# ON HYPER KÄHLER MANIFOLDS ASSOCIATED TO LAGRANGEAN KÄHLER SUBMANIFOLDS OF $T^*\mathbb{C}^n$

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ABSTRACT. For any Lagrangean Kähler submanifold $M \subset T^*\mathbb{C}^n$, there exists a canonical hyper Kähler metric on $T^*M$. A Kähler potential for this metric is given by the generalized Calabi Ansatz of the theoretical physicists Cecotti, Ferrara and Girardello. This correspondence provides a method for the construction of (pseudo) hyper Kähler manifolds with large automorphism group. Using it, a class of pseudo hyper Kähler manifolds of complex signature (2,2n) is constructed. For any hyper Kähler manifold $N$ in this class a group of automorphisms with a codimension one orbit on $N$ is specified. Finally, it is shown that the bundle of intermediate Jacobians over the moduli space of gauged Calabi Yau 3-folds admits a natural pseudo hyper Kähler metric of complex signature (2,2n).

## INTRODUCTION

The generalized Calabi Ansatz of Cecotti, Ferrara and Girardello was discovered in the context of super string theory [C-F-G]. Nevertheless, it provides a simple method for the construction of such classical geometric structures as hyper Kähler metrics.

In the first part of this paper a self contained presentation of this construction is given. We explain how to a Lagrangean (pseudo) Kähler submanifold $M \subset T^*\mathbb{C}^n$ one canonically associates a (pseudo) hyper Kähler metric on the complex symplectic manifold $T^*M$, s. Thm. 1.5.

Then we study natural group actions on $M$ and on its cotangent bundle $T^*M$ preserving the special geometric structures. This opens the way for a systematic construction of (pseudo) hyper Kähler manifolds of small cohomogeneity, s. Prop. 2.1 and Cor. 2.2. Using the classification of certain Lagrangean cones given in [dW-VP] and [C2], we construct examples of pseudo hyper Kähler manifolds of complex signature (2,2n) admitting a group of automorphisms of cohomogeneity one, s. Thm. 2.3. If $M$ is a cone of appropriate signature, then $M$ can be interpreted as formal moduli space of gauged Calabi Yau 3-folds, s. Prop. 3.2 and Thm. 3.4. In particular, the Lagrangean cones classified in [dW-VP] and [C2] are models for moduli spaces with large automorphism group, s. Prop. 3.3 and Rem. 5.

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Finally, we prove that the pseudo hyper Kähler structure on the cotangent bundle $T^*M$ of a Lagrangean pseudo Kähler submanifold $M \subset T^*\mathbb{C}^n$ is always defined on a discrete fibre preserving quotient of $T^*M$ which is a torus bundle over $M$, s. Thm. 3.1. As a consequence, we obtain a natural pseudo hyper Kähler structure of complex signature (2,2n) on the bundle of intermediate Jacobians over the moduli space of gauged Calabi Yau 3-folds, where $n = h^{2,1}$, s. Thm. 3.5.

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## Contents



## 1. The canonical pseudo hyper Kähler metric on the cotangent bundle of a Lagrangean pseudo Kähler submanifold $M \subset T^*\mathbb{C}^n$

Consider the following **fundamental algebraic data**:

1) A complex symplectic vector space $(V, \omega)$, $\dim_{\mathbb{C}} V = 2n$.
2) A compatible real structure $\tau : V \to V$, i.e. a $\mathbb{C}$-antilinear involution such that the restriction of $\omega$ to its fix point set $V^\tau$ is a real symplectic structure.

Up to isomorphism, we can assume that $V = T^*\mathbb{C}^n$, $V^\tau = T^*\mathbb{R}^n$ and that $\omega$ is the standard symplectic structure of $T^*\mathbb{C}^n$, which is a real symplectic structure when restricted to $T^*\mathbb{R}^n$.

Recall that a linear subspace of a (real or complex) symplectic vector space is called **Lagrangean** if it is maximally isotropic and that a submanifold $M$ of a symplectic vector space is called **Lagrangean submanifold** if $T_mM$ is a Lagrangean subspace for every point $m \in M$.

Given $(V, \omega, \tau)$ we can define a Hermitian form $\gamma$ of signature (n,n) on $V$ by

$$\gamma(u, v) = \sqrt{-1}\omega(u, \tau v)\,, \quad u, v \in V\,. \tag{1}$$

$(V, \gamma)$ is a pseudo Kähler manifold (of complex signature (n,n)) and hence the notion of pseudo Kähler submanifold is defined. In fact, a complex submanifold $M \subset V$ is called a **pseudo Kähler submanifold** of $(V, \gamma)$ if $\gamma|M$ is nondegenerate.

Recall that a **complex symplectic structure** on a complex manifold is a holomorphic, closed and nondegenerate 2-form.

**Definition 1.1.** *A (pseudo) Kähler manifold will be called* **(pseudo) hyper Kähler manifold** *if it admits a parallel complex symplectic structure.*

It follows from this definition that a Kähler manifold of complex dimension $2n$ is hyper Kähler if and only if the holonomy group (of the canonical connection) is a subgroup of $Sp(n) = U(2n) \cap Sp(n, \mathbb{C})$, cf. [Be]. Here $Sp(n, \mathbb{C})$ denotes the symplectic group of $\mathbb{C}^{2n}$.

**Remark 1:** The complex symplectic structure $\omega$ defines on the (flat) pseudo Kähler manifold $(V,\gamma)$ the structure of pseudo hyper Kähler manifold.

**Proposition 1.1.** *Given $(V,\omega,\tau)$ and $\gamma$ as above, a Lagrangean submanifold $M \subset (V,\omega)$ is a pseudo Kähler submanifold $M \subset (V,\gamma)$ if and only if $T_mM \cap \tau T_mM = 0$ for all $m \in M$. In particular, $V^\tau \cap T_mM = 0$ is necessary.*

**Proof:** Let $L \subset (V,\omega)$ be a Lagrangean subspace. Then $\gamma|L$ is nondegenerate if and only if $L \cap \tau L = 0$. $\square$

A connected Lagrangean pseudo Kähler submanifold $M \subset (V,\omega,\tau)$ has a well defined complex signature $(k,l)$, $k+l=n$, namely the signature of the Hermitian form $\gamma|T_mM$, $m \in M$ arbitrary. If $l=0$, then $\gamma|T_mM$ is positively defined and $M$ is a Kähler submanifold of $(V,\gamma)$.

**Proposition 1.2.** *For any pseudo Kähler submanifold $M \subset (V,\gamma)$ the function $K^M(u) := \gamma(u,u)$, $u \in M$, is a pseudo Kähler potential. Any subgroup of $Aut(V,\gamma)$ which preserves $M$ acts on $M$ by holomorphic isometries.*

**Proof:** Let $u : U \to M$, $U \subset \mathbb{C}^m$, $m = \dim_{\mathbb{C}} M$, be a local holomorphic parametrization of $M$, then

$$\frac{\partial^2(K^M \circ u)}{\partial z^i \partial \overline{z}^j} \quad = \quad \gamma(\frac{\partial u}{\partial z^i}, \frac{\partial u}{\partial z^j})\,.$$

This proves the first claim. The second claim follows from the fact that $Aut(V,\gamma) \cong U(n,n)$ acts holomorphically and isometrically on the pseudo Kähler manifold $(V,\gamma)$ and that $M$ is a pseudo Kähler submanifold of $(V,\gamma)$. $\square$

The linear automorphism group $Aut(V,\omega,\tau) \cong Aut(V^\tau,\omega|V^\tau) \cong Sp(n,\mathbb{R})$ of our fundamental algebraic data acts on the hyper Kähler manifold $(V,\gamma,\omega)$ by automorphisms, i.e. by holomorphic isometries preserving the complex symplectic structure. Here $Sp(n,\mathbb{R})$ denotes the real symplectic group in 2n variables.

