TORIC INTERSECTION THEORY FOR AFFINE ROOT COUNTING[1]

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This paper is dedicated to Basil Gordon on the occassion of his $2^{\underline{6\text{th}}}$ birthday.

Abstract. Given any polynomial system with fixed monomial term structure, we give explicit formulae for the generic number of roots with specified coordinate vanishing restrictions. For the case of affine space minus an arbitrary union of coordinate hyperplanes, these formulae are also the tightest possible upper bounds on the number of isolated roots. We also characterize, in terms of sparse resultants, precisely when these upper bounds are attained. Finally, we reformulate and extend some of the prior combinatorial results of the author on which subsets of coefficients must be chosen generically for our formulae to be exact.

Our underlying framework provides a new toric variety setting for computational intersection theory in affine space minus an arbitrary union of coordinate hyperplanes. We thus show that, at least for root counting, it is better to work in a naturally associated toric compactification instead of always resorting to products of projective spaces.

1. Introduction

We give a new toric variety context for convex geometric root counts for polynomial systems. Our results also improve prior extensions to affine space [Kho78, DK87, Roj92, Roj94, RW96, LW96, HS96] of the seminal works [Kus75, Ber75, Kus76, Kho77] on root counting in the algebraic torus. In addition to their combinatorial appeal, there has been growing excitement about these methods in the computational algebra community due to their efficiency and applicability in many industrial problems [PC94, Emi94, EC95, VGC96].

Let us begin with some notation: Let $E_1, \ldots, E_n$ be nonempty finite subsets of $(\mathbb{N} \cup \{0\})^n$. For any $e = (e_1, \ldots, e_n) \in (\mathbb{N} \cup \{0\})^n$ let x^e denote the monomial $x_1^{e_1} \cdots x_n^{e_n}$. In this way we will let $f_1, \ldots, f_n$ be polynomials in the variables $\{x_1, \ldots, x_n\}$ with (algebraically independent) indeterminate coefficients, such that the set of exponent vectors occuring in f_i is precisely E_i. The set E_i is called the *support* of f_i and this representation specifies exactly which monomials can appear in f_i. All of our root counts will make maximal use of this monomial term information — not just the degrees of the f_i. A convenient short-hand will be the following: Let $E := (E_1, \ldots, E_n)$ and $F := (f_1, \ldots, f_n)$. Then E is the *support* of F and we call F an $n \times n$ *indeterminate polynomial system.* We also let $\mathcal{C}_E$ denote the vector (or sometimes the set) consisting of all the indeterminate coefficients of all the f_i. If we *specialize* some of the coefficients (that is, give them values chosen from some field)[2] then we say that F has support *contained* in E.

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[2] We will leave the case of more general specializations, e.g., non-trivial polynomial relations amongst the coefficients, for future work.

Let K be an algebraically closed field of arbitrary characteristic. For instance, K can be the complex numbers or the algebraic closure of a finite field. Also let $|E|$ denote the sum of the cardinalities of the E_i. Our first definition focuses our attention on the generic number of roots a polynomial system has in a given region W, when the monomial term structure is determined by E.

Proposition and Definition 1. *Let F be an $n \times n$ indeterminate polynomial system with support E, $\mathcal{C}_E$ the vector of coefficients of F, and W a constructible subset of K^n. For any $\mathcal{C} \in K^{|E|}$ let $\mathcal{N}_K(E;W;\mathcal{C})$ denote the number of roots of $F|_{\mathcal{C}_E=\mathcal{C}}$ lying in W, counting multiplicities.*[3] *Then there exists a* proper *algebraic subset $\Delta \subset K^{|E|}$, depending on E and W, such that $\mathcal{N}_K(E;W;\cdot)$ is a constant function on $K^{|E|} \setminus \Delta$. We let $\mathcal{N}_K(E;W)$ denote the value of this constant function. We will also refer to $\mathcal{N}_K(E;W)$ as the* generic value *of $\mathcal{N}_K(E;W;\cdot)$ or the* generic number of roots *of F in W.* □

Definition 2. *Let $[a..b]$ be the set of integers $\{a, a+1, \ldots, b\}$ and for any (possibly empty) $J \subseteq [1..n]$ define $O_J := \{x \in K^n \mid x_j \neq 0 \iff j \in J\}$. We call O_J an* orbit.

Note that O_J is a relatively open subset of a $|J|$-dimensional coordinate subspace of K^n.

Recall that the (n-dimensional) mixed volume, $\mathcal{M}(\cdot)$, takes as input an n-tuple of nonempty compact convex sets in $\mathbb{R}^n$ and always outputs a nonnegative real number [BF34, Grü69, Roj94, Sch94, HS95, EC95, DGH96, VGC96, DRS96].

Main Result. *We will express $\mathcal{N}_K(E;W)$ in terms of mixed volume for W an arbitrary union of orbits, K algebraically closed, and any E. We will also give a computational algebraic criterion for precisely when this generic number of isolated roots is attained, i.e., explicit algebraic equations for Δ. Our algebraic criterion is then refined to a more practical computational result: a combinatorial classification of the sets of coefficients (subvectors of $\mathcal{C}_E$) whose genericity guarantees that F indeed has exactly $\mathcal{N}_K(E;W)$ isolated roots lying in W, counting multiplicities.*

The above result is contained in Main Theorems 1–3, the Affine Point Theorem II, and Corollary 1 of the next section. Examples of our main results appear in section 3 and the remaining sections are devoted to proving our main theorems. Two useful tools applied in our proofs may be of independent interest: the Antipodality Theorem ([Roj96d] and cf. section 5) and a toric variety version of Bernshtein's Theorem (cf. section 5.1). The former tells us how curves behave at toric infinity, while the latter collects some folkloric facts relating Bernshtein's famous theorem on root counting [Ber75] to intersection theory on toric varieties [Ful84b, Ful93].

2. Summary of Our Main Results

We will make the natural restriction of considering only those E for which $\mathcal{N}_K(E;W) < \infty$. Such E, which we will call *W-nice,* are completely characterized combinatorially in the appendix. It will also be helpful to describe certain subspace unions and cones concisely.

Definition 3. *For any $I \subseteq [1..n]$ let $\mathrm{Hyper}(I) \subset K^n$ be the union of coordinate hyperplanes $\bigcup_{j \in I}\{x \mid x_j = 0\}$. Also let $\mathrm{Lin}(I) \subseteq \mathbb{R}^n$ be the coordinate subspace generated by the subset $\{\hat{e}_j \mid j \in I\}$ of the standard basis, and let σ_I be the cone defined by the intersection of $\mathrm{Lin}([1..n] \setminus I)$ with the nonnegative orthant. Finally, let $\mathbf{O}$ denote the origin in whatever module we work in. In particular, $\mathrm{Hyper}(\emptyset) = \emptyset$ and $\mathrm{Lin}(\emptyset) = \sigma_{[1..n]} = O_\emptyset = \mathbf{O}$.*

[3] See remark 5 of section 3.2 for the definition of intersection multiplicity.

2.1. **Explicit Formulae.** We give the following recursive formula for $\mathcal{N}_K(E;W)$. Although perhaps cumbersome at first glance, our formula contains important intersection theoretic information that helps extend certain algorithms for solving polynomial systems [Roj96d] and is also quite practical in low dimensions (cf. section 3.3 and remark 12). Our result also generalizes, and makes more explicit, an algorithm for computing $\mathcal{N}_{\mathbb{C}}(E;\mathbb{C}^n)$ (for a smaller class of E) alluded to in [Kho78, DK87].

Main Theorem 1. *Let K be any algebraically closed field and suppose $E:=(E_1,\ldots,E_n)$ is an n-tuple of finite subsets of $(\mathbb{N}\cup\{0\})^n$ which is nice for W, where W is a union of orbits in K^n. Also, for all $i,j\in[1..n]$, define $m_{ij}:=\min\{y_j \mid (y_1,\ldots,y_n)\in E_i\}$ and let $m_1,\ldots,m_n$ be the rows of the matrix $[m_{ij}]$. Then $\mathcal{N}_K(E;W)$ is precisely*

$$\sum_{J\subseteq[1..n]}\ \sum_{\rho: J^c\hookrightarrow[1..n]}\left[\left(\prod_{j\in J^c} m_{j\rho(j)}\right)\mathcal{N}_K(E_{(J,\rho)};W\cap \mathrm{Lin}(\rho(J^c)^c))\right],$$

where $(\cdot)^c$ denotes set-theoretic complement within $[1..n]$, $E_{(J,\rho)}:=((E_i-m_i)\cap\mathrm{Lin}(\rho(J^c)^c)\mid i\in J)$, and $\mathcal{N}_K(\emptyset;W)$ is defined as 1 or 0 according as W is $\mathbf{O}$ or $\emptyset$. Furthermore, if $W=K^n\setminus\mathrm{Hyper}(I)$ for some $I\subseteq[1..n]$, then $\mathcal{N}_K(E;W)$ is also the maximum *number of isolated roots in W, counting multiplicities.*

Remark 1. *Our root counting formulae also hold when E is not nice for W, provided one counts* embedded [Eis95, pg. 90] *zero-dimensional components as well.*

A simple example of the above formula is given in section 3.3 and its proof appears in section 6.2. Main Theorem 1 is recursive in the sense that every term on the right-hand side is a lower-dimensional or *cornered* [RW96] case of $\mathcal{N}_K(\cdot)$. In particular, the following definition and main result take care of the "first" term $\mathcal{N}_K(E_{([1..n],\cdot)};W)$.

Definition 4. *Call a k-tuple $C:=(C_1,\ldots,C_k)$ of nonempty subsets of $\mathbb{R}^n$* cornered *iff C_i lies in the nonnegative orthant and $C_i\cap\{(y_1,\ldots,y_n)\in\mathbb{R}^n \mid y_j=0\}\neq\emptyset$ for all $i\in[1..k]$ and $j\in[1..n]$. Also, for any $a_1,\ldots,a_k\in\mathbb{R}^n$, define $a\cup C$ to be the k-tuple of convex hulls $(\mathrm{Conv}(\{a_1\}\cup C_1),\ldots,\mathrm{Conv}(\{a_k\}\cup C_k))$.*

Affine Point Theorem II. *Fix $I\subseteq[1..n]$ and suppose E is an n-tuple of finite subsets of $(\mathbb{N}\cup\{0\})^n$ which is nice for $K^n\setminus\mathrm{Hyper}(I)$ and cornered. For each $i\in[1..n]$ let $a_i\in E_i\cap\mathrm{Lin}(I)$ or set $a_i:=\mathbf{O}$ if $E_i\cap\mathrm{Lin}(I)$ is empty. Then $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))=\mathcal{M}(a\cup E)$ and this generic number is also the* maximum *number of isolated roots in $K^n\setminus\mathrm{Hyper}(I)$, counting multiplicities. More generally, if E is instead nice for O_J and cornered, then $\mathcal{N}_K(E;O_J)=\sum_{J'\supseteq J}(-1)^{|J'\setminus J|}\mathcal{M}_{J'}$, where $\mathcal{M}_{J'}$ is the mixed volume corresponding to the case $W=K^n\setminus\mathrm{Hyper}(J')$.*

The above result is proved in section 6.1 and complements the author's Affine Point Theorem I which first appeared in [RW96].

Remark 2. *Note that proposition 1 directly implies that $\mathcal{N}_K(E;W)$ is additive with respect to disjoint unions in W. So we can compute $\mathcal{N}_K(E_{([1..n],\cdot)},W)$ for general W simply by summing various instances of the Affine Point Theorem II.*

Remark 3. *Separating into cornered and non-cornered cases also simplifies Khovanskii's earlier notion of attached, weakly attached, and strongly attached hyperplanes* [Kho78].

The reader need not be alarmed at the prospect of computing an alternating sum of mixed volumes since a more efficient way to compute $\mathcal{N}_K(E;O_J)$ is given by the following corollary of Main Theorem 1. This result, which generalizes a formula for $\mathcal{N}_{\mathbb{C}}(E;\mathbb{C}^n\setminus\mathrm{Hyper}(I))$ due to Huber and Sturmfels [HS96], also seems to yield a more efficient way to compute $\mathcal{N}_K(E;W)$ for general W when $n>2$.

Corollary 1. *Following the notation of Main Theorem 1, fix $I\subseteq[1..n]$ and suppose E is nice for $K^n\setminus\mathrm{Hyper}(I)$. Then $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))=\mathcal{SM}_{I^c}(E)$. Furthermore, if $J\subseteq[1..n]$, E is instead O_J-nice, and Ω is a stable subdivision of E, then $\mathcal{N}_K(E;O_J)=\sum_C\mathcal{M}(C)$, where the sum is over all stable cells $C\in\Omega$ such that the inner normal of the lifted cell $\widehat{C}$ has support J^c.*

The quantity $\mathcal{SM}_J(E)$ is a new convex geometric entity called the *J-stable mixed volume*. We refer the reader to [HS96] for its definition, and to [HS95, HS96] for the definitions of subdivisions, lifted cells, and stable cells. The support of a vector is simply the set of indices corresponding to its nonzero coordinates. Corollary 1 is proved in section 6.2.

Better still, we can determine precisely when our formulae count the number of roots exactly, even when some of the coefficients are fixed and only a few coefficients are generic.

2.2. **Algebraic and Combinatorial Criteria for Exactness.** For any $w\in\mathbb{R}^n$, let E^w_i be the set of points $y\in E_i$ which minimize the standard inner product $w\cdot y$. Similarly, for any polytope $P\subset\mathbb{R}^n$, let P^w denote the face of P with inner normal w. Also let $E^w:=(E^w_1,\ldots,E^w_n)$ and recall that a *facet* is a polytope face of codimension 1. A key innovation of Bernshtein's seminal work on root counting is the algebraic condition he gave for his formula to be the *exact* number of roots. Sadly, this "second half" of Bernshtein's Theorem is not sufficiently explored in the literature. So we give the following generalization, proved in section 6.3.

Main Theorem 2. *Following the notation of definitions 1 and 3 and Main Theorem 1, suppose $W=K^n\setminus\mathrm{Hyper}(I)$ and that the coefficients of F have all been specialized to constants in K. Let S be the polytope $\sum_{i=1}^n\mathrm{Conv}(E_i)$. Then the following condition implies that F has exactly $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))$ roots, counting multiplicities, in $K^n\setminus\mathrm{Hyper}(I)$:*

- (a$_2$) *$\prod\mathrm{Res}_{E^w}(F)\neq 0$, where the product is over all unit inner facet normals $w\in\mathbb{R}^n\setminus\sigma_I$ of S, and*
- (b$_2$) *if $n>1$ then for all $J\subsetneqq[1..n]$ containing I, and all injections $\rho:J^c\hookrightarrow[1..n]$ such that $\rho(J^c)\cap I=\emptyset$ and $\prod_{j\in J^c}m_{j\rho(j)}>0$,*

$$\mathcal{N}_K\left(E_{(J,\rho)};\mathrm{Lin}(\rho(J^c)^c)\cap(K^n\setminus\mathrm{Hyper}(I));\mathcal{C}_E\right)=\mathcal{N}_K\left(E_{(J,\rho)};\mathrm{Lin}(\rho(J^c)^c)\cap(K^n\setminus\mathrm{Hyper}(I))\right).$$

Furthermore, the converse implication holds as well if $\mathcal{N}_K(E_{([1..n],\cdot)};K^n\setminus\mathrm{Hyper}(I))>0$. In particular, (a$_2$) and (b$_2$) together imply that the zero set of F in $K^n\setminus\mathrm{Hyper}(I)$ is zero-dimensional or empty.

The sparse resultant $\mathrm{Res}_*(F)$ is described at length in [GKZ90, PS93, CE93, Stu94, GKZ94, EC95] and our notation is explained in section 6.3. Sharper criteria for the cases $\mathcal{N}_K(E_{([1..n],\cdot)};K^n\setminus\mathrm{Hyper}(I))=0$ are also discussed in section 6.3.

