TOWARDS A MORI THEORY ON COMPACT KÄHLER THREEFOLDS, II

THOMAS PETERNELL

Abstract. This paper, a continuation of "Towards a Mori theory on compact Kähler manifolds, 1" (written with F. Campana), introduces a non-algebraic analogue to Mori theory in dimension 3.

Introduction

This is the second part to my joint paper "Towards a Mori theory on compact Kähler manifolds, 1", with F. Campana. With the techniques developped in that paper we shall prove here the following result.

Main Theorem *Let X be a non-algebraic compact Kähler threefold satisfying one of the following conditions.*
(I) X can be approximated algebraically.
(II) $\kappa(X) = 2$.
(III) X has a good minimal model.

Assume that K_X is not nef. Then
(1) X contains a rational curve C with $K_X \cdot C < 0$;
(2) There exists a surjective holomorphic map $\varphi : X \longrightarrow Y$ to a normal complex space Y with $\varphi_*(\mathcal{O}_X) = \mathcal{O}_Y$ of one of the following types.
(a) φ is a $\mathbf{P}_1$- bundle or a conic bundle over a non-algebraic surface (this can happen only in case (1))
(b) φ is bimeromorphic contracting an irreducible divisor E to a point, and E together with its normal bundle N is one of the following

$$(\mathbf{P}_2, \mathcal{O}(-1)), (\mathbf{P}_2, \mathcal{O}(-2)), (\mathbf{P}_1 \times \mathbf{P}_1, \mathcal{O}(-1,-1)), (Q_0, \mathcal{O}(-1)),$$

where Q_0 is the quadric cone.
(c) Y is smooth and φ is the blow-up of Y along a smooth curve.

φ is called an extremal contraction.

Y is (a possibly singular) Kähler space in all cases except possibly (2c). Moreover in all cases but possibly (2c), φ is the contraction of an extremal ray in the cone $\overline{NE}(X)$.

Most parts of this paper have been worked out during a stay at the MSRI in Berkeley. I would like to thank the institute for its hospitality and the excellent working conditions. Research at MSRI is supported in part by NSF grant DMS-9022140.

This is therefore a non-algebraic analogue to Mori theory in dimension 3. Of course one expects that in case (2c) we can arrange things such that Y is Kähler, too, and that φ is the contraction of an extremal ray. In principle we would of course like to prove the theorem without any of the assumptions (I),(II) or (III).

We will now explain the theorem and the method of the proof. Nefness of a line bundle L on an arbitrary compact manifold X is defined via metrics : L is nef, if for every positive ϵ there is a metric h_ϵ on L whose curvature satisfies $\Theta_{L,h_\epsilon} \geq -\omega$, for a fixed positive (1,1)-form ω. Passing to the special case $L = K_X$, it is not at all clear that there is any curve C with $K_X \cdot C < 0$, for arbitrary L this is even false. In order to circumvent this difficulty, we introduce the additional alternative assumptions (I),(II) and (III). We first focus on case (I), i.e. X admits an algebraic approximation, This means that there is a family of compact Kähler manifolds (X_t) over the unit disc in some $\mathbf{C}^m$ with $X_0 \simeq X$ and with a sequence t_ν converging to 0 such that all X_{t_ν} are projective. Assume now that K_X is not nef. Then we will prove that $K_{X_{t_\nu}}$ is not nef for large ν. Therefore a fixed X_{t_ν} admits a contraction of an extremal ray. This extremal ray is given by a non-splitting family of rational curves. The family can be deformed to the nearby fibers and hence also to X_0, however the limit family may split. Therefore one has to extract from the limit family a non-splitting family which in turn by Part 1 of this paper will define the map φ we are looking for. However it is now not clear whether φ is induced by contractions of extremal rays in the nearby (projective) fibers. This causes the difficulty in case (2c) together with some projective problem in the limit which will be explained in full detail in sect. 4; see also below for more comments.

The proof that the bundles $K_{X_{t_\nu}}$ are not nef is done by arguing by contradiction. If all $K_{X_{t_\nu}}$ would be nef, then a suitable multiple would be generated by global sections by Miyaoka-Kawamata's abundance theorem. Now we examine the limit linear system. It turns out that in case of the presence of a base locus, we can find a curve C, in the base locus, with $K_X \cdot C < 0$. Here we make heavily use of the fact that we are working in dimension three (also abundance works at the moment only in this dimension) because we have to analyse line bundles which are not nef in the analytic sense but which are nef in the algebraic sense on non-algebraic (possibly singular) surfaces. This is possible since the structure of surfaces of algebraic dimension ≤ 1 is not complicated.

If $\kappa(X) = 2$ and K_X is not nef, we can directly show that there is a rational curve C with $K_X \cdot C < 0$, using the meromorphic map (Iitaka reduction) attached to the linear system $|mK_X|$ for large m. If $\kappa(X) = 1$, we can at least prove that there is a curve C with $K_X \cdot C < 0$. Once we have a rational curve C with $K_X \cdot C < 0$, we obtain a non-splitting family of rational curves with the same property and if $\kappa(X) \geq 0$, we can apply the main result of [CP94] to obtain an extremal contraction.