**Definition 1.2.** *(cf.* [C-F-G]*) The group*

$$Aut_d(M) = \{\varphi \in Aut(V,\omega,\tau) | \varphi M = M\}$$

*is called the* **duality group** *of the Lagrangean pseudo Kähler submanifold $M \subset (V,\omega,\gamma)$.*

The next proposition follows from Prop. 1.2.

**Proposition 1.3.** *The duality group $Aut_d(M)$ acts on $M$ holomorphically and isometrically.*

To define the generalized Calabi Ansatz of [C-F-G], we have to choose a **Lagrangean splitting** for $V^\tau$, i.e. a decomposition

$$V^\tau = L_0 \oplus L_0' \tag{2}$$

into two Lagrangean subspaces. A Lagrangean splitting for $V^\tau$ induces a Lagrangean splitting for $V$:

$$V = L \oplus L' , \tag{3}$$

where $L$ and $L'$ are determined by

$$L = \tau L , \quad L' = \tau L' \quad \text{and} \quad L_0 = L^\tau , \quad L'_0 = (L')^\tau . \tag{4}$$

Moreover, we have canonical isomorphisms associated to (2) and (3):

$$V^\tau \cong T^* L_0 , \quad V \cong T^* L .$$

**Definition 1.3.** *A Lagrangean submanifold $M \subset (V, \omega)$ is in* **general position** *with respect to the splitting $V = L \oplus L'$ if the projection $p : V \to L$ induces an isomorphism of $M$ onto its image.*

**Proposition 1.4.** *A Lagrangean Kähler submanifold $M \subset (V, \omega, \gamma)$ is in general position with respect to any splitting $V = L \oplus L'$ induced by a Lagrangean splitting of $V^\tau$, s. (2), (3) and (4).*

**Proof:** It is sufficient to show that $T_m M \cap L' = 0$ for all $m \in M$. This follows from the fact that $\gamma$ is positively defined on $T_m M$ and zero on $L' = \tau L'$. □

Given fundamental algebraic data $(V, \omega, \tau)$, compatible Lagrangean splittings (2)–(4) and a Lagrangean pseudo Kähler submanifold $M \subset (V, \omega, \gamma)$ in general position with respect to the given splitting, there is a **canonical real structure** $\rho'$ on $TM$. In fact, the projection $p : V \to L$ induces an isomorphism $T_m M \xrightarrow{\sim} L$ and hence we can define a real form of $(T_m M)^{\rho'}$ of $T_m M$ by the equation

$$dp\, (T_m M)^{\rho'} = L_0 .$$

Now we can define $\rho'$ (at the point m) as the $\mathbb{C}$-antilinear involution of $T_m M$ with fix point set $(T_m M)^{\rho'}$.

We denote by $\rho$ the real structure on $T^* M$ which is dual to $\rho'$ and by $g_m^{-1}$ the Hermitian metric on $T_m^* M$ which is inverse to $g_m = \gamma | T_m M$. Then the **generalized Calabi Ansatz** of [C-F-G] is given by the following potential on $T^* M$:

$$K(\sigma) = K^M(\pi(\sigma)) + g^{-1}_{\pi(\sigma)}(\sigma + \rho(\sigma), \sigma + \rho(\sigma)) , \quad \sigma \in T^* M , \tag{5}$$

where $\pi : T^* M \to M$ is the natural projection and $K^M$ is the pseudo Kähler potential of $M$, s. Prop. 1.2.

**Theorem 1.5.** *Let $(V, \omega, \tau)$ be fundamental algebraic data, s. p. 1, $\gamma$ the Hermitian form defined in (1), $V = L \oplus L'$ a Lagrangean splitting as in (2)–(4) and $M \subset (V, \omega, \gamma)$ a Lagrangean Kähler submanifold (resp. pseudo Kähler submanifold of complex signature $(k, l)$ in general position, s. Def. 1.3). Then the function $K$ on $T^* M$ associated to these data, s. (5), is the Kähler potential of a hyper Kähler metric $G$ on $T^* M$ (resp. the pseudo Kähler potential of a pseudo hyper Kähler metric $G$ of complex signature $(2k, 2l)$ on $T^* M$). More precisely, the standard complex symplectic structure*

*$\Omega$ on the cotangent bundle $T^*M$ is parallel with respect to the canonical connection of the Kähler (resp. pseudo Kähler) manifold $N = (T^*M, G)$.*

**Remark 2:** In [C-F-G] the correspondence $M \mapsto N$ is called *the c-map in rigid super symmetry.* There is also a *c-map in local super symmetry* $M \mapsto N$ as was proven in [F-S]. In the latter case, $N$ is a quaternionic Kähler manifold of negative Ricci curvature.

**Proof:** First we derive the local coordinate expression for the field $G$ of Hermitian forms defined by the potential $K$ on $T^*M$. Using this expression, we show that $G$ is nondegenerate and has complex signature (2k,2l). Then we prove that the standard complex symplectic structure $\Omega$ on $T^*M$ is parallel, also by a direct computation.

Let us choose a linear isomorphism $L_0 \cong \mathbb{R}^n$. Then we can identify $V^\tau = T^*\mathbb{R}^n$ and $V = T^*\mathbb{C}^n$. We denote by $(q^1, \dots, q^n, p_1, \dots, p_n)$ the complex coordinates on $T^*\mathbb{C}^n$ which correspond to the standard coordinates on $\mathbb{R}^n$. In these coordinates $\omega = \sum_{i=1}^n dq^i \wedge dp_i$.

Since $M$ is a Lagrangean submanifold in general position, s. Prop. 1.4, it is the image of a closed and hence locally exact section of $T^*\mathbb{C}^n$. In other words, we can describe $M$ locally by equations of the form

$$p_i = \frac{\partial F(q^1, \dots, q^n)}{\partial q^i}, \quad i = 1, \dots, n, \tag{6}$$

where $F(q^1, \dots q^n)$ is a locally defined holomorphic function of n variables. Remark that for any locally defined holomorphic function F the equations (6) define a Lagrangean submanifold in general position, namely the image of the exact section $dF$ of $T^*\mathbb{C}^n$.