Remark 4. *Parallel to Main Theorem 1, Main Theorem 2 is also recursive since for any $\vartheta\subsetneqq[1..n]$, $\mathrm{Lin}(\vartheta)\cap(K^n\setminus\mathrm{Hyper}(I))$ can be naturally identified with the complement of a union of coordinate hyperplanes in $K^{|\vartheta|}$.*

Alternatively, we can give combinatorial criteria for exactness which are always sufficient and necessary. More precisely, let $c_{i,e}$ denote the (indeterminate) coefficient of the x^e term of f_i. If an n-tuple $D:=(D_1,\ldots,D_n)$ satisfies $D_i\subseteq E_i$ for all $i\in[1..n]$ then we simply abbreviate this as $D\subseteq E$. For any such D define $\mathcal{C}_D=\{c_{i,e}\mid i\in[1..n],\ e\in D_i\}$.

Definition 5. *We say that D W-*counts *E iff (0) $D\subseteq E$, (1) D and E are nice for W, and (2) for any specialization over K of the coefficients $\mathcal{C}_E\setminus\mathcal{C}_D$, a generic specialization of the remaining coefficients $\mathcal{C}_D$ suffices to make F have exactly $\mathcal{N}_K(E;W)$ roots lying in W, counting multiplicities.*

So by proposition 1 we at least know that E always W-counts E if E is W-nice.

Define $\mathrm{Supp}(D):=\{i\mid D_i\neq\emptyset\}$, $D\cap\mathrm{Lin}(J)=(D_1\cap\mathrm{Lin}(J),\ldots,D_n\cap\mathrm{Lin}(J))$, and $D\cap E^w:=(D_1\cap E^w_1,\ldots,D_n\cap E^w_n)$. Our final main theorem gives an exact convex geometric criterion for when D $(K^n\setminus\mathrm{Hyper}(I))$-counts E.

Main Theorem 3. *Following the notation of definitions 1 and 3 and Main Theorem 1, suppose that $W = K^n \setminus \mathrm{Hyper}(I)$ and let S be the polytope $\sum_{i=1}^n \mathrm{Conv}(E_i)$. Then D $(K^n\setminus\mathrm{Hyper}(I))$-counts $E \Longleftrightarrow$ one of the following exclusive conditions holds:*

1. $\mathcal{N}_K(E; K^n \setminus \mathrm{Hyper}(I)) = 0$ *and for all* $J \subseteq [1..n]$ *containing* I, $\mathrm{Supp}(D \cap \mathrm{Lin}(J))$ *contains a subset essential for* $E \cap \mathrm{Lin}(J)$.
2. $\mathcal{N}_K(E; K^n \setminus \mathrm{Hyper}(I)) > 0$ *and*
   (a$_3$) *for each face of S with an inner normal $w \in \mathbb{R}^n \setminus \sigma_I$, pick a single such w. Then for all of these w,* $\mathrm{Supp}(D \cap E^w)$ *contains a subset essential for* E^w, *and*
   (b$_3$) *if $n > 1$ then for all $J \subsetneqq [1..n]$ containing I, and all injections $\rho : J^c \hookrightarrow [1..n]$ such that $\rho(J^c) \cap I = \emptyset$ and $\prod_{j \in J^c} m_{j\rho(j)} > 0$, the $|J|$-tuple*
   $$((D_i - m_i) \cap \mathrm{Lin}(\rho(J^c)^c) \mid i \in J)$$
   $W_{(J,\rho)}$*-counts* $E_{(J,\rho)}$, *where* $W_{(J,\rho)} := \mathrm{Lin}(\rho(J^c)^c) \cap (K^n \setminus \mathrm{Hyper}(I))$.

The definition of essentiality, which is a combinatorial geometric condition, appears in the appendix. We thus obtain a recursive combinatorial condition for when the zero set of F in $K^n\setminus\mathrm{Hyper}(I)$ consists of exactly $\mathcal{N}_K(E; K^n\setminus\mathrm{Hyper}(I))$ points, counting multiplicities. Our final main theorem is proved in section 6.3 as well. Here we deal mainly with genericity conditions for *global* root counting, so we will leave the classification of O_ϑ-counting (when $\vartheta \neq [1..n]$) for another paper.

3. Examples

In the following examples, any mixed volume computation will follow easily (even by hand) from the definition or basic properties of the mixed volume [BF34, Grü69, Sch94]. In particular, it useful to recall the following formula for the $n=2$ case: $\mathcal{M}(P_1, P_2) = \mathrm{Area}(P_1 + P_2) - \mathrm{Area}(P_1) - \mathrm{Area}(P_2)$.

3.1. Comparisons to the Generalized Bézout Theorems.

Although mixed volume bounds can be hard to compute for some extremely large polynomial systems, they do have the advantage that they are always at least as good as any Bézout-type bound. Also, current mixed volume software is already fast enough to have been useful in many industrial problems, e.g., [Emi94, VGC96]. Here we will give an example of a family of polynomial systems whose mixed volume root counts are significantly better than any generalized Bézout bound.

However, let us first recall what is meant by a generalized Bézout bound. A good reference is [Wam92] so we will only quickly outline the most general (zero-dimensional) version of Bézout's Theorem: Given a partition of $\{x_1, \ldots, x_n\}$ into sets of cardinality $n_1, \ldots, n_\lambda$, the corresponding *multihomogeneous* Bézout Theorem gives an explicit formula for $\mathcal{N}_K(E; \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_\lambda}_K)$ as a polynomial expression involving the degrees of the f_i with respect to the chosen sets of variables.[4] Implicit in the grouping of variables chosen is an embedding $K^n \hookrightarrow \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_\lambda}_K$ and in this way we obtain an upper bound on $\mathcal{N}_K(E; K^n)$.

One can then try to group variables so that this method gives as tight an upper bound on $\mathcal{N}_K(E; K^n)$ as possible, but the following example shows that this bound can be very loose, no matter how one groups variables.

Example 1. (Spiky Newton Polytopes) *Consider the indeterminate polynomial system* $F := (c_{10} + c_{11}x_1^d + c_{12}x_2 + \cdots + c_{1n}x_n, \ldots, c_{n0} + c_{n1}x_1^d + c_{n2}x_2 + \cdots + c_{nn}x_n)$, *where* $d \in \mathbb{N}$ *and* $n \geq 2$. *Clearly, the Newton polytopes of F are all identical and equal to the "spike"* $P := \mathrm{Conv}(\mathbf{O}, d\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n)$. *The Affine Point Theorem II then tells us that* $\mathcal{N}_K(E; K^n) = \mathcal{M}(P, \ldots, P) = n!\mathrm{Vol}(P) = d$, *where E is the support of F.*

[4] Polynomial roots in a toric compactification are described in sections 5 and 6.1, and are formalized in definition 17 and lemma 4. In particular, products of projective spaces are special cases of toric compacta.

However, the usual Bézout Theorem tells us that $\mathcal{N}_K(E;K^n) \leq d^n$. Can this be significantly improved by going to a multi-homogeneous version? The answer is "yes, but not enough:" the best one can do is $\mathcal{N}_K(E;K^n) \leq nd$. This bound can be obtained by using two groups of variables: $\{x_1\}$ and $\{x_2,\ldots,x_n\}$. That this is the best one can do with any multihomogeneous version of Bézout's Theorem is most easily proved geometrically: It is easy to see that computing the optimal generalized Bézout bound is equivalent to finding a product of scaled standard simplices, with smallest volume, which contains P. (This reduction is described more explicitly, but in a different context, in [Roj96a]*.)*

More generally, one can use Main Theorem 3 to determine when a particular Bézout Theorem generically matches (or exceeds) a mixed volume root count: One simply lets E be the n-tuple of vertex sets of the corresponding products of simplices, and lets D be the n-tuple of vertex sets of the Newton polytopes in question. From there, one checks the corresponding *counting* criterion (cf. definition 5).

3.2. **Generic Local Intersection Multiplicity.** Setting $W=\mathbf{O}$ in Main Theorem 1 we immediately obtain a method for computing the *generic* intersection multiplicity, at the origin, of a general sparse system of n polynomials in n unknowns. An alternative general algorithm, potentially more efficient in higher dimensions, is the special case $J=\emptyset$ of Corollary 1. For example, if F is a 2×2 polynomial system with *cornered* support E, we obtain from the Affine Point Theorem II that $\mu(\mathbf{O};F)=\mathcal{N}_K(E;\mathbf{O})=\mathcal{M}(\mathbf{O}\cup E)-\mathcal{M}(E)$ for generic $\mathcal{C}_E$. This last formula already generalizes an earlier result of Warren [War94, Theorem 3] for the *unmixed* case ($E_1=E_2$) over $\mathbb{C}$.

However, it is important to note that $\mathcal{N}_K(E;\mathbf{O})$ is *not*, in general, an upper bound on intersection multiplicity at the origin: For example, it is easily verified that the polynomial system $(x+y^2, x+x^2+y^2)$ has an isolated root at $\mathbf{O}$ with multiplicity 4. (One simply notes that this system has no roots other than $\mathbf{O}$ and concludes by Bézout's Theorem in $\mathbb{P}^2_K$.) However, setting $E:=(\{(1,0),(0,2)\},\{(1,0),(2,0),(0,2)\})$, our last paragraph implies that $\mathcal{N}_K(E;\mathbf{O})=\mathcal{M}(\mathbf{O}\cup E)-\mathcal{M}(E)=4-2=2$.

Remark 5. *More generally, if F is an $n\times n$ polynomial system over K, then the intersection multiplicity $\mu(\zeta,F)$ of a zero-dimensional component $\zeta\in K^n$ of $Z(F)$ can be defined as the dimension of the quotient ring $\{\frac{r(x)}{s(x)}\in K(x) \mid \gcd(r,s)=1,\ s(\zeta)\neq 0\}/\langle F\rangle$ as a K-vector space* [Ful84b, Example 7.1.10 (b)]. *For the purposes of definition 1, we will set $\mu(\zeta,F)=+\infty$ when ζ lies in a* positive-*dimensional component of $Z(F)$.*

3.3. **Our Main Theorems in Two Dimensions.** Let $n=2$ and consider the following bivariate polynomial system:

$$\begin{aligned} f_1(x,y) &:= \alpha_1 y^2+\alpha_2 y^4+\alpha_3 xy^5+\alpha_4 x^2y^5+\alpha_5 x^2y^7+\alpha_6 x^4y^8 \\ f_2(x,y) &:= \beta_1 x+\beta_2 x^2+\beta_3 xy+\beta_4 x^2y+\beta_5 x^6y^2+\beta_6 x^6y^3+\beta_7 x^7y^3+\beta_8 x^9y^5 \end{aligned}$$

How do we get a tight upper bound on the number of isolated affine roots of $F:=(f_1,f_2)$? One way is to set $E:=\mathrm{Supp}(F)$ and apply the Affine Point Theorem I [RW96]. In which case we obtain that F has no more than $\mathcal{M}(\mathbf{O}\cup E)=53$ isolated roots. (It is also easily verified that the best generalized Bézout bound is $(\deg_x f_1)(\deg_y f_2)+(\deg_x f_2)(\deg_y f_1)=92$.) However, it is clear that E is nice for K^2 (by lemma 7 of the appendix) and not cornered, so let us see if Main Theorem 1 can do better. Figure 1 below clarifies the preceding (and upcoming) calculations.

Following the notation of Main Theorem 1, we obtain $m_1=(0,2)$, and $m_2=(1,0)$. So Main Theorem 1 asserts that

$$\begin{aligned} \mathcal{N}_K(E;K^2) &= \mathcal{N}_K(E_{\{1,2\}};K^2)+\mathcal{N}_K(\{\mathbf{O},(0,2)\},\{0\}\times K)+2\mathcal{N}_K(\{\mathbf{O},(1,0)\},K\times\{0\})+2\mathcal{N}_K(\emptyset,\mathbf{O}) \\ &= 32+\mathcal{N}_K(\{0,2\},K)+2\mathcal{N}_K(\{0,1\},K)+2. \end{aligned}$$

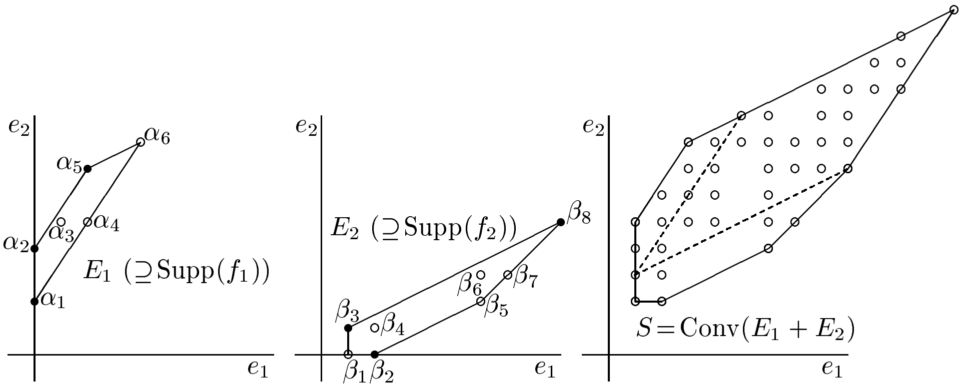

FIGURE 1. The sets E_1, E_2, and E_1+E_2, along with their underlying convex hulls. (The points of E_1 and E_2 are labelled according to their corresponding polynomial coefficients.) The subdivision of the Minkowski sum shows that, in this example, $\mathcal{M}(E)$ can actually be expressed as a single determinant.

(The last equality follows from the Affine Point Theorem II.) The remaining unknown terms add up to 4 (by the Affine Point Theorem II again, or simply the fundamental theorem of algebra), so we finally obtain the tight upper bound $\mathcal{N}_K(E; K^2)=38$.

We can also give a precise algebraic condition for when this F has *exactly* thirty eight affine roots, counting multiplicities. Applying Main Theorem 2 (and figure 1) to our example, we see that the only w we need worry about in condition (a_2) are (in counter-clockwise order) $(-1,2)$, $(-1,-1)$, $(-3,2)$, $(1,-2)$, and $(3,-2)$. Furthermore, the corresponding sparse resultants are easily seen to be 1, 1, 1, $\alpha_5\beta_8-\alpha_6\beta_3$, and 1. (The papers [Stu92, PS93, Stu94] and the book [GKZ94] contain some very nice examples of how to compute low-dimensional sparse resultants.) Condition (b_2) then clearly specializes to two one-dimensional cases of condition (a_2). (More conservatively, the fundamental theorem of algebra could also be applied to (b_2).) So it is not hard to see that condition (b_2) is equivalent to α_2 and β_2 being nonzero. Since each individual term of the summation from Main Theorem 1 was positive, we thus obtain that F has exactly thirty eight affine roots, counting multiplicities, **iff** $(\alpha_5\beta_8-\alpha_6\beta_3)\alpha_1\alpha_2\beta_2\beta_3\beta_8\neq 0$.

Similarly, by Main Theorem 3 (and making use of figures 1 and 2 (cf. the appendix)) we obtain that the genericity of $\mathcal{C}_D$ implies F has exactly thirty eight affine roots (counting multiplicities) $\Longleftrightarrow$ $\mathcal{C}_D$ contains at least one coefficient from *each* of the following sets: $\{\alpha_1,\beta_2\}$, $\{\alpha_1\}$, $\{\alpha_1,\beta_5\}$, $\{\alpha_1\}$, $\{\alpha_8\}$, $\{\alpha_6,\beta_8\}$, $\{\alpha_5,\alpha_6\}$,$\{\beta_3,\beta_8\}$, $\{\alpha_5,\beta_3\}$, $\{\beta_3\}$, and $\{\alpha_2,\beta_3\}$ (in counter-clockwise order, from condition (a_3)); and $\{\alpha_2\}$ and $\{\beta_2\}$ (from condition (b_3)). For example, regardless of how the other eight coefficients have been specialized, it suffices to choose the vector $(\alpha_1,\alpha_2,\alpha_5,\beta_2,\beta_3,\beta_8)\in K^6$ generically for F to have exactly thirty eight roots (counting multiplicities). In other words, D K^2-counts E, where $D=(\{(0,2),(0,4),(2,7)\},\{(2,0),(1,1),(9,5)\})$. The dark points in figure 1 represent D.

4. Background and Terminology

Aside from a few variations, we will follow the same notation as [Roj94, EC95, RW96, HS96, VGC96]. In those papers one can also find some of the definitions below described at a more leisurely pace. We will also liberally quote, e.g., from [Grü69, Har77, Sha80, CLO92, Sch94], various simple facts from convex and algebraic geometry that we will use. However, for the convenience of the reader, we will review a few notions.

