CANONICAL SYSTEMS AND FINITE RANK PERTURBATIONS OF SPECTRA

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Abstract. We use Rokhlin's Theorem on the uniqueness of canonical systems to find a new way to establish connections between Function Theory in the unit disk and rank one perturbations of self-adjoint or unitary operators. In the n-dimensional case, we prove that for any cyclic self-adjoint operator A, operator $A_\lambda = A + \Sigma_{k=1}^{n}\lambda_k(\cdot,\phi_k)\phi_k$ is pure point for a.e. $\lambda = (\lambda_1,\lambda_2,...,\lambda_n) \in \mathbb{R}^n$ iff operator $A_\eta = A + \eta(\cdot,\phi_k)\phi_k$ is pure point for a.e. $\eta \in \mathbb{R}$ for $k = 1,2,...,n$. We also show that if A_λ is pure point for a.e. $\lambda \in \mathbb{R}^n$ then A_λ is pure point for a.e. $\lambda \in \gamma$ for any analytic curve $\gamma \in \mathbb{R}^n$.

Introduction

This note analyzes the spectral properties of finite rank perturbations of self-adjoint and unitary operators. In Section 1 we will try to put some well-known results on rank one perturbations into a different prospective. The new approach will allow us to shorten some of the existing proofs. Section 2 deals with rank n perturbations and contains the main results of this paper.

Our approach in Section 1 will be based on the notion of canonical systems of measures. This notion was introduced by Kolmogorov and Rokhlin [R], and, independently, by Ambrose, Halmos and Kakutani [A-H-K, H1-2] more than half a century ago. It was originally used to study partitions, homeomorphisms and factor spaces of various measure spaces. The term "canonical" belongs, most likely, to Kolmogorov. We will first discuss examples of canonical systems of measures naturally appearing in certain problems of Function Theory in the unit disk and Perturbation Theory. We will then use Rokhlin's uniqueness Theorem to show that different problems yield the same families of measures.

In our first example of canonical systems (see Example 1 in Section 1) we are going to discuss families of measures on the unit circle generated by inner functions in the unit disk. Such families first appeared in [C] and then were further studied in [A1-5], [P1-3] and [S]. Our second example (see Example 2 in Section 1) deals with families of spectral measures of rank one perturbations of self-adjoint or unitary

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operators. These families were extensively studied by many authors. We refer to [A], [A-D] and [D] for the basic results on this subject, and to [J-L-R-S], [R-M-S], [P3] and [Si] for the latest developments and further references.

As was proven by D. Clark in [C], any family of measures generated by an inner function (as in Example 1) is a family of spectral measures of all rank one perturbations of a model contraction. In Section 1 we will show how Rokhlin's Theorem can help establish this connection for families of spectral measures of cyclic self-adjoint or unitary operators.

Section 2 is devoted to n-dimensional analogies of Simon-Wolff Theorem on the pure point spectrum of a random rank one perturbation. The one-dimensional result was applied in [S-W] to prove the the existence of Anderson localization for the wave propagation described by the discrete Schrödinger operator with random potential in one dimension. It is still an open question if Anderson localization takes place in dimension more than one.

As we will find out, the n-dimensional case adds certain new and somewhat unexpected features to the general picture. First, using the observations made in Section 1, we will prove that for any cyclic self-adjoint operator A and its cyclic vectors $\phi_1, ..., \phi_n$, operator $A_\lambda = A + \sum_{k=1}^n \lambda_k(\cdot, \phi_k)\phi_k$ is pure point for a.e. $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n$ iff operator $A_\eta = A + \eta(\cdot, \phi_k)\phi_k$ is pure point for a.e. $\eta \in \mathbb{R}$ for $k = 1, 2, ..., n$. I. e. random perturbation A_λ is pure point almost surely iff it is pure point almost surely on the skeleton S (the union of coordinate axes) in $\mathbb{R}^n$. Using the terminology of Function Theory, this result means that S is, in some sense, a *sampling set* for this perturbation problem.

We will also prove that if A_λ is pure point for a.e. $\lambda \in \mathbb{R}^n$ then A_λ is pure point for a.e. $\lambda \in \gamma$ for any analytic curve $\gamma \in \mathbb{R}^n$.

This statement reveals a certain fine structure of the set

$$E = \{\lambda \in \mathbb{R}^n \,|\, A_\lambda \text{is not pure point } \}.$$

Such a property of E may seem surprising, since various examples of perturbations of singular spectra show that E can be almost "arbitrarily bad". For instance, as follows from the results of [R-M-S] and [G], if the spectrum of A contains an interval, then, even if A_λ is pure point a. s., E is everywhere dense and G_δ on every line parallel to the coordinate axes.

The results of Section 2 allow some infinite-dimensional generalizations which are to be published elsewhere.

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1. Canonical systems of measures in function theory and functional analysis

Let us start with the following

Definition. Consider a measure space $(X, \mathcal{A}, \mu)$ where X is a set of points, $\mathcal{A}$ is a σ-algebra of subsets of X and μ is a finite measure on $\mathcal{A}$. Space $(X, \mathcal{A}, \mu)$ is called Lebesgue space if there exist a 1-1 measure preserving map from $(X, \mathcal{A}, \mu_c)$ onto the unit interval with Lebesgue measure.

Here μ_c denotes the continuous part of μ and the term "measure preserving" means in particular that the map induces a 1-1 correspondence between $\mathcal{A}$ and Lebesgue σ-algebra on the unit interval.

Let ξ be a measurable partition of a Lebesgue space $(X, \mathcal{A}, \mu)$: $\xi = \{S_\alpha\}_{\alpha \in Y}$ where Y is some set of parameters, $S_\alpha \in \mathcal{A}$, $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$ and $\bigcup_Y S_\alpha = X$. Then we can consider the factor space $(X/\xi, \mathcal{A}_\xi, \mu_\xi)$ where σ-algebra $\mathcal{A}_\xi$ and measure μ_ξ are induced by $\mathcal{A}$ and μ via the factor map.

Suppose each set S_α from our partition ξ is itself a Lebesgue space $(S_\alpha, \mathcal{A}_\alpha, \mu_\alpha)$ for some σ-algebra $\mathcal{A}_\alpha$ and measure μ_α.