Denote by $z^i := q^i|M$, $i = 1, \dots, n$, the natural complex coordinates on $M$ and by $(z^1, \dots, z^n, w_1, \dots, w_n)$ the corresponding complex coordinate system for $T^*M$. To unify the notation put $z^{i'} := w_i$ and $(z^I) := ((z^i), (z^{i'}))$. Recall that the (pseudo) Kähler metric on $M$ is $g = \gamma|M$, so in our coordinates $(z^i)$:

$$g_{ij} = g_{ji} = \gamma(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) = \sqrt{-1}(\overline{F}_{ij} - F_{ij}), \tag{7}$$

where $F_i = p_i|M$, $F_{ij} = \frac{\partial F_i}{\partial z^j}$, $F_{ijk} = \frac{\partial F_{ij}}{\partial z^k}$ etc. and we have used that

$$\frac{\partial}{\partial z^i} \quad = \quad \frac{\partial}{\partial q^i} + \sum_j F_{ij} \frac{\partial}{\partial p_j}.$$

The Hermitian fibre metric $g^{-1} = (g^{ij})$ on $T^*M$ is defined by the equation

$$\sum_j g^{ij} g_{jk} = \delta^i_k. \tag{8}$$

With respect to the coordinates $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ on $T^*M$ the potential $K$ reads

$$K(z^1, \ldots, z^n, w_1, \ldots, w_n) = K^M(z^1, \ldots, z^n) + \sum_{ij} g^{ij}(w_i + \overline{w}_i)(w_j + \overline{w}_j)\,, \tag{9}$$

where

$$K^M(z^1, \ldots, z^n) = \gamma(dF|_{(z^1,\ldots,z^n)}, dF|_{(z^1,\ldots,z^n)}) = \sqrt{-1}\sum_{ij}(z^i\overline{F}_i - \overline{z}^i F_i)\,. \tag{10}$$

Now we compute $G = (G_{IJ})$, where

$$G_{IJ} = \frac{\partial^2 K}{\partial z^I \partial \overline{z}^J}\,, \quad I, J \in \{1, \ldots, n, 1', \ldots, n'\}\,.$$

In the following, lower indices separated by komma will always denote partial derivatives, e.g. $g_{ij,k\bar{l}m'} = \frac{\partial^3 g_{ij}}{\partial z^k \partial \overline{z}^l \partial w_m}$. The equations

$$\begin{aligned}
G_{ij} &= g_{ij} + \sum_{kl} g^{kl}{}_{,i\bar{j}}(w_k + \overline{w}_k)(w_l + \overline{w}_l) \\
&= g_{ij} + 2\sum_{krspql} g^{kr}g^{sp}g^{ql}F_{rsi}\overline{F}_{pqj}(w_k + \overline{w}_k)(w_l + \overline{w}_l) \\
G_{ij'} &= \overline{G_{j'i}} = 2\sqrt{-1}\sum_{kpq} g^{kp}F_{pqi}g^{qj}(w_k + \overline{w}_k) \\
G_{i'j'} &= 2g^{ij}
\end{aligned}$$

follow immediately from the basic formulas

$$g^{kl}{}_{,i} = -\sum_{pq} g^{kp}g_{pq,i}g^{ql} = \sqrt{-1}\sum_{pq} g^{kp}F_{pqi}g^{ql} \tag{11}$$

$$g^{kl}{}_{,\bar{j}} = -\sum_{pq} g^{kp}g_{pq,\bar{j}}g^{ql} = -\sqrt{-1}\sum_{pq} g^{kp}\overline{F}_{pqj}g^{ql}\,. \tag{12}$$

If we put $b = (b^i_j)_{i,j=1,\ldots,n}$, $b^i_j := G_{ji'}$, then

$$G = (G_{IJ}) = \begin{pmatrix} g + \frac{1}{2}b^t g\overline{b} & b^t \\ \overline{b} & 2g^{-1} \end{pmatrix}$$

and we can easily invert this matrix:

$$G^{-1} = (G^{IJ}) = \begin{pmatrix} g^{-1} & -\frac{1}{2}b \\ -\frac{1}{2}\overline{b}^t & \frac{1}{2}(g + \frac{1}{2}\overline{b}^t gb) \end{pmatrix}.$$

This shows that $G$ is a pseudo Kähler metric on $T^*M$. In particular, $(T^*M, G)$ has a well defined signature over each connected component of $M$. We may assume that $M$ is connected and $(M, g)$ has complex signature (k,l). Then $(T^*M, G)$ has signature (2k,2l) near the zero section $M \subset T^*M$ (i.e. for $b \to 0$) and hence everywhere.

Now we show that $N = (T^*M, G)$ is a pseudo hyper Kähler manifold, s. Def. 1.1. The complex manifold $T^*M$ has the canonical complex symplectic structure $\Omega = \sum dz^i \wedge dw_i$. We will show that $\nabla\Omega = 0$ for the covariant derivative $\nabla$ of the pseudo Kähler manifold $N$. Let us denote by $\Gamma^I_{JK}$ the Christoffel symbols of the pseudo Kähler metric $G$ on $N$. We recall that $\Gamma^I_{JK} = \sum_L G^{LI} G_{JL,K}$, s. e.g. [K-N2] for the basic theory of Kähler manifolds.

**Lemma 1.6.** *The complex symplectic structure $\Omega$ of $T^*M$ is parallel with respect to the pseudo Kähler metric $G$ if and only if the Christoffel symbols $\Gamma^I_{JK}$ have the following symmetries.*

(i) $\Gamma^i_{Jk} = -\Gamma^{k'}_{Ji'}$,
(ii) $\Gamma^i_{Jk'} = \Gamma^k_{Ji'}$, $\Gamma^{i'}_{Jk} = \Gamma^{k'}_{Ji}$.

**Proof:** It is straightforward to check that equations (i) and (ii) are equivalent to

$$\nabla_{\frac{\partial}{\partial z^l}} \Omega = 0\,, \quad l = 1, \ldots, n\,,$$

if $J = j = 1, \ldots, n$ and equivalent to

$$\nabla_{\frac{\partial}{\partial w_l}} \Omega = 0\,, \quad l = 1, \ldots, n\,,$$

if $J = j' = 1', \ldots, n'$. □

The equations (i) and (ii) of Lemma 1.6 are verified by a direct computation of the Christoffel symbols $\Gamma^I_{JK}$. This finishes the proof of Thm. 1.5. □

**Remark 3:** Instead of Lemma 1.6 one can also use [H] Lemma 6.8, cf. [C-F-G]. We remark that to the pseudo hyper Kähler manifold $(T^*M, G, \Omega)$ constructed in Thm. 1.5 we can canonically associate a parallel hypercomplex structure $(J_1, J_2, J_3)$, which is Hermitian with respect to the pseudo Riemannian metric $\langle\cdot,\cdot\rangle = Re\, G$. Here $J_1$ is the standard complex structure of the (holomorphic) cotangent bundle $T^*M$; the complex structures $J_2$ and $J_3$ are defined by the equation

$$\Omega(v, w) \;=\; \langle J_2 v, w\rangle + \sqrt{-1}\,\langle J_3 v, w\rangle\,, \quad v, w \in T(T^*M)\,.$$

**Example 1:** The simplest example of Lagrangean pseudo Kähler submanifold $M \subset T^*\mathbb{C}^n$ is a Lagrangean subspace $L$ such that $L \cap \tau L = 0$. If e.g.

$$L = span_{\mathbb{C}}\{\frac{\partial}{\partial q^1} + i\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial q^k} + i\frac{\partial}{\partial p_k}, \frac{\partial}{\partial q^{k+1}} - i\frac{\partial}{\partial p_{k+1}}, \ldots, \frac{\partial}{\partial q^n} - i\frac{\partial}{\partial p_n}\}\,,$$

$i = \sqrt{-1}$, then $\gamma|L$ has complex signature (k,n-k) and $(T^*L, G)$ is the flat model of pseudo hyper Kähler manifold of quaternionic signature (k,n-k), i.e. of complex signature (2k,2n-2k).

