THE ALGEBRAIC DIMENSION OF COMPACT COMPLEX THREEFOLDS WITH VANISHING SECOND BETTI NUMBER

FRÉDÉRIC CAMPANA, JEAN-PIERRE DEMAILLY, AND THOMAS PETERNELL

Abstract. We investigate compact complex manifolds of dimension three and second Betti number $b_2(X) = 0$. We are interested in the algebraic dimension $a(X)$, which is by definition the transcendence degree of the field of meromorphic functions over the field of complex numbers. The topological Euler characteristic $\chi_{\mathrm{top}}(X)$ equals the third Chern class $c_3(X)$ by a theorem of Hopf. Our main result is that, if X is a compact 3-dimensional complex manifold with $b_2(X) = 0$ and $a(X) > 0$, then $c_3(X) = \chi_{\mathrm{top}}(X) = 0$, that is, we either have $b_1(X) = 0,\ b_3(X) = 2$ or $b_1(X) = 1,\ b_3(X) = 0$.

Introduction

In this note we shall investigate compact complex manifolds of dimension three and second Betti number $b_2(X) = 0$. Of course such a manifold cannot be algebraic. Therefore we will be interested in the algebraic dimension $a(X)$ which is by definition the transcendence degree of the field of meromorphic functions over the field of complex numbers. Note that $a(X) > 0$ if and only if X admits a non-constant meromorphic function. The topological Euler characteristic will be denoted $\chi_{\mathrm{top}}(X)$ which is also the third Chern class $c_3(X)$ by a theorem of Hopf. Our main result is

Theorem

Let X be a compact 3-dimensional complex manifold with $b_2(X) = 0$ and $a(X) > 0$. Then $c_3(X) = \chi_{\mathrm{top}}(X) = 0$, i.e. we either have $b_1(X) = 0,\ b_3(X) = 2$ or $b_1(X) = 1,\ b_3(X) = 0$.

Notice that if $a(X) = 3$, i.e. X is Moishezon, then we have $b_2(X) > 0$, but examples of compact threefolds X with $a(X) = 1$ or 2 and with the above Betti numbers exist (see sect. 4). We also show by an example that the assumption $a(X) > 0$ cannot be omitted.

The following corollary was actually our motivation for the Theorem:

Parts of this work were done during one of the author's stay at MSRI. Research at MSRI is supported in part by NSF grant DMS-9022140.

Corollary

Let X be a compact complex manifold homeomorphic to the 6-dimensional sphere S^6. Then $a(X) = 0$.

In other words, S^6 does not admit a complex structure with a non-constant meromorphic function.

The proof of the Theorem consists in showing that for generic $\mathcal{L} \in \mathrm{Pic}^0(X)$, we have

$$h^0(X, B \otimes \mathcal{L}) = h^2(X, B \otimes \mathcal{L}) = 0$$

for B either the tangent bundle T_X or the cotangent bundle Ω^1_X. This forces $\frac{c_3(X)}{2} = \chi(X, T_X)$ to be 0. If $a(X) = 2$ or if $a(X) = 1$ and the algebraic reduction is not holomorphic, we can even prove the last vanishing statements for all vector bundles B on X.

In the last section we study more closely the structure of threefolds X with $b_2(X) = 0$ and algebraic dimension 1 whose algebraic reduction is holomorphic. We show e.g. that smooth fibers can only be Inoue surfaces, Hopf surface with algebraic dimension 0 or tori.

1. Preliminaries and Criteria for the vanishing of H^0.

1.0 Notations

(1) Let X be a compact complex manifold, always assumed to be connected. The algebraic dimension, denoted $a(X)$, is the transcendence degree of the field of meromorphic functions over $\mathbb{C}$.

(2) $b_i(X) = \dim H^i(X, \mathbb{R})$ denotes the $i-$th Betti number of X.

(3) If G is a finitely generated abelian group, then rkG will denote its rank (over $\mathbb{Z}$).

(4) If X is a compact space, then $h^q(X, \mathcal{F})$ denotes the dimension of $H^q(X, \mathcal{F})$.

1.1 Proposition

Let Y be a connected compact complex space (not necessarily reduced), every component Y_i of Y being of positive dimension. Let D be an effective divisor on Y such that $D|Y_i \neq 0$ for all i. Let $\mathcal{F}$ be a locally free sheaf on Y. Then there exists $k_0 \in \mathbb{N}$ such that

$$H^0(Y, \mathcal{F} \otimes \mathcal{O}_Y(-kD)) = 0$$

for $k \geq k_0$.

Proof. We have natural inclusions

$$H^0(Y, \mathcal{F} \otimes \mathcal{O}_Y(-(k+1)D)) \subset H^0(Y; \mathcal{F} \otimes \mathcal{O}_Y(-kD)).$$

Take k_0 such that this sequence is stationary for $k \geq k_0$. Then s has to vanish at any order along $D|Y_i$ for every i (s can be thought of locally as a tuple of holomorphic functions), hence $s|Y_i = 0$, and $s = 0$.

1.2 Corollary

Let S be a smooth compact complex surface containing an effective divisor C such that $c_1(\mathcal{O}_S(C)) = 0$. Let B be a vector bundle on S. Then for a generic $\mathcal{L} \in \mathrm{Pic}^0(X)$ we have

$$H^0(S, B \otimes \mathcal{L}) = H^2(S, B \otimes \mathcal{L}) = 0.$$

In particular $\chi(S, B) = -h^1(S, B \otimes \mathcal{L}) \leq 0$.

Proof. The vanishing $H^0(S, B \otimes \mathcal{O}_S(-kC)) = 0$ for large k follows from (1.1). Since $\mathcal{O}_S(kC)$ is topologically trivial for large suitable k, the required H^0-vanishing follows from semi-continuity. The H^2-vanishing follows by applying the previous arguments to $B^* \otimes K_S$ and Serre duality.

1.3 Corollary

Let X be a smooth compact threefold with $b_2(X) = 0$ carrying an effective divisor D. Then $H^0(X, B \otimes \mathcal{L}) = 0$ for generic $\mathcal{L}$ in $\mathrm{Pic}^0(X)$ and every vector bundle B on X.

Proof. Since $c_1(\mathcal{O}_X(mD)) = 0$ in $H^2(X, \mathbb{Z})$, we can apply (1.1) to obtain

$$H^0(X, B \otimes \mathcal{O}_X(-kmD)) = 0$$

for k large. Now we conclude again by semi-continuity.