Definition. The system of measures $\{\mu_\alpha\}_{\alpha \in Y}$ is called canonical if for any $A \in \mathcal{A}$ we have $A \cap S_\alpha \in \mathcal{A}_\alpha$ for μ_ξ-a.e. α and

$$\mu(A) = \int_{X/\xi} \mu_\alpha(A \cap S_\alpha) d\mu_\xi(\alpha). \tag{1}$$

Note that the factor space X/ξ can be naturally identified with the set Y of parameters α. This justifies the expression "μ_ξ-a.e. α" and integration over $d\mu_\xi(\alpha)$ in (1).

Suppose we have a measurable partition ξ of $(X, \mathcal{A}, \mu)$. Does there exist a corresponding canonical system of measures? If ξ is countable the answer is obviously yes. In this case μ_ξ in (1) is discrete and μ_α can be chosen as the restriction of μ on S_α. However when ξ is uncountable, most of S_α have measure 0 and the canonical system can not be constructed so easily. Nevertheless we have the following

Theorem 1 (Rokhlin [R]). *For any measurable partition ξ of Lebesgue measure space $(X, \mathcal{A}, \mu)$ there exists a canonical system $\{\mu_\alpha\}$ satisfying (1). Such a system is essentially unique i. e. for any two such canonical systems $\{\mu_\alpha\}$ and $\{\mu'_\alpha\}$ $\mu_\alpha = \mu'_\alpha$ for μ_ξ-a.e. α.*

In the rest of this section we are going to use the uniqueness part of this result to show how different problems in function theory and functional analysis yield the same families of measures.

Note that condition $\bigcup S_\alpha = X$ can be replaced by a weaker condition

$$\mu(\bigcup S_\alpha) = \mu(X) \tag{2}$$

in all the above definitions and in Theorem 1.

Our first example of a canonical system is related to Analytic Function Theory in the unit disk.

Example 1. Let θ be an inner function in the unit disk $\mathbb{D}$. We will denote by $M_+(\mathbb{T})$ the set of all finite positive Borel measures on the unit circle $\mathbb{T} = \partial\mathbb{D}$.

For each $\alpha \in \mathbb{T}$ function $\frac{\alpha+\theta}{\alpha-\theta}$ has positive real part in $\mathbb{D}$. Therefore for each $\alpha \in \mathbb{T}$ there exists $\mu_\alpha \in M_+(\mathbb{T})$ such that its Poisson integral satisfies

$$P\mu_\alpha = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2} d\mu_\alpha(\xi) = Re\frac{\alpha+\theta}{\alpha-\theta}. \tag{3}$$

We will denote by $M_\theta(\mathbb{T})$ the family $\{\mu_\alpha\}_{\alpha\in\mathbb{T}}$ of all such measures corresponding to θ. Note that since $Re\frac{\alpha+\theta}{\alpha-\theta} = 0$ a.e. on $\mathbb{T}$, all measures μ_α are singular.

If θ is defined in the upper half plane $\mathbb{C}_+$ then one can replace the Poisson integral in (3) with the Poisson integral in $\mathbb{C}_+$ and define an analogous family $M_\theta(\hat{\mathbb{R}})$ on the completed real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (see [P3]).

Families M_θ possess many interesting properties (see [A1-5], [C], [P1-3] and [Sa]). Our next task is to show that all such families are *canonical systems*.

First let us notice that the defining formula (3) implies that measures μ_α are concentrated on disjoint sets $S_\alpha = \{\xi \in \mathbb{T} | \theta(\xi) = \alpha\}$ (all analytic functions in this paper are defined on the boundary by their nontangential boundary values). Also if $\mathcal{P}_z$ is a Poisson kernel for some $z \in \mathbb{D}$ then

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \mathcal{P}_z d\mu_\alpha dm(\alpha) = \int_{\mathbb{T}} Re\frac{\alpha+\theta(z)}{\alpha-\theta(z)} dm(\alpha) =$$

$$= 1 = \int_{\mathbb{T}} \mathcal{P}_z dm. \tag{4}$$

Note that since linear combinations of Poisson kernels are dense in the space of all continuous functions on $\mathbb{T}$, one can replace $\mathcal{P}_z$ in (4) with an arbitrary continuous function. Further, by taking the limit over continuous functions, one can replace $\mathcal{P}_z$ in (4) with a characteristic function of any Borel set $B \subset \mathbb{T}$ to obtain the following analogy of formula (1) (see [A1]):

$$\int_{\mathbb{T}} \mu_\alpha(B) dm(\alpha) = m(B). \tag{5}$$

Therefore family M_θ defined by formula (3) is indeed a canonical system on $(\mathbb{T}, \mathcal{B}, m)$ for the partition $S_\alpha = \{\theta = \alpha\}$ where $\mathcal{B}$ is Borel σ-algebra on $\mathbb{T}$.

The same argument holds in the case of the upper half plane for the families $M_\theta(\hat{\mathbb{R}})$.

Our second example concerns Perturbation Theory of self-adjoint and unitary operators.

Example 2. Let A_0 be a singular bounded cyclic self-adjoint operator acting on a separable Hilbert space, ϕ its cyclic vector. Consider the family of rank one perturbations of A_0:

$$A_\lambda = A_0 + \lambda(\cdot,\phi)\phi \qquad \lambda \in \mathbb{R}.$$

Let μ_λ be the spectral measure of ϕ for A_λ. We will denote by $M_{A_0,\phi}$ the family $\{\mu_\lambda\}_{\lambda\in\mathbb{R}}$.

For basic results and references on families $M_{A_0,\phi}$ the reader can consult [A], [A-D], [D] or [S].

Here we will show that family $M_{A_0,\phi} = \{\mu_\lambda\}_{\lambda\in\mathbb{R}}$ is a *canonical system.*

Note that Cauchy integral of μ_λ satisfies:

$$K\mu_\lambda(z) = \int_{-\infty}^{\infty} \frac{d\mu_\lambda(t)}{t-z} = ((A_\lambda - z)^{-1}\phi,\phi). \tag{6}$$

Also for the resolvents of operators A_λ we have:

$$(A_0 - z)^{-1} - (A_\lambda - z)^{-1} = \left[\lambda(\cdot,\phi)(A_\lambda - z)^{-1}\phi\right](A_0 - z)^{-1}. \tag{7}$$

Combining (6) and (7) we obtain

$$K\mu_\lambda(z) = \frac{K\mu_0(z)}{1+\lambda K\mu_0(z)} \tag{8}$$

(see [A]).