## 2. A class of pseudo hyper Kähler manifolds admitting a group of automorphisms of cohomogeneity one

Let $(V, \omega, \tau)$ be our fundamental algebraic data, s. p. 1, $\gamma$ the Hermitian form defined in (1), $V = L \oplus L'$ a Lagrangean splitting as in (2)–(4) and $M \subset (V, \omega, \gamma)$ a Lagrangean pseudo Kähler submanifold in general position, s. Def. 1.3 and Prop. 1.4. By Thm. 1.5 to these data we canonically associate the pseudo hyper Kähler manifold $N = (T^*M, G)$ with standard complex symplectic structure $\Omega$.

Now we are interested in natural group actions on $M$ and $N$ preserving the given geometric structures. Recall that the duality group $Aut_d(M)$ of $M$ consists of those linear automorphisms of $(V, \omega, \tau)$ which preserve $M$. It acts holomorphically and isometrically on the pseudo Kähler manifold $(M, g)$, s. Prop. 1.3. Let us consider the subgroup $Aut_{sd}(M)$ of $Aut_d(M)$ preserving also the Lagrangean splittings (2)–(4), i.e.

$$Aut_{sd}(M) \;=\; \{\varphi \in Aut_d(M) | \varphi L = L \,,\; \varphi L' = L'\} \,.$$

Since $Aut_{sd}(M) \subset Aut_d(M) \hookrightarrow Sp(V^\tau, \omega|V^\tau) \subset GL(V^\tau) \subset Aff(V^\tau) = V^\tau \rtimes GL(V^\tau)$, we can also consider the corresponding affine group

$$V^\tau \rtimes Aut_{sd}(M) \subset V^\tau \rtimes Aut_d(M) \hookrightarrow Aff(V^\tau) \,.$$

We will show that the affine group $V^\tau \rtimes Aut_{sd}(M)$ acts on $N$ preserving the pseudo hyper Kähler structure. First we define a fibre preserving action of the vector group $V^\tau \cong \mathbb{R}^{2n}$ on $T^*M$. For this we consider the quotient map $V \to V/T_mM$. Using the inclusion $V^\tau \subset V$ and the isomorphism $V/T_mM \cong T_m^*M$ given by the symplectic structure $\omega$, the quotient map induces an isomorphism of real vector spaces, cf. Prop. 1.1:

$$\psi_m : V^\tau \to T_m^*M \,. \tag{13}$$

Now we define the action of an element $v \in V^\tau$ on $T^*M$ by

$$T_m^*M \ni \sigma \;\mapsto\; \sigma + \sqrt{-1}\psi_m(v) \in T_m^*M \,. \tag{14}$$

Choosing linear coordinates $(v^1, \ldots, v^n)$ for $L_0 \cong \mathbb{R}^n$ induces coordinates $(v_1, \ldots, v_n)$ for $L_0' \cong L_0^*$ and $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ for $T^*M$ as in the previous section. In these coordinates the action of $v = (v^1, \ldots, v^n, v_1, \ldots, v_n)$ on $T^*M$ defined in (14) reads (cf. [C-F-G]):

$$\begin{aligned} (z^1, \ldots, z^n, w_1, \ldots, w_n) \quad &\mapsto \quad (\tilde{z}^1, \ldots, \tilde{z}^n, \tilde{w}_1, \ldots, \tilde{w}_n) \\ \tilde{z}^i \;=\; z^i \,, \qquad \tilde{w}_i \;&=\; w_i - \sqrt{-1}(v_i - \sum_j F_{ij} v^j) \,, \end{aligned} \tag{15}$$

where we recall that $F_{ij} = \frac{\partial^2 F}{\partial z^i \partial z^j}$ are the second derivatives of the local holomorphic function $F(z^1, \ldots, z^n)$ defining the Lagrangean submanifold $M$, s. (6).

**Proposition 2.1.** *The group $V^\tau \rtimes Aut_{sd}(M) \hookrightarrow Aff(V^\tau)$ acts naturally on $N$ by automorphisms of the pseudo hyper Kähler structure (s. Thm. 1.5), i.e. by holomorphic isometries preserving the complex symplectic structure $\Omega$. The action of $Aut_{sd}(M)$ is by point transformations of $N = T^*M$ and that of $V^\tau$, given by (14), is fibre preserving and simply transitive on each fibre.*

**Proof:** $Aut_d(M)$ and thereby its subgroup $Aut_{sd}(M)$ acts holomorphically on $M$ and hence by complex symplectomorphisms on $T^*M$, namely by holomorphic point transformations. Moreover, $Aut_{sd}(M)$ preserves the pseudo Kähler potential (5) on $T^*M$ associated to the choice of Lagrangean splitting (2). This shows that $Aut_{sd}(M)$ acts on $N$ by automorphisms of the pseudo hyper Kähler structure.

Consider the map

$$\psi : M \times V^\tau \to T^*M\,, \quad \psi(m, v) := \psi_m(v)\,.$$

¿From the fact that $\psi$ is $Aut_d(M)$-equivariant it follows that (14) extends the action of $Aut_d(M)$ on $N$ to a holomorphic action of $V^\tau \rtimes Aut_{sd}(M)$ on $N$. We check that the action of $v \in V^\tau$, s. (15), preserves the complex symplectic structure $\Omega = \sum_i dz^i \wedge dw_i$:

$$\sum_i (d\tilde{z}^i \wedge d\tilde{w}_i - dz^i \wedge dw_i) = \sqrt{-1}\sum_{ij} v^j dz^i \wedge dF_{ij} = \sqrt{-1}\sum_{ijk} v^j F_{ijk} dz^i \wedge dz^k = 0\,.$$

Next we check that under the action of $v \in V^\tau$ the pseudo Kähler potential $K$ of $N$, s. (5), changes only by a pluriharmonic function. Using (7)-(10) and (15) we obtain:

$$K(\tilde{z}^1, \ldots, \tilde{w}_n) - K(z^1, \ldots, w_n) = -2\sum_j v^j(w_j + \overline{w}_j) + \sum_{ij} g_{ij} v^i v^j\,.$$

The result is pluriharmonic, since the $v^i$ are constants and $g_{ij}$ is pluriharmonic as sum of a holomorphic and an antiholomorphic function, s. (7). Now it only remains to show that $V^\tau$ acts simply transitively on each fibre of $T^*M \to M$. This is clear since (13) is an isomorphism of real vector spaces. □

**Corollary 2.2.** *If a subgroup $A \subset Aut_{sd}(M)$ has an orbit of codimension $r$ on $M$ then the subgroup $V^\tau \rtimes A \subset V^\tau \rtimes Aut_{sd}(M)$ has an orbit of codimension $r$ on $N$.*

Our aim is to use Cor 2.2 for the construction of pseudo hyper Kähler manifolds $N$ with the smallest possible cohomogeneity of the group $V^\tau \rtimes Aut_{sd}(M)$. (Recall that the cohomogeneity of a Lie group acting on a manifold is the minimal codimension of its orbits.)

The pseudo Kähler potential $K^M(u) = \gamma(u, u)$, $u \in M$, defines an $Aut_d(M)$-invariant function on $M$ and hence $K^M \circ \pi$, $\pi : T^*M \to M$ the projection, is an $V^\tau \rtimes Aut_d(M)$-invariant function on $N$. This function cannot be constant on an open set, since $K^M$ is the potential of a (nondegenerate) metric. Therefore, $V^\tau \rtimes Aut_d(M) \supset V^\tau \rtimes Aut_{sd}(M)$ has no open orbit on $N$ and hence is of cohomogeneity at least one.

