}^n(D) \to \mathcal{H}^n(A) \to \mathcal{H}^n(B) \to \ldots; \tag{3.6}$$

(II) *there exists an associated long exact sequence of Banach Bar homology groups*

$$\cdots \to \mathcal{HR}_n(B) \to \mathcal{HR}_n(A) \to \mathcal{HR}_n(D) \to \mathcal{HR}_{n-1}(B) \to \cdots \to \mathcal{HR}_0(D) \to 0 \tag{3.7}$$

if and only if there exists an associated long exact sequence of Banach Bar cohomology groups

$$0 \to \mathcal{HR}^0(D) \to \cdots \to \mathcal{HR}^{n-1}(B) \to \mathcal{HR}^n(D) \to \mathcal{HR}^n(A) \to \mathcal{HR}^n(B) \to \ldots. \tag{3.8}$$

Proof. (I) By Lemma 3.4 (iii), for $n = 0, 1, \ldots,$ the extension (3.4) induces a complex of Banach spaces and continuous operators

$$0 \to C_n(B) \stackrel{i^{\hat{\otimes}(n+1)}}{\to} C_n(A) \stackrel{j^{\hat{\otimes}(n+1)}}{\to} C_n(D) \to 0$$

which is exact at $C_n(D)$ and $C_n(B)$. It is routine to check that the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & C_0(B) & \stackrel{i}{\to} & C_0(A) & \stackrel{j}{\to} & C_0(D) & \to & 0 \\
 & & \uparrow d_0 & & \uparrow d_0 & & \uparrow d_0 & & \\
0 & \to & C_1(B) & \stackrel{i\hat{\otimes}i}{\to} & C_1(A) & \stackrel{j\hat{\otimes}j}{\to} & C_1(D) & \to & 0 \\
 & & \uparrow d_1 & & \uparrow d_1 & & \uparrow d_1 & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & C_n(B) & \stackrel{i\hat{\otimes}\ldots\hat{\otimes}i}{\to} & C_n(A) & \stackrel{j\hat{\otimes}\ldots\hat{\otimes}j}{\to} & C_n(D) & \to & 0 \\
 & & \uparrow d_n & & \uparrow d_n & & \uparrow d_n & & \\
0 & \to & C_{n+1}(B) & \stackrel{i\hat{\otimes}\ldots\hat{\otimes}i}{\to} & C_{n+1}(A) & \stackrel{j\hat{\otimes}\ldots\hat{\otimes}j}{\to} & C_{n+1}(D) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

is commutative. The same is true for cochains. Thus the extension (3.4) induces a short exact sequence of chain complexes

$$0 \to \mathcal{C}_{\sim}(A, D) \to \mathcal{C}_{\sim}(A) \to \mathcal{C}_{\sim}(D) \to 0$$

and a short exact sequence of cochain complexes

$$0 \to \mathcal{C}^{\sim}(D) \to \mathcal{C}^{\sim}(A) \to \mathcal{C}^{\sim}(A; D) \to 0$$

in the category of Banach spaces and continuous operators, where $\mathcal{C}_{\sim}(A, D)$ is the subcomplex $\mathrm{Ker}(j\hat{\otimes} \dots \hat{\otimes} j)$ of $\mathcal{C}_{\sim}(A)$; $\mathcal{C}^{\sim}(A; D)$ is the subcomplex $[\mathrm{Ker}(j\hat{\otimes} \dots \hat{\otimes} j)]^*$ of $\mathcal{C}^{\sim}(A)$. Hence, by [He1, Chapter 0, Section 5.5], there exist a long exact sequence of homology groups

$$\dots \to \mathcal{H}_{n+1}(D) \to H_n(\mathcal{C}_{\sim}(A; D)) \to \mathcal{H}_n(A) \to \mathcal{H}_n(D) \to \dots$$

and a long exact sequence of cohomology groups

$$\dots \to H^{n-1}(\mathcal{C}^{\sim}(A; D)) \to \mathcal{H}^n(D) \to \mathcal{H}^n(A) \to H^n(\mathcal{C}^{\sim}(A; D)) \to \dots .$$