The last lemma is of course well-known; we include it for the convenience of the reader.

1.4 Lemma

Let X be a compact manifold of dimension n with $a(X) = n$. Then $b_2(X) > 0$.

Proof. Choose a birational morphism $\pi : \hat{X} \longrightarrow X$ such that $\hat{X}$ is a projective manifold. Take a general very ample divisor $\hat{D}$ on $\hat{X}$ and a general curve $\hat{C} \in \hat{X}$. Let $D = \pi(\hat{D})$ and $C = \pi(\hat{C})$. Then D meets C in finitely many points, hence $D \cdot C > 0$, in particular $c_1(\mathcal{O}_X(D))$ is not torsion in $H^2(X; \mathbb{Z})$.

1.5 Lemma

Let X be a smooth compact threefold and $f : X \longrightarrow C$ be a surjective holomorphic map to the smooth curve C. Let $\mathcal{F}$ be a locally free sheaf on X. Then $R^i f_*(\mathcal{F})$ is locally free for all i.

Proof. (a) Note that local freeness is equivalent to torsion freeness, since $\dim C = 1$. Hence the claim is clear for $i = 0$.

(b) Next we treat the case $i = 2$. We shall use relative duality (see [RRV71], [We85]); it states in our special situation (f is flat with even Gorenstein fibers) thatif $R^j f_*(\mathcal{G})$ is locally free for a given locally free sheaf $\mathcal{G}$ and fixed j, then

$$R^{2-j} f_*(\mathcal{G}^* \otimes \omega_{X|C}) \simeq R^j f_*(\mathcal{G})^*,$$

in particular $R^{2-j} f_*(\mathcal{G}^* \otimes \omega_{X|C})$ is locally free. Here $\omega_{X|C} = \omega_X \otimes f^*(\omega_C^*)$ is the relative dualising sheaf. Applying this to $j = 0$ and $\mathcal{G} = \mathcal{F}^* \otimes \omega_{X|C}^*$ our claim for $i = 2$ follows.

(c) Finally we prove the freeness of $R^1 f_*(\mathcal{F})$. By a standard theorem of Grauert it is sufficient that $h^1(X_y, \mathcal{F}|X_y)$ is constant, X_y the analytic fiber over $y \in C$. By flatness, $\chi(X_y, \mathcal{F}|X_y)$ is constant, hence it is sufficient that $h^j(X_y, \mathcal{F}|X_y)$ is constant for $j = 0$ and $j = 2$. By the vanishing $R^3 f_*(\mathcal{F}) = 0$, we have (see e.g. [BaSt76])

$$R^2 f_*(\mathcal{F})|\{y\} \simeq H^2(X_y, \mathcal{F}|X_y).$$

Therefore $h^2(X_y, \mathcal{F}|X_y)$ is constant by (b). Finally

$$h^0(X_y, \mathcal{F}|X_y) = h^2(X_y, \mathcal{F}^*|X_y \otimes \omega_{X_y}) = h^2(X_y, \mathcal{F}^* \otimes \omega_X|X_y)$$

is constant by applying the same argument, now to $\mathcal{F}^* \otimes \omega_X$.

2. Criteria for the vanishing of $H^2(X, B)$.

2.0 Set-up

We fix in this section the following situation:

(1) X is a smooth compact threefold with $b_2(X) = 0$.

(2) V is a normal projective variety of dimension $d = 1$ or 2

(3) $f : X \dashrightarrow V$ is a meromorphic map

(4) $\hat{X}$ is a normal compact threefold with a birational map $\sigma : \hat{X} \longrightarrow X$ such that the induced map $\hat{f} : \hat{X} \longrightarrow V$ is holomorphic; $\hat{X}$ is smooth in case $d = 1$

(5) $\hat{f}$ is equidimensional.

In our applications f will be an algebraic reduction.

2.1 Lemma

Let $f : X \longrightarrow V$ be as in 2.0,(1)-(3). Then (4) and (5) can always be achieved, possibly after changing V birationally.

Proof. The claim is obvious if $d = 1$, by elimination of indeterminacies. So let $d = 2$. Take a sequence of blow-ups $\rho : X_1 \longrightarrow X$ such that the induced map $f_1 : X_1 \longrightarrow V$ is holomorphic. Then however f_1 will in general not be equidimensional. To achieve this we could either apply Hironaka's flattening theorem or we can argue in a more elementary way as follows. Let $V_0 = \{y \in V | f_1^{-1}(y) \text{ is } 1-\text{dimensional}\}$. We obtain a holomorphic map $\mu : V_0 \longrightarrow \mathcal{C}_1(X_1)$, the space of 1-cycles on X_1. It is a standard fact in the theory of cycle spaces that μ extends to a meromorphic map $\mu : V \dashrightarrow \mathcal{C}_1(X_1)$. Now let $V' = \overline{\mu(V)} \subset \mathcal{C}_1(X_1)$ and let $X' \longrightarrow V'$ be the induced family. Modify V' to V'' so that V'' is projective; then take $X'' = X' \times_{V'} V''$ (base change). Finally let $\hat{X}$ be the normalisation of X'' and $\hat{V}$ the normalisation of V''. Then the induced map $\hat{f} : \hat{X} \longrightarrow \hat{V}$ has the properties we are looking for.

From now on we substitute $\hat{V}$ by V.

First we will treat the case $d = 2$. We fix an ample effective divisor A on V in the set-up (2.0) and let $\mathcal{L} = \sigma_*(\hat{f}^*(A))^{**}$. Then

$$\hat{f}^*(A) = \sigma^*(\mathcal{L}) \otimes \mathcal{O}_{\hat{X}}(-E)$$

with an effective divisor E on $\hat{X}$ supported on the exceptional locus of σ.

2.2 Proposition

$H^2(X, B \otimes \mathcal{L}^{\otimes m}) = 0$ for every vector bundle B and $m \gg 0$ (depending on B.)

Proof. First we claim

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{L}^{\otimes m})(-mE)) = 0 \tag{1}$$

for large m.