Let us observe that, as follows from (6), measures μ_λ are concentrated on the sets $S_\lambda = \{x\in\mathbb{R} | ((A_\lambda - x)^{-1}\phi,\phi) = \infty\}$. Via (7) S_λ can be redefined as

$$S_\lambda = \{x\in\mathbb{R} | ((A_0 - x)^{-1}\phi,\phi) = -\frac{1}{\lambda}\}$$

which shows that S_λ are pairwise disjoint.

To prove the integral formula notice that if $\mathcal{P}_z = \frac{y}{(x-t)^2+y^2}$ is the Poisson kernel for $z = x+iy \in \mathbb{C}_+$ then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{P}_z(t)d\mu_\lambda(t)d\lambda = \int_{-\infty}^{\infty} Re\frac{K\mu_0(z)}{1+\lambda K\mu_0(z)}d\lambda.$$

Since μ_λ is a positive measure, $\omega = -1/(K\mu_0(z))$ belongs to $\mathbb{C}_+$. Therefore

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{P}_z(t)d\mu_\lambda(t)d\lambda = \int_{-\infty}^{\infty} \mathcal{P}_\omega(\lambda)d\lambda = \pi = \int_{-\infty}^{\infty} \mathcal{P}_z(\lambda)d\lambda.$$

Since every continuous $L^1(\mathbb{R})$ function can be approximated by linear combinations of Poisson kernels, in the same way as in the previous example we can conclude that

$$\int_{-\infty}^{\infty} \mu_\lambda(B)d\lambda = |B| \tag{9}$$

for any Borel set B (see [S]).

Therefore family $M_{A_0,\phi}$ is a canonical system on $(\mathbb{R},\mathcal{B},dx)$ for the partition $S_\lambda=\{((A_0-x)^{-1}\phi,\phi)=-\frac{1}{\lambda}\}$, where $\mathcal{B}$ denotes Borel σ-algebra.

An analogous argument can be applied in the case of unitary perturbations.

Let U_1 be a unitary cyclic singular operator, v, $||v||=1$ its cyclic vector. Consider the family of unitary rank one perturbations of U_1 :

$$U_\alpha=U_1+(\alpha-1)(\cdot,U_1^{-1}v)v,$$

$\alpha\in\mathbb{T}$. For the resolvents we have

$$(U_1-z)^{-1}-(U_\alpha-z)^{-1}=(U_\alpha-z)^{-1}[(\alpha-1)(\cdot,U_1^{-1}v)v](U_1-z)^{-1}.$$

Denote by ν_α the spectral measure of v for U_α. Let $K\nu_\alpha$ be the Cauchy integral of measure ν_α in the unit disk $\mathbb{D}$:

$$K\nu_\alpha=\int_{\mathbb{T}}\frac{1}{1-\bar{\xi}z}d\nu_\alpha(\xi).$$

Since

$$K\nu_\alpha=((U_\alpha-z)^{-1}v,U_\alpha^{-1}v)$$

for $z\in\mathbb{D}$, we have that

$$K\nu_\alpha=\frac{\alpha K\nu_1}{1+(\alpha-1)K\nu_1}. \tag{10}$$

Using (10) in the same way as we used (7) we can obtain that family $M_{U_1,v}=\{\nu_\alpha\}_{\alpha\in\mathbb{T}}$ is a canonical system on $(\mathbb{T},\mathcal{B},m)$ for the partition

$$S_\alpha=\{((U_1-z)^{-1}v,U_1^{-1}v)=\frac{1}{1-\alpha}\}.$$

Now we will use Rokhlin's Theorem to show that every family of spectral measures of rank one perturbations (Example 2) is in fact a family generated by an inner function (Example 1) and vice versa.

Indeed, if we consider the following inner function in $\mathbb{C}_+$:

$$\theta(z)=\frac{1-i((A_0-z)^{-1}\phi,\phi)}{1+i((A_0-z)^{-1}\phi,\phi)} \tag{11}$$

then the partitions from Examples 1 and 2 coincide: $S_\alpha=S_\lambda$ for $\alpha=\frac{\lambda+i}{\lambda-i}$. Hence by Rokhlin Theorem canonical system M_θ essentially coincides with canonical system $M_{A_0,\phi}$. But in both systems the measures depend on parameter continuously (with respect to the $*$-weak topology of the space of measures). Since the systems coincide *almost everywhere* and are continuous, they must coincide *everywhere*:

$$M_\theta = M_{A_0,\phi}. \tag{12}$$

Conversely, for each inner θ we can choose A_0 to be the operator of multiplication by z in $L^2(\mu_1)$, $\mu_1 \in M_\theta$ and $\phi = 1 \in L^2(\mu_1)$. Then θ, A_0, and ϕ will satisfy (12).

In a similar way, for unitary operators, if we choose

$$\theta(z) = \frac{1}{K\nu_1(z)} - 1$$

for any $z \in \mathbb{D}$ then we will have

$$M_\theta = M_{U_1,v}. \tag{13}$$

with $\mu_\alpha = \nu_\alpha$ for any $\alpha \in \mathbb{T}$, $\mu_\alpha \in M_\theta(\mathbb{T})$ and $\nu_\alpha \in M_{U_1,v}$ (cf. [C]).

The families of measures from Examples 1 and 2 are not the only natural examples of canonical systems. For instance, in [D] one can find two more families of operators: the family of all self-adjoint extensions of a given symmetric operator with deficiency indices (1,1) and the family of all self-adjoint extensions associated with a given limit-point Stourm-Liouville problem. Applying the argument, similar to the one we used here, one can prove that the corresponding families of spectral measures are canonical systems generated by inner functions.

If A_0 and ϕ are as in Example 2, then $A_i = A_0 + i(\cdot,\phi)\phi$ is a dissipative operator. Its Cayley transform T is a completely nonunitary C_0 contraction. Sz.Nagy-Foias model theory provides T with a characteristic function I in the unit disk. Function I is related to our function θ, defined by (11), via the equation $I(z) = \theta(\omega(z))$ where ω is the standard conformal map from the unit disk onto the upper half-plane.