For any $q_1,\ldots,q_n \in \mathbb{R}^n$, let $[q_1,\ldots,q_n]$ denote the $n\times n$ matrix whose $i^{\underline{\text{th}}}$ column is q_i. It will be useful to recall the following facts concerning the Smith normal form of an integral matrix [Jac85, Chap. 3.7]

Proposition and Definition 6. [Ili89, HM91] *An* integral basis *for $\mathbb{R}^n$ is a vector space basis for $\mathbb{R}^n$ which is also a $\mathbb{Z}$-module basis for $\mathbb{Z}^n$. Equivalently, a basis $\{u_1,\ldots,u_n\}$ for $\mathbb{R}^n$ is integral iff $[u_1,\ldots,u_n]\in\mathbb{GL}_n(\mathbb{Z})$. Such a basis* respects *a rational subspace L of $\mathbb{R}^n$ iff $\{u_1,\ldots,u_{\dim L}\}$ is a $\mathbb{Z}$-module basis for $L\cap\mathbb{Z}^n$. Furthermore, given any rational bases for L and a complementary subspace, an integral basis for $\mathbb{R}^n$ respecting L can be found within $O(n^{11})$ arithmetic operations.* □

Defining $x^{\mathcal{U}} := (x_1^{u_{11}}\cdots x_n^{u_{n1}},\ldots,x_1^{u_{1n}}\cdots x_n^{u_{nn}})$ for any $n\times n$ matrix $\mathcal{U}:=[u_{ij}]$ with integer entries, we then see that the above proposition tells us how we can find a *monomial* change of variables which converts a polynomial into a form involving as few variables as possible.

We will use $\mathrm{Supp}(f)$ and $\mathrm{Newt}(f)$ for, respectively, the support and Newton polytope (the convex hull of $\mathrm{Supp}(f)$) of any $f\in K'[x_1^{\pm1},\ldots,x_n^{\pm1}]$, where $K' := K[\Lambda]$ and Λ is any set of algebraically independent indeterminates.

Definition 7. *For any* weight $w\in\mathbb{R}^n$ *the* initial term polynomial $\mathrm{in}_w(f)(x)$ *is* $\sum_{e\in\mathrm{Supp}(f)^w} c_e x^e$. *More generally, if $B\supseteq\mathrm{Supp}(f)$, we define the* relativized *initial term polynomial* $\mathrm{in}_{w,B}(f)(x) := \sum_{e\in B^w\cap\mathrm{Supp}(f)} c_e x^e$. *Also, any c_e with e on the boundary of* $\mathrm{Newt}(f)$ *is called a* boundary coefficient *of f.*

Alternatively, when $w\in\mathbb{Z}^n$, we can simply substitute $x\mapsto t^w x := (t^{w_1}x_1,\ldots,t^{w_n}x_n)$ into f and define $\mathrm{in}_w(f)$ as the coefficient of the term of *lowest* degree in t.

More generally, for any $k\times n$ polynomial system F (with constant or indeterminate coefficients), we define the *initial term system* $\mathrm{in}_w(F)$ to be $(\mathrm{in}_w(f_1),\ldots,\mathrm{in}_w(f_k))$. Also, if a k-tuple $C := (C_1,\ldots,C_k)$ satisfies $C_i\supseteq\mathrm{Supp}(f_i)$ for all $i\in[1..k]$, then we say that C *contains the support of* F and we define the *relativized* initial term system $\mathrm{in}_{w,C}(F)$ to be $(\mathrm{in}_{w,C_1}(f_1),\ldots,\mathrm{in}_{w,C_k}(f_k))$. An especially important property of initial term systems is the following.

Proposition 8. [Roj94] *Suppose F is an $n\times n$ indeterminate polynomial system with support $E=(E_1,\ldots,E_n)$. In particular, we assume that each E_i is nonempty. Then for generic $\mathcal{C}_E$ and any $w\neq\mathbf{O}$, the polynomial system* $\mathrm{in}_{w,C}(F)$ *has* no *roots in* $(K^*)^n$. □

Note that for any polynomial system F with support contained in C, the set $\{\mathrm{in}_{w,C}(F)\mid w\in\mathcal{S}^{n-1}\}$ is finite: When $\mathrm{Supp}(f_i)=C_i$ for all i, we can construct a bijection between the set of initial term systems and the face lattice of $\mathrm{Conv}(\sum C_i)$, simply by picking a single inner normal w for each face.

There is a rich interplay between the combinatorial geometric structure of $\mathrm{Newt}(f)$ and the topology of the zero set of f and we will see again (in section 5 and beyond) that initial term polynomials are extremely valuable in this respect.

4.1. **Algebraic Geometry.** As usual, we will let $Z(F)$ denote the zero scheme of F in K^n. We will make some use of algebraic cycles (e.g., finite formal $\mathbb{Z}$-linear combinations of closed subvarieties of some toric variety), rational equivalence, and intersection theory, so let us also recall the following facts and definitions [Har77, Sha80, Ful84a, Ful84b, Ful93]:

1. For any cycle $\mathcal{A}$, $\mathrm{Supp}(\mathcal{A})$ is the union of all closed subvarieties V such that the coefficient of V within $\mathcal{A}$ is nonzero. Also, a divisor is said to be *effective* iff all its coefficients are nonnegative.
2. There is a natural intersection product "$\cap$" on the group of all cycles on a variety $\mathcal{X}$ giving this group a (commutative) ring structure called the *Chow ring* of $\mathcal{X}$, $\mathrm{Chow}(\mathcal{X})$. This product is also compatible with *rational equivalence* [Ful84b, pp. 10, 15–17].
3. A 0-*cycle* on $\mathcal{X}$ is a cycle of the form $\mathcal{D}=\sum n_\zeta\{\zeta\}$ where each ζ is a point and $n_\zeta\in\mathbb{Z}$. When $\mathcal{X}$ is complete, the homorphism from the group of 0-cycles on $\mathcal{X}$ to $\mathbb{Z}$ defined by $\sum n_\zeta\{\zeta\}\mapsto\sum n_\zeta$ is invariant under rational equivalence and is called the *degree map*, $\deg(\cdot)$.

4. Any intersection, $\mathcal{D}$, of $\dim \mathcal{X}$ many divisors in $\mathrm{Chow}(\mathcal{X})$ is rationally equivalent to some 0-cycle, and thus has a well-defined (*cycle class*) degree. Furthermore, if each divisor is effective, the coefficient of any zero-dimensional component ζ of such an intersection is its *intersection multiplicity*, or *intersection number*, $\mu(\zeta;\mathcal{D})$ [Ful84b, Example 7.1.10 (b)].

The most advanced prerequisites we will require from algebraic geometry will be the belief in certain theorems dealing with divisor intersections on toric varieties. Good general references are [Dan78, Oda88, Ful84b, Ful93]. The toric variety facts we'll need are covered in the next section so we now state the main intersection theoretic result we'll use.

Theorem 1. [Ful84b, Theorem 12.2] *Suppose $\mathcal{T}$ is a complete n-dimensional variety over an algebraically closed field, and $\mathcal{D}_1,\ldots,\mathcal{D}_n$ are effective Cartier divisors such that each line bundle $\mathcal{O}(\mathcal{D}_i)$ is generated by its sections. Also let $\mathcal{D}$ denote the intersection product of these divisors in* $\mathrm{Chow}(\mathcal{T})$. *Then the intersection number of any distinguished component of $\mathcal{D}$ is nonnegative. Furthermore, if we let $\mathcal{I}$ denote the sum of the intersection numbers of the distinguished components of $\mathcal{D}$, then*

$$\mathcal{I} \leq \deg \mathcal{D},$$

and equality holds if $\mathcal{D}$ is zero-dimensional or empty. □

Remark 6. *See* [Ful84b] *for the definition of a distinguished component. For our purposes, suffice it to say that a zero-dimensional irreducible component is a distinguished component, and the converse holds as well when $\mathcal{D}$ is zero-dimensional or empty. Also, embedded* [Eis95, pg. 90] *zero-dimensional components are distinguished components.*

Precise conditions for equality in the above inequality are subtle and difficult to find in the literature. However, we conjecture that equality always holds in the cases where we will apply this theorem. This has already been verified in a particular case, giving a refinement of Bézout's Theorem over $\mathbb{C}$ [Shu93].

4.2. **Toric Varieties.** We will assume the reader to be familiar with fans and the construction of toric varieties from fans and polytopes. Excellent references are [KKMS73, Dan78, Oda88, Ful93, GKZ94, Stu95].

Let $T := (K^*)^n$, which is sometimes called the *algebraic torus.*

Definition 9. *Let $P \subset \mathbb{R}^n$ be an n-dimensional rational polytope. We will associate to P its* (inner) normal fan $\mathrm{Fan}(P)$ *as follows: The rays of this fan are generated by the inner facet normals of P, and to each (not necessarily proper) face P^w of P we associate the cone σ_w generated by the rays corresponding to the facets containing P^w. Each σ_w is also called a* (inner) normal cone *of P.*

It is useful to think of the *duals* of the cones of $\mathrm{Fan}(P)$ as "angle" cones. In fact, it easy to show that for any w there is a small ball $B \subset \mathbb{R}^n$, centered at the origin, such that $B \cap \sigma_w^\vee = B \cap (P - v)$, for some $v \in \mathrm{RelInt}\, P^w$.

We will be working with the following class of toric varieties.

Definition 10. *Following the notation of definition 9, we will let $\mathcal{T}_P$ be the toric variety over K corresponding to the normal fan of P. We call $\mathcal{T}_P$ the* toric compactification *of T corresponding to P.*

It follows that $\mathcal{T}_P$ is n-dimensional, rational, projective, normal, integral, separated, and complete [Ful93]. The toric variety $\mathcal{T}_P$ also has a naturally embedded copy of T (cf. theorem 2). For certain P the toric variety $\mathcal{T}_P$ is also nonsingular but we will not need this fact. We will also say that any point of $\mathcal{T}_P \setminus (K^*)^n$ is *at infinity* and sometimes refer to $\mathcal{T}_P \setminus (K^*)^n$ as *toric infinity.* Since our polynomial systems will have a priori specified supports, F will usually have far fewer extraneous roots in an appropriately chosen $\mathcal{T}_P$ than in $\mathbb{P}^n_K$. Hence toric compactifications are the spaces where we will actually be counting roots of polynomial systems. Toward this end, it will be useful to recall the correspondence between the topology of $\mathcal{T}_P$ and the face structure of P. However, we will need a little more notation before stating this correspondence as a theorem.

Definition 11. *Given any $w\in\mathbb{R}^n$, we will use the following notation:*

$$\begin{aligned}
U_w &= \mathrm{Spec}(K[x^e \mid e\in\sigma_w^\vee\cap\mathbb{Z}^n]) = \textit{The affine chart of } \mathcal{T}_P \textit{ corresponding to the cone } \sigma_w \textit{ of } \mathrm{Fan}(P)\\
L_w &= \textit{The } \dim(P^w)\textit{-dimensional subspace of } \mathbb{R}^n \textit{ parallel to the face } P^w \textit{ of } P\\
x_w &= \textit{The point in } U_w \textit{ corresponding to the semigroup homomorphism } \sigma_w^\vee\cap\mathbb{Z}^n \longrightarrow \{0,1\} \textit{ mapping}\\
&\quad\ p\mapsto\delta_{w\cdot p,0}, \textit{ where } \delta_{ij} \textit{ denotes the Kronecker delta}\\
O_w &= \textit{The } T\textit{-orbit of } x_w = \textit{The } T\textit{-orbit corresponding to } \mathrm{RelInt} P^w\\
V_w &= \textit{The closure of } O_w \textit{ in } \mathcal{T}_P\\
p_w &= \textit{The first integral point } \text{not } \textit{equal to } \mathbf{O} \textit{ met along the ray generated by } w \textit{ (when } w\in\mathbb{Q}^n\setminus\{\mathbf{O}\})
\end{aligned}$$

Note that L_w is a face of the cone $\sigma_w^\vee$ so x_w is indeed well-defined. Also, recalling that a closed point in an affine toric variety can be identified with a semigroup homomorphism [Ful93, Chap. 1.3], it is clear that any point $x\in O_w$ is completely determined by w and the (nonzero) values of $x(\cdot)$ on any $\mathbb{Z}$-module basis of $L_w\cap\mathbb{Z}^n$. Note that our characterization of x_w is a slight variation of that of [Ful93] but is easily seen to be equivalent. In particular, our x_w is the same as Fulton's x_σ when $\sigma=\sigma_w$.

Example 2. **(Certain Cornered Polytopes)** *Suppose P is an n-dimensional rational polytope with a vertex* v *such that the edges emanating from* v *generate the nonnegative orthant as a cone. Suppose further that the coordinates of w are all* non*negative. Then $O_w\cong O_J$, where $J=\mathrm{Supp}(w)^c$. In particular, we see that $\dim P^w = \dim O_w = n - |\mathrm{Supp}(w)|$ and $\dim\sigma_w = |\mathrm{Supp}(w)|$. Note that here x_w is the 0-1 vector with support $\mathrm{Supp}(w)^c$. Furthermore, $U_{(1,\dots,1)}\cong K^n$ so we can thus conclude that K^n embeds naturally within such a $\mathcal{T}_P$. This example will be especially important in our approach to affine root counting.*

Example 3. *Suppose $w,w'\in\mathbb{R}^n$. Then, relative to $\mathcal{T}_P$, the defining ideal $I_{w'}\subset K[x^e \mid e\in\sigma_w^\vee\cap\mathbb{Z}^n]$ of $V_{w'}\cap U_w\subset U_w$ is $K[x^e \mid e\in(\sigma_w^\vee\setminus L_{w'})\cap\mathbb{Z}^n]$.*

With our orbit notation in place, we can now state the following important result.

Theorem 2. [Ful93, Chap. 3.1] *The toric variety $\mathcal{T}_P$ is the disjoint union $\coprod O_w$, where a single inner normal w is chosen for each (not necessarily proper) face of P. Also, for all $w\in\mathbb{R}^n$,*

1. $\dim O_w = \dim V_w = \dim P^w$
2. $U_w = \coprod O_v$, *where a single inner normal v is chosen for each (not necessarily proper) face containing P^w. In particular, U_w is always an n-dimensional open subvariety of $\mathcal{T}_P$.*
3. $O_w=\mathrm{Spec}(K[x^e \mid e\in L_w\cap\mathbb{Z}^n])\cong(K^*)^{\dim P^w}$.
4. *If $d=\dim P^w$ then V_w is isomorphic to any toric compactification of $(K^*)^d$ corresponding to a polytope $Q\subset\mathbb{R}^d\subseteq\mathbb{R}^n$ which is $\mathbb{GL}_n(\mathbb{Z})$-similar to a translate of P^w.* □

In particular, there is an order-preserving correspondence between the faces (resp. face interiors) of P and the orbit closures (resp. orbits) of $\mathcal{T}_P$. Also, there is an order-*reversing* correspondence between the affine charts of $\mathcal{T}_P$ and the faces of P. The above result is also contained in [KSZ92, GKZ94] but in the setting where $\mathcal{T}_P$ is defined via an explicit projective embedding.

Since toric compactifications will be the spaces in which we analyze the roots of F, it will be useful to embed the support of F within an n-tuple of nonempty integral polytopes $\mathcal{P}:=(P_1,\dots,P_n)$ and define P as a function of $\mathcal{P}$. We can then consider the roots of F within $\mathcal{T}_P$ as follows: Each (nonzero) polynomial f_i defines a Weil divisor $\mathrm{Div}(f_i)$ in $\mathcal{T}_P$ [Ful93, Chap. 3.3]. The closure (in $\mathcal{T}_P$) of the zero scheme of f_i in $(K^*)^n$ is a summand of $\mathrm{Div}(f_i)$ and is the portion of $\mathrm{Div}(f_i)$ we are actually interested in. To isolate this portion of $\mathrm{Div}(f_i)$ we will add another specially defined divisor (depending on P and P_i) to $\mathrm{Div}(f_i)$. This will cancel out the negative part of $\mathrm{Div}(f_i)$ but sometimes introduce extraneous components. In any case, the zero scheme of F in $(K^*)^n$ is thus embedded in an intersection of effective divisors in $\mathcal{T}_P$. In section 6.2 we will show how to eliminate some of these extra components, modify P so that $\mathcal{T}_P$ has a naturally embedded copy of $K^n \setminus \mathrm{Hyper}(I)$, and thus derive our method for affine root counting. The construction of our divisors is detailed in the next section.