By Lemma 3.4 (i),

$$i^{\hat{\otimes}(n+1)} : B^{\hat{\otimes}(n+1)} \to \mathrm{Ker}\; j^{\hat{\otimes}(n+1)}$$

is injective and $\mathrm{Im}(i^{\hat{\otimes}(n+1)})$ is closed. Therefore, by Proposition 3.2, for the two complexes $\mathcal{C}_{\sim}(B)$ and $\mathcal{C}_{\sim}(A; D)$ and the continuous morphism $\{i^{\hat{\otimes}(n+1)}\}$ of complexes, we have $H_n(\mathcal{C}_{\sim}(A; D)) \cong \mathcal{H}_n(B)$ for $n = 0, 1, \dots$ if and only if $H^n(\mathcal{C}^{\sim}(A; D)) \cong \mathcal{H}^n(B)$ for $n = 0, 1, \dots$. The result follows from the five lemma [Ma, Lemma 1.3.3].

(II) follows by the same arguments as in the case (I). □

The following lemma is widely known.

3.6 Lemma. *Let*

$$0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0 \tag{3.9}$$

be an extension of Banach algebras. Suppose that B has a left or right bounded approximate identity $(e_\alpha)_{\alpha\in\Lambda}$. Then (3.9) is a weakly admissible.

Proof. Consider the Fréchet filter F on Λ, with base $\{Q_\lambda : \lambda \in \Lambda\}$, where $Q_\lambda = \{\alpha \in \Lambda : \alpha \geq \lambda\}$. Thus

$$F = \{E \subset \Lambda : \text{ there is a } \lambda \in \Lambda \text{ such that } Q_\lambda \subset E\}.$$

Let U be an ultrafilter on Λ which refines F. One can find information on filters in [Bo].

Suppose that $(e_\alpha)_{\alpha\in\Lambda}$ is a right bounded approximate identity. For $f \in B^*$ we define $g_f \in A^*$ by

$$g_f(a) = \lim_{\alpha\to U} f(i^{-1}(ai(e_\alpha))) \text{ for all } a \in A.$$

It is easy to check that g_f is a bounded linear functional, the operator

$$L : B^* \to A^* : f \mapsto g_f$$

is a bounded linear operator and $i^* \circ L = id_{B^*}$. □

Let

$$0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0$$

be an extension of Banach algebras. Note that $(id_{A^{\hat{\otimes}(n+1)}} - t_{\underline{n}}) \circ i^{\hat{\otimes}(n+1)} = i^{\hat{\otimes}(n+1)} \circ (id_{B^{\hat{\otimes}(n+1)}} - t_{\underline{n}})$ and $(id_{D^{\hat{\otimes}(n+1)}} - t_{\underline{n}}) \circ j^{\hat{\otimes}(n+1)} = j^{\hat{\otimes}(n+1)} \circ (id_{A^{\hat{\otimes}(n+1)}} - t_{\underline{n}})$. Thus there is a complex

$$0 \to CC_n(B) \overset{\widetilde{i^{\hat{\otimes}(n+1)}}}{\to} CC_n(A) \overset{\widetilde{j^{\hat{\otimes}(n+1)}}}{\to} CC_n(D) \to 0,$$

where $\widetilde{i^{\hat{\otimes}(n+1)}}$ and $\widetilde{j^{\hat{\otimes}(n+1)}}$ are induced by $i^{\hat{\otimes}(n+1)}$ and $j^{\hat{\otimes}(n+1)}$ respectively.

3.7 Lemma. *Let*

$$0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0$$

be an extension of Banach algebras. Suppose B has a left or right bounded approximate identity $(e_\alpha)_{\alpha\in\Lambda}$. Then, for every n, the complex

$$0 \to CC_n(B) \overset{\widetilde{i^{\hat{\otimes}(n+1)}}}{\to} CC_n(A) \overset{\widetilde{j^{\hat{\otimes}(n+1)}}}{\to} CC_n(D) \to 0,$$

is exact at $CC_n(B)$ and $CC_n(D)$; $\widetilde{i^{\hat{\otimes}(n+1)}}$ is injective and $\mathrm{Im}(\widetilde{i^{\hat{\otimes}(n+1)}})$ is closed; and $\mathrm{Ker}\ \widetilde{j^{\hat{\otimes}(n+1)}} = \{u + \mathrm{Im}\ (id_{A^{\hat{\otimes}(n+1)}} - t_{\underline{n}})$; *where* $u \in \mathrm{Ker}\ j^{\hat{\otimes}(n+1)}\}$ *and so it is the closure of the linear span of* $\{a_0 \otimes a_1 \otimes \cdots \otimes a_n + \mathrm{Im}\ (id_{A^{\hat{\otimes}(n+1)}} - t_{\underline{n}})$: at least one a_i belongs to $i(B)\}$.