For the proof notice $\sigma^*(B \otimes \mathcal{L}^{\otimes m})(-mE)) \simeq \sigma^*(B) \otimes \hat{f}^*(mA)$. We consider the Leray spectral sequence associated to $\hat{f}$. Then

$$E_2^{2,0} = H^2(V, \hat{f}_*(\sigma^*(B)) \otimes mA) = 0, m >> 0,$$

and

$$E_2^{1,1} = H^1(V, R^1\hat{f}_*(\sigma^*(B)) \otimes mA) = 0, m >> 0$$

by Serre's vanishing theorem. Moreover $E_2^{0,2} = 0$ simply because $R^2\hat{f}_*(\sigma^*(B)) = 0$, all fibers of $\hat{f}$ being 1-dimensional. Now the claim (1) follows from the spectral sequence.

Let A_m be the complex subspace of X defined by the ideal sheaf $\sigma_*(-mE)$. Then we consider the following commutative diagram, the vertical arrows just being pull-back maps (as arising in the Leray spectral sequence)

$$\begin{array}{ccccc} H^2(\hat{X}, \sigma^*(B \otimes \mathcal{L}^m)(-mE)) & \longrightarrow & H^2(\hat{X}, \sigma^*(B \otimes \mathcal{L}^m)) & \longrightarrow & H^2(mE, \sigma^*(B \otimes \mathcal{L}^m)) \\ \uparrow & & \uparrow & & \uparrow \\ H^2(X, B \otimes \mathcal{L}^m \otimes \sigma_*(-mE)) & \longrightarrow & H^2(X, B \otimes \mathcal{L}^m) & \longrightarrow & H^2(X, B \otimes \mathcal{L}^m \otimes \mathcal{O}_{A_m}) \end{array}$$

Note that

$$H^2(X, B \otimes \mathcal{L}^m \otimes \mathcal{O}_{A_m}) = 0 \tag{2}$$

since $\dim A_m \leq 1$.

We claim that the pull-back map

$$\alpha : H^2(X, B \otimes \mathcal{L}^m) \longrightarrow H^2(\hat{X}, \sigma^*(B \otimes \mathcal{L}^m))$$

is injective. In fact, let $\tau : \tilde{X} \longrightarrow \hat{X}$ be a desingularisation. Since $R^1(\sigma \circ \tau)_*(\mathcal{O}_{\tilde{X}}) = 0$, the Leray spectral sequence yields the injectivity of the pull-back map

$$\beta : H^2(X, B \otimes \mathcal{L}^m) \longrightarrow H^2(\tilde{X}, (\sigma \circ \tau)^*(B \otimes \mathcal{L}^m)).$$

Since β factors through α, also α must be injective. Now the diagram together with (1) and (2) gives our claim (2.2).

2.3 Corollary

Let X be a compact threefold with $b_2(X) = 0$ and with $a(X) = 2$. Let B be a vector bundle on X. Then

(a) $H^2(X, B \otimes \mathcal{L}) = 0$ for $\mathcal{L} \in \mathrm{Pic}^0(X)$ generic.

(b) $\chi(X, B) \leq 0$

(c) $c_3(X) = 0$.

Proof. Since $b_2(X) = 0$, some multiple $\mathcal{L}^{\otimes k}$ with $\mathcal{L}$ as in (2.2) is in $\mathrm{Pic}^0(X)$. Apply (2.2) to see that $H^2(X, B \otimes \mathcal{L}^{mk}) = 0$ for large m. Hence by semi-continuity (a) holds. By Serre duality $H^1(X, B \otimes \mathcal{L}) = 0$ for generic $\mathcal{L}$. From (1.3) we therefore deduce $\chi(X, B \otimes \mathcal{L}) = 0$. Hence $\chi(X, B) = 0$. Now (c) follows from Riemann-Roch for $B = T_X$.

(2.4) We now treat the case that $d = 1$ and that f is not holomorphic. Again we fix an ample divisor A on V and let $\mathcal{L} = \sigma_*(\hat{f}^*(\mathcal{O}_V(A))^{**}$. We will prove :

2.5 Proposition

Let X be a smooth compact threefold with $a(X) = 1$, $b_2(X) = 0$ and with non-holomorphic algebraic reduction $f : X \dashrightarrow V$. Then

$$\chi(X, B) \leq 0$$

for every vector bundle B on X.

Proof. As in (2.3) it is sufficient to prove

$$H^q(X, B \otimes \mathcal{H}) = 0$$

for $q = 0$ and $q = 2$ and some $\mathcal{H} \in \mathrm{Pic}^0(X)$ (which of course can be taken individually in each case).

For $q = 0$ the claim follows from (1.3). Turning to $q = 2$, we fix a positive integer k such that

$$H^2(\hat{F}, \sigma^*(B \otimes \mathcal{L}^k)) = 0 \qquad (*),$$

where $\hat{F}$ is the general fiber of $\hat{f}$. This k can be found since

$$H^2(\hat{F}, \sigma^*(B \otimes \mathcal{L}^k)) = H^0(\hat{F}, \sigma^*(B^* \otimes \mathcal{L}^{-k}) \otimes \omega_{\hat{F}}^{-1}),$$

which vanishes by (1.1), having in mind that $\sigma^*(\mathcal{L})|\hat{F} = \hat{f}^*(A) \otimes \mathcal{O}_{\hat{X}}(E)\hat{F}$ is effective and non-zero.

Now let

$$\mathcal{H} = \mathcal{L}^{m+k}$$

with m large to be specified in a moment. Arguing as in (2.2) we must prove

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{L}^{m+k})(-mE)) = H^2(\hat{X}, \sigma^*(B \otimes \mathcal{L}^k) \otimes \hat{f}^*(mA)) = 0.$$

By the Leray spectral sequence for $\hat{f}$ we need for that

$$H^1(V, R^1\hat{f}_*(\sigma^*(B \otimes \mathcal{L}^k)) \otimes mA) = 0$$

and

$$H^0(V, R^2\hat{f}_*(\sigma^*(B \otimes \mathcal{L}^k)) \otimes mA) = 0.$$

The first is just Serre vanishing; for the second we claim

$$R^2\hat{f}_*(\sigma^*(B \otimes \mathcal{L}^k)) = 0.$$

But this is just (*) together with (1.5), saying that $R^2\hat{f}_*(\sigma^*(B\otimes\mathcal{L}^k))$ is locally free. This finishes the proof.

3. The case of a holomorphic algebraic reduction

3.0 Setup

In this section we fix a smooth compact 3-fold X with *holomorphic* algebraic reduction $f : X \longrightarrow V$ to the smooth curve V of genus g. We shall always assume $b_2(X) = 0$. Of course f has connected fibers.

3.1 Theorem

Let B be a holomorphic vector bundle on X. Then $\chi(X, B) = 0$.