5. The Importance of Roots at Toric Infinity

Here we point out two, more or less folkloric results on toric divisors. Combined with the Antipodality Theorem [Roj96d], these two results considerably simplify the proof of our toric compactification version (theorem 3 in the next subsection) of the BKK bound [Ber75].

First we give the following definition to help us find the right $\mathcal{T}_P$, and the right divisor to add to $\mathrm{Div}(f_i)$, for our root counting theory to go through.

Definition 12. *Let $Q \subset \mathbb{R}^n$ be an* integral *polytope. We will say that a fan $\mathcal{F}$ is* compatible *with Q iff every normal cone of Q is a union of cones of $\mathcal{F}$. We will also say that a* rational *polytope $P \subset \mathbb{R}^n$ is* compatible *with Q iff $\mathrm{Fan}(P)$ is compatible with Q. Also, following the notation of definition 11, we define the integer $\gamma_w(Q) := -\min\limits_{v \in Q}\{v \cdot p_w\}$ for any $w \in \mathbb{Q}^n \setminus \{\mathbf{O}\}$.*

Example 4. *It is easily shown that $\sum P_i$ is always compatible with $P_1, \ldots, P_n$. Compatibility was applied earlier in* [Kho77, Ful93] *and the terminology "sufficiently fine decomposition" was used in the first reference.*

Next we describe precisely which divisors we will be intersecting.

Definition 13. *Assuming $P \subset \mathbb{R}^n$ is a rational polytope compatible with an integral polytope $Q \subset \mathbb{R}^n$, let $\mathcal{E}_P(Q) := \sum \gamma_w(Q) V_w$, where w ranges over all the inner facet normals of P. We call $\mathcal{E}_P(Q)$* the torus-invariant divisor of $\mathcal{T}_P$ corresponding to Q. *Also, set $\mathcal{D}_P(0, Q) := \mathcal{T}_P$ and, for any polynomial f with $\emptyset \neq \mathrm{Supp}(f) \subseteq Q$, define $\mathcal{D}_P(f, Q) := \mathrm{Div}(f) + \mathcal{E}_P(Q)$.*

It follows by definition that $\mathcal{D}_P(f, Q)$ is always effective [Ful93] and invariant under translations of P and identical translations of Q and f. This turns out to be good for root counting in $(K^*)^n$ but bad for root counting in K^n. Hence we will modify the definition of $\mathcal{D}_P(f, Q)$ in section 6.2.

Definition 14. *Suppose $F = (f_1, \ldots, f_k)$ is a $k \times n$ polynomial system over K with support contained in a k-tuple of nonempty integral polytopes $\mathcal{P} = (P_1, \ldots, P_k)$. Then $P \subset \mathbb{R}^n$ is* compatible *with $\mathcal{P} \iff P$ is compatible with $P_1, \ldots, P_k$. Furthermore, when this is the case, we define $\mathcal{D}_P(F, \mathcal{P})$ to be the intersection product $\bigcap_{i=1}^k \mathcal{D}_P(f_i, P_i) \in \mathrm{Chow}(\mathcal{T}_P)$.*

It is easy to see that even as schemes $\overline{(K^*)^n \cap Z(F)} = (K^*)^n \cap \mathcal{D}_P(F, \mathcal{P})$ when $\dim P = n$. Also note that $\mathcal{D}_P(f, Q)$ is precisely the closure $\overline{(K^*)^n \cap Z(f)}$ in $\mathcal{T}_P$ if $\mathrm{Newt}(f) = Q$ and $\dim P = n$. By our last observation, we could just set $P_i := \mathrm{Newt}(f_i)$ for all i in the construction of $\mathcal{D}_P(F, \mathcal{P})$ in order to work directly with $\overline{(K^*)^n \cap Z(F)}$. However, this is not always advantageous computationally and it actually behooves us to fully understand the cases where $\mathrm{Newt}(f_i)$ is not compatible with P or $\mathrm{Newt}(f_i) \neq P_i$. One reason is that for precise *sparse* affine root counting, it is necessary to know precisely what happens to $\mathrm{Div}(f_i)$ as lots of coefficients of F are specialized to 0.

So let us now find explicitly the behavior of $\mathcal{D}_P(F, \mathcal{P})$ within a neighborhood of toric infinity. The following lemma, which is a direct consequence of the development followed in [Ful93] or [GKZ94], shows us that $U_w \cap \mathcal{D}_P(F, \mathcal{P})$ can be described by a relatively simple ideal.

Lemma 1. *Let $w \in \mathbb{R}^n$. Then, following the notation of definitions 11 and 14, the defining ideal in $K[x^e \mid e \in \sigma_w^\vee \cap \mathbb{Z}^n]$ of $U_w \cap \mathcal{D}_P(F, \mathcal{P})$ is $\langle x^{b_1} f_1, \ldots, x^{b_k} f_k \rangle$, for any $b_1, \ldots, b_k \in \mathbb{Z}^n$ such that $b_i + P_i^w \subseteq L_w$ for all $i \in [1..k]$.* $\square$

Should one be so inclined, the intersection multiplicity of a component of $\mathcal{D}_P(F,\mathcal{P})$ can be computed by restricting to an appropriate chart U_w and this lemma gives one an explicit coordinate ring to work in.

The following is a more computational version of the above lemma and is easily proved by localization.

Corollary 2. *Following the notation of lemma 1, the underlying topological spaces of $O_w \cap \mathcal{D}_P(F,\mathcal{P})$ and*

$$\mathrm{Spec}(K[x^e \mid e \in L_w \cap \mathbb{Z}^n]/\langle \mathrm{in}_{w,b_1+P_1}(x^{b_1}f_1), \ldots, \mathrm{in}_{w,b_k+P_k}(x^{b_k}f_k)\rangle)$$

are homeomorphic.

In particular, if $\{u_1,\ldots,u_n\} \subseteq L_w \cap \mathbb{Z}^n$ is a generating set, then the map defined by $x \mapsto (x(u_1),\ldots,x(u_n))$ is an isomorphism from O_w onto a subvariety of $(K^*)^n$, and $[z \in O_w \cap \mathcal{D}_P(F,\mathcal{P}) \Longleftrightarrow \mathrm{in}_{w,\mathcal{P}}(F)$ vanishes at the point $(z(u_1),\ldots,z(u_n))]$. □

It is a frequent misconception that the last equivalence should begin with $z \in O_w \cap \overline{(K^*)^n \cap Z(F)}$. This is false unless, for instance, $P_i = \mathrm{Newt}(f_i)$ for all i. (A simple counterexample is $(n,P,f,w) := (1,[0,2],x^2+x,1)$.) Also, it is extremely important to note that the intersection multiplicity of a component of $\mathcal{D}_P(F,\mathcal{P})$ lying in O_w can *not* always be determined by this corollary when $w \neq \mathbf{O}$. (Simply observe the case $P_1 = P_2 = P = 3\mathrm{Conv}\{\mathbf{O},\hat{e}_1,\hat{e}_2\}$, $F = (x_1^3 - x_1^2, x_2^3 - x_2^2)$, and $w := (1,1)$.) Thus there is a loss of information as we intersect with O_w and pass from F to its initial term systems.

The last two results thus tell us that the relativized initial term systems $\mathrm{in}_{w,\mathcal{P}}(F)$ (for $w \neq \mathbf{O}$) describe the topological behavior of a particular divisor intersection (canonically defined by F, $\mathcal{P}$, and P) at a piece of toric infinity. This generalizes the classical construction of how the terms of highest total degree depict the closure of the zero scheme of F at the projective hyperplane at infinity.

Our toric variety $\mathcal{T}_P$ also gives us an interesting way to detect excess components in the zero set of F in $(K^*)^n$.

Antipodality Theorem. [Roj96d] *Suppose $P \subset \mathbb{R}^n$ is an n-dimensional rational polytope, Y is a curve in $(K^*)^n$, and $\overline{Y}$ is the closure of Y in $\mathcal{T}_P$. Also, identify $\{O_w \subset \mathcal{T}_P \mid w \neq \mathbf{O}\}$ with a partition of $\mathcal{S}^{n-1}$ into* cells *by letting w and w' in $\mathcal{S}^{n-1}$ belong to the same cell iff $O_w = O_{w'}$. Then*

1. *$\overline{Y}$ must touch two (possibly identical) cells whose union does not lie in any closed coordinate hemisphere.*
2. *If $\overline{Y}$ touches a cell intersecting some open hemisphere $\mathcal{H}$ then it must also touch a cell intersecting $\mathcal{S}^{n-1} \setminus \mathcal{H}$.*

Furthermore, $\overline{Y}$ must touch at least two topologically separated cells. □

Remark 7. *The cells described above are in fact stereographic images of the facets of the dual (or polar) of P.*

5.1. **A Toric Variety Version of Bernshtein's Theorem.** Before stating our generalization of Bernshtein's Theorem, we point out a very useful immediate corollary of the Antipodality Theorem and corollary 2.

Corollary 3. *Following the notation of corollary 2, $\mathcal{D}_P(F,\mathcal{P})$ has positive dimension $\Longrightarrow \mathrm{in}_{w,\mathcal{P}}(F)$ has a root in $(K^*)^n$ for some $w \neq \mathbf{O}$.* □

D. N. Bernshtein proved the case of corollary 3 where $(K^*)^n \cap \mathcal{D}_P(F,\mathcal{P})$ is positive-dimensional and $K = \mathbb{C}$ [Ber75]. His proof used an ingenious Puiseux series construction, but unfortunately Puiseux series expansions are not always defined for algebraic curves over a field of *positive* characteristic. Hence our need for antipodality theorems.

By combining the following lemma with theorem 1, we see that working within $\mathcal{T}_P$ allows us to reduce the computation of intersection numbers (generically) to the evaluation of a mixed volume.

Lemma 2. [Ful93, Chap. 5.4] *Following the notation of definitions 13 and 14, the cycle class degree of $\mathcal{E}_P(P_1) \cap \cdots \cap \mathcal{E}_P(P_n) \in \mathrm{Chow}(\mathcal{T}_P)$ is precisely $\mathcal{M}(E)$. Furthermore, for all i, the line bundle $\mathcal{O}(\mathcal{D}_P(f_i, P_i))$ is generated by its sections.* □

Putting all our machinery together, we can derive the following toric variety version of the BKK bound.

Theorem 3. *Suppose F is an $n \times n$ polynomial system over K with support contained in an n-tuple $\mathcal{P}$ of nonempty integral polytopes in $\mathbb{R}^n$. Further suppose that $P \subset \mathbb{R}^n$ is an n-dimensional rational polytope compatible with $\mathcal{P}$. Then the zero scheme of F in $(K^*)^n$ embeds naturally as a subscheme of the toric cycle $\mathcal{D}_P(F, \mathcal{P})$ and:*

1. *If $\mathcal{D}_P(F, \mathcal{P})$ is zero-dimensional or empty then $\mathcal{D}(F, \mathcal{P})$ consists of exactly $\mathcal{M}(\mathcal{P})$ points, counting multiplicities.*
2. *If $\mathcal{D}_P(F, \mathcal{P})$ is positive-dimensional and $\mathcal{M}(\mathcal{P}) = 0$ then $\mathcal{D}_P(F, \mathcal{P})$ has* no *zero-dimensional irreducible components in $\mathcal{T}_P$.*
3. *If $\mathcal{D}_P(F, \mathcal{P})$ is positive-dimensional and $\mathcal{M}(\mathcal{P}) > 0$ then $\mathcal{D}_P(F, \mathcal{P})$ has strictly less than $\mathcal{M}(\mathcal{P})$ zero-dimensional irreducible components in $\mathcal{T}_P$, counting multiplicities.*

Remark 8. *Assertion (3) appears to be new for the case* $\mathrm{char} K \neq 0$. *The case $(K, P_1, \ldots, P_n) = (\mathbb{C}, \mathrm{Newt}(f_1), \ldots, \mathrm{Newt}(f_n))$ first appeared in* [Ber75] *and was stated as a root count over $(\mathbb{C}^*)^n$ instead of $\mathcal{T}_P$. Assertions (1) and (2) (over a general algebraically closed field) then appeared implicitly in* [Dan78]*, but this was not as well known as it should have been.*

Proof: That $(K^*)^n \cap Z(F)$ embeds as a subscheme of $\mathcal{D}_P(F, \mathcal{P})$ follows immediately from our previous observations regarding definitions 13 and 14.

Assertion (1) then follows immediately from theorem 1 and lemma 2 since $\mathcal{D}_P(f_i, P_i)$ and $\mathcal{E}_P(P_i)$ are rationally equivalent. Assertion (2) follows similarly since the intersection multiplicity of a zero-dimensional irreducible component of $\mathcal{D}_P(F, \mathcal{P})$ is positive [Ful84b]. Fulton stated this concise argument in [Ful93] for the case $K = \mathbb{C}$ and his proof has the added benefit that it is independent of the (algebraically closed) field where one is working.

For the last assertion (3) we will generalize a novel homotopy proof due to D. N. Bernshtein [Ber75]. First note that if any f_i is identically zero then there can be no zero-dimensional components and we are done. Thus we may assume that no f_i is identically zero. Our generalization of Bernshtein's argument can then be outlined as follows:

(i) Pick a point $y \in O_w$, for some $w \in \mathbb{R}^n \setminus \{\mathbf{O}\}$, which lies in a positive-dimensional component of $\mathcal{D}_P(F, \mathcal{P})$.
(ii) Construct a generic polynomial system G with n-tuple of Newton polytopes $\mathcal{P}$ and distinguished root $z \in (K^*)^n$ such that $F(z) \neq \mathbf{O}$.
(iii) Construct a rational algebraic curve $L \subset \mathcal{T}_P \times \mathbb{P}^1_K$ with parameterization $\bar{l} : \mathbb{P}^1_K \longleftrightarrow L$ such that $\bar{l}(0) = y$ and $\bar{l}(1) = z$.
(iv) For all $i \in [1..n]$, define $h_i(x, t) \in K[t, x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ to be the polynomial obtained by clearing denominators from the reduced form of the rational function $f_i(x) g_i(\bar{l}(t)) - g_i(x) f_i(\bar{l}(t))$. Show that $h_i(x, 0)$ (resp. $h_i(x, 1)$) is a nonzero scalar multiple of $f_i(x)$ (resp. $g_i(x)$).
(v) Let $H(x, t) := (h_1(x, t), \ldots, h_n(x, t))$ and consider the subscheme $Z := \overline{((K^*)^n \times K) \cap Z(H)}$ of $\mathcal{T}_P \times \mathbb{P}^1_K$. Show that $\mathcal{D}_P(F, \mathcal{P}) \cong (\mathcal{T}_P \times \{0\}) \cap Z$.
(vi) Show that the natural $(n+1)^{\underline{\text{st}}}$ coordinate projection defined on $(K^*)^n \times K$ extends to a proper morphism $\pi : \mathcal{T}_P \times \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ with $\pi(L) = \mathbb{P}^1_K$.

(vii) Define $\mathcal{Y}$ to be the union of all 1-dimensional irreducible components of Z with surjective image under π. Show that the support of the zero-dimensional part of $\mathcal{D}_P(F,\mathcal{P})$ is contained in $\mathrm{Supp}((\mathcal{T}_P\times\{0\})\cap\mathcal{Y})$.

(viii) Show that $\mathcal{Y}\cap(\mathcal{T}_P\times\{0\})$ consists of exactly $\mathcal{M}(\mathcal{P})$ points, counting multiplicities.

Assuming the above steps, (3) then follows immediately since $y\in L$, $L\subseteq\mathcal{Y}$, and thus $y\in\mathrm{Supp}(\mathcal{Y}\cap(\mathcal{T}_P\times\{0\}))$, i.e., the zero-dimensional part of $\mathcal{D}_P(F,\mathcal{P})$ consists of strictly fewer than $\mathcal{M}(\mathcal{P})$ points, counting multiplicities.