Proof. Since $j^{\hat{\otimes}(n+1)}$ is surjective and $(id_{D^{\hat{\otimes}(n+1)}} - t_{\underline{n}}) \circ j^{\hat{\otimes}(n+1)} = j^{\hat{\otimes}(n+1)} \circ (id_{A^{\hat{\otimes}(n+1)}} - t_{\underline{n}})$, the mapping $\widetilde{j^{\hat{\otimes}(n+1)}} : CC_n(A) \to CC_n(D)$ is surjective too.

Suppose that $(e_\alpha)_{\alpha\in\Lambda}$ is a right bounded approximate identity. Let us check that $(\widetilde{i^{\hat{\otimes}(n+1)}})^*$ is surjective. For

$$f \in CC_n(B)^* = \{f \in (B^{\hat{\otimes}(n+1)})^* : f|_{\mathrm{Im}\ (id_{B^{\hat{\otimes}(n+1)}} - t_{\underline{n}})} = 0\},$$

we define $g_f \in (A^{\hat{\otimes}(n+1)})^*$ by

$$g_f(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \lim_{\alpha \to U} f(i^{-1}(a_0 i(e_\alpha)) \otimes \cdots \otimes i^{-1}(a_n i(e_\alpha))),$$

where $a_i \in A$; $i = 0, \ldots, n$ and the ultrafilter U is defined as in Lemma 3.6. It is easy to check that g_f is a bounded linear functional, $g_f|_{\mathrm{Im}\ (id_{A^{\hat{\otimes}(n+1)}} - t_{\underline{n}})} = 0$ and $f = (i^{\hat{\otimes}(n+1)})^* g_f$. Thus, by [Ed; 8.6.15], $\widetilde{i^{\hat{\otimes}(n+1)}} : CC_n(B) \to CC_n(A)$ is injective and its image is closed.

Now let us show that there exists a continuous operator $\alpha_n : C_n(A)^* \to C_n(D)^*$ such that $\alpha_n \circ (j^{\hat{\otimes}(n+1)})^* = id_{C_n(D)^*}$ and $\alpha_n(CC_n(A)^*) \subset CC_n(D)^*$. For $g \in C_n(A)^*$, we define $f_g \in C_n(D)^*$, by

$$f_g(d_0 \otimes d_1 \otimes \cdots \otimes d_n) =$$

$$g(a_0 \otimes a_1 \otimes \cdots \otimes a_n) - \lim_{\alpha \to U} \sum_{k=0}^{n} g(a_0 \otimes \cdots \otimes a_{k-1} \otimes a_k i(e_\alpha) \otimes a_{k+1} \otimes \cdots \otimes a_n) +$$

$$\lim_{\alpha \to U} \sum_{0 \le k < t \le n} g(a_0 \otimes \cdots \otimes a_{k-1} \otimes a_k i(e_\alpha) \otimes a_{k+1} \otimes \cdots \otimes a_t i(e_\alpha) \otimes a_{t+1} \otimes \cdots \otimes a_n) - \cdots +$$

$$(-1)^{n+1} \lim_{\alpha \to U} g(a_0 i(e_\alpha) \otimes a_1 i(e_\alpha) \otimes \cdots \otimes a_n i(e_\alpha)),$$

where $d_i = j(a_i) \in D;\ i = 0, \ldots, n$. It is easy to check that the operator

$$\alpha_n : C_n(A)^* \to C_n(D)^* : g \mapsto f_g$$

is a bounded linear operator, $\alpha_n \circ (j^{\hat{\otimes}(n+1)})^* = id_{C_n(D)^*}$ and $\alpha_n(CC_n(A)^*) \subset CC_n(D)^*$.