3.2 Corollary

$c_3(X) = 0$.

Proof. This is just Riemann-Roch for $\chi(X, T_X)$.

Proof of 3.1 Let $W = H^1(X, \mathcal{O}_X)$. Then every element in W is represented as a topologically trivial line bundle. Let A be an ample line bundle on V. Then for $m \gg 0$ we have

$$H^0(V, R^j f_*(B) \otimes A^{-m}) = 0 \tag{1}$$

for all $j \geq 0$. We claim that (1) implies

$$H^0(V, R^j f_*(B \otimes \mathcal{L})) = 0 \tag{2}$$

for $\mathcal{L} \in W$ general. In fact, consider locally the universal bundle $\hat{\mathcal{L}}$ on $X \times W$. Let $F = f \times \mathrm{id} : X \times P \longrightarrow V \times P$ and $\hat{B} = \mathrm{pr}_X^*(B)$. The coherent sheaf $R^jF_*(\hat{B} \otimes \hat{\mathcal{L}})$ satisfies

$$R^jF_*(\hat{B} \otimes \hat{\mathcal{L}})|V \times \{t\} \simeq R^j f_*(B \otimes \hat{\mathcal{L}}_t),$$

where $\hat{\mathcal{L}}_t$ is the line bundle corresponding to $t \in W$. Choose $m \gg 0$ and t_0 such that $f^*(A^{-m}) = \hat{\mathcal{L}}_{t_0}$. By (1) and the assumption $b_2(X) = 0$ we have

$$H^0(V, R^j f_*(B \otimes \hat{\mathcal{L}}_{t_0})) = 0.$$

Hence it is sufficient to show that $R^jF_*(\hat{B}\otimes\hat{\mathcal{L}})$ is flat with respect to the projection $q : V \times P \longrightarrow W$, over a Zariski open set of W, then the usual semi-continuity theorem gives the claim (2). Now $R^j f_*(B \otimes \hat{\mathcal{L}}_t)$ is locally free on $V = V \times t$ for every t by (1.5), hence it is clear that there is a Zariski open set $U \subset W$ such that

$R^j F_*(\hat{B} \otimes \hat{\mathcal{L}})$ has constant rank over U, hence is locally free over U (observe just that the set where the rank of a coherent sheaf is not minimal is analytic).

On the other hand we have for $m \gg 0$ and all $i \geq 1, j \geq 0$ by Serre's vanishing theorem

$$H^i(V, R^j f_*(B) \otimes A^m) = 0 \qquad (3).$$

In the same way as (2) follows from (1) we conclude from (3) that

$$H^i(V, R^j f_*(B \otimes \mathcal{L})) = 0 \qquad (4)$$

for general $\mathcal{L} \in P$.

Now (2) and (4) give by the Leray spectral sequence $H^i(X, B \otimes \mathcal{L}) = 0$ for general $\mathcal{L} \in P$ and all $i \geq 0$. In particular $\chi(X, B \otimes \mathcal{L}) = 0$. Since $c_k(B \otimes \mathcal{L}) = c_k(B)$, Riemann-Roch gives our claim $\chi(X, B) = 0$.

4. The Main Theorem and Examples

Putting (2.6), (2.8) and (3.2) together, we obtain

4.1 Theorem

Let X be a compact complex manifold of dimension 3. If $b_2(X) = 0$ and if $c_3(X) \neq 0$, then every meromorphic function on X is constant.

Proof. We only have to notice that $a(X) < 3$ by (1.2) and then to apply the above mentioned propositions.

4.2 Corollary

Let X be a compact complex threefold homeomorphic to the sphere S^6. Then every meromorphic function on X is constant.

Proof. Note that $c_3(X) = \chi_{\text{top}}(S^6) = 2$ and apply (4.1).

The condition $c_3(X) \neq 0$ in (4.1) can be translated into $2b_1(X) + b_3(X) \neq 2$.

4.3 Corollary

Let X be a compact threefold with $b_2(X) = 0$ and $a(X) = 1$ or 2. Let $X \dashrightarrow V$ be an algebraic reduction to the smooth variety V.

(1) If $\dim V = 1$, then $V = \mathbb{P}_1$.

(2) If $\dim V = 2$ and $\kappa(X) = -\infty$, then V is rational.

Proof. By the Main Theorem we have $b_1(X) \leq 1$. Let $\hat{X} \longrightarrow X$ be a sequence of blow-ups with smooth centers such that the induced map $\hat{f} : \hat{X} \longrightarrow V$ is holomorphic. Notice $b_1(\hat{X}) \leq 1$. Now the Leray spectral sequence gives $b_1(V) \leq b_1(\hat{X})$, since $\hat{f}$ has connected fibers and therefore $\hat{f}_*(\mathbb{C}) = \mathbb{C}$. So $V = \mathbb{P}_1$ in case $\dim V = 1$.

If dim $V = 2$ we still need to know that $\kappa(V) = -\infty$. This comes from [Ue87].

In the rest of this section we give examples of threefolds with $a(X) > 0$, $b_2(X) = 0$ and $c_3(X) = 0$ so that (4.1) is sharp.

4.4 Example

The so-called Calabi-Eckmann threefolds are compact threefolds homeomorphic to $S^3 \times S^3$, see [Ue75]. They can be realised as elliptic fiber bundles over $\mathbb{P}_1 \times \mathbb{P}_1$. Hence $a(X) = 2$, $b_1(X) = b_2(X) = 0$ and $b_3(X) = 2$.

We now show that Calabi-Eckmann manifolds can be deformed to achieve $a(X) = 1$ or $a(X) = 0$ and $b_1 = b_2 = 0$, $b_3 = 2$. We choose positive real numbers a, b, c and let $B = \mathbb{C}^2 \setminus \{(0,0)\}$. We define the following action of $\mathbb{C}$ on $B \times B$:

$$(t, x, y, u, v) \mapsto (\exp(t)x, \exp(at)y, \exp(ibt)u, \exp(ict)v).$$

One checks easily that this action is holomorphic, free and almost proper so that the quotient X exists and is a compact manifold. If $a = 1$ and $b = c$, then X is a Calabi-Eckmann manifold. If however $\log a \neq \mathbb{Q}$ and $b = c$ resp. $\log a \neq \mathbb{Q}$ and $\log b, \log c$ are $\mathbb{Q}$-linearly independent, then $a(X) = 1$ resp. $a(X) = 0$.