To complete our proof, we now proceed to prove each individual step.

(i): Easy, by the Antipodality Theorem.

(ii): By generic we will specifically mean that $f_1,\ldots,f_n$ are all nonzero at all roots of G in $(K^*)^n$ and that G has exactly $\mathcal{M}(\mathcal{P})$ roots in $(K^*)^n$ (counting multiplicities). That such G occur generically follows easily from proposition 8, corollary 3, assertion (1) (which we've already proved), and the fact that the intersection of any two generic conditions is again a generic condition.

(iii): We will first construct the parameterization $\bar{l}$ and then the corresponding complete curve L.

Since y is completely determined by the (nonzero) values of $y(\cdot)$ on any $\mathbb{Z}$-module basis of $L_w\cap\mathbb{Z}^n$, let $\{u_1,\ldots,u_n\}$ be any basis for $\mathbb{Z}^n$ respecting $L_w\cap\mathbb{Z}^n$. (Such a basis is guaranteed to exist by proposition 6.) Let $\mathcal{U}:=[u_1,\ldots,u_n]$ and $[v_1,\ldots,v_n]:=\mathcal{V}:=\mathcal{U}^{-1}$. By assumption $\mathcal{U}\in\mathbb{GL}_n(\mathbb{Z})$ so clearly $\mathcal{V}\in\mathbb{GL}_n(\mathbb{Z})$. Now let $l:K^*\longrightarrow(K^*)^n$ be the following parameterization of a toric line:

$$((1-t)y^{\mathcal{U}}+tz^{\mathcal{U}})^{\mathcal{V}}$$

where, quite naturally, $y^{\mathcal{U}}:=(y(u_1),\ldots,y(u_n))$ and t is a new variable. Then it is easily verified that $l(1)=z$ (via the general identity $(x^A)^B=x^{AB}$). Note that l naturally defines a rational function from $\mathbb{P}^1_K$ to $\mathcal{T}_P\times\mathbb{P}^1_K$ via $[t:1]\mapsto(l_1(t),\ldots,l_n(t))\times[t:1]$. So by [Sil86, Prop. 2.1] this rational function extends uniquely to a morphism $\bar{l}$. This is our desired $\bar{l}$ and, of course, $\bar{l}(1)=z$.

Now let L be the closure in $\mathcal{T}_P\times\mathbb{P}^1_K$ of the subvariety of $(K^*)^{n+1}$ defined by the ideal

$$\langle(1-t)y(u_1)+tz^{u_1}-x^{u_1},\ldots,(1-t)y(u_n)+tz^{u_n}-x^{u_n}\rangle.$$

It is then easily verified that the hyperplane $\mathcal{T}_P\times\{t_0\}$ intersects L in the unique point $l(t_0)\in(K^*)^n$ for all but finitely many $t_0\in K$. (Simply solve the resulting binomial equation by exponentiating by $\mathcal{V}$.) It is also clear that $\mathcal{T}_P\times\{t_0\}$ does not meet L within $(K^*)^n\times\{t_0\}$ for the remaining values of t_0. Thus L must indeed be a curve. Since L is closed, it must also be complete and equal to the graph of $\bar{l}$ in $\mathcal{T}_P\times\mathbb{P}^1_K$. By corollary 2 it easily follows that $L\cap O_{(w,1)}=(z,0)$ (for w as in (i)) and thus $\bar{l}(0)=z$.

(iv): First note that the definition makes sense since we can just substitute l for $\bar{l}$. Now to verify that $h_i(x,t)$ satsifies our desired properties, note that $l(t)^{u_i}=(1-t)y(u_i)+tz^{u_i}$ by a straightforward calculation. If we define $d:=\dim L_w$ and write $e=\alpha_1u_1+\cdots+\alpha_nu_n$, it then becomes clear that $\mathrm{ord}_t(l(t)^e)=\alpha_{d+1}+\cdots+\alpha_n$. By changing the signs of the columns of $\mathcal{U}$ where necessary, we can then assume that $w\cdot u_i\geq 0$ for all $i\in[1..n]$ and write

$$f_i(l(t))=t^{b_i}(\mathrm{in}_{w,P_i}(f_i)|_{x=(y(\pi_w(\hat{e}_1)),\ldots,y(\pi_w(\hat{e}_n)))}+\text{higher order terms in } t)$$

$$g_i(l(t))=t^{b_i}(\mathrm{in}_{w,P_i}(g_i)|_{x=(y(\pi_w(\hat{e}_1)),\ldots,y(\pi_w(\hat{e}_n)))}+\text{higher order terms in } t)$$

where $b_i\in\mathbb{Z}$ and $\pi_w:\mathbb{R}^n\longrightarrow L_w$ is the natural projection defined by the basis $\mathcal{U}$. In particular, note that $t^{-b_i}f_i(l(t))|_{t=0}=0$ (by corollary 2 and the definition of l) and $t^{-b_i}g_i(l(t))|_{t=0}\neq 0$ (by corollary 2 and the definition of G). Thus $h_i(x,t)=t^{-b_i}\kappa(t)(f_i(x)g_i(l(t))-g_i(x)f_i(l(t)))$, for some $\kappa(t)\in K(t)$ satisfying $\mathrm{ord}_t(\kappa)=0$. So we are done.

(v): First note that for any $\omega\in\mathbb{R}^{n+1}$, there is a $b\in\mathbb{Z}^{n+1}$ such that $\mathrm{Newt}(x^bh_i)$ touches every facet of the cone $\sigma^\vee_\omega$. (This follows easily since all the f_i are nonzero and G has n-tuple of Newton polytopes $\mathcal{P}$.) Thus by definition 13, $\overline{((K^*)^n\times K)\cap Z(h_i)}=\mathcal{D}_{P\times[0,1]}(h_i,P_i\times[0,d_i])$, where d_i is the t-degree of h_i. Lemma 1 then implies that for any $w' \in \mathbb{R}^n \setminus \{\mathbf{O}\}$, the defining ideal of $Z \cap U_{(w',1)}$ is generated by $x^{b_1} h_1, \ldots, x^{b_n} h_n$, for suitable $b_1, \ldots, b_n \in \mathbb{Z}^n$. By theorem 2 we know that $\mathcal{T}_P \times \{0\} \cong V_{\hat{e}_{n+1}}$, so by lemma 1 the defining ideal of $(\mathcal{T}_P \times \{0\}) \cap U_{(w',1)}$ is principal and generated by t. Since $h_i \equiv f_i \pmod t$, we thus see that the defining ideal of $Z \cap (\mathcal{T}_P \times \{0\}) \cap U_{(w',1)}$ is generated by t and $x^{b_1} f_1, \ldots, x^{b_n} f_n$. Lemma 1 then implies that $\mathcal{D}_P(F, \mathcal{P}) \cap U_{w'}$ has a defining ideal with generators $x^{b_1} f_1, \ldots, x^{b_n} f_n$. Patching together charts, we are done.

(vi): This follows easily from [Ful93, Chap. 2.4].

(vii:) Clearly, any zero-dimensional irreducible component ζ of $\mathcal{D}_P(F, \mathcal{P})$ must be contained in some positive-dimensional irreducible component Σ of Z. If $\dim \Sigma > 1$ then ζ must lie in a positive-dimensional component of $\Sigma \cap (\mathcal{T}_P \times \{0\})$ — a contradiction. Thus $\dim \Sigma = 1$. Clearly $0 \in \pi(\Sigma)$ and $\pi(\Sigma) \neq \{0\}$ so by properness we must have $\pi(\Sigma) = \mathbb{P}^1_K$. So Σ must be a component of $\mathcal{Y}$.

(viii:) Let $t_0 \in \mathbb{P}^1_K$. If $\mathcal{Y}$ is irreducible then it follows directly from the definition of intersection multiplicity that $\sum \mu(\zeta)$, where ζ ranges over all zero-dimensional irreducible components of $\mathcal{Y} \cap (\mathcal{T}_P \times \{t_0\})$, is precisely the sum of the ramification indices [Sil86] of $\pi^{-1}(t_0)$. The latter number in turn is precisely the degree of the map π and is thus independent of the point t_0 [Ful84b, Examples 4.3.7 and 7.1.15].[5] If $\mathcal{Y}$ is reducible then we can extend the definition of degree simply by summing the degrees of $\pi|_{\mathcal{Y}_j}$ over all the irreducible components $\mathcal{Y}_j$ of $\mathcal{Y}$ and then our preceding identity still holds.

Thus it suffices to compute $\sum \mu(\zeta)$ for *any* t_0. In particular, by construction, we already know that this number is precisely $\mathcal{M}(\mathcal{P})$ when $t_0 = 1$. □

The above theorem is quite useful for root counting in $(K^*)^n$ but still has the nagging problem that it doesn't give the exact number of roots when the intersections are ill-behaved — more precisely, when $\mathcal{D}_P(F, \mathcal{P})$ intersects toric infinity. However, our theorem (when combined with corollary 2) at least provides us with a computational method for knowing exactly when this happens. (Indeed, Main Theorem 2 is based on this very fact!) Also, when $\mathcal{D}_P(F, \mathcal{P})$ is zero-dimensional, the precise number of roots, counting multiplicities, can *still* be obtained as follows.

Corollary 4. *Following the notation of theorem 3, suppose further that $\mathcal{D}_P(F, \mathcal{P})$ is zero-dimensional or empty. Let $\mathcal{I} := \sum \mu(\zeta)$ where the sum ranges over all components ζ of $\mathcal{D}_P(F, \mathcal{P}) \setminus (K^*)^n$. Then the number of roots of F in $(K^*)^n$ is precisely $\mathcal{M}(\mathcal{P}) - \mathcal{I}$, counting multiplicities.* □

This approach to *exact* (as opposed to generic) root counting is pursued further in [Roj96b, Roj96c] and was independently suggested in [MSW95] (in the special case of multihomogeneous systems) and [Ver96, pp. 180–185 and 215–216] (not counting some intersection multiplicities).

Intuitively, it is a weaker condition to require $\mathcal{D}_P(F, \mathcal{P})$ to be zero-dimensional than to require all the roots of F to be isolated *and* lie in $(K^*)^n$. This statement is made more precise in the next section and in section 6.3 we will also give a combinatorial characterization of the stronger hypothesis.

Another natural question which still remains is how to extend our analysis to other spaces — for example, K^n. We do this in the next section.

Remark 9. *We point out that* **all** *of the results of this section hold for more general complete toric varieties as well — in particular, toric varieties corresponding to (compatible) complete fans. The modifications are minor and we have omitted them simply because toric compacta corresponding to polytopes are sufficiently powerful for our particular root counting problems in affine space.*

Remark 10. *A more elementary (but longer) proof of parts (1) and (2) of theorem 3 can be obtained by generalizing Huber and Sturmfels' proof of Bernshtein's Theorem* [HS95] *to arbitrary algebraically closed fields: One first replaces the use of theorem 1 and lemma 2 in our intersection theoretic proof*

[5] Note that in our definitions of ramification index and degree, we are including the inseparability degree. This is relevant when the characteristic of K is positive.

by the combinatorics of mixed subdivisions. *Then, the use of Puiseux series in their proof is replaced by some algebraic curve theory à la the proof of (viii). The resulting polyhedral proof requires no more machinery than that already used in the proof of part (3). However, for the sake of brevity we will omit this alternative proof.*

6. Proofs of Our Five Main Results

We now expand our applications of toric compacta to root counting in affine space. We will begin by proving the Affine Point Theorem II and then proceed to prove Main Theorem 1, Corollary 1, and Main Theorems 3 and 2.

6.1. **Affine Embeddings.**

Contrary to what one might expect, a toric compactification $\mathcal{T}_P$ does *not* always contain a naturally embedded copy of K^n. This technicality forces us to require P to satisfy an additional hypothesis before we apply $\mathcal{T}_P$ to root counting in K^n. The following definition is the first of our two main tricks for applying toric intersection theory to affine root counting.

Definition 15. *We say a rational polytope $P\subset\mathbb{R}^n$ is* cornered *iff* $\mathrm{Fan}(P)$ *contains the nonnegative orthant as one of its cones. More generally, for any $I\subseteq[1..n]$, P is I-*cornered *iff σ_I is one of the cones of* $\mathrm{Fan}(P)$ *(following the notation of definition 3).*

Note that cornering is different for polytopes and k-tuples of point sets: For polytopes, $\emptyset$-cornering is easily seen to be equivalent to a translate of P being identical to the nonnegative orthant in a neighborhood of $\mathbf{O}$. For a k-tuple $(C_1,\ldots,C_k)$, cornering refers to the *position* of each C_i within the nonnegative orthant $\sigma_\emptyset$.

Our last definition is well-motivated for the following reason.

Proposition 16. *If $P\subset\mathbb{R}^n$ is I-cornered then $\mathcal{T}_P$ has a naturally embedded copy of $K^n\backslash\mathrm{Hyper}(I)$. More precisely, for such a $\mathcal{T}_P$, $K^n\backslash\mathrm{Hyper}(I)\cong U_w$, where w is the 0-1 vector with support I^c.* □

We now show how to construct a special I-cornered $P^{\llcorner}$ from any given k-tuple of polytopes in $\mathbb{R}^n$.

Algorithm 1.

Input: *A positive integer n, a k-tuple of* nonempty *integral polytopes $\mathcal{P}=(P_1,\ldots,P_k)$ lying in the nonnegative orthant of $\mathbb{R}^n$, and a subset $I\subseteq[1..n]$.*

Output: *An n-dimensional rational polytope $P^{\llcorner}\subset\mathbb{R}^n$, and points $a_1,\ldots,a_k\in\mathbb{Z}^n$, such that $P^{\llcorner}$ is n-dimensional, I-cornered and compatible with $a\cup\mathcal{P}$.*

Description:
1. *For all $i\in[1..k]$ and $j\in[1..n]$, define $m_{ij}:=\min\{e_j \mid (e_1,\ldots,e_n)\in P_i\}$, and let $m_1,\ldots,m_k$ be the rows of the matrix $[m_{ij}]$.*
2. *For each $i\in[1..k]$ let $a_i\in P_i\cap\mathbb{Z}^n\cap(m_i+\mathrm{Lin}(I))$ or set $a_i:=m_i$ if $P_i\cap(m_i+\mathrm{Lin}(I))=\emptyset$.*
3. *Define $Q:=\sum_{i=1}^k\mathrm{Conv}(\{a_i\}\cup P_i)$. If $\dim Q<n$ then set $Q:=Q+\mathrm{Conv}(\mathbf{O}\cup\mathcal{B})$, where $\mathcal{B}$ is a generic set of $n-\dim Q$ rational points in the nonnegative orthant (so that $\dim Q=n$).*
4. *For all $i\in[1..k]$, let $\varepsilon_i:=\frac{1}{2}\min\{\mathrm{v}_i\}$, where the minimum ranges over all vertices $\mathrm{v}=(\mathrm{v}_1,\ldots,\mathrm{v}_n)$ of Q incident to (but not lying in) $(\mathrm{Hyper}(i)+\sum_{j=1}^n a_j)\cap Q$.*
5. *Define $P^{\llcorner}:=((\varepsilon_1,\ldots,\varepsilon_n)+\sigma_I^\vee)\cap Q$.*[6]

From the last step of our construction it is easily verified that $P^{\llcorner}$ is I-cornered. Also, since Q is already n-dimensional and compatible with $a\cup\mathcal{P}$, it is clear that our choice of $(\varepsilon_1,\ldots,\varepsilon_n)$ keeps $P^{\llcorner}$ n-dimensional and compatible with $a\cup\mathcal{P}$. Thus intersecting a translate of $\sigma_I^\vee$ with Q in step (5) is somewhat reminiscent of refining the fan of a polytope by, quoting [GKZ94, pg. 190], "cutting out (as with a knife)... any face of codimension at least 2."

[6]It is useful to note that the dual cone $\sigma_I^\vee$ is precisely $\bigcap\limits_{j\in I^c}\{(y_1,\ldots,y_n)\in\mathbb{R}^n \mid y_j\geq 0\}$.