Hence we can see that the dual map of

$$\sigma_n : \text{Ker } j^{\hat{\otimes}(n+1)} \to \text{Ker } \widetilde{j^{\hat{\otimes}(n+1)}} : u \mapsto u + \text{Im } (id_{A^{\hat{\otimes}(n+1)}} - t_{\underline{n}})$$

is injective and Im σ_n is closed. Thus, by [Ed; 8.6.15], σ_n is surjective, that is, $(\text{Ker } \widetilde{j^{\hat{\otimes}(n+1)}})^* = \text{Im } \sigma_n$. It follows from Lemma 3.4(ii) that the Ker $\widetilde{j^{\hat{\otimes}(n+1)}}$ is the closure of the linear span of $\{a_0 \otimes a_1 \otimes \cdots \otimes a_n + \text{Im } (id_{A^{\hat{\otimes}(n+1)}} - t_{\underline{n}})$: at least one a_i belongs to $i(B)\}$. □

3.8 Theorem. *Let*

$$0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0$$

be an extension of Banach algebras. Suppose B has a left or right bounded approximate identity. Then there exists an associated long exact sequence of Banach cyclic homology groups

$$\cdots \to \mathcal{HC}_n(B) \to \mathcal{HC}_n(A) \to \mathcal{HC}_n(D) \to \mathcal{HC}_{n-1}(B) \to \cdots \to \mathcal{HC}_0(D) \to 0 \tag{3.10}$$

if and only if there exists an associated long exact sequence of Banach cyclic cohomology groups

$$0 \to \mathcal{HC}^0(D) \to \cdots \to \mathcal{HC}^{n-1}(B) \to \mathcal{HC}^n(D) \to \mathcal{HC}^n(A) \to \mathcal{HC}^n(B) \to \ldots. \tag{3.11}$$

The proof requires only minor modifications of that of Theorem 3.5 in view of Lemmas 3.6 and 3.7. □

4. The existence of long exact sequences of cohomology groups of Banach algebras

As stated above, Wodzicki remarked that his result on the existence of long exact sequences of cyclic and simplicial homology groups can be extended to the continuous case. In Corollary 4 [Wo2] he considered extensions of Banach algebras $0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0$ where B has a left or right bounded approximate identity and stated the existence of long exact sequences associated with the extension under the explicit hypothesis of the admissibility of the extension. Subsequently, he pointed out that this hypothesis is unnecessary. Here we state his result in a new formulation.

4.1 THEOREM. *Let*

$$0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0$$

be an extension of Banach algebras. Suppose B has a left or right bounded approximate identity. Then there exist associated long exact sequences of Banach simplicial homology groups (3.5), of Banach cyclic homology groups (3.10) and of Banach Bar homology groups (3.7), and so $\mathcal{HR}_n(A) = \mathcal{HR}_n(D)$ for all $n \geq 0$.

Proof. Let us introduce the following notation. Let $\underline{n} = (n_1, \ldots, n_{l+1})$ be an $(l+1)$-tuple of integers such that $n_1, n_{l+1} \geq 0$ and the others are > 0 and let $\underline{k} = (k_1, \ldots, k_l)$ be an l-tuple of integers such that $k_1 \geq 0$ and the others are > 0. Put $|\underline{n}| := n_1 + \cdots + n_{l+1}$ and $l(\underline{n}) := l$. In view of Lemmas 3.3, 3.4, 3.6 and 3.7, we can see that, up to algebraic isomorphism,

$$C_m(A) = \bigoplus_{\underline{n},\underline{k}:|\underline{n}|+|\underline{k}|=m+1,\ l(\underline{n})\geq 1,\ l(\underline{k})=l(\underline{n})-1} D^{\hat{\otimes} n_1} \hat{\otimes} B^{\hat{\otimes} k_1} \hat{\otimes} \ldots \hat{\otimes} D^{\hat{\otimes} n_{l+1}},$$

$$\text{Ker } j^{\hat{\otimes}(m+1)} = \bigoplus_{\underline{n},\ \underline{k}:k_1>0,\ |\underline{n}|+|\underline{k}|=m+1,\ l(\underline{n})\geq 1,\ l(\underline{k})=l(\underline{n})-1} D^{\hat{\otimes} n_1} \hat{\otimes} B^{\hat{\otimes} k_1} \hat{\otimes} \ldots \hat{\otimes} D^{\hat{\otimes} n_{l+1}}$$