4.5 Example

Hopf threefolds of the form

$$\mathbb{C}^3 \setminus \{(0,0,0)\}/\mathbb{Z},$$

with the action of $\mathbb{Z} \simeq \{\lambda^k \,;\, k \in \mathbb{Z}\}$ being defined by

$$\lambda(x, y, z) = (\alpha x, \beta y, \gamma z), \qquad 0 < |\alpha|, |\beta|, |\gamma| < 1$$

are homeomorphic to $S^1 \times S^5$. They have $a(X) = 0, 1$ or 2 and $b_1(X) = 1$, while $b_2(X) = b_3(X) = 0$. This realises the other possibility for the pair (b_1, b_3) when $b_2 = 0$ and $a(X) > 0$, see the Main Theorem.

Notice that the algebraic reduction is holomorphic in (4.4) but not in (4.5).

4.6 Example

We finally give other examples of compact threefolds X with $a(X) = 0$ and $b_1 = b_2 = 0$, $b_3 = 2$. Let $\Gamma \subset \mathrm{Sl}(2, \mathbb{C})$ be a torsion free cocompact lattice in such a way that $X := \mathrm{Sl}(2, \mathbb{C})/\Gamma$ has $b_1(X) = 0$.

This last condition is not automatic. Let $Y = \mathrm{SU}(2)\backslash \mathrm{Sl}(2, \mathbb{C})/\Gamma$; then Y is a compact differentiable manifold admitting a differentiable fibration $\pi : X \longrightarrow Y$ with S^3 is fiber. Since $b_1(X) = 0$, we also have $b_1(Y) = 0$, hence $b_2(Y) = 0$ by Poincaré duality. Now the Leray spectral sequence immediately gives $b_2(X) = 0$, the fibers of π being 3-spheres. $b_3(X) = 2$ is again clear from the Leray spectral sequence. Finally the fact that X does not carry any non-constant meromorphic function results from [HM83] from which we even deduce that X does not carry any hypersurface (anyway X is homogeneous).

5. On the finer structure of threefolds with $b_2(X) = 0$ and $a(X) = 1$.

In this section we investigate more closely threefolds X with $b_2(X) = 0$ and holomorphic algebraic reduction $f : X \longrightarrow V$ to a curve V. We know already that V is rational. We fix these notations. The key to our investigations is

5.1 Theorem

The restriction $r : H^1(X, \mathcal{O}_X) \longrightarrow H^1(F, \mathcal{O}_F)$ is surjective where F is the general smooth fiber of f.

We need some preparations for the proof of (5.1). Let $\Delta \subset V$ be a finite **non empty** set such that $A = f^{-1}(\Delta) \subset X$ contains all singular fibers of f. Let $W = V \setminus \Delta, U = f^{-1}(W)$ so that $g = f|U$ has only smooth fibers, hence is a C^∞-fibration. Let D_i, $1 \le i \le r$ be the irreducible components of A and let $s = \text{card}\,\Delta$ be the number of connected components of A. Furthermore we set $t = b_1(F)$, F the general smooth fiber of f and recall that g is the genus of V. For a non-compact space Z we let $b_i(Z) = \dim H_i(Z, \mathbb{R})$, if this space is finite-dimensional. We prepare the proof of (3.1) by three lemmas.

5.2 Lemma

(1) The natural exact sequence of groups

$$1 = \pi_2(W) \longrightarrow \pi_1(F) \longrightarrow \pi_1(U) \longrightarrow \pi_1(W) \longrightarrow 1$$

is exact and (non-canonically) split.

(2) $b_1(U) = b_1(W) + b_1(F) = 2g + s - 1 + t \le b_1(X) + s - 1 + t$.

(3) $r = s - 1 + t + 2g - b_1(X)$.

Proof. (1) Since W is a non-compact Riemann surface, $\pi_1(W)$ is a free group of $2g + s - 1$ generators and since W is uniformised by either $\mathbb{C}$ or by the unit disc, we have $\pi_2(W) = 1$. Hence the exact homotopy sequence of the fibration $g : U \longrightarrow W$ gives the exact sequence of groups stated in (1). Since $\pi_1(W)$ is a free group, the sequence splits.

(2) From (1) we deduce that

$$H_1(U, \mathbb{Z}) \simeq H_1(W, \mathbb{Z}) \oplus H_1(F, \mathbb{Z}).$$

Moreover $f_* : \pi_1(X) \longrightarrow \pi_1(V)$ is surjective since the fibers of f are connected. Hence (2) follows.

(3) The cohomology sequence with rational coefficients of the pair (X, A) gives

$$0 = H^4(X) \longrightarrow H^4(A) \longrightarrow H^5(X, A) \longrightarrow H^5(X) \longrightarrow H^5(A) = 0.$$

By duality we have $H^5(X, A) \simeq H_1(U)$ and $H^5(X) \simeq H_1(X)$. Hence

$$r = \dim H^4(A) = b_1(U) - b_1(X) = 2g + s - 1 + t - b_1(X)$$

by (2), as claimed.

Now choose an integer $m > 0$ such that $c_1(\mathcal{O}_X(mD_i)) = 0$ in $H^2(X, \mathbb{Z})$ for all $1 \le i \le r$. Let $G = \bigoplus \mathbb{Z}[mD_i] \simeq \mathbb{Z}^r$ be the free abelian group of r generators and let $\phi : G \longrightarrow \mathrm{Pic}^0(X)$ be given by sending $D = \sum_i a_i mD_i$ to $\mathcal{O}_X(D)$.

5.3 Lemma

Let $K = \mathrm{Ker}\phi$ and $H = \mathrm{Im}\phi$. Then $\mathrm{rk}\, K \le s - 1$ and $\mathrm{rk}\, H \ge r - s + 1$.