2. Finite rank perturbations of singular spectra

In this section we will need the following Lemma, which can be obtained directly from the definition of families M_θ:

Lemma 2. *Let θ be an inner function in $\mathbb{D}$ and $\xi \in \mathbb{T}$. Then*

1) measure $\mu_\alpha \in M_\theta$ has a point mass at ξ iff functions θ and θ' have non-tangential limits $\theta(\xi)$ and $\theta'(\xi)$ at ξ and $\theta(\xi) = \alpha$;

2) function θ' has a non-tangential limit $\theta'(\xi)$ at ξ iff there exists $\mu_\beta \in M_\theta$ such that

$$\int_{\mathbb{T}} \frac{d\mu_\beta(\omega)}{|\xi - \omega|^2} < \infty; \tag{14}$$

3) measure $\mu_\alpha \in M_\theta$ has a point mass at ξ iff the non-tangential limit $\theta(\xi)$ is equal to α and (14) holds for any $\beta \in \mathbb{T}$, $\beta \neq \alpha$.

An analogous statement holds true for families M_θ on the real line.

Suppose $\theta'(\xi)$ (the non-tangential limit of θ' at ξ) exists everywhere on $\mathbb{T}$ except for $\xi \in E,\ E \subset \mathbb{R}$. If $m(E) = 0$ then, by formula (5), for measures from M_θ we have $\mu_\alpha(E) = 0$ for almost every $\alpha \in \mathbb{T}$. Since μ_α is concentrated on $\{\theta = \alpha\}$, statement 1) from Lemma 2 implies that μ_α is pure point for a.e. α. Now we can use statement 2) from Lemma 2 and connections with operator theory discussed in the previous section (formula (12)) to obtain the following

Theorem 3 [S-W]. *Let A be a cyclic self-adjoint operator, ϕ its cyclic vector and μ the spectral measure of ϕ for A. Then operator*

$$A_\lambda = A + \lambda(\cdot, \phi)\phi$$

is pure point for a.e. $\lambda \in \mathbb{R}$ iff

$$\int_{\mathbb{R}} \frac{d\mu(x)}{|x-y|^2} < \infty \tag{15}$$

for a.e. $y \in \mathbb{R}$.

The purpose of this section is to find an n-dimensional analogy of this result. We will prove the following

Theorem 4. *Let A be a cyclic self-adjoint operator, $\phi_1, \phi_2, ..., \phi_n$ its cyclic vectors. The following two conditions are equivalent:*
1) operator

$$A_\lambda = A + \sum_{k=1}^{n} \lambda_k(\cdot, \phi_k)\phi_k$$

is pure point for a.e. $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n$.
2) operator

$$A_\lambda = A + \lambda(\cdot, \phi_k)\phi_k$$

is pure point for a.e. $\lambda \in \mathbb{R}$ for $k = 1, 2, ..., n$;
i. e.

$$\int_{\mathbb{R}} \frac{d\mu_k(x)}{|x-y|^2} < \infty$$

for a.e. $y \in \mathbb{R}$ for $k = 1, 2, ..., n$, where μ_k is the spectral measure of ϕ_k for A.

The theory of unitary rank n perturbations is slightly more complicated due to the fact that a sum of two unitary perturbations may not be unitary. Because of that, we will have to define the rank n perturbation of a unitary operator U corresponding to vectors $\phi_1, ..., \phi_n$ recursively. I. e., let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{T}^n$. Denote

$$\alpha^1 = \alpha_1 \in \mathbb{R},\ \alpha^2 = (\alpha_1, \alpha_2) \in \mathbb{R}^2, ...$$

$$..., \alpha^{n-1} = (\alpha_1, ..., \alpha_{n-1}) \in \mathbb{R}^{n-1}, \quad \text{and} \quad \alpha^n = \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n.$$

Put

$$U_{\alpha^1} = U + (\alpha_1 - 1)(\cdot, U^{-1}\phi_1)\phi_1,$$

and

$$U_{\alpha^k} = U_{\alpha^{k-1}} + (\alpha_k - 1)(\cdot, U^{-1}_{\alpha^{k-1}}\phi_k)\phi_k$$

for $k = 2, 3, ..., n$. Finally, denote $U_\alpha = U_{\alpha^n}$.

If vectors $\phi_1, \phi_2, ..., \phi_n$ are pairwise orthogonal, then our definition yields

$$U_\alpha = U + \sum_{k=1}^{n} \alpha_k(\cdot, U^{-1}\phi_k)\phi_k.$$

One can show that the following Theorem is equivalent to Theorem 4:

Theorem 4'. *Let U be a cyclic unitary operator, $\phi_1, \phi_2, ..., \phi_n$ its cyclic vectors. Then the following two conditions are equivalent:*

1) operator U_α (defined as above) is pure point for a.e. $\alpha \in \mathbb{T}^n$;

2) operator $U_\alpha = U + \alpha(\cdot, U^{-1}\phi_k)\phi_k$ is pure point for a.e. $\alpha \in \mathbb{T}$ for $k = 1, 2, ..., n$.

Since it is more convenient for us to operate in the unit disk, we will prove the result in the form of Theorem 4' rather than Theorem 4. Our proof will be based on the following approach.

Let U_1 be a cyclic singular unitary operator. Suppose ϕ_1 and ϕ_2 are its cyclic vectors, $||\phi_1|| = ||\phi_2|| = 1$. Let μ_α be the spectral measure of ϕ_1 for

$$U_\alpha = U_1 + (\alpha - 1)(\cdot, U_1^{-1}\phi_1)\phi_1. \tag{16}$$

Then, as was shown in Section 1, there exists an inner function θ, $\theta(0) = 0$ such that $\{\mu_\alpha\}_{\alpha\in\mathbb{T}} = M_\theta$. By the Spectral Theorem, each operator U_α is unitarily equivalent to the operator Y_α of multiplication by z in $L^2(\mu_\alpha)$.

We will denote by $\theta^*(H^p), p > 0$ the invariant subspace of the backward shift operator in H^p corresponding to θ:

$$\theta^*(H^p) = \overline{Span}_{H^p}\left[\left\{\frac{\theta(\lambda) - \theta(z)}{\lambda - z}\right\}_{\lambda\in\mathbb{D}}\right].$$

In the case $p = 2$ we have $\theta^*(H^2) = H^2 \ominus \theta H^2$.