and $\text{Ker } \widetilde{j^{\hat{\otimes}(m+1)}} = \{u + \text{Im } (id_{A^{\hat{\otimes}(m+1)}} - t_{\underline{m}});$ where $u \in \text{Ker } j^{\hat{\otimes}(m+1)}\}$. Thus the proof is the same as in the algebraic case in Theorem 3.1 [Wo3] and Theorem 3 [Wo2] with filtrations in which algebraic tensor products are replaced by the projective tensor products. □

4.2 THEOREM. *Let*

$$0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0$$

be an extension of Banach algebras. Suppose B has a left or right bounded approximate identity. Then there exist associated long exact sequences of Banach simplicial cohomology groups (3.6), of Banach cyclic cohomology groups (3.11) and of Banach Bar cohomology groups (3.8), and so $\mathcal{HR}^n(D) = \mathcal{HR}^n(A)$ for all $n \geq 0$.

Proof. The existence of the long exact sequences follows from Theorems 4.1, 3.5 and 3.8 and Lemma 3.6.

By Theorem 16 [He2], $\mathcal{HR}^n(D) = \{0\}$ for every Banach algebra A with left or right bounded approximate identity and for all $n \geq 0$. Hence the final statement follows. □

Since C^*-algebras have bounded approximate identities, Theorem 4.2 applies whenever B is a C^*-algebra. By virtue of the main result of [Dix], the Banach algebra $\mathcal{K}(E)$ of compact operators on a Banach space E with the bounded compact approximation property has a bounded left approximate identity. For more examples of Banach algebras with a left or right bounded approximate identity see, for example [Pa; Section 5.1].

4.3 Remark. Let us consider the extension of Banach algebras $0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0$, when B has a left or right bounded approximate identity. The existence of the Connes-Tsygan sequence for one of Banach algebras A or D implies the existence of the same exact sequence for the other. This follows from Theorems 15, 16 [He2] and Theorem 4.2.

4.4 Proposition. *Let*

$$0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0,$$

be an extension of Banach algebras. Then

(i) *if B is an amenable Banach algebra, then $\mathcal{H}^n(A) = \mathcal{H}^n(D)$ for all $n \geq 2$ and there exist exact sequences*

$$0 \to D^{tr} \to A^{tr} \to B^{tr} \to \mathcal{H}^1(D) \to \mathcal{H}^1(A) \to 0$$

and

$$0 \to \mathcal{HC}^{2n}(D) \to \mathcal{HC}^{2n}(A) \to B^{tr} \to \mathcal{HC}^{2n+1}(D) \to \mathcal{HC}^{2n+1}(A) \to 0$$

for every $n \geq 0$;

(ii) *if B is a C^*-algebra without non-zero bounded traces then*

$$\mathcal{H}^n(A) = \mathcal{H}^n(D) \text{ and } \mathcal{HC}^n(A) = \mathcal{HC}^n(D) \text{ for all } n \geq 0.$$

Proof. (i) By assumption B is amenable, and so B has a bounded approximate identity. Hence, by Theorem 4.2, there exist associated long exact sequences of Banach simplicial and cyclic cohomology groups (3.6) and (3.11). Recall that, for an amenable Banach algebra B, $\mathcal{H}^n(B) = \{0\}$ for all $n \geq 1$ and, by Theorem 25 [He2], $\mathcal{HC}^n(B)$ is equal to B^{tr} for even n and to 0 for odd n. The result follows in view of Lemma 7.1.32 [BS].

(ii) By Proposition 1.7.3 [Di], any C^*-algebra has a bounded approximate identity and so Theorem 4.2 applies. By Theorem 4.1 and Corollary 3.3 [ChS], for every C^*-algebras without non-zero bounded traces, the Banach simplicial and cyclic cohomology groups vanish for all $n \geq 0$. □

Note that some results of this proposition were proved by a different approach in [Ly].