Next we will present a class of pseudo hyper Kähler manifolds $N$ for which the cohomogeneity of $V^\tau \rtimes Aut_{sd}(M) \subset V^\tau \rtimes Aut_d(M)$ is in fact one. This class is associated, via the correspondence of Thm. 1.5, to an interesting class of Lagrangean pseudo Kähler submanifolds $M \subset T^*\mathbb{C}^{n+1}$ in general position (with respect to the standard Lagrangean splitting $T^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \oplus (\mathbb{C}^{n+1})^*$ into positions and momenta). The latter class can be defined by the following additional conditions

(i) $M$ is a cone, i.e. $\lambda M = M$ for all $\lambda \in \mathbb{C} - \{0\}$.
(ii) The third fundamental form of $M \subset T^*\mathbb{C}^{n+1}$ is given by a homogeneous cubic polynomial $h(x^1, \dots, x^n)$ with real coefficients, s. remarks below.
(iii) The real hypersurface $\{h = 1\} \subset \mathbb{R}^n$ admits an open orbit $\mathcal{H} \subset \{h = 1\}$ of a subgroup of $GL(n, \mathbb{R})$. Moreover, the second fundamental form of $\mathcal{H}$ is negatively defined.

The complete classification of such manifolds $M$ was first obtained in [dW-VP]; an alternative, conceptual approach was developped in [C2].

First of all let us explain the meaning of (ii). The notion of 3rd fundamental form of a Lagrangean cone was introduced by Bryant and Griffiths [B-G] in the study of period maps for general Calabi Yau 3-folds, s. Section 3. The **third fundamental form** $\theta$ of a Lagrangean submanifold $M \subset T^*\mathbb{C}^{n+1}$ in general position satisfying (i) is by definition the 3rd fundamental form of the immersion

$$\phi : (q^1, \dots, q^n) \ \mapsto \ (\pi|M)^{-1}(1, q^1, \dots, q^n)\,,$$

where $\pi : T^*\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the natural projection. More precisely, $\theta$ is the section of $S^3T^*\mathbb{C}^n$ whose value $\theta_q \in S^3T^*_q\mathbb{C}^n \cong S^3(\mathbb{C}^n)^*$ at $q = (q^1, \dots, q^n)$, $(1, q) \in \pi(M)$, is given by the cubic form

$$\theta_q(x^1, \dots, x^n) \ = \ \sum_{ijk=1}^{n} x^i x^j x^k \frac{\partial^3 \phi(q)}{\partial q^i \partial q^j \partial q^k}\,.$$

So the condition (ii) is satisfied if and only if the homogeneous cubic polynomial $\theta_q$ is independent of $q$, i.e. $\theta_q(x^1, \dots, x^n) = h(x^1, \dots, x^n)$, and has (constant) real coefficients $\frac{\partial^3 \phi(q)}{\partial q^i \partial q^j \partial q^k} \in \mathbb{R}$.

In the following discussion we recall all needed basic facts about the class of Lagrangean submanifolds defined above; for complete details the reader is referred to [dW-VP] and [C2]. Under the condition (i), we can consider the complex manifold $P(M) \subset P(T^*\mathbb{C}^{n+1}) \cong P^{2n+1}_{\mathbb{C}}$, which is a projectivized cone. If the potential $K^M$ does not vanish on $M$, then $P(M)$ admits a canonical Hermitian form $g^{P(M)}$ invariant under the projective action of the duality group $Aut_d(M)$, cf. (17). (The corresponding potential is $\log|K^M|$.) Under the conditions (ii) and (iii), there exists an open subcone $\mathcal{C} \subset M \subset T^*\mathbb{C}^{n+1}$ such that $Aut_{sd}(\mathcal{C})$ acts transitively on $P(\mathcal{C}) \subset P(M)$, the Hermitian form $\gamma$ has complex signature (1,n) on $\mathcal{C}$ and $-g^{P(\mathcal{C})}$ is a (positively defined) Kähler metric on $P(\mathcal{C})$. Without restriction of generality we assume $M = \mathcal{C}$.

From the preceding remarks it follows that $Aut_{sd}(M)$ has an orbit of codimension one on $M$ and by Prop. 2.1 and Cor. 2.2 the group $V^\tau \rtimes Aut_{sd}(M)$, $V^\tau = T^*\mathbb{R}^{n+1} \cong \mathbb{R}^{2n+2}$, of automorphisms of the hyper Kähler manifold $N$ has cohomogeneity one.

Now we describe the group $Aut_{sd}(M)$ in more detail. Remark that the subgroup of $Sp(n+1,\mathbb{R}) \subset GL(2n+2,\mathbb{R})$ preserving the standard Lagrangean splitting $T^*\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \oplus (\mathbb{R}^{n+1})^*$ is the group of linear point transformations of $T^*\mathbb{R}^{n+1}$, which is canonically isomorphic to $GL(n+1,\mathbb{R})$. This gives rise to the embedding

$$\iota : Aut_{sd}(M) \hookrightarrow GL(n+1,\mathbb{R})\,, \quad \varphi \mapsto \varphi|\mathbb{R}^{n+1}\,,$$

where $\mathbb{R}^{n+1} \subset T^*\mathbb{R}^{n+1}$ is the zero section. $GL(n+1,\mathbb{R})$ contains the affine group $Aff(n,\mathbb{R}) = \mathbb{R}^n \rtimes GL(n,\mathbb{R})$ as a subgroup:

$$Aff(n,\mathbb{R}) \ni (v,B) \ \mapsto \ \begin{pmatrix} 1 & 0 \\ v & B \end{pmatrix} \in GL(n+1,\mathbb{R})\,.$$

So we can define the subgroup

$$A(M) \ := \ \iota^{-1}(Aff(n,\mathbb{R})) \subset Aut_{sd}(M)\,,$$

which is embedded in the affine group $Aff(n,\mathbb{R})$ via the map $\iota$.

By construction, s. [dW-VP] and [C2], the class of submanifolds $M \subset T^*\mathbb{C}^{n+1}$ defined above has the following property: The group $A(M)$ has an open orbit on $P(M)$ and (by restricting to an open subcone of $M$ if necessary) we can assume that $A(M)$ acts transitively on $P(M)$. Moreover, it was proven in [C2] that $(P(M), -g^{P(M)})$ is isomorphic, as $A(M)$-Kähler manifold, to a Kählerian Siegel domain $(U,g)$ of first type. The group $\iota(A(M)) \cong A(M)$ acts naturally on $(U,g)$ by affine transformations which are (holomorphic) isometries for the Kähler metric $g$.

The complete list of Kählerian Siegel domains $(U,g)$ corresponding to the class of submanifolds $M \subset T^*\mathbb{C}^{n+1}$ defined above, s. (i)-(iii), was given in [C2] Thm. 2.8. For convenience of the reader, we recall that $U$ has rank 2 or 3. The rank 2 domains in the list are numerated by the nonnegative integers and the rank 3 domains by special isometric maps or, equivalently, by $\mathbb{Z}_2$-graded Clifford modules (up to equivalence defined in [C1] Def. I.10 and Def. II.5, cf. Def. II.4, Prop. II.25 and Prop. II.26). Recall that a $\mathbb{Z}_2$-**graded Clifford module of order k** is a $\mathbb{Z}_2$-graded module $\Psi = \Psi_0 \oplus \Psi_1$ of the real Clifford algebra $C\ell_k$, $k = 0,1,2,\ldots$, s. e.g. [L-M]. Summarizing our discussion and applying Cor. 2.2 to $A = A(M)$, we obtain the following theorem.