The following operator V_α acting from $L^2(\mu_\alpha)$ to $\theta^*(H^2)$ was studied in [A2], [C] and [P1]:

$$V_\alpha f = \frac{Kf\mu_\alpha}{K\mu_\alpha}.$$

We will need the following

Theorem 5 [C]. *For each $\alpha \in \mathbb{T}$ operator V_α maps $L^2(\mu_\alpha)$ onto $\theta^*(H^2)$ unitarily. Operator V_α sends operator Y_α of multiplication by z in $L^2(\mu_\alpha)$ into the following operator T_α in $\theta^*(H^2)$:*

$$T_\alpha = V_\alpha Y_\alpha V_\alpha^* f = z\left(f - \left(f, \frac{\theta}{z}\right)\frac{\theta}{z}\right) + \left(f, \frac{\theta}{z}\right)\alpha. \tag{17}$$

Operators T_α are all possible unitary rank one perturbations of the model contraction $T_\theta = P_\theta S$, where S is the operator of multiplication by z in $\theta^*(H^2)$ and P_θ is the orthogonal projector from H^2 to $\theta^*(H^2)$ (see [C]). Also note, that in our situation $\theta(0) = 0$, otherwise (17) would not be valid.

The conjugate operator V_α^* in (17) can also be defined quite naturally:

Theorem 6 [P1]. *For each function $f \in \theta^*(H^2)$ and each $\alpha \in \mathbb{T}$ the non-tangential boundary values of f exist μ_α-a.e. Function $V_\alpha^* f \in L^2(\mu_\alpha)$ coincides with the non-tangential boundary values of f μ_α-a.e.*

Via the original Fourier transform and formula (17), each unitary rank one perturbation U_α given by (16) is identified with an operator in $\theta^*(H^2)$. Vector ϕ_1 corresponds to $1 \in \theta^*(H^2)$ and vector ϕ_2 corresponds to some function $f \in \theta^*(H^2)$. As μ_α was chosen to be the spectral measure of ϕ_1 for U_α, the measure $|f|^2\mu_\alpha$ is the spectral measure of ϕ_2 for U_α (note that by Theorem 6 the non-tangential boundary values of $|f|^2$ exist μ_α-a.e., therefore the notation $|f|^2\mu_\alpha$ makes sense).

Lemma 2 and Theorem 3 imply that operator $U_{(\alpha,\beta)} = U_\alpha + \beta(\cdot, U_\alpha^{-1}\phi_2)\phi_2$ is pure point for a.e. $\beta \in \mathbb{T}$ iff the (non-tangential boundary values of the) derivative of $K|f|^2\mu_\alpha$ exist a.e. on $\mathbb{T}$. We will use this fact in the proof of Theorem 4'.

By one of the properties of $\theta^*(H^2)$, if $f \in \theta^*(H^2)$ is equal to 0 at 0, then the function $\hat{f}$ such that $\hat{f} = \theta\overline{f}$ a.e. on $\mathbb{T}$ also belongs to $\theta^*(H^2)$.

We will need the following

Lemma 7. *Let $f \in \theta^*(H^2), ||f|| = 1, f_0 = f - f(0)$. Then*

1) there exist $g, h \in \cap_{0<p<1}\theta^*(H^p)$ such that $f_0\hat{f}_0 = g + \theta h$;

2) $\hat{f}_0$ is equal to $\alpha\overline{f}_0$ μ_α-a.e. for every $\alpha \in \mathbb{T}$;

3)

$$K|f|^2\mu_\alpha = \frac{g + \alpha h + f(0)\hat{f}_0 + \alpha\overline{f(0)}f_0 + \alpha|f(0)|^2}{\alpha - \theta} \tag{18}$$

for every $\alpha \in \mathbb{T}$

Remark. Recall, that $\theta(0) = 0$, which implies $f_0 \in \theta^*(H^2)$.

Proof. Since $f_0\hat{f}_0 \in \theta^*(H^1)$, 1) follows from Riesz' Theorem.

To prove 2) it is enough to show that if f_0 is real μ_1-a.e., then $\hat{f}_0 = f_0$. But if f_0 is real μ_1-a.e. then

$$\theta\overline{\left(\frac{Kf_0\mu_1}{K\mu_1}\right)} = \frac{1}{2}\frac{\theta}{1-\overline{\theta}}(Pf_0\mu_1 - iImKf_0\mu_1) =$$

$$= \frac{1}{2}\frac{1}{1-\theta}(Pf_0\mu_1 - iImKf_0\mu_1) = -\frac{Pf_0\mu_1 - iImKf_0\mu_1}{K\mu_1}$$

a.e. on $\mathbb{T}$. Since $f_0\mu_1$ is a singular measure, Poisson integral $Pf_0\mu_1$ is equal to 0 a.e. on $\mathbb{T}$. Therefore

$$\hat{f}_0 = \theta\overline{\left(\frac{Kf_0\mu_1}{K\mu_1}\right)} = -\frac{Pf_0\mu_1 - iImKf_0\mu_1}{K\mu_1} = \frac{Kf_0\mu_1}{K\mu_1} = f_0$$

a.e. on $\mathbb{T}$.

To prove 3), let us first assume that $f \in H^\infty$. Then g, h and $g + \alpha h$ belong to $\theta^*(H^2)$. Since $\hat{f}_0 = \alpha\overline{f_0}$ μ_α-a.e., $(f_0 + f(0))(\hat{f}_0 + \alpha\overline{f(0)}) = \alpha|f|^2$ μ_α-a.e. Since $\theta = \alpha$ μ_α-a.e., $f_0\hat{f}_0 = g + \theta h = g + \alpha h$ μ_α-a.e. Hence

$$g + \alpha h + f(0)\hat{f}_0 + \alpha\overline{f(0)}f_0 + \alpha|f(0)|^2 = \alpha|f|^2$$

μ_α-a.e. Thus, by Theorem 6,

$$\frac{K|f|^2\mu_\alpha}{K\mu_\alpha} = \overline{\alpha}(g + \alpha h + f(0)\hat{f}_0 + \alpha\overline{f(0)}f_0 + \alpha|f(0)|^2).$$

Since $K\mu_\alpha = \frac{1}{1-\overline{\alpha}\theta}$ (recall that $\theta(0) = 0$), we obtain (18).