As one may have already guessed, $P^{\llcorner}$ is especially useful for root counting in $K^n\setminus\mathrm{Hyper}(I)$ and the points $a_1,\ldots,a_k$ will also be quite important. As a warm-up, the following lemma is easily verified from theorem 2, definition 13, and proposition 16.

Lemma 3. *Following the notation of definition 14, assume further that $P_1,\ldots,P_k$ all lie in the nonnegative orthant of $\mathbb{R}^n$. Fix $I\subseteq[1..n]$ and define $a_1,\ldots,a_k$ and $P^{\llcorner}$ via algorithm 1. Then $\mathcal{P}$ cornered $\Longrightarrow$ $(K^n\setminus\mathrm{Hyper}(I))\cap Z(F)=(K^n\setminus\mathrm{Hyper}(I))\cap\mathcal{D}_{P^{\llcorner}}(F,a\cup\mathcal{P})$ as schemes. Furthermore, if $\mathrm{Newt}(f_i)=P_i$ for all i as well, then $\overline{(K^n\setminus\mathrm{Hyper}(I))\cap Z(F)}=\mathcal{D}_{P^{\llcorner}}(F,a\cup\mathcal{P})$.* $\square$

We emphasize that $K^n\setminus\mathrm{Hyper}(I)$ is **not** always naturally embedded in $\mathcal{T}_P$, hence our need for $P^{\llcorner}$. Thus, under certain assumptions, the above lemma allows us to embed an affine hypersurface into a toric divisor. In fact, we can do even better: We are now in a position to apply our framework to proving the Affine Point Theorem II.

Proof of the Affine Point Theorem II: Focusing on the first part of the theorem, the case $\mathcal{M}(a\cup E)=0$ is easiest to prove so we dispose of it first: By the author's Affine Point Theorem I [RW96], we obtain that a polynomial system with support contained in E can have *no* isolated roots in $K^n\setminus\mathrm{Hyper}(I)$. Since E is $(K^n\setminus\mathrm{Hyper}(I))$-nice by assumption, we are done.

So let us now assume that $\mathcal{M}(a\cup E)>0$. Set $\mathcal{P}:=(\mathrm{Conv}(E_1),\ldots,\mathrm{Conv}(E_n))$ and, applying algorithm 1, define $\mathcal{D}:=\mathcal{D}_{P^{\llcorner}}(F,a\cup\mathcal{P})$. We will need the following important fact:

$$\star:\ O_w\cap\mathcal{D}=\emptyset \text{ for all } w\in\mathbb{R}^n\setminus\sigma_I \Longleftrightarrow [\mathcal{D}\subset K^n\setminus\mathrm{Hyper}(I) \text{ and } \dim\mathcal{D}\leq 0].$$

That the left-hand side is equivalent to $\mathcal{D}\subset K^n\setminus\mathrm{Hyper}(I)$ follows easily from theorem 2 and proposition 16. Furthermore, it follows easily from part (2) of the Antipodality Theorem and proposition 16 that $\dim\mathcal{D}>0\Longrightarrow O_w\cap\mathcal{D}\neq\emptyset$ for some $w\in\mathbb{R}^n\setminus\sigma_I$. So $\star$ is true.

Now note that the left-hand side of $\star$ is generically true by proposition 8 and corollary 2. So by theorem 3, lemma 3, and $\star$, all but the last sentence of the Affine Point Theorem II is now verified. Note also that we may drop the assumption that E be $(K^n\setminus\mathrm{Hyper}(I))$-nice, as long as we also count *embedded* zero-dimensional components.

To prove the final part, first note the following identity of weighted set unions:

$$\dagger:\ O_J=(K^n\setminus\mathrm{Hyper}(J))\setminus\left(\bigcup_{|J'\setminus J|=1}(K^n\setminus\mathrm{Hyper}(J'))\right)\cup\left(\bigcup_{|J'\setminus J|=2}(K^n\setminus\mathrm{Hyper}(J'))\right)\setminus\cdots$$

which terminates in the appropriate union or set difference according as $n-|J|$ is even or odd. This follows easily from the principle of inclusion-exclusion [GKP94] since, for any $\vartheta\subseteq[1..n]$, $K^n\setminus\mathrm{Hyper}(\vartheta)$ is precisely the disjoint union $\coprod_{\vartheta'\supseteq\vartheta}O_{\vartheta'}$. The key to proving our alternating mixed volume formula is then to simply find an intersection theoretic analogue of $\dagger$.

To do this we must work in a new *lifted* compactification depending on J. So let $P(J')$ denote the $P^{\llcorner}$ corresponding to the $I=J'$ case of algorithm 1 and define $\widetilde{\mathcal{T}}$ to be the toric compactification corresponding to $\widetilde{P}:=\sum_{J'\supseteq J}P(J')$. Also let $a_1(J'),\ldots,a_n(J')$ respectively denote the integral points $a_1,\ldots,a_n$ from the $I=J'$ case of algorithm 1. By example 4, $\widetilde{P}$ is compatible with $a(J')\cup\mathcal{P}$ for all $J'\supseteq J$, so define $\widetilde{\mathcal{D}}(J'):=\mathcal{D}_{\widetilde{P}}(F,a(J')\cup\mathcal{P})$. Then, by [Ful93, Chap. 2.4] and our construction, there is a proper morphism $\varphi:\widetilde{\mathcal{T}}\twoheadrightarrow\mathcal{T}_{P(J)}$ with no fibers of infinite cardinality. Similar to proposition 16 and lemma 3, it is also easily checked that $O_J\cap\varphi(\widetilde{\mathcal{D}}(J))=O_J\cap Z(F)$.

More importantly, it is easily verified from definition 13 and expanding in $\mathrm{Chow}(\widetilde{\mathcal{T}})$ that (as cycles) $\widetilde{\mathcal{D}}(J'')$ is a *summand* of $\widetilde{\mathcal{D}}(J')$ for all $J''\supseteq J'\supseteq J$. Also, it immediately follows from theorem 3 that $\deg\widetilde{\mathcal{D}}(J')=\mathcal{M}_{J'}$ for all $J'\supseteq J$. So by inclusion-exclusion once again, we have the following equality of cycles:

$$\varphi^{-1}(O_J)\cap\widetilde{\mathcal{D}}(J)=\widetilde{\mathcal{D}}(J)-\left(\sum_{|J'\setminus J|=1}\widetilde{\mathcal{D}}(J')\right)+\left(\sum_{|J'\setminus J|=2}\widetilde{\mathcal{D}}(J')\right)-\cdots$$

provided $\varphi^{-1}(O_J)\cap\widetilde{\mathcal{D}}(J)$ is zero-dimensional or empty.[7] So then $\deg(\varphi^{-1}(O_J)\cap\widetilde{\mathcal{D}}(J))$ is precisely our alternating mixed volume formula. Note that $\widetilde{\mathcal{D}}(J)$ generically has exactly $\deg(\varphi^{-1}(O_J)\cap\widetilde{\mathcal{D}}(J))$ points (counting multiplicities) in $\varphi^{-1}(O_J)$, by the portion of the Affine Point Theorem II that we've already proved and since E is O_J-nice. The last cycle class degree is also precisely $\deg(O_J\cap\varphi(\widetilde{\mathcal{D}}(J)))$ [Ful84b, Example 7.1.9], so we are done. □

Of course, the assumption that E be cornered is quite restrictive. We relax this assumption in the following section by refining lemma 3, and then Main Theorem 1 follows easily by explicitly expanding a different intersection product in the Chow ring of $\mathcal{T}_{P^{\llcorner}}$.

6.2. **Chow Rings and Main Theorem 1.** Our second and final trick for applying special $\mathcal{T}_P$'s to affine root counting is a bit more abstract. Whereas our first trick ("cornering") consisted of a convex geometric construction, the construction we give now amends a difficulty with the divisors $\mathcal{D}_P(f,Q)$ we used earlier. In particular, for noncornered $(P_1,\ldots,P_n)$, it is possible that $\overline{(K^n\setminus\mathrm{Hyper}(I))\cap Z(f_i)}$ and $\mathcal{D}_{P^{\llcorner}}(f_i,\mathrm{Conv}(\{a_i\}\cup P_i))$ differ in the coefficients corresponding to the coordinate hyperplanes. This is remedied by the following definition and lemma.

Definition 17. *Following the notation of Main Theorem 1, definition 11, and lemma 3, define* $\mathcal{X}_j:=V_{\hat{e}_j}\subset\mathcal{T}_{P^{\llcorner}}$ *for any* $j\in I^c$. *Also, set* $\mathcal{D}_{\mathrm{shift}}(0,P_i):=\mathcal{T}_{P^{\llcorner}}$ *and, if* $\mathrm{Supp}(f_i)\neq\emptyset$, *define* $\mathcal{D}_{\mathrm{shift}}(f_i,P_i):=\mathcal{D}_{P^{\llcorner}}(f_i,\mathrm{Conv}(\{a_i\}\cup P_i))+\sum_{j\in I^c}m_{ij}\mathcal{X}_j\in\mathrm{Chow}(\mathcal{T}_{P^{\llcorner}})$. *Finally, let* $\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P}):=\bigcap_{i=1}^k\mathcal{D}_{\mathrm{shift}}(f_i,P_i)\in\mathrm{Chow}(\mathcal{T}_{P^{\llcorner}})$. *The* roots of f_i within $\mathcal{T}_{P^{\llcorner}}$ *are then, formally,* $\mathcal{D}_{\mathrm{shift}}(f_i,\mathrm{Newt}(f_i))$.

Lemma 4. *Following the notation of definition 17,* $\overline{(K^n\setminus\mathrm{Hyper}(I))\cap Z(F)}=\overline{(K^n\setminus\mathrm{Hyper}(I))}\cap\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$ *as schemes. Furthermore, if* $\mathrm{Newt}(f_i)=P_i$ *then* $\overline{(K^n\setminus\mathrm{Hyper}(I))\cap Z(f_i)}=\mathcal{D}_{\mathrm{shift}}(f_i,P_i)$, *within* $\mathcal{T}_{P^{\llcorner}}$. □

In particular, for any $j\in I^c$, $\mathcal{X}_j$ is the closure of the hyperplane $\{x\mid x_j=0\}\cap(K^n\setminus\mathrm{Hyper}(I))$ in $\mathcal{T}_{P^{\llcorner}}$. Keeping this in mind, the proof of the lemma is then straightforward from theorem 2, proposition 16, and lemma 3. So by "shifting" our toric divisors, we now at last have a completely general way of embedding an affine hypersurface into a toric compactification. As an application, we will prove Main Theorem 1.

Proof of Main Theorem 1: Set $\mathcal{P}:=(\mathrm{Conv}(E_1),\ldots,\mathrm{Conv}(E_n))$. We will first prove the case $W=K^n\setminus\mathrm{Hyper}(I)$ and, to do so, it will clearly suffice to demonstrate the following two statements:

$\mathrm{A}_{\mathrm{deg}}$: $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))=\deg\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$.

$\mathrm{A}_{\mathrm{sum}}$: $\deg\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$ is precisely the double summation stated in Main Theorem 1.

Consider also the following auxiliary statement:

$\mathrm{A}_{\mathrm{gen}}$: for fixed E and generic $\mathcal{C}_E$, $\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$ is zero-dimensional and supported entirely within $K^n\setminus\mathrm{Hyper}(I)$.

To prove $\mathrm{A}_{\mathrm{deg}}$ and $\mathrm{A}_{\mathrm{sum}}$, we will actually first prove $(\mathrm{A}_{\mathrm{deg}})\wedge(\mathrm{A}_{\mathrm{gen}})$ by induction on n, and then $\mathrm{A}_{\mathrm{sum}}$ will follow easily.

[7] We should remark that the left-hand intersection is *set*-theoretic, and *not* a Chow product.

First note that by the definition of $\mathcal{D}_{\mathrm{shift}}(\cdot)$ we may formally expand $\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$ in $\mathrm{Chow}(\mathcal{T}_{P^{\llcorner}})$ as a polynomial in the $\mathcal{X}_j$. More explicitly,

$$\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P}) = \bigcap_{i=1}^{n}\left(\mathcal{D}_{P^{\llcorner}}(f_i, \mathrm{Conv}(\{a_i\}\cup P_i)) + \sum_{j\in I^c} m_{ij}\mathcal{X}_j\right) =$$

$$\sum_{[1..n]\supseteq J\supseteq I}\ \sum_{\rho: J^c \hookrightarrow [1..n]} \left[\left(\prod_{j\in J^c} m_{j\rho(j)}\right)\left(\bigcap_{i\in J}\mathcal{D}_{P^{\llcorner}}(f_i, \mathrm{Conv}(\{a_i\}\cup P_i))\right)\cap\left(\bigcap_{j\in\rho(J^c)}\mathcal{X}_j\right)\right]$$

This is where the shape of our asserted formula comes from. Note that $j\in I \Longrightarrow \mathcal{X}_j \cap (K^n \setminus \mathrm{Hyper}(I)) = \emptyset$, thus allowing the slight simplification of the outer summation.

Now note that $\bigcap_{j\in\rho(J^c)}\mathcal{X}_j$ is itself isomorphic to the toric variety corresponding to a face $P_{(J,\rho)}$ of $P^{\llcorner}$, à la theorem 2. In particular, letting $\mathcal{P}_{(J,\rho)} := ((P_j - m_j)\cap \mathrm{Lin}(\rho(J^c)^c) \mid j\in J)$, it easy to see that $P_{(J,\rho)}$ can occur as the output of the $(k,n,\mathcal{P},I) \rightsquigarrow (|J|,|J|,\mathcal{P}_{(J,\rho)},\rho(J^c)^c\cap I)$ case of algorithm 1.

Let $F_{(J,\rho)}$ be the polynomial system obtained by setting the variables $\{x_j \mid j\in\rho(J^c)\}$ to 0 in the $|J|$-tuple $(x_1^{-m_{i1}}\cdots x_n^{-m_{in}} f_i \mid i\in J)$. Also note that $\bigcap_{j\in\vartheta}\mathrm{Hyper}(j) = \mathrm{Lin}(\vartheta^c)$ for any $\vartheta\subseteq[1..n]$. We may then say that

$$\left(\bigcap_{i\in J}\mathcal{D}_{P^{\llcorner}}(f_i, \mathrm{Conv}(\{a_i\}\cup P_i))\right)\cap\left(\bigcap_{j\in\rho(J^c)}\mathcal{X}_j\right) \cong \mathcal{D}_{\mathrm{shift}}(F_{(J,\rho)},\mathcal{P}_{(J,\rho)})$$

where the underlying compactification for the right-hand cycle is $\mathcal{T}_{P_{(J,\rho)}}$. This last identity follows from definitions 13 and 17, and our preceding observations.

Note that $E_{([1..n],\cdot)}$ is cornered. So then the proof of the Affine Point Theorem II (and definition 13) immediately implies that $\deg\mathcal{D}_{P^{\llcorner}}(F, a\cup\mathcal{P}) = \mathcal{M}(a\cup\mathcal{P}) = \mathcal{N}_K(E_{([1..n],\cdot)}; K^n\setminus\mathrm{Hyper}(I))$ and, generically, $\mathcal{D}_{P^{\llcorner}}(F,a\cup\mathcal{P})$ is zero-dimensional and supported entirely within $K^n\setminus\mathrm{Hyper}(I)$. As for the remaining intersection terms with $J\neq[1..n]$, our induction hypothesis (with $n=|J|$) implies that

$$\mathcal{N}_K\left(E_J; \mathrm{Lin}(\rho(J^c)^c)\cap(K^n\setminus\mathrm{Hyper}(I))\right) = \deg\mathcal{D}_{\mathrm{shift}}(F_{(J,\rho)},\mathcal{P}_{(J,\rho)}).$$

Furthermore, our induction hypothesis also implies that, generically, $\mathcal{D}_{\mathrm{shift}}(F_{(J,\rho)},\mathcal{P}_{(J,\rho)})$ is zero-dimensional and supported entirely within $\mathrm{Lin}(\rho(J^c)^c)\cap(K^n\setminus\mathrm{Hyper}(I))$.