**Theorem 2.3.** *To any $p \in \{0,1,2,\ldots\}$ (resp. $\mathbb{Z}_2$-graded Clifford module $\Psi$ of order $k \in \{0,1,2,\ldots\}$) we can canonically associate a Lagrangean pseudo Kähler submanifold in general position $M(p) \subset T^*\mathbb{C}^{n+1}$, $n = 2+p$, (resp. $M(\Psi) \subset T^*\mathbb{C}^{n+1}$, $n = k+3+\dim\Psi$) satisfying conditions (i)-(iii) on p. 8. Conversely, any Lagrangean pseudo Kähler submanifold of $T^*\mathbb{C}^{n+1}$ in general position satisfying (i)-(iii) contains one of the manifolds $M$ above (i.e. $M = M(p)$ or $M = M(\Psi)$) as an open subcone. For the manifolds $M$ above, the affine group $A(M) \overset{\iota}{\hookrightarrow} Aff(n,\mathbb{R})$*

*acts transitively and isometrically on the Kähler manifold* $(P(M), -g^{P(M)})$ *by projective linear transformations. Moreover,* $(P(M), -g^{P(M)})$ *is isomorphic as* $A(M)$*-Kähler manifold to a Kählerian Siegel domain of type I with transitive affine action of* $\iota(A(M)) \subset Aff(n, \mathbb{R})$*.*

*The pseudo hyper Kähler manifold* $N = (T^*M, G)$ *associated to any of these manifolds* $M$ *by Thm. 1.5 has complex signature* $(2, 2n)$*. Finally, the affine group* $\mathbb{R}^{2n+2} \rtimes A(M) \subset Aff(2n+2, \mathbb{R})$ *acts on* $N$ *by automorphisms of the pseudo hyper Kähler structure with an orbit of codimension one.*

**Remark 4:** The manifolds $(T^*M, G)$, $M = M(p)$ or $M = M(\Psi)$, should be thought of as natural pseudo hyper Kählerian versions of Alekseevsky's homogeneous quaternionic Kähler manifolds, cf. [A], [Ce], [dW-VP], [dW-V-VP], [C1], [A-C] and [C2].

## 3. The pseudo hyper Kähler metric of the bundle of intermediate Jacobians over the moduli space of gauged Calabi Yau 3-folds

Let $(V, \omega, \tau)$ be our fundamental algebraic data, s. p. 1, $\gamma$ the Hermitian form defined in (1), $V = L \oplus L'$ a Lagrangean splitting as in (2)-(4), $M \subset (V, \omega, \gamma)$ a Lagrangean pseudo Kähler submanifold in general position, s. Prop. 1.4, and, finally, $\Gamma \subset V^\tau$ a (cocompact) lattice. Using the isomorphism $V/T_mM \cong T_m^*M$ induced by the symplectic form $\omega$, we can identify $T^*M$ with the normal bundle $\mathcal{N} \to M$ of $M$ in $V$, $\mathcal{N}_m = V/T_mM$. Since $V^\tau \supset \Gamma$ has zero intersection with $T_mM$, s. Prop. 1.1, $\Gamma$ projects to a lattice $[\Gamma]$ in $\mathcal{N}_m = V/T_mM$ and $\mathcal{N}_m/[\Gamma]$ is a complex torus. Let us denote by $\mathcal{N}/\Gamma$ the corresponding (holomorphic) torus bundle over $M$. Remark that via the isomorphism $\mathcal{N}_m \cong T_m^*M$ the lattice $[\Gamma] \subset \mathcal{N}_m$ corresponds to the lattice $\psi_m(\Gamma) \subset T_m^*M$, s. (13), and we can identify $\mathcal{N}/\Gamma$ with the quotient of $T^*M$ by the action of $\Gamma \subset V^\tau$ defined in equation (14).

**Theorem 3.1.** *The pseudo hyper Kähler structure on* $T^*M \cong \mathcal{N}$ *constructed in Thm. 1.5 induces a pseudo hyper Kähler structure on the torus bundle* $T^*M/\Gamma \cong \mathcal{N}/\Gamma$ *for any lattice* $\Gamma \subset V^\tau$*.*

**Proof:** It was proven in Prop. 2.1 that the action of $V^\tau \supset \Gamma$ preserves the complex symplectic structure $\Omega$ and the pseudo hyper Kähler metric $G$ on $T^*M$. $\square$

The purpose of this section is to use Thm. 3.1 for the construction of a pseudo hyper Kähler structure on the bundle of intermediate Jacobians over the moduli space of gauged Calabi Yau 3-folds. For this we have to review some known facts about the moduli space, thereby relating it to Thm. 3.1.

Let $X$ be a **(general) Calabi Yau 3-fold**, i.e. a compact Kähler 3-fold with holonomy group $SU(3)$. This implies that $X$ has a holomorphic volume form $vol_X \in H^{3,0}(X)$, unique up to scaling. Such a pair $(X, vol_X)$ is called a **gauged** Calabi Yau 3-fold. The **(Kuranishi) moduli space** $S$ of $X$ is smooth and can be identified with a neighborhood of zero in $H^{2,1}(X) \cong H^1(X, T)$, s. [Bo], [Ti] and [To]. Denote by $\mathcal{X} = (X_s)_{s\in S} \to S$, $X_0 = X$, the corresponding deformation of complex structure. The **"intersection" form**

$$\omega(\xi,\eta) := \int_X \xi \wedge \eta\,, \quad \xi,\eta \in H^3(X,\mathbb{Z})\,, \tag{16}$$

defines an integral nondegenerate skew symmetric bilinear form on $H^3(X,\mathbb{Z})$. The corresponding complex symplectic form on $H^3(X,\mathbb{C})$ will be denoted by the same letter. Consider the holomorphic line bundle $H^{3,0}(\mathcal{X}) \to S$ with fibre $H^{3,0}(X_s)$ at $s \in S$. Denote by $H^{3,0}(\mathcal{X}) - S$ the $\mathbb{C}^*$-bundle over $S$ which is obtained from the complex line bundle $H^{3,0}(\mathcal{X})$ by removing the zero section $S \ni s \mapsto 0 \in H^{3,0}(X_s)$. We think of it as **moduli space of gauged Calabi Yau 3-folds** $(X_s, vol_s)$, $vol_s \in H^{3,0}(X_s) - \{0\}$, $s \in S$.

The holomorphic vector bundle $H^3(\mathcal{X},\mathbb{C}) \to S$ has a canonical flat connection defined by the lattice bundle $H^3(\mathcal{X},\mathbb{Z}) \subset H^3(\mathcal{X},\mathbb{C})$, which is known as Gauß-Manin connection. Since the moduli space $S$ is local, we can assume that $S$ is simply connected and that the bundle $H^3(\mathcal{X},\mathbb{C}) \to S$ is trivial. In particular, we have canonical identifications $H^3(X_s,\mathbb{Z}) \cong H^3(X,\mathbb{Z})$ and $H^3(X_s,\mathbb{C}) \cong H^3(X,\mathbb{C})$. So we can define the **period map**