Proof. Write $\Delta = \{x_1, \ldots, x_s\}$ and define $G_0 = \bigoplus_j \mathbb{Z}[mx_j] \subset \mathrm{Div}(V)$. Let $K_0 = \{g \in G | \deg(g) = 0\}$. Then $K_0 \simeq \mathbb{Z}^{s-1}$. Consider the pull-back map $f^* : \mathrm{Div}(V) \longrightarrow \mathrm{Div}(X)$ and let $K_1 = f^*(K_0)$. We claim that $K \subset K_1$, which will prove the lemma, since then $\mathrm{rk}\, K \le \mathrm{rk}\, K_1 \le \mathrm{rk}\, K_0 \le s - 1$. To show $K \subset K_1$, consider $D = \sum_i a_i mD_i = \sum \alpha_j mD_j - \sum \beta_k mD_k = D' - D''$ with $\alpha_j, \beta_k > 0$ and $D_j \ne D_k$ for all $j \ne k$. If $D \in K$, then $\mathcal{O}_X(D) \simeq \mathcal{O}_X$, hence $\mathcal{O}_X(D') \simeq \mathcal{O}_X(D'')$. Letting $\mathcal{L} = \mathcal{O}_X(D')$, we see that $h^0(X, \mathcal{L}) \ge 2$, therefore $\mathcal{L} \in f^*(\mathrm{Pic}(V)$ since f is the algebraic reduction of X. Thus $D_i \in K_1$, hence $D \in K_1$, proving the claim.

The last ingredient in the proof of (5.1) is provided by

5.4 Lemma

The kernel of the restriction map $\alpha : H^1(X, \mathcal{O}_X) \longrightarrow H^1(U, \mathcal{O}_U)$ equals the image of $f^* : H^1(V, \mathcal{O}_V) \longrightarrow H^1(X, \mathcal{O}_X)$.

Proof. The Leray spectral sequence gives the following commutative diagram with exact rows :

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(V, \mathcal{O}_V) & \xrightarrow{f^*} & H^1(X, \mathcal{O}_X) & \longrightarrow & H^0(V, R^1 f_*(\mathcal{O}_X)) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 = & H^1(W, \mathcal{O}_W) & \longrightarrow & H^1(U, \mathcal{O}_U) & \longrightarrow & H^0(W, R^1 g_*(\mathcal{O}_W)) & \longrightarrow 0
\end{array}$$

Now $R^1 f_*(\mathcal{O}_X)$ is locally free by (1.5). This implies that the restriction map

$$\tilde{\alpha} : H^0(V, R^1 f_*(\mathcal{O}_X)) \longrightarrow H^0(W, R^1 g_*(\mathcal{O}_W))$$

is surjective, hence also $\alpha : H^1(X, \mathcal{O}_X) \longrightarrow H^1(U, \mathcal{O}_U)$ by the diagram. This proves (5.4).

We are now able to finish the proof of (5.1).

We consider again $H \subset \mathrm{Pic}^0(X)$, the image of $\phi : G \longrightarrow \mathrm{Pic}^0(X)$. H has rank $\geq r - s + 1 = t + (2g - b_1(X))$ by (5.2). Let $\tilde{H}$ be the inverse image of H under the natural map $H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}^0(X)$. Then

$$\mathrm{rk}\,\tilde{H} = \mathrm{rk}\,H + b_1(X) \geq 2g + t.$$

Let $H_1 = \alpha(\tilde{H}) \subset H^1(U, \mathcal{O}_U)$. We claim that

$$\mathrm{rk}\,H_1 \geq t. \tag{$*$}$$

In fact, by (5.4) $\tilde{H}_0 := \mathrm{Ker}(\alpha|\tilde{H}) = \tilde{H} \cap f^*(H^1(V, \mathcal{O}_V))$. Now $\tilde{H}_0$ is a discrete subgroup of $f^*(H^1(V, \mathcal{O}_V) \simeq \mathbb{C}^g$, so that $rk\tilde{H}_0 \leq 2g$. Hence $\mathrm{rk}\,H_1 \geq 2g + t - 2g = t$, proving (*).

Let $\zeta : H^1(U, \mathbb{Z}) \longrightarrow H^1(U, \mathcal{O}_U)$ be the canonical map; then $H_1 \subset \zeta(H^1(U, \mathbb{Z}))$ since $\mathcal{O}_U(D) \simeq \mathcal{O}_U$ for all $D \in G$. We therefore conclude that the group $H_2 := \beta(H_1)$ with

$$\beta : H^1(U, \mathcal{O}_U) \longrightarrow H^0(W, R^1 g_*(\mathcal{O}_U))$$

the canonical isomorphism, is of finite index in

$$\Gamma := \mathrm{Im}(H^0(W, R^1 f_* \mathbb{Z})) \longrightarrow H^0(W, R^1 g_*(\mathcal{O}_U)).$$

By 5.2(2) we conclude that $\mathrm{rk}\,H^0(W, R^1 f_*(\mathbb{Z})) = t$, hence H_2 has finite index in Γ. Now take a general smooth fiber F and let $\lambda : H^0(W, R^1 g_*(\mathcal{O}_U)) \longrightarrow H^1(F, \mathcal{O}_F)$ be the restriction map. Then we conclude that $\lambda(H_2)$ is of finite index in

$$\mathrm{Im}(H^1(F, \mathbb{Z}) \longrightarrow H^1(F, \mathcal{O}_F)).$$

Since the linear span of this image in $H^1(F, \mathcal{O}_F)$ is $H^1(F, \mathcal{O}_F)$, we deduce from the construction of H_2 that the restriction map $H^1(X, \mathcal{O}_X) \longrightarrow H^1(F, \mathcal{O}_F)$ is surjective proving finally (5.1).

(5.5) We now study the structure of the smooth fibers F of f. Since $K_F \equiv 0$ and $c_2(F) = c_2(X)|F = 0$, we conclude from the classification of surfaces that F is one of the following: a Hopf surface, an Inoue surface, a Kodaira surface (primary or secondary), a torus or hyperelliptic (see e.g. [BPV84]). By [Ka69] however, F cannot be hyperelliptic. The reason is the existence of a relative Albanese reduction in that case.

5.6 Theorem

The general fiber F of f cannot be a Kodaira surface nor a Hopf surface with algebraic dimension 1.

Proof. Let F_0 be a fixed smooth fiber and assume that F_0 is a Kodaira surface or a Hopf surface with $a(F_0) = 1$. Let $g_0 : F_0 \longrightarrow C_0$ be "the" algebraic reduction which is an elliptic fiber bundle. Let $\mathcal{L}_0 = g_0^*(\mathcal{G}_0)$ with $\mathcal{G}_0$ very ample on C_0.

(1) There exists a line bundle $\tilde{\mathcal{L}}$ on X with $\tilde{\mathcal{L}}|F_0 = \mathcal{L}_0$.