In the general case, consider $f_n \to f$ in $\theta^*(H^2)$ where $f_n \in H^\infty$. Then for each f_n formula (18) will hold with some functions g_n and h_n on the right hand side such that $g_n + \alpha h_n \to g + \alpha h$ pointwise in $\mathbb{D}$. Since by Theorem 5 $|f_n|^2 \to |f|^2$ in $L^1(\mu_\alpha)$,

$$\frac{K|f_n|^2\mu_\alpha}{K\mu_\alpha} \to \frac{K|f|^2\mu_\alpha}{K\mu_\alpha}$$

pointwise in $\mathbb{D}$. ▲.

We will only prove Theorem 4' and other results from this section for the case $n = 2$. This restriction will allow us to significantly shorten the proofs without losing any essential ideas.

Proof of Theorem 4' for n=2. WLOG $||\phi_i|| = 1$. Let $f \in \theta^*(H^2)$ be the function corresponding to ϕ_2 (we keep the notation introduced before Lemma 7). Denote by μ_α and ν_α the spectral measures of ϕ_1 and ϕ_2 for

$$U_\alpha = U + (\alpha - 1)(\cdot, U^{-1}\phi_1)\phi_1$$

respectively. Then $\nu_\alpha = |f|^2\mu_\alpha$. To simplify (18) and other formulas we will assume that $f(0) = 0$. The general case can be treated in the same way.

If $f(0) = 0$ then Lemma 7 implies that there exist $g, h \in \cap_{0<p<1}\theta^*(H^p)$ such that $f\hat{f} = g + \theta h$ and for every $\alpha \in \mathbb{T}$

$$K\nu_\alpha = \frac{g + \alpha h}{\alpha - \theta}. \tag{19}$$

Let $\nu_{(\alpha,\beta)}$ be the spectral measure of ϕ_2 for

$$U_{(\alpha,\beta)} = U_\alpha + (\beta - 1)(\cdot, U_\alpha^{-1}\phi_2)\phi_2.$$

Using formulas (10) and (19) one can conclude that

$$K\nu_{(\alpha,\beta)} = \frac{\beta\frac{g+\alpha h}{\alpha-\theta}}{1+(\beta-1)\frac{g+\alpha h}{\alpha-\theta}}. \tag{20}$$

1) $\Rightarrow$ 2). As we discussed earlier, condition 1) means that the derivative of the function on the right hand side of (19) exists a.e. on $\mathbb{T}$ for a.e. $\alpha \in \mathbb{T}$. Let us first show that this implies that the derivatives of f, g and θ exist a.e. on $\mathbb{T}$.

Indeed,

$$\left(\frac{g+\alpha h}{\alpha-\theta}\right)' = \frac{1}{\alpha-\theta}g' + \frac{\alpha}{\alpha-\theta}h' + + \frac{g+\alpha h}{(\alpha-\theta)^2}\theta'. \tag{21}$$

Choose $\xi \in \mathbb{T}$ such that there exist finite non-tangential limits $g(\xi) = a$, $h(\xi) = b$, $\theta(\xi) = c$ and $\left(\frac{g+\alpha h}{\alpha-\theta}\right)'(\xi)$ for a.e. α (note that for a.e. ξ such limits exist for a.e. α). Suppose that at least one of the non-tangential limits $g'(\xi)$, $h'(\xi)$ or $\theta'(\xi)$ does not exist. WLOG we can assume that there exists a sequence $\{z_n\}$ tending to ξ non-tangentially, such that the limit

$$\lim_{n\to\infty} g'(z_n) \tag{22}$$

does not exist and both sequences $\{h'(z_n)\}$ and $\{\theta'(z_n)\}$ are $O(g'(z_n))$ (such sequence $\{z_n\}$ exists for either g', h' or θ').

Consider the vector function

$$v(\alpha) = \left(\frac{1}{\alpha-c}, \frac{\alpha}{\alpha-c}, \frac{a+\alpha b}{(\alpha-c)^2}\right)$$

whose coordinate functions represent the limits of the coefficients on the right hand side of (21) at ξ. Pick α_1, α_2 and α_3 such that the vectors $v(\alpha_i)$ are linearly independent and the limit

$$\lim_{n\to\infty}\left(\frac{g+\alpha_i h}{\alpha_i-\theta}\right)'(z_n) \tag{23}$$

exists for $i = 1, 2, 3$. Let constants d_1, d_2, d_3 be such that $\sum_{i=1}^3 d_i v(\alpha_i) = (1, 0, 0)$. Then the sum $\sum_{i=1}^3 d_i\left(\frac{g+\alpha_i h}{\alpha_i-\theta}\right)'$ can be represented as

$$\sum_{i=1}^3 d_i\left(\frac{g+\alpha_i h}{\alpha_i-\theta}\right)'(z) = k_1(z)g'(z) + k_2(z)h'(z) + k_3(z)\theta'(z)$$

where $k_1(z_n) \to 1$, $k_2(z_n) = o(1)$ and $k_3(z_n) = o(1)$. Since the limit (21) does not exist and $h'(z_n), \theta(z_n) = O(g'(z_n))$, the limit

$$\lim_{n\to\infty} \sum_{i=1}^{3} d_i \left(\frac{g + \alpha_i h}{\alpha_i - \theta} \right)' (z_n)$$

does not exist. But by the choice of α_i the limit (23) exists for $i = 1, 2, 3$ and we have a contradiction. Hence the non-tangential limits $g'(\xi)$, $h'(\xi)$ and $\theta'(\xi)$ exist for a.e. $\xi \in \mathbb{T}$.

Since

$$K\mu_1 = \frac{1}{1-\theta}, \tag{24}$$

the derivative $(K\mu_1)'$ exists a.e. on $\mathbb{T}$. Therefore $U_\alpha = U + \alpha(\cdot, \phi_1)\phi_1$ is pure point for a.e. $\alpha \in \mathbb{T}$. Also, since

$$K\nu_1 = \frac{g+h}{1-\theta}, \tag{25}$$

the derivative $(K\nu_1)'$ exists a.e. on $\mathbb{T}$. Hence $U_\alpha = U + \alpha(\cdot, \phi_2)\phi_2$ is pure point for a.e. $\alpha \in \mathbb{T}$.