Now note that our Chow expansion also immediately implies that

$$\mathrm{Supp}(\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})) = \bigcup_{I\subseteq J\subseteq[1..n]}\ \bigcup_{\rho:J^c\hookrightarrow[1..n]} \mathrm{Supp}(\mathcal{D}_{\mathrm{shift}}(F_{(J,\rho)},\mathcal{P}_{(J,\rho)})),$$

modulo some isomorphisms fixing $K^n\setminus\mathrm{Hyper}(I)$. Since a finite conjunction of generic conditions is again a generic condition, we thus arrive at $\mathrm{A}_{\mathrm{gen}}$.

Recall that lemma 4 states that $(K^n\setminus\mathrm{Hyper}(I))\cap Z(F)$ is naturally embedded in $\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$. Thus $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I)) \leq \deg\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$ and, by $\mathrm{A}_{\mathrm{gen}}$, we arrive at $\mathrm{A}_{\mathrm{deg}}$. Noting that the $n=1$ case of $(\mathrm{A}_{\mathrm{deg}})\wedge(\mathrm{A}_{\mathrm{gen}})$ is true simply via the fundamental theorem of algebra over K, our induction is complete.

Finally, $\mathrm{A}_{\mathrm{sum}}$ follows simply by taking degrees of both sides of our Chow expansion. Note also that our embedding, along with $\mathrm{A}_{\mathrm{deg}}$, implies that $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))$ is indeed the maximal number of isolated roots. So the case $W=K^n\setminus\mathrm{Hyper}(I)$ is proved.

The general case then follows easily from inclusion-exclusion, much like our proof of the Affine Point Theorem II. This method goes through because our asserted formula is additive with respect to disjoint unions in W, and already true for $W=K^n\setminus\mathrm{Hyper}(I)$. $\square$

Remark 11. *Since our proof of Main Theorem 1 computes $\mathcal{N}_K(E;W)$ as the degree of an algebraic cycle, remark 1 is just a straightforward abstract extension of our preceding proof.*

Remark 12. *The double summation of Main Theorem 1 can of course simplify considerably when W is smaller than K^n. For instance, there is only 1 term when $W=(K^*)^n$. Also, it is a simple combinatorial exercise to show that the double summation of Main Theorem 1 has at most $\prod_{i=1}^n(|\mathrm{Supp}(m_i)|+1)$ terms. Equality occurs, for example, when $W=K^n$. Note also that by our recursive formula, the matrix $[m_{ij}]$ can be assumed to have at most 1 nonzero entry per column if E is W-nice. So, by the fact that $(a+1)(b+1)\geq a+b+1$ for positive integers, we also obtain that our double summation has at most 2^n terms. Furthermore, this maximum is attained iff $[m_{ij}]$ has the same support as a permutation matrix.*

Remark 13. *If one would like a formula closer to the number of* distinct *roots in $K^n\setminus\mathrm{Hyper}(I)$, a useful trick is the following: Use Main Theorem 1, but replacing $[m_{ij}]$ with the 0-1 matrix having the same support. From our last proof, it is easy to see that this new formula has the effect of (generically) counting isolated roots lying on $\bigcup\limits_{J\supseteq I,\ |J|=n-1} O_J$* without *multiplicity. Thus, if one continues to propogate this trick throughout the recursion of Main Theorem 1, we can count (omitting multiplicities due to inseparability degree) the generic number of distinct roots of F in W. This is important because for many E, a sparse system with support contained in E* always *has roots of multiplicity >1 lying in $K^n\setminus(K^*)^n$.*

A useful corollary of our proof of Main Theorem 1 is the following concise generalization of theorem 3.

Corollary 5. *Following the notation of lemma 4, assume further that $k=n$ and $\mathcal{P}=(\mathrm{Conv}(E_1),\ldots,\mathrm{Conv}(E_n))$. Then $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))=\deg\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$. Furthermore, if both $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))$ and $\dim\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$ are positive, then $\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$ has strictly less than $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))$ zero-dimensional components, counting multiplicities.*

Proof: The first portion follows immediately from our proof of the $W=K^n\setminus\mathrm{Hyper}(I)$ case of Main Theorem 1. As for the remaining portion, by the Chow expansion from our last proof, it suffices to prove the cornered case and then simply mimic the earlier descent by induction. Since the cornered case of our present corollary is already contained in the $(\mathcal{P},P)\rightsquigarrow(a\cup\mathcal{P},P^{\llcorner})$ case of theorem 3, we are done. $\square$

So theorem 3 is just the $I=[1..n]$ case of collary 5. In intersection theoretic terms, the above result establishes the *numerical positivity* [Ful84b] of any positive-dimensional component of the new shifted cycle $\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$. This will allow us to derive precise algebraic conditions for what "generic" means in the context of affine root counting.

Corollary 1 then follows easily from Main Theorem 1 as follows:

Proof of Corollary 1: Although Huber and Sturmfels did not explicitly mention intersection multiplicities in [HS96], an examination of their proof of the stable mixed volume formula shows that multiplicities were at least included implicitly. In particular, we may safely assume that the first portion of Corollary 1 is true for $K=\mathbb{C}$. The remaining portion (for $K=\mathbb{C}$) is already implicit in Huber and Sturmfels' proof[8] of the stable mixed volume formula, so we may safely assume that *all* of Corollary 1 is true for $K=\mathbb{C}$.

Generalizing to arbitrary algebraically closed K is then almost trivial: The right-hand sides (of both asserted formulae) are clearly independent of K. By Main Theorem 1 and the Affine Point Theorem II, the left-hand sides are also independent of K, provided K is algebraically closed. Since both formulae are already true for $K=\mathbb{C}$, we are done. $\square$

[8] Note that our O_J is actually $O_{\{1,\ldots,n\}\setminus J}$ in the notation of [HS96].

Similar to remark 10, a more elementary (but longer) proof of Corollary 1 can be derived by generalizing Huber and Sturmfels' proof of their stable mixed formula.

6.3. **Sparse Resultants, Roots at Infinity, and Main Theorems 2 and 3.** We conclude with an analysis of conditions under which our (global) generic root counts are exact. The conditions we give can be split into two types: algebraic and combinatorial. In the combinatorial case our conditions are always both sufficient and necessary, while in the algebraic case our criteria are always sufficient but fail to be necessary for certain systems which generically have no roots. However, we fully classify the cases where our algebraic criteria are necessary. These results will rely on the following technical result relating our shifted toric divisors with toric infinity.

Lemma 5. *Following the notation of the proof of Main Theorem 1, let $\mathcal{D}_{[1..n]}$ ($\in \mathrm{Chow}(\mathcal{T}_{P^{\llcorner}})$) be the $J=[1..n]$ term of the Chow expansion of $\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P})$. Then the following conditions imply that F has exactly $\mathcal{N}_K(E;K^n\setminus \mathrm{Hyper}(I))$ roots, counting multiplicities, in $K^n\setminus\mathrm{Hyper}(I)$:*

(a) *$O_w\cap\mathcal{D}_{[1..n]}=\emptyset$ for all $w\in\mathbb{R}^n\setminus\sigma_I$, and*
(b) *if $n>1$ then for all $J\subsetneqq[1..n]$ containing I, and all injections $\rho: J^c\hookrightarrow[1..n]$ such that $\rho(J^c)\cap I=\emptyset$ and $\prod_{j\in J^c} m_{j\rho(j)}>0$,*

$$\mathcal{N}_K\left(E_{(J,\rho)};\mathrm{Lin}(\rho(J^c)^c)\cap(K^n\setminus\mathrm{Hyper}(I));\mathcal{C}_E\right)=\mathcal{N}_K\left(E_{(J,\rho)};\mathrm{Lin}(\rho(J^c)^c)\cap(K^n\setminus\mathrm{Hyper}(I))\right).$$

Furthermore, the converse implication holds as well if $\mathcal{N}_K(E_{([1..n],\cdot)};K^n\setminus\mathrm{Hyper}(I))>0$. In particular, (a) and (b) together imply that the zero set of F in $K^n\setminus\mathrm{Hyper}(I)$ is zero-dimensional or empty.

Proof of the Lemma: Note that $\mathrm{Supp}(\mathcal{D}_{\mathrm{shift}}(F,\mathcal{P}))=\bigcup\mathrm{Supp}(\mathcal{D}_{\mathrm{shift}}(F_{(J,\rho)},\mathcal{P}_{(J,\rho)}))$ where the union ranges over $([1..n],\cdot)$ and all pairs (J,ρ) described above. This follows immediately from our Chow expansion from the proof of Main Theorem 1, and the fact that the terms with $\rho(J^c)\cap I$ nonempty or $\prod_{j\in J^c} m_{j\rho(j)}$ zero simply aren't there (by definition 17). Recall also that $E_{([1..n],\cdot)}$ is cornered. So it suffices to prove the cornered case and descend by induction, just as we did in the proofs of Main Theorem 1 and corollary 5. But the cornered case, minus the partial converse, is already contained in assertion $\star$ from our proof of the Affine Point Theorem II. One also observes that the ($\Longrightarrow$) portion of $\star$ continues to hold even when $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))=0$. So we are done. $\square$

Remark 14. *Note that the converse of the main assertion of lemma 5 can fail if $\mathcal{N}_K(E_{([1..n],\cdot)};K^n\setminus \mathrm{Hyper}(I))=0$: Consider the polynomial system $F=(1+x,1+x,(1+x)(y+z)+1)$ where $E:=\mathrm{Supp}(F)$ and note that condition (b) is violated when $w=(0,-1,-1)$.*

Remark 15. *However, the converse doesn't always fail if $\mathcal{N}_K(E_{([1..n],\cdot)};K^n\setminus\mathrm{Hyper}(I))=0$: For instance, the converse* always *holds for $n=2$ when $E=\mathrm{Supp}(F)$. More generally, for any $n>2$, setting $f_1:=1$ and $E_1:=\{\mathbf{O}\}$ gives an entire family of examples. Basically, when $\mathcal{N}_K(E;K^n\setminus \mathrm{Hyper}(I))=0$, the converse of the main assertion of lemma 5 fails precisely when E is sufficiently complicated to allow specializations of $\mathcal{C}_E$ where F has roots at toric infinity while having* none *within $K^n\setminus\mathrm{Hyper}(I)$.*

We are now ready to prove Main Theorem 3.

Proof of Main Theorem 3: We will first dispose of case (1) which is the easiest. Recall that $K^n\setminus\mathrm{Hyper}(I)=\coprod_{J\supseteq I} O_J$ and that a finite conjunction of generic conditions is again a generic condition. Since E is null for $K^n\setminus\mathrm{Hyper}(I)$ (and thus for every O_J with $J\supseteq I$) it suffices to show that our condition from case (1) is equivalent to $D\cap\mathrm{Lin}(J)$ O_J-counting $E\cap\mathrm{Lin}(J)$ for all $J\supseteq I$. This, in essence, is the statement of Lemma 3 of [Roj94]. So case (1) is complete.

As for case (2), note that E^w depends only on the face S^w. So by corollary 2, the same is true of $O_w\cap\mathcal{D}_{[1..n]}$. Then by lemma 5, the definition of W-counting, and since any finite conjunction of generic conditions is again a generic condition, we need only prove the case where E is cornered and then descend by induction just as in three of our last four proofs. For $I=\emptyset$, the cornered case is just case (2) of Theorem 7 of [RW96]. Applying algorithm 1, generalizing the proof there to arbitrary I is simple. (In fact, the proof of [RW96, Theorem 7] already contains what is essentially the $I=\emptyset$ case of algorithm 1.) So we are done. □

We now recall the *sparse resultant* (also known as the $(\mathcal{A}_1,\ldots,\mathcal{A}_k)$*-resultant*, *mixed* resultant, *Newton* resultant, or *toric* resultant), which is an extremely important operator on overdetermined polynomial systems. It is defined for any $k\times n$ **indeterminate** polynomial system F with support E, provided that all the E_i can be translated into a common $(k-1)$-dimensional subspace of $\mathbb{R}^n$. Since we can always identify such a subspace with a rational hyperplane in $\mathbb{R}^k$, we will consider only the case of $n\times(n-1)$ systems and monomial transformations (involving an extra variable) of such systems.

More explicitly, suppose E is an n-tuple of nonempty finite subsets of $\mathbb{Z}^n$ which can be translated into a common $(n-1)$-plane in $\mathbb{R}^n$. Then the sparse resultant, with respect to E, will be a (homogeneous) polynomial $\mathrm{Res}_E(\cdot)$ in the coefficients $\mathcal{C}_E$ satisfying the following property: If $\mathcal{C}\in K^{|E|}$ and $F|_{\mathcal{C}_E=\mathcal{C}}$ has a root in $(K^*)^n$, then $\mathrm{Res}_E(\mathcal{C})=0$. For fixed E, the polynomial $\mathrm{Res}_E(\cdot)$ can then be **defined** (up to a nonzero scalar multiple) as the unique polynomial in $\mathcal{C}_E$ of least total degree satisfying this last property. The computation of $\mathrm{Res}_E(\cdot)$ is a deep subject and we refer the reader to [GKZ90, PS93, CE93, SZ94, Stu94, GKZ94, EC95, Roj96c] for further background on sparse resultants.

For convenience, we will use $\mathrm{Res}_E(F)$ in place of $\mathrm{Res}_E(\mathcal{C})$ whenever the coefficients of F have been specialized to some $\mathcal{C}\in K^{|E|}$. We also point out the following important fact: $\mathrm{Res}_E(F)=0$ does **not** necessarily imply that F has a root in $(K^*)^n$. The correct statement, at least for initial term systems, is the following.

Theorem 4. [Roj96c] *Following the notation of theorem 3, suppose* $\mathcal{P}=(\mathrm{Conv}(E_i)\mid i\in[1..n])$ *and* w *is an inner facet normal of* P. *Then* $\mathrm{Res}_{E^w}(F)=0 \Longleftrightarrow V_w\cap\mathcal{D}_P(F,\mathcal{P})\neq\emptyset$. □

Remark 16. *Since* $\mathrm{in}_{w,E}(F)$ *involves a subset of the coefficients* $\mathcal{C}_E$, *we've further simplified notation by writing* $\mathrm{Res}_{E^w}(F)$ *in place of* $\mathrm{Res}_{E^w}(\mathrm{in}_{w,E}(F))$.

Our algebraic condition for F to have generically many roots is based on the above useful property of the sparse resultant.

Proof of Main Theorem 2: Recall from the proof of Main Theorem 3 that E^w and $O_w\cap\mathcal{D}_{[1..n]}$ depend only on the face S^w. So then, by theorem 2, $V_w\cap\mathcal{D}_{[1..n]}$ also depends only on S^w. To conclude, by theorems 2 and 4, it is clear that conditions (a) and (b) of lemma 5 are respectively equivalent to conditions (a_2) and (b_2) of Main Theorem 2. So we are done. □

Remark 17. *In practice, we would not actually construct the Minkowski sum* S *stated in our last two main theorems. Instead, we would actually work with the individual normal fans of* $\mathrm{Conv}(E_1),\ldots,\mathrm{Conv}(E_n)$ *and dynamically update the smallest common refinement necessary for our search space. The deeper stages in the recursion would then simply consist of slicing cones in the search fans by appropriate coordinate subspaces.*

The case $(K,I)=(\mathbb{C},[1..n])$ of Main Theorem 2 was independently discovered and presented in [HS95, Theorem 6.1]. However, the statement there is false in the case where the mixed volume is zero: Simply consider the counter-example from remark 14. Moreover, parallel to lemma 5, the converse of the main assertion of Main Theorem 2 doesn't depend completely on the positivity of $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))$: The examples given in remarks 14 and 15 also work here in an analogous way. Sharper computational conditions for exactness in the cases $\mathcal{N}_K(E;K^n\setminus\mathrm{Hyper}(I))=0$ will be addressed in future work.

Remark 18. *Following the notation of definition 1, we see that Main Theorem 2 allows us to express* Δ *as a union of hypersurfaces, via condition (a_2) and the recursion of condition (b_2). However, the* minimal *delta (containing all* $\mathcal{C}$ *such that* $F|_{\mathcal{C}_E=\mathcal{C}}$ *does* not *have the generic number of roots) need not be a hypersurface or even an algebraic variety. In general, this "generic counting discriminant" is a* constructible *variety of codimension* >1. *However, we can at least explicitly compute the codimension by combining Main Theorem 2 with Theorem 1.3 of* [Stu94].