Proof. By passing to some power $\mathcal{L}_0^m$ if necessary, we have $c_1(\mathcal{L}_0) = 0$ in $H^2(X, \mathbb{Z})$. Let

$$\lambda_1 : H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}(X)$$

and

$$\lambda_2 : H^1(F_0, \mathcal{O}_{F_0}) \longrightarrow \mathrm{Pic}(F_0)$$

be the canonical maps, $r : H^1(X, \mathcal{O}_X) \longrightarrow H^1(F_0, \mathcal{O}_{F_0})$ the restriction map. Choose $\alpha \in H^1(F_0, \mathcal{O}_{F_0})$ with $\lambda_2(\alpha) = \mathcal{L}_0$. Since r is surjective by (4.2), we find $\beta \in H^1(X, \mathcal{O}_X)$ with $r(\beta) = \alpha$. Now let $\tilde{\mathcal{L}} = \lambda_1(\beta)$.

(2) Let F be any smooth fiber of f. Then $\kappa(\tilde{\mathcal{L}}|F) = 1$. In fact, it follows from the local freeness of $R^j f_*(\tilde{\mathcal{L}})$ (1.5) that

$$f_*(\tilde{\mathcal{L}}^\mu)|\{y\} \simeq H^0(F, \tilde{\mathcal{L}}^\mu),$$

where $F = f^{-1}(y)$, see [BaSt76, chap.3, 3.10].

(3) From the generically surjective morphism

$$f^* f_*(\tilde{\mathcal{L}}^m) \longrightarrow \tilde{\mathcal{L}}^m,$$

($m \gg 0$), we obtain a meromorphic map

$$g : X \dashrightarrow \mathbb{P}(f_*(\tilde{\mathcal{L}}^m)),$$

which, restricted to F is holomorphic and just gives the algebraic reduction of F. Let Z be the closure of the image of g. Then f factors via the meromorphic map $h_1 : X \dashrightarrow Z$ and the holomorphic map $h_2 : Z \longrightarrow V$. Now h_2 is the restriction of the canonical projection $\mathbb{P}(f_*(\tilde{\mathcal{L}}^m)) \longrightarrow V$, therefore h_2 is a projective morphism and Z is projective. Hence $a(X) \geq 2$, contradiction.

From (5.6) it follows that F can only be an Inoue surface, a Hopf surface without meromorphic functions or a torus. In order to exclude by a similar method as in (5.6) also tori of algebraic dimension 1, we would need the existence of a relative algebraic reduction (the analogue of $h_2 : X \dashrightarrow Z$) in that case, too.

We now look more closely to the structure of f.

5.7 Proposition

Assume that F is not a torus. Then
(1) $R^1 f_*(\mathcal{O}_X) = \mathcal{O}_V$
(2) $R^2 f_*(\mathcal{O}_X) = 0$
(3) $\dim H^1(X, \mathcal{O}_X) = 1$
(4) $H^2(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = 0$.

Proof. (2) Since $H^2(F, \mathcal{O}_F) = 0$, the sheaf $R^2 f_*(\mathcal{O}_X)$ is torsion, hence 0 by (1.5). Then $H^3(X, \mathcal{O}_X) = 0$ is immediate from the Leray spectral sequence.

(1) Since $h^1(\mathcal{O}_F) = 1$, $R^1 f_*(\mathcal{O}_X)$ is a line bundle on V. Let

$$d = \deg R^1 f_*(\mathcal{O}_X).$$

Then Riemann-Roch gives $\chi(R^1 f_*(\mathcal{O}_X)) = d + 1$. On the other hand the Leray spectral sequence together with $H^3(X, \mathcal{O}_X) = 0$ and (2) yields

$$\chi(R^1 f_*(\mathcal{O}_X)) = h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X) = -\chi(\mathcal{O}_X) + 1.$$

We conclude $d = -\chi(\mathcal{O}_X) = 0$. This proves (1). Now (3) and the second part of (4) are obvious.

5.8 Remark In case F is a torus, $R^1 f_*(\mathcal{O}_X)$ is a rank 2 bundleand $R^2 f_*(\mathcal{O}_X)$ is a line bundle. Using (4.2) it is easy to see that
(1) $R^1 f_*(\mathcal{O}_X) = \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a, b \geq 0$
(2) $R^2 f_*(\mathcal{O}_X) = \mathcal{O}(a + b)$.

Note that (2) gives dually $f_*(\omega_{X|V}) = \mathcal{O}(-a - b)$. Usually one expects the degree of $f_*(\omega_{X|V})$ to be semi-positive, but here we are in a highly non-Kähler situation where it might happen that the above degree is negative, see [Ue87].

5.9 Proposition

Assume that F is not a torus. Then $H^0(X, \Omega^i_X) = 0$, $1 \leq i \leq 3$.

Proof. For $i = 3$ the claim follows already from (5.7) and (5.8).

(1) First we treat the case $i = 1$. Let ω be a holomorphic 1-form. Let $j : F \longrightarrow X$ be the inclusion. Then $j^*(\omega) = 0$, hence at least locally near F we have $\omega = f^*(\eta)$, hence $d\omega = 0$ near F and therefore the holomorphic 2-form $d\omega = \partial\omega = 0$ everywhere. Now the space of closed holomorphic 1-forms can be identified with $H^0(X, d\mathcal{O}_X)$ and as it is well known (see e.g. [Ue75]), we have the inequality

$$2h^0(X, d\mathcal{O}_X) \leq b_1(X),$$

hence $b_1(X) \leq 1$ gives our claim.

(2) In case $i = 2$, we again have $j^*(\omega) = 0$. Let U be a small open set in V such that $f|f^{-1}(U)$ is smooth. z be a coordinate on U and $h = f^*(z)$. Then we conclude that

$$\omega|f^{-1}(U) = dh \wedge \alpha$$

with some holomorphic 1-form α. Now again $j^*(\alpha) = 0$ and therefore $\alpha = f^*(\beta)$. In total $\omega = f^*(dz \wedge \beta) = 0$.

5.10 Corollary

Assume that F is not a torus. Then there exists some x such that $f_*(\Omega^1_X) = \mathbb{C}_x$, i.e. a sheaf supported on x with a 1-dimensional stalk at x. In particular f has at most one normal singular fiber and such a fiber has exactly one singularity which has embedding dimension 3.