2) $\Rightarrow$ 1) Since $U_\alpha = U + \alpha(\cdot, \phi_1)\phi_1$ is pure point for a.e. $\alpha \in \mathbb{T}$, the derivative $(K\mu_1)'$ exists a.e. on $\mathbb{T}$. Hence by (24), θ' exist a.e. on $\mathbb{T}$. Also since $U_\alpha = U + \alpha(\cdot, \phi_2)\phi_2$ is pure point for a.e. $\alpha \in \mathbb{T}$, the derivative $(K\mu_2)'$ exists a.e. on $\mathbb{T}$. Together with (25) this implies that $(g+h)'$ exists a.e. on $\mathbb{T}$.

The condition that the derivatives $(K\mu_1)'$ and $(K\nu_1)'$ exist a.e. on $\mathbb{T}$ is equivalent to

$$\int_{\mathbb{T}} \frac{d(\mu_1 + \nu_1)(\xi)}{|\psi - \xi|^2} < \infty \tag{26}$$

for a.e. $\psi \in \mathbb{T}$. Since $\mu_1 + \nu_1 = (1 + |f|^2)\mu_1$ and $|f| < 1 + |f|^2$, (26) implies

$$\int_{\mathbb{T}} \frac{|f| d\mu_1(\xi)}{|\psi - \xi|^2} < \infty.$$

Therefore the derivatives $(Kf\mu_1)'$ and $(K\overline{f}\mu_1)'$ exist a.e. on $\mathbb{T}$. Since $f = (1-\theta)Kf\mu_1$ and $\hat{f} = (1-\theta)K\overline{f}\mu_1$, f' and $\hat{f}'$ also exist a.e. on $\mathbb{T}$. Thus $(\hat{f}f)' = (g+\theta h)'$ exist a.e. on $\mathbb{T}$. Since $(g+h)'$ and $(g+\theta h)'$ exist a.e. on $\mathbb{T}$, $((1-\theta)h)'$ exists a.e. on $\mathbb{T}$. Since θ' and $(g+h)'$ exist a.e. on $\mathbb{T}$, this implies that g' and h' exist a.e. on $\mathbb{T}$. Therefore by (19), for each $\alpha \in \mathbb{T}$ $(K\nu_\alpha)'$ exists a.e. on $\mathbb{T}$. By Lemma 2 this means that the operator $U_{(\alpha,\beta)}$ is pure point for a.e. $\beta \in \mathbb{T}$ for every $\alpha \in \mathbb{T}$. ▲

Theorem 4 implies that if operator A_λ is pure point for a.e. $\lambda \in \mathbb{R}^n$, then for any line $L \subset \mathbb{R}^n$ parallel to one of the coordinate axis, A_λ is pure point for a.e. $\lambda \in L$. We will now show that this statement holds true even if one replaces the line L with an arbitrary analytic curve.

Definition. Let $I_1, I_2, ..., I_n$ be inner functions in the unit disk $\mathbb{D}$. Let $\Sigma \in \mathbb{T}$, $m(\Sigma) = 1$ be a set such that $I_k(\xi)$ exists for every $\xi \in \Sigma$ for any $1 \leq k \leq n$. Let γ be a subset of the n-dimensional torus $\mathbb{T}^n$ such that

$$\gamma = \{\alpha \in \mathbb{T}^n | \alpha = \gamma(\xi) = (I_1(\xi), I_2(\xi), ..., I_n(\xi)),\ \xi \in \Sigma\}.$$

We will call such γ an analytic curve in $\mathbb{T}^n$.

Similarly, if $J_1, J_2, ..., J_n$ are analytic functions in $\mathbb{C}_+$ whose boundary values are real a. e. on $\mathbb{R}$ and imaginary parts are non-negative in $\mathbb{C}_+$, then we can consider $\gamma \in \mathbb{R}^n$ such that

$$\gamma = \{\lambda \in \mathbb{R}^n | \lambda = \gamma(x) = (J_1(x), J_2(x), ..., J_n(x)),\ x \in \Sigma\}$$

where $\Sigma \in \mathbb{R}$ is a set of full measure such that $J_k(x)$ exists for every $x \in \Sigma$ for any $1 \geq k \geq n$. We will call such γ an analytic curve in $\mathbb{R}^n$.

Let ω be the standard conformal map from $\mathbb{C}_+$ to $\mathbb{D}$: $\omega(z) = \frac{z-i}{z+i}$. As one can see, for any analytic curve $\gamma \in \mathbb{R}^n, \gamma(t) = (J_1(t), ..., J_n(t))$ there exists an analytic curve $\eta \in \mathbb{T}^n, \eta(\xi) = (I_1(\xi), ..., I_n(\xi))$ such that $J_k(z) = \omega^{-1} I_k(\omega(z))$. Conversely, any analytic curve in $\mathbb{T}^n$ is an "image" of an analytic curve in $\mathbb{R}^n$.

We will need the following generalization of formula (9):

Lemma 8. *Let $U, \phi_i (i = 1, ..., n)$ and U_α be as in Theorem 4'. Denote by ν_α the spectral measure of ϕ_n for U_α. Let $\gamma = (I_1, I_2, ..., I_n)$ be an analytic curve in $\mathbb{T}^n$. Then there exists a bounded analytic in $\mathbb{D}$ function φ such that*

$$\int_{\mathbb{T}} \nu_{\gamma(\xi)} dm(\xi) = \varphi m$$

i. e. for any Borel set $B \subset \mathbb{T}$

$$\int_{\mathbb{T}} \nu_{\gamma(\xi)}(B) dm(\xi) = \int_B \varphi dm. \tag{27}$$

If $I_k(0) = 0$ for $k = 1, ..., n$ then $\varphi = 1$.

In the proof we will obtain an explicit formula for the function φ for the case $n = 2$, see (32).