"Sparse" techniques have recently been applied quite succesfully to *solving* many polynomial systems occuring in industrial problems [PC94, Emi94, EC95, VGC96]. Software implementations of resultant-based algorithms are also discussed in almost all of these papers. Thus Main Theorem 2 presents another potentially useful application of the sparse resultant.

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References

[Ber75] Bernshtein, D. N., *"The Number of Roots of a System of Equations,"* Functional Analysis and its Applications (translated from Russian), Vol. 9, No. 2, (1975), pp. 183–185.

[BF34] Bonnessen, T. and Fenchel, W., *Convex Bodies,* translated from German, TCS Associates, 1934.

[CE93] Canny, John F. and Emiris, Ioannis Z., *"An Efficient Algorithm for the Sparse Mixed Resultant,"* Proceedings AAECC, Puerto Rico, May, Lect. Notes in Comp. Science 263, pp. 89–104, Springer-Verlag, 1993.

[CLO92] Cox, D., Little, J., and O'Shea, D., *Ideals, Varieties, and Algorithms,* Undergraduate Texts in Mathematics, Springer-Verlag, 1992.

[Dan78] Danilov, V. I., *"The Geometry of Toric Varieties,"* Russian Mathematical Surveys, 33 (2), pp. 97–154, 1978.

[DK87] Danilov, V. I., and Khovanskii, A., *"Newton Polyhedra and an Algorithm for Computing Hodge-Deligne Numbers,"* Math. USSR Ivzestiya, Vol. 29 (1987), No. 2.

[DGH96] Dyer, M., Gritzmann, P., and Hufnagel, A., *"On the Complexity of Computing Mixed Volumes,"* SIAM J. Comput., to appear (1996).

[DRS96] Dalbec, J., Rojas, J. M., and Sturmfels, B., *"How to Fill a Mixed Volume,"* manuscript, Massachusetts Institute of Technology, 1996.

[EC95] Emiris, Ioannis Z. and Canny, John F., *"Efficient Incremental Algorithms for the Sparse Resultant and the Mixed Volume,"* Journal of Symbolic Computation, vol. 20 (1995), pp. 117–149.

[Eis95] Eisenbud, David, *"Commutative Algebra with a View Toward Algebraic Geometry,"* Graduate Texts in Mathematics 150, Springer-Verlag.

[Emi94] Emiris, Ioannis Z., *"Sparse Elimination and Applications in Kinematics,"* Ph.D. dissertation, Computer Science Division, U. C. Berkeley (December, 1994), available on-line at `http://www.inria.fr/safir/SAFIR/Ioannis.html`.

[Ful84a] Fulton, William, *Introduction to Intersection Theory in Algebraic Geometry,* no. 54, Regional Conference Series in Mathematics, American Mathematical Society, 1984.

[Ful84b] ______________, *Intersection Theory,* Springer-Verlag, 1984.

[Ful93] ______________, *Introduction to Toric Varieties*, Annals of Mathematics Studies, no. 131, Princeton University Press, Princeton, New Jersey, 1993.

[GKP94] Graham, R. L., Knuth, D. E., and Patashnik, O., *Concrete Mathematics: A Foundation for Computer Science*, $2^{\underline{\text{nd}}}$ edition, Addison-Wesley, 1994.

[GKZ90] Gel'fand, I. M., Kapranov, M. M., and Zelevinsky, A. V., *"Discriminants of Polynomials in Several Variables and Triangulations of Newton Polytopes"* Algebra and Analysis (translated from Russian) 2, 1–62, 1990.

[GKZ94] Gel'fand, I. M., Kapranov, M. M., and Zelevinsky, A. V., *Discriminants, Resultants and Multidimensional Determinants,* Birkhauser, Boston, 1994.

[Grü69] Grünbaum, Branko, *Convex Polytopes*, Interscience, London, New York, Sydney, 1969.

[Har77] Hartshorne, Robin, *Algebraic Geometry,* Springer-Verlag, 1977.

[HM91] Hafner, James L. and McCurley, Kevin S., *"Asymptotically Fast Triangularization of Matrices Over Rings,"* SIAM Journal on Computing, 20 (1991), no. 6, pp. 1068–1083.

[HS95] Huber, Birkett and Sturmfels, Bernd, *"A Polyhedral Method for Solving Sparse Polynomial Systems,"* Mathematics of Computation, 64, pp. 1541–1555, 1995.

[HS96] ________________, *"Bernshtein's Theorem in Affine Space,"* Discrete and Computational Geometry, to appear, 1996.

[Ili89] Iliopoulos, Costas S., *"Worst Case Complexity Bounds on Algorithms for Computing the Canonical Structure of Finite Abelian Groups and the Hermite and Smith Normal Forms of an Integer Matrix,"* SIAM Journal on Computing, 18 (1989), no. 4, pp. 658–669.

[Jac85] Jacobson, Nathan, *Basic Algebra I*, $2^{\underline{\text{nd}}}$ edition, W. H. Freeman and Company, 1985.

[Kho77] Khovanskii, Askold G., *"Newton Polyhedra and Toroidal Varieties,"* Functional Anal. Appl., 11 (1977), pp. 289–296.

[Kho78] ________________, *"Newton Polyhedra and the Genus of Complete Intersections,"* Functional Analysis (translated from Russian), Vol. 12, No. 1, January–March (1978), pp. 51–61.

[KKMS73] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B., *Toroidal Embeddings I*, Lecture Notes in Mathematics 339, Springer-Verlag, 1973.

[KSZ92] Kapranov, M. M., Sturmfels, B., and Zelevinsky, A. V., *"Chow Polytope and General Resultants,"* Duke Mathematical Journal, Vol. 67, No. 1, pp. 189–218 (1992).

[Kus75] Kushnirenko, A. G., *"A Newton Polytope and the Number of Solutions of a System of k Equations in k Unknowns,"* Usp. Matem. Nauk., 30, No. 2, pp. 266–267 (1975).

[Kus76] ________________, *"Newton Polytopes and the Bézout Theorem,"* Functional Analysis and its Applications (translated from Russian), vol. 10, no. 3, July–September (1976), pp. 82–83.

[LW96] Li, T. Y. and Wang, Xiaoshen, *"The BKK Root Count in $\mathbb{C}^n$,"* Mathematics of Computation, October, 1996.

[MSW95] Morgan, A. P., Sommese, A. J., and Wampler, C. W., *"A Product-Decomposition Theorem for Bounding Bézout Numbers,"* SIAM J. Numer. Anal., 32(4): 1308–1325, 1995.

[Mum76] Mumford, David *Algebraic Geometry I: Complex Algebraic Varieties*, Springer-Verlag, 1976.

[Oda88] Oda, Tadeo, *Convex Bodies and Algebraic Geometry: an Introduction to the Theory of Toric Varieties*, Springer-Verlag, 1988.

[PC94] Parsons, David and Canny, J. F., *"Geometric Problems in Molecular Biology and Robotics,"* In Proceedings of the Second International Conference on Intelligent Systems for Molecular Biology, Palo Alto, CA, August, to appear (1994).

[PS93] Pedersen, P. and Sturmfels, B., *"Product Formulas for Sparse Resultants and Chow Forms,"* Mathematische Zeitschrift, 214: 377–396, 1993.

[Roj91] Rojas, J. Maurice, *"An Optimal Condition for Determining the Exact Number of Roots of a Polynomial System,"* M.S. thesis, C.S. Division, U. C. Berkeley, 1991.

[Roj92] ________________, *"When Can One Easily Find the Number of Roots of a Nonhomogeneous Polynomial System?,"* extended abstract presented at the $40^{\underline{\text{th}}}$ Annual Meeting of the Society for Industrial and Applied Mathematics, Los Angeles, July 15–19, 1992.

[Roj94] ________________, *"A Convex Geometric Approach to Counting the Roots of a Polynomial System,"* Theoretical Computer Science (1994), vol. 133 (1), pp. 105–140. (Additional notes and corrections available on-line at `http://www-math.mit.edu/~rojas`.)

[Roj96a] ________________, *"On the Average Number of Real Roots of Certain Random Sparse Polynomial Systems,"* pp. 689–699, *The Mathematics of Numerical Analysis*, Lectures in Applied Mathematics, vol. 32 (1996), edited by Jim Renegar, Mike Shub, and Steve Smale, American Mathematical Society.

[Roj96b] ________________, *"Toric Laminations, Sparse Generalized Characteristic Polynomials, and a Refinement of Hilbert's Tenth Problem,"* Proceedings of the Rio de Janeiro Foundations of Computational Mathematics Conference (January 1997), Springer-Verlag, to appear.

[Roj96c] ________________, *"When do Resultants Really Vanish? Applications to Diophantine Complexity,"* manuscript, MIT.

[Roj96d] ________________, *"Affine Elimination Theory and Solving Certain Diophantine Equations Quickly,"* manuscript, MIT.

[RW96] Rojas, J. M., and Wang, Xiaoshen, *"Counting Affine Roots of Polynomial Systems Via Pointed Newton Polytopes,"* Journal of Complexity, vol. 12, June (1996), pp. 116–133.

[Sha80] Shafarevich, I. N., *Basic Algebraic Geometry,* Springer study editions, Springer-Verlag, 1980.

[Sch94] Schneider, Rolf, *Convex Bodies: The Brunn-Minkowski Theory,* Encyclopedia of Mathematics and its Applications, v. 44, Cambridge University Press, 1994.

[Shu93] Shub, Mike, *"Some Remarks on Bézout's Theorem and Complexity Theory,* From Topology to Computation: Proceedings of the Smalefest, pp. 443–455, Springer-Verlag, 1993.

[Sil86] Silverman, Joseph, *The Arithmetic of Elliptic Curves,* Springer Graduate Texts in Mathematics 106, Springer-Verlag, 1986.

[Stu92] Sturmfels, Bernd, *"More Examples of $\mathcal{A}$-Resultants,"* letter to Misha Kapranov, 1992.
[Stu94] ______________, *"On the Newton Polytope of the Resultant,"* Journal of Algebraic Combinatorics, 3: 207–236, 1994.
[Stu95] ______________, *Gröbner Bases and Convex Polytopes,* Lectures presented at the Holiday Symposium at New Mexico State University, December 27–31, 1994.
[SZ94] Sturmfels, B. and Zelevinsky, A., *"Multigraded Resultants of Sylvester Type"* Journal of Algebra, 163(1): 115–127, 1994.
[Ver96] Verschelde, Jan, *"Homotopy Continuation Methods for Solving Polynomial Systems,"* Doctoral Dissertation, Katholieke Universiteit Leuven, Leuven, Belgium, May 1996.
[VGC96] Verschelde, J., Gatermann, K., and Cools, R., *"Mixed Volume Computation by Dynamic Lifting Applied to Polynomial System Solving,"* Discrete and Computational Geometry, Vol. 16, No. 1, pp. 69–112 (1996).
[Wam92] Wampler, Charles W., *"Bezout Number Calculations for Multi-Homogeneous Polynomial Systems,"* Applied Mathematics and Computation 51, pp. 143–157, 1992.
[War94] Warren, *"A Bound on the Implicit Degree of Polygonal Bézier Surfaces,"* Algebraic Geometry and its Applications, edited by Chandrajit L. Bajaj, pp. 513–525

Appendix: Niceness and Genericity

Here we briefly recount some earlier results on W-counting and some related concepts. Some of the material below is covered at greater length in [Roj94] (for the case $I=[1..n]$) and [RW96] (for the case $I=\emptyset$ and E cornered). The paper [Stu94] is also a useful reference but deals more with the sparse resultant than with root counting. The results below form the basis for our combinatorial conditions for when a "partially" generic polynomial system has generically many isolated roots in a given union of orbits.

Recall that the *dimension* of any $B\subseteq\mathbb{R}^n$, $\dim B$, is the dimension of the smallest subspace of $\mathbb{R}^n$ containing a translate of B. The following two definitions are fundamental to our development.

Definition 18. *Suppose $C:=(C_1,\ldots,C_n)$ is an n-tuple of polytopes in $\mathbb{R}^n$* or *an n-tuple of finite subsets of $\mathbb{R}^n$. We will allow any C_i to be empty and say that a nonempty subset $J\subseteq[1..n]$ is* essential *for C (or C* has essential subset *J) $\Longleftrightarrow$ (0)* $\mathrm{Supp}(C)\supseteq J$, *(1)* $\dim(\sum_{j\in J}C_j)=|J|-1$, *and (2)* $\dim(\sum_{j\in J'}C_j)\geq|J'|$ *for all nonempty* proper $J'\subsetneq J$.

Definition 19. *We say that C* has an almost essential subset *$J\Longleftrightarrow$ (0)* $\mathrm{Supp}(C)\supseteq J$, *(1)* $\dim(\sum_{j\in J}C_j)=|J|$, *and (2)* $\dim(\sum_{j\in J'}C_j)\geq|J'|$ *for all nonempty $J'\subseteq J$. Also, $\emptyset$ is defined to be almost essential for C iff* $\mathrm{Supp}(C)=\emptyset$.

Equivalently, J is essential for $C\Longleftrightarrow$ the $|J|$-dimensional mixed volume of $(C_j \mid j\in J)$ is 0 and no smaller subset of J has this property. The following figure shows some simple examples of essential subsets for C, for various C in the case $n=2$. It also worth noting that $J\cup\{j\}$ is essential for $C\Longrightarrow J$ is almost essential for C, provided $|J\cup\{j\}|>|J|>0$.

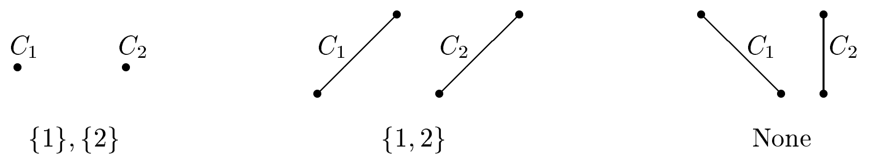

Figure 2. The essential subsets for 3 different pairs of plane polygons. (The segments in the middle pair are meant to be parallel.)

Making use of the fact that $K^n\setminus\mathrm{Hyper}(I)$ is the disjoint union $\coprod_{J\supseteq I}O_J$, our combinatorial results for $(K^n\backslash\mathrm{Hyper}(I))$-counting and $(K^n\backslash\mathrm{Hyper}(I))$-niceness will follow easily upon partitioning $K^n\backslash\mathrm{Hyper}(I)$ into orbits. In particular, it will be useful to refine W-niceness slightly as follows.

Definition 20. *Suppose E is an n-tuple of finite subsets of $(\mathbb{N} \cup \{0\})^n$. We then call E* null *for $W \Longleftrightarrow$ a generic polynomial system with support contained in E has* no *roots in W.*

For any $J \subseteq [1..n]$, define $E \cap \mathrm{Lin}(J) := (E_1 \cap \mathrm{Lin}(J), \ldots, E_n \cap \mathrm{Lin}(J))$. We may now quote the following useful result.

Lemma 6. [RW96, Corollary 2] *Suppose E is an n-tuple of finite subsets of $(\mathbb{N} \cup \{0\})^n$. Then E is nice for $O_J \Longleftrightarrow E \cap \mathrm{Lin}(J)$ has an almost essential subset of cardinality $|J|$ or an essential subset. In particular, E is null for $O_J \Longleftrightarrow E \cap \mathrm{Lin}(J)$ has an essential subset.* □

The following characterization of $(K^n \setminus \mathrm{Hyper}(I))$-niceness then follows almost immediately.

Lemma 7. *An n-tuple E of finite subsets of $(\mathbb{N} \cup \{0\})^n$ is nice for $K^n \setminus \mathrm{Hyper}(I) \Longleftrightarrow$ for all $J \supseteq I$, $E \cap \mathrm{Lin}(J)$ has an almost essential subset of cardinality $|J|$ or an essential subset. In particular, E is null for $K^n \setminus \mathrm{Hyper}(I) \Longleftrightarrow$ for all $J \supseteq I$, $E \cap \mathrm{Lin}(J)$ has an essential subset.* □

The characterization of W-niceness for W an arbitrary union of orbits is then completely analogous.

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