Proof. The exact sequence

$$0 \longrightarrow f^*(\Omega_V) \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X|V} \longrightarrow 0$$

together with (5.7) and the projection formula leads to an exact sequence

$$0 \longrightarrow \Omega^1_V \longrightarrow f_*(\Omega^1_X) \longrightarrow f_*(\Omega^1_{X|V} \xrightarrow{\alpha} \Omega^1_V \otimes R^1 f_*(\mathcal{O}_X) \simeq \Omega^1_V.$$

Since F has no holomorphic 1-forms, $f_*(\Omega^1_{X|V})$ is a torsion sheaf, therefore $\alpha = 0$. Now observe the following facts.
(1) $H^0(X, \Omega^1_X) = 0$ by (5.9)
(2) $H^1(X, f^*(\Omega_X)) = H^1(V, \Omega^1_V)$
(3) $H^1(\Omega^1_V) \longrightarrow H^1(X, \Omega^1_X)$ is 0 (since $b_2(X) = 0$).
Then taking cohomology of the first sequence, it follows that

$$h^0(X, \Omega^1_{X|V}) = 0,$$

hence the second sequence gives our claim.

We can say something more about the structure of the singular fibers of f.

5.11 Proposition

Assume that F is not a torus. Let A be a union of fibers containing all singular fibers of f. Let $s = \mathrm{card}(f(A))$ and r the number of irreducible components of A. Then $r = s$, i.e. all fibers of f are irreducible and $b_1(X) = 0$.

Proof. Since F is an Inoue surface or a Hopf surface, we have $b_1(F) = 0$, moreover $V = \mathbb{P}_1$, hence the claim follows from (5.2).

5.12 Remark In case F is a torus, the same proof gives $r = s + 3$. It seems rather reasonable to expect that tori actually cannot appear as fibers of f. Observe that f must have a singular fiber in this case because of $r = s + 3$. So a study of the singular fibers is needed to exclude tori as fibers of f. We hope to come back to this question in a later paper.

5.13 Proposition

Assume that F is not a torus. Then $h^{1,1} = h^{1,2} = h^{2,1} = 1$ (so that we know all Hodge numbers of X.)

Proof.(1) $h^{1,2} = h^{2,1}$ is of course Serre duality.

(2) By (5.9) and $\chi(X, \Omega^1_X) = 0$ it suffices to see $h^{1,3} = 0$ in order to get $h^{1,1} = h^{1,2}$. But this follows again from Serre duality and (5.9).

(3) From the exact sequence

$$0 \longrightarrow f^*(\Omega^1_V) \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X|V} \longrightarrow 0 \tag{S}$$

we deduce that it suffices to show
(a) $h^2(X, f^*(\Omega^1_V)) = 1$
(b) $h^2(X, \Omega^1_{X|V}) = 0$

in order to get $h^{1,2} \leq 1$.

(a) By the Leray spectral sequence and (5.7) we have $h^2(X, f^*(\Omega^1_V)) = h^2(V, \Omega^1_V) = 1$.

(b) Again we argue by the Leray spectral sequence. Since $R^1 f_*(\Omega^1_{X|V})$ is a torsion sheaf, we need only to show that

$$R^2 f_*(\Omega^1_{X|V}) = 0.$$

But, taking f_* of (S), $R^2 f_*(\Omega^1_{X|V})$ is a quotient of $R^2 f_*(\Omega^1_X)$ which is 0 by $H^2(\mathcal{O}_F) = H^2(\Omega^1_F) = 0$ and (1.5).

(4) We finally show $h^{1,1} \neq 0$ to conclude the proof. Let (E^{pq}_r) be the Frölicher spectral sequence on X. Since $b_1(X) = 0$ by (5.11), we get $E^{0,1}_\infty = 0$. Hence $E^{0,1}_2 = 0$. On the other hand

$$E^{0,1}_2 = \mathrm{Ker}\partial : E^{0,1}_1 \longrightarrow E^{1,1}_1.$$

Since $E^{p,q}_1 = H^{pq}(X)$, we conclude that $H^1(X, \mathcal{O}_X)$ injects into $H^{1,1}(X)$. So by (5.7) $H^{1,1}(X) \neq 0$.

We finally collect all our knowledge in the case the general fiber of f is not a torus.

5.14 Theorem

Let X be a smooth compact threefold with $b_2(X) = 0$ and holomorphic algebraic reduction $f : X \longrightarrow V$ to the smooth curve V. Assume that the general smooth fiber is not a torus. Then:

(1) $b_1(X) = 0, b_3(X) = 2$

(2) any smooth fiber of f is a Hopf surface without meromorphic functions or an Inoue surface.

(3) the Hodge numbers of X are as follows: $h^{1,0} = 0$, $h^{0,1} = 0$, $h^{2,0} = 0$, $h^{1,1} = 1$, $h^{0,2} = 0$, $h^{3,0} = 0$, $h^{2,1} = 1$ (the others are determined by these via Serre duality).

(4) all fibers of f are irreducible. There is at most one normal singular fiber.

References

[BaSt76] Banica, C.; Stanasila, O.: Algebraic methods in the global theory of complex spaces. Wiley 1976

[BPV84] Barth, W.; Peters, C.; van de Ven,: Compact complex surfaces. Erg. d. Math. (3), Band 4, Springer 1984

[HM83] Huckleberry, A.T.; Margulis, G.: Invariant analytic hypersurfaces. Inv. Math. **71**, 235-240 (1983)

[Ka69] Kawai, S.: On compact complex analytic manifolds of complex dimension 3, II. J. Math. Soc. Japan **21**, 604-616 (1969)

[RRV71] Ramis, J.; Ruget, G.; Verdier, J.L.: Dualité relative en géométrie analytique complexe. Inv. Math. **13**, 261-283 (1971)

[Ue75] Ueno, K.: Classification theory of algebraic varieties and compact complex spaces. Lecture Notes in Math. 439, Springer 1975

[Ue87] Ueno, K.: On compact analytic threefolds with non-trivial Albanese tori. Math. Ann. **278**, 41-70 (1987)

[We85] Wehler, J.: Der relative Dualitätssatz für Cohen-Macaulay-Räume. Schriftenr. des Math. Instituts der Univ. Münster, 2.Serie, Heft **35** (1985)

Frédéric Campana, Département de mathématiques, Université de Nancy, BP 239, 54506 Vandoeuvre les Nancy, France (`frederic.campana@iecn.u-nancy.fr`)

Jean-Pierre Demailly, Université de Grenoble, Institut Fourier, BP 74, U.R.A. 188 du C.N.R.S., 38402 Saint-Martin d'Heres, France (`demailly@ujf-grenoble.fr`)

Thomas Peternell, Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany (`peternel@btm8x1.mat.uni-bayreuth.de`)