$$Per : S \to P(H^3(X,\mathbb{C}))\,, \quad s \mapsto H^{3,0}(X_s)\,.$$

It follows from Kodaira and Spencer's deformation theory (s. e.g. [M-K]) that

$$dPer(T_sS) \;=\; d\pi(H^{3,0}(X_s) + H^{2,1}(X_s))\,,$$

where $\pi : H^3(X,\mathbb{C}) \to P(H^3(X,\mathbb{C}))$ is the canonical projection. This implies that the period map is an immersion and since $S$ is local we can assume that $Per : S \to Per(S) \subset P(H^3(X,\mathbb{C}))$ is an isomorphism. As a consequence, the cone $M_X = \cup_{s\in S} Per(s) - \{0\} \subset H^3(X,\mathbb{C})$ over $Per(S) = P(M_X)$ is canonically identified with the moduli space $H^{3,0}(\mathcal{X}) - S$ of gauged Calabi Yau 3-folds. By the first Hodge-Riemann bilinear relations the tangent space

$$T_uM_X \;=\; H^{3,0}(X_s) + H^{2,1}(X_s)\,, \quad u \in Per(s) - \{0\}\,,$$

is a Lagrangean subspace of $H^3(X,\mathbb{C})$ with respect to the intersection form $\omega$. Remark that, using the standard real structure $\tau$ on $H^3(X,\mathbb{C})$ with fix point set $H^3(X,\mathbb{R})$, we can define the Hermitian form $\gamma$ of complex signature $(n+1,n+1)$, $n = h^{2,1}(X)$, on $H^3(X,\mathbb{C})$ by equation (1). The first and second Hodge-Riemann bilinear relations imply that $M_X$ is a pseudo Kähler submanifold $M_X \subset (H^3(X,\mathbb{C}),\gamma)$ of complex signature $(1,n)$. More precisely, we have the relations: $\gamma(u,u) > 0$, $\gamma(v,v) < 0$ and $\gamma(u,v) = 0$ for all $u \in H^{3,0}(X_s) - \{0\}$ and $v \in H^{2,1}(X_s) - \{0\}$. This motivates the following definition.

**Definition 3.1.** *Given fundamental algebraic data $(V,\omega,\tau)$ as on p. 1 and $\gamma$ defined in (1), a Lagrangean pseudo Kähler submanifold $M \subset (V,\omega,\gamma)$ is called a* **formal**

**moduli space (of gauged Calabi Yau 3-folds)** *if the following conditions are satisfied:*

(i) *$M$ is a cone, i.e. $\lambda M = M$ for all $\lambda \in \mathbb{C} - \{0\}$.*
(ii) *$\gamma(u,u) > 0$ for all $u \in M$.*
(iii) *$\gamma(v,v) < 0$ for all $0 \neq v \in T_uM$ such that $\gamma(u,v) = 0$.*

To a Calabi Yau 3-fold $X$ we have associated the following algebraic data: $V = H^3(X,\mathbb{C})$, $\omega$ the intersection form (16), $\tau$ the real structure with fix point set $V^\tau = H^3(X,\mathbb{R})$ and $\gamma = \sqrt{-1}\omega(\cdot,\tau\cdot)$. With this understood the next proposition gives a summary of the preceding discussion.

**Proposition 3.2.** *The cone $M_X \subset (V,\omega,\gamma)$ over the image $Per(S) = P(M_X)$ of the period map is a formal moduli space of gauged Calabi Yau 3-folds in the sense of Def. 3.1.*

**Proposition 3.3.** *The class of Lagrangean pseudo Kähler cones defined in the previous section on p. 8 and classified in* [dW-VP] *and* [C2] *consists of formal moduli spaces of gauged Calabi Yau 3-folds.*

Remark that any statement which is true for formal moduli spaces $M$ is true when $M = M_X (\cong H^{3,0}(\mathcal{X}) - S)$ is the (actual) moduli space of gauged Calabi Yau 3-folds associated to the Kuranishi moduli space $S$ of a Calabi Yau 3-fold $X$.
**Remark 5:** For any formal moduli space $M \subset (V,\omega,\gamma)$ there is a canonical Kähler metric $-g^{P(M)}$ on $P(M)$ known as **special Kähler metric**, cf. p. 9 and [C2], which can be defined by

$$g^{P(M)}_{\pi u}(d\pi v, d\pi v) = \frac{\gamma(v,v)}{\gamma(u,u)} - \left|\frac{\gamma(u,v)}{\gamma(u,u)}\right|^2 , \tag{17}$$

for $u \in M$, $v \in T_uM$, where $\pi : M \to P(M)$ is the canonical projection. In the case of actual moduli spaces of gauged Calabi Yau 3-folds $M = M_X$ the metric $g^{P(M)}$ is known as **Weil-Petersson metric**. The formal moduli spaces $M$ of Prop. 3.3 provide all the known examples of **homogeneous** special Kähler manifolds, i.e. with transitive isometry group. It is noteworthy that "most" of these examples are not Hermitian symmetric, s. [dW-VP] and [C2]. We may ask the following natural question:
*Which of the homogeneous special Kähler manifolds can be realized as moduli spaces of Calabi Yau 3-folds (equipped with the Weil-Petersson metric)?*

To round up our presentation, we place the concept of formal moduli space in the context of infinitesimal variations of Hodge structure, s. [G] and [B-G]. Given fundamental algebraic data $(V,\omega,\tau)$ as on p. 1, a **polarized Hodge structure of weight 3** on $V$, $\dim_{\mathbb{C}} V = 2n+2$, with **Hodge numbers** $h^{3,0} = 1$ and $h^{2,1} = n$ is given by a decomposition into complex subspaces

$$V \;=\; H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} , \tag{18}$$

such that $H^{p,q} = \tau H^{q,p}$, $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ and satisfying the first and second Hodge-Riemann bilinear relations. Moreover, one assumes that a lattice $\Gamma \subset V^\tau$ is given and that $\omega$ restricts to an *integral* nondegenerate skew symmetric bilinear form on $\Gamma$. Let us denote by $\mathcal{D}$ the classifying space for such Hodge structures on $V$. Remark that an element of $\mathcal{D}$, i.e. a Hodge structure as above, is determined by the pair $(H^{3,0}, H^{2,1})$.

To any formal moduli space $M \subset (V, \omega, \gamma)$, s. Def. 3.1, we associate a map $M \to \mathcal{D}$ by

$$u \;\mapsto\; (H^{3,0}(u), H^{2,1}(u))\,,$$
$$H^{3,0}(u) \;:=\; \mathbb{C}u\,, \quad H^{2,1}(u) \;:=\; T_uM \cap u^\perp\,,$$

where the orthogonal complement $\perp$ is to be taken with respect to the Hermitian form $\gamma$. Obviously this map factorizes to a map $\varphi_M : P(M) \to \mathcal{D}$. If we denote by $p : \mathcal{D} \to P(V)$ the projection $(H^{3,0}, H^{2,1}) \mapsto H^{3,0}$ then $p \circ \varphi_M : P(M) \to P(V)$ is the trivial inclusion. For any Hodge decomposition (18) one has also the corresponding **Hodge filtration**

$$F^3 \subset F^2 \subset F^1 \subset F^0 = V\,, \quad F^p \;=\; \oplus_{p \le k \le 3} H^{k,3-k}\,.$$

Consider now a variation of Hodge structure $u \mapsto (H^{p,q}(u))_{p,q}$, $u \in U$, and denote by $(F^p(u))_p$ the corresponding Hodge filtrations. If the variation of Hodge structure arises from a local deformation of complex structure $(X_u)_{u \in U}$, i.e. $H^{p,q}(u) = H^{p,q}(X_u)$, then it must satisfy Griffiths' infinitesimal period relations [G]

$$\partial F^p(u) \;\subset\; F^{p-1}(u)\,. \tag{19}$$

This means that holomorphic partial derivatives of a (local) holomorphic section of the vector bundle $(F^p(u))_{u \in U}$ are sections of $(F^{p-1}(u))_{u \in U}$. The next theorem follows from the work of Bryant and Griffiths [B-G].