4.5 Remark. Let us consider the extension of Banach algebras $0 \to B \xrightarrow{i} A \xrightarrow{j} D \to 0$, where B has a left or right bounded approximate identity. Then the condition that D be an amenable Banach algebra implies, by the same arguments as in Theorem 4.4, the following: $\mathcal{H}^n(A) = \mathcal{H}^n(B)$ for all $n \geq 1$ and there exist exact sequences

$$0 \to D^{tr} \to A^{tr} \to B^{tr} \to 0$$

and

$$0 \to \mathcal{HC}^{2n+1}(A) \to \mathcal{HC}^{2n+1}(B) \to D^{tr} \to \mathcal{HC}^{2n+2}(A) \to \mathcal{HC}^{2n+2}(B) \to 0$$

for every $n \geq 0$.

As in Theorem 4.4, the condition that D be a C^*-algebra without non-zero bounded traces implies $\mathcal{H}^n(A) = \mathcal{H}^n(B)$ and $\mathcal{HC}^n(A) = \mathcal{HC}^n(B)$ for all $n \geq 0$.

4.6 Examples. Some examples of C^*-algebras without non-zero bounded traces are: (i) The C^*-algebra $\mathcal{K}(H)$ of compact operators on an infinite-dimensional Hilbert space H; see [An; Theorem 2]. We can also show that $C(\Omega, \mathcal{K}(H))^{tr} = 0$, where Ω is a compact space. (ii) Properly infinite von Neumann algebras $\mathcal{U}$. By Proposition 2.2.4 [Sa] in $\mathcal{U}$ there exists a sequence (p_m) of mutually orthogonal, equivalent projections with $p_m \sim e$. Thus, by Theorem 2.1 [Fa], each hermitian element of $\mathcal{U}$ is the sum of five commutators. Hence there are no non-zero traces on $\mathcal{U}$. This class includes the C^*-algebra $\mathcal{B}(H)$ of all bounded operators on an infinite-dimensional Hilbert space H; see also [Hal] for the statement $\mathcal{B}(H)^{tr} = 0$.

4.7 Examples. Each nuclear C^*-algebra is amenable [Ha]. Some examples of amenable C^*-algebras are: (i) GCR C^*-algebras; in particular, commutative C^*-algebras and the C^*-algebra of compact operators $\mathcal{K}(H)$ on a Hilbert space H; (ii) Uniformly hyperfinite algebras (UHF-algebras) [Mu; Remark 6.2.4].

The group algebra $L^1(G)$ of Haar integrable functions on an amenable locally compact group G with convolution product is amenable too [Jo1]. For more examples of amenable Banach algebras see, for example [GJW].

4.8 Examples. In [GJW] it is shown that the Banach algebra $\mathcal{K}(E)$ of compact operators on a Banach space E with property ($\mathbb{A}$) which was defined in [GJW] is amenable. Property ($\mathbb{A}$) implies that $\mathcal{K}(E)$ contains a bounded sequence of projections of unbounded finite rank, and from this it is easy to show (via embedding of matrix algebras) that there is no non-zero bounded trace on $\mathcal{K}(E)$. Thus we can see from Theorem 4.4 that, for every extension of Banach algebras

$$0 \rightarrow \mathcal{K}(E) \xrightarrow{i} A \xrightarrow{j} D \rightarrow 0,$$

we have

$$\mathcal{H}^n(A) = \mathcal{H}^n(D) \text{ and } \mathcal{HC}^n(A) = \mathcal{HC}^n(D) \text{ for all } n \geq 0.$$

In particular, for the Banach algebra $A = \mathcal{B}(E)$ of all bounded operators on a Banach space E, $\mathcal{H}^n(\mathcal{B}(E)) = \mathcal{H}^n(\mathcal{B}(E)/\mathcal{K}(E))$ and $\mathcal{HC}^n(\mathcal{B}(E)) = \mathcal{HC}^n(\mathcal{B}(E)/\mathcal{K}(E))$ for all $n \geq 0$. Several classes of Banach spaces have the property ($\mathbb{A}$): $l_p; 1 < p < \infty; C(K)$, where K is a compact Hausdorff space; $L_p(\Omega, \mu); 1 < p < \infty$ (for details and more examples see [Jo1] and [GJW]).

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