Proof for n=2. WLOG $||\phi_i|| = 1$. Starting as in the proof of Theorem 4' (see (20)) we can observe that

$$K\nu_{(I_1(\xi), I_2(\xi))} = \frac{\frac{\overline{I_1(\xi)}g+h}{1-\overline{I_1(\xi)}\theta}}{\overline{I_2(\xi)} + (1 - \overline{I_2(\xi)})\frac{\overline{I_1(\xi)}g+h}{1-\overline{I_1(\xi)}\theta}} \tag{28}$$

where $g, h \in \cap_{0<p<1}\theta^*(H^p)$ (to simplify formula (18) we will again assume that the function from $\theta^*(H^2)$ corresponding to ϕ_2 is 0 at the origin). If $\mathcal{K}_z$ is Cauchy kernel for $z \in \mathbb{D}$, then

(29)
$$\int_{\mathbb{T}}\int_{\mathbb{T}} \mathcal{K}_z(\omega)d\nu_{(I_1(\xi),I_2(\xi))}(\omega)dm(\xi) = \int_{\mathbb{T}} \frac{\overline{I_1(\xi)}\frac{g(z)+h(z)}{1-\overline{I_1(\xi)}\theta(z)}}{\overline{I_2(\xi)} + (1-\overline{I_2(\xi)})\frac{\overline{I_1(\xi)}g(z)+h(z)}{1-\overline{I_1(\xi)}\theta(z)}}dm(\xi).$$

Note that if we fix ξ such that $|I_1(\xi)| = 1$ then

$$Re\frac{\overline{I_1(\xi)}g(z)+h(z)}{1-\overline{I_1(\xi)}\theta(z)} > \frac{1}{2} \tag{30}$$

in $\mathbb{D}$ since fraction

$$\frac{\overline{I_1(\xi)}g(z)+h(z)}{1-\overline{I_1(\xi)}\theta(z)} \tag{31}$$

is the Cauchy integral of a probability measure (namely it is the Cauchy integral of the spectral measure of ϕ_2 for $U + I_1(\xi)(\cdot, U^{-1}\phi_1)\phi_1$, see formula (19)).

Now if we fix $z \in \mathbb{D}$ then fraction (31) is a bounded antianalytic function of ξ in $\mathbb{D}$. Hence (30) must hold true for every $\xi \in \mathbb{D}$ because it holds a.e. on $\mathbb{T}$. This implies that for any fixed $z \in \mathbb{D}$ the denominator

$$\overline{I_2(\xi)} + (1-\overline{I_2(\xi)})\frac{\overline{I_1(\xi)}g(z)+h(z)}{1-\overline{I_1(\xi)}\theta(z)}$$

on the right hand side of (29) is an antianalytic function in $\mathbb{D}$ whose absolute value is bounded away from 0. Hence for every $z \in \mathbb{D}$ the whole fraction on the right hand side of (29) is a bounded antianalytic function of ξ in $\mathbb{D}$. Thus

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \mathcal{K}_z(\omega)d\nu_{(I_1(\xi),I_2(\xi))}(\omega)dm(\xi) = \frac{\overline{I_1(0)}\frac{g(z)+h(z)}{1-\overline{I_1(0)}\theta(z)}}{\overline{I_2(0)} + (1-\overline{I_2(0)})\frac{\overline{I_1(0)}g(z)+h(z)}{1-\overline{I_1(0)}\theta(z)}} = \tag{32}$$

$$= \varphi(z) = \int_{\mathbb{T}} \mathcal{K}_z\varphi dm.$$

Inequality (30) implies that $|\varphi| \leq \frac{1}{1-|I_2(0)|}$. Proceeding as in Examples 1 and 2 from the previous section, we can replace $\mathcal{K}_z$ in (32) with the characteristic function of any Borel set $B \subset \mathbb{T}$ to obtain (26). ▲

We are now ready to generalize 1) ⇒ 2) part of Theorems 4 and 4' in the following way:

Theorem 9. *Let A be a cyclic self-adjoint operator, $\phi_1, \phi_2, ..., \phi_n$ its cyclic vectors. Let γ be an analytic curve in $\mathbb{R}^n$. Suppose that operator*

$$A_\lambda = A + \sum_{k=1}^{n} \lambda_k(\cdot, \phi_k)\phi_k$$

is pure point for a.e. $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n$. Then $A_{\gamma(t)}$ is pure point for a.e. $t \in \mathbb{R}$.

Theorem 9'. *Let U, $\phi_1, \phi_2, ..., \phi_n$ and U_α be the same as in Theorem 4'. Let γ be an analytic curve in $\mathbb{T}^n$. Suppose that operator U_α is pure point for a.e. $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{T}^n$. Then $U_{\gamma(\xi)}$ is pure point for a.e. $\xi \in \mathbb{T}$.*

Proof of Theorem 9' for n=2. If $\nu_{(I_1(\xi), I_2(\xi))}$ is the spectral measure of ϕ_2 for $U_{(I_1(\xi), I_2(\xi))}$ then, as we showed in the previous proof, its Cauchy integral must satisfy

$$K\nu_{(I_1(\xi),I_2(\xi))} = \frac{\frac{\overline{I_1(\xi)}g+h}{1-\overline{I_1(\xi)}\theta}}{\overline{I_2(\xi)} + (1 - \overline{I_2(\xi)})\frac{\overline{I_1(\xi)}g+h}{1-\overline{I_1(\xi)}\theta}} \tag{33}$$

By Lemma 2 to show that $\nu_{(I_1(\xi),I_2(\xi))}$ is pure point it is enough to show that function

$$\frac{\overline{I_1(\xi)}g + h}{1 - \overline{I_1(\xi)}\theta} \tag{34}$$

has a non-tangential derivative $\nu_{(I_1(\xi),I_2(\xi))}$-a.e.

Denote by E the subset of $\mathbb{T}$ where the non-tangential derivative of (34) does not exist. As was shown in the proof of Theorem 4', the condition that U_α is pure point for a.e. α implies that functions f, g and θ have nontangential derivatives a.e. on $\mathbb{T}$. Hence $m(E) = 0$. By Lemma 8 (formula (27)) this means that $\nu_{(I_1(\xi),I_2(\xi))}(E) = 0$ for a.e. $\xi \in \mathbb{T}$. Therefore $\nu_{(I_1(\xi),I_2(\xi))}$ is pure point for a.e. $\xi \in \mathbb{T}$. ▲

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