**Theorem 3.4.** *Let $M \subset (V, \omega, \gamma)$ be a formal moduli space, $\dim_{\mathbb{C}} V = 2n+2$ and $\mathcal{D}$ the classifying space for Hodge structures as above. Then the map $\varphi_M : P(M) \to \mathcal{D}$ is a solution to the differential system on $\mathcal{D}$ defined by the infinitesimal period relations (19). Conversely, any solution $\varphi : U \to \mathcal{D}$, $U$ a complex $n$-fold, to this differential system for which $p \circ \varphi : U \to P(V)$ is an immersion is locally of the form $\varphi_M$.*

Let $X$ be a Calabi Yau 3-fold, $S$ its Kuranishi moduli space and $M_X = \cup_{s \in S} Per(s) - \{0\} \subset H^3(X, \mathbb{C})$ the cone over the image of the period map $Per : S \to P(H^3(X, \mathbb{C}))$. Recall that $M_X$ is the moduli space of gauged Calabi Yau 3-folds associated to $X$. The **intermediate Jacobian** of $X_s$, $s \in S$, is the complex torus

$$\mathcal{J}(X_s) \;=\; \frac{H^3(X, \mathbb{C})}{H^{3,0}(X_s) + H^{2,1}(X_s) + H^3(X, \mathbb{Z})}\,.$$

The **bundle of intermediate Jacobians** $\mathcal{J} \to M_X$ over $M_X$ is the holomorphic torus bundle whose fibre at $u \in Per(s) - \{0\} \subset M_X$ is $\mathcal{J}_u = \mathcal{J}(X_s)$.

**Theorem 3.5.** *For any Lagrangean splitting $H^3(X,\mathbb{R}) = L_0 \oplus L_0'$ such that $M_X \subset H^3(X,\mathbb{C})$ is in general position, there is a pseudo hyper Kähler structure of complex signature $(2,2n)$, $n = h^{2,1}(X)$, on the bundle of intermediate Jacobians $\mathcal{J} \to M_X$.*

**Proof:** By Prop. 3.2 the moduli space of gauged Calabi Yau 3-folds $M_X$ is a formal moduli space. In particular, s. Def. 3.1, it is a Lagrangean pseudo Kähler submanifold of $(H^3(X,\mathbb{C}),\omega,\gamma)$, where $\omega$ is the intersection form (16) and $\gamma = \sqrt{-1}\omega(\cdot,\tau\cdot)$ is defined with the help of the standard real structure $\tau$, i.e. $H^3(X,\mathbb{C})^\tau = H^3(X,\mathbb{R})$. By Thm. 1.5, we can associate to these data together with the Lagrangean splitting of $H^3(X,\mathbb{R})$ a pseudo hyper Kähler structure on $T^*M_X$. Using the intersection form $\omega$ we can identify the cotangent bundle $T^*M_X \to M_X$ with the normal bundle $\mathcal{N} \to M_X$ of $M_X \subset H^3(X,\mathbb{C})$. The normal bundle has fibre

$$\mathcal{N}_u = \frac{H^3(X,\mathbb{C})}{T_u M_X} = \frac{H^3(X,\mathbb{C})}{H^{3,0}(X_s) + H^{2,1}(X_s)}$$

at $u \in Per(s) - \{0\} \subset M_X$ and $\mathcal{J} \to M_X$ is precisely the torus bundle $\mathcal{N}/\Gamma \to M$ considered in Thm. 3.1 with $M = M_X$, $V = H^3(X,\mathbb{C})$ and $\Gamma = H^3(X,\mathbb{Z})$. Now the theorem is an immediate consequence of Thm. 3.1. $\Box$

**Remark 6:** As checked in the proof of Prop. 2.1, the standard complex symplectic structure $\Omega$ on $T^*M_X \cong \mathcal{N}$ is invariant under the action of the lattice $\Gamma = H^3(X,\mathbb{Z}) \subset H^3(X,\mathbb{R}) = V^\tau$. So it factorizes to a complex symplectic structure on $\mathcal{J} = \mathcal{N}/\Gamma \cong T^*M/\Gamma$ independent of any choice of Lagrangean splitting $H^3(X,\mathbb{R}) = L_0 \oplus L_0'$. This is the parallel complex symplectic structure associated to the pseudo hyper Kähler structure of Thm. 3.5 and it coincides with the complex symplectic structure which was recently constructed by Donagi and Markman [D-M].

**Remark 7:** Given a Lagrangean subspace $L_1 \subset (V,\omega,\tau)$ such that $L_1 \cap \tau L_1 = \{0\}$, there exists a Lagrangean subspace $L$ such that $L = \tau L$ and $L \cap L_1 = \{0\}$. This shows that locally one can always find a Lagrangean splitting of $H^3(X,\mathbb{R})$ such that $M_X$ is in general position. However, such a choice is not unique. In fact, the Lagrangean splittings of $H^3(X,\mathbb{R}) \cong \mathbb{R}^{2n+2}$ are parametrized by the coset space $Sp(n+1,\mathbb{R})/GL(n+1,\mathbb{R})$. To reduce this arbitrariness to only a countable number of allowed choices, we can proceed as follows. Any isomorphism $H^3(X,\mathbb{Z}) \cong \mathbb{Z}^{n+1} \oplus (\mathbb{Z}^{n+1})^*$ mapping the intersection form to the standard nondegenerate integral skew symmetric bilinear form on $\mathbb{Z}^{n+1} \oplus (\mathbb{Z}^{n+1})^*$ induces a Lagrangean splitting of $H^3(X,\mathbb{R})$. In other words, we allow only Lagrangean splittings of $H^3(X,\mathbb{R})$ which are induced by the choice of a symplectic basis $(\xi^i,\eta_i)_{i=0,\dots,n}$ for the integral cohomology, i.e. $\xi^i,\eta_i \in H^3(X,\mathbb{Z})$, $\omega(\xi^i,\xi^j) = \omega(\eta_i,\eta_j) = 0$ and $\omega(\xi^i,\eta_j) = \delta^i_j$. The Lagrangean splittings of this type are parametrized by $Sp(n+1,\mathbb{Z})/GL(n+1,\mathbb{Z})$.

**Remark 8:** The pseudo hyper Kähler metrics $G$ on the cotangent bundle $T^*M$ of a formal moduli space of gauged Calabi Yau 3-folds $M \subset V$, s. Def. 3.1 and Thm. 1.5, are not complete. In fact, the open line segment $l$ joining a point $u \in M$ to the origin $0 \in V - M$ is a geodesic arc of finite length in the totally geodesic zero section

$M \subset T^*M$. We may consider the natural blow up $\sigma : \tilde{M} \to M$ of the cone $M$ at the origin. $\tilde{M} = M \cup P(M)$ is identified with the universal bundle of the projectivized cone $P(M)$. It is easy to see that $\sigma^*(G|M) = \sigma^* g$ extends smoothly to the divisor $P(M) \subset \tilde{M}$. However, this extension gives only a degenerate metric on $\tilde{M} \subset T^*\tilde{M}$, unless $M$ is a complex line. (Of course the same remarks apply in the case of actual moduli spaces of gauged Calabi Yau 3-folds $M = M_X$ as considered in Thm. 3.5.)

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