An important point for further developments is of course the question whether Y is again Kähler in case φ is birational. For the notion of a singular Kähler space we refer to sect.3. As stated already in the Main Theorem, we are able to prove this in case that the exceptional divisor E is contracted to a point, using various techiques of the theory currents. In case $\dim\varphi(E) = 1$, this is however not the case; at least there is no a priori reason why that should be true. Therefore this case remains open but one would expect that Y is Kähler if we have chosen the "correct" φ. The "correct" φ should be given by an extremal ray in the cone of effective curves or the dual cone to the Kähler cone.

A Motivation

The final aim of a "Mori theory" of compact Kähler threefolds X would of course be

(a) to construct minimal models unless X is uniruled

(b) to prove abundance for minimal models: if X' is a compact Kähler threefold with at most terminal singularities and $K_{X'}$ nef, then $mK_{X'}$ is generated by global sections for suitable large m.

We will discuss these topics in section 6. Besides its independent interest, the solution of these problems would give some new insight in the structure of Kähler threefolds which is not connected a priori to Mori theory. The statement we want to deduce is the following

If the problems (a) and (b) have a positive solution, then simple Kähler threefolds are Kummer.

The relevant definitions are :

(1) a compact Kähler manifold is *simple*, if there is no covering family of positive dimensional subvarieties (hence through a very general point of X there is no positive dimensional irreducible subvariety);

(2) a compact Kähler manifold is Kummer, if it is bimeromorphic to a variety T/G, where T is a torus and G a finite group acting on T.

Proof. Since X is simple, it cannot be uniruled. Hence by (a) X has a minimal model X'. By (b), $mK_{X'}$ is generated by global sections for some m. Hence the Kodaira dimension $\kappa(X') \geq 0$. Since X' is simple, we conclude $\kappa(X') = 0$, hence

$$mK_{X'} = \mathcal{O}_{X'}.$$

Now there exists a covering

$$\tilde{X} \longrightarrow X',$$

unramified over the regular part reg$X$, such that $K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$, in particular $\tilde{X}$ is Gorenstein, see e.g. [KMM87]. By the Riemann-Roch theorem for Gorenstein threefolds (cp. e.g.[Ka86]), we obtain

$$\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0. \qquad (*)$$

Now let $\hat{X} \longrightarrow \tilde{X}$ be a desingularisation. Since $\hat{X}$ is non-algebraic we have by Kodaira's theorem $H^2(\hat{X}, \mathcal{O}_{\hat{X}}) \neq 0$. Because $\tilde{X}$ has only rational singularities, we obtain

$$H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \neq 0.$$

Since $h^3(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$, we obtain from (*):

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \neq 0.$$

Again working with $\hat{X}$ we therefore have a non-trivial Albanese map

$$\alpha : \tilde{X} \longrightarrow \mathrm{Alb}(\tilde{X}) = \mathrm{Alb}(\hat{X}).$$

Since $\tilde{X}$ is simple, α is surjective. Having in mind that $K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$, we immediately see that α is etale outside the singular locus of $\tilde{X}$. If x_0 is a singularity of $\tilde{X}$, then some 1-form would vanish at x_0 and hence we obtain by wedging 1-forms a 3-form with an isolated singularity which is absurd. Hence $\tilde{X}$ is smooth and therefore $\tilde{X}$ is a torus (α is an isomorphism). Compare [Ka85,sect. 8]. Now $\tilde{X}$ being a torus, X is Kummer.

Preliminaries

(0.1) Let X be a compact Kähler manifold. We say that X can be approximated algebraically, if the folowing holds. There exists a family $\mathcal{X} = (X_t)_{t \in \Delta}$ of compact Kähler manifolds over the unit disc $\Delta \subset \mathbf{C}^m$ such that $X_0 \simeq X$ and there is a sequence t_ν converging to 0 such that all X_{t_ν} are projective. The projection map $\mathcal{X} \longrightarrow \Delta$ is usually denoted π.

By a conjecture of Kodaira (or Andreotti?) every compact Kähler manifolds can be approximated algebraically, however at the moment this only known in dimension 2 (via Enriques-Kodaira classification).

(0.2) Here we collect some standard notations. If X is a compact manifold, we let $\rho(X)$ denote the Picard number of X. Define $N_1(X)$ to be the linear subspace of $H_2(X, \mathbf{R})$ generated by the classes of irreducible curves. We let $\overline{NE}(X) \subset N_1(X)$

denote the closed cone generated by the classes of irreducible curves. For details on this and on Mori theory in the algebraic case in general we refer e.g. to [KMM87].

(0.3) The algebraic dimension of an irreducible reduced compact complex space X is by definition the transcendence degree of its field $\mathcal{M}(X)$ of meromorphic functions over $\mathbf{C}$.

An algebraic reduction of X is a meromorphic map $f : X \rightharpoonup Y$ inducing an isomorphism $f^* : \mathcal{M}(Y) \longrightarrow \mathcal{M}(X)$.

(0.4) For the convenience of the reader we state here the main result of [CP94] which we shall use several times. We shall use the notion of a non-splitting family $(C_t)_{t \in T}$ of rational curves. This means that T is compact and irreducible and every C_t is an irreducible and reduced rational curve. Now the main result of [CP94] reads as follows

Theorem *Let X be a compact Kähler threefold and $(C_t)_{t\in T}$ a nonsplitting family of rational curves.*

(1) If $K_X \cdot C_t = -4$ and $\dim T = 4$, then $X \simeq \mathbf{P}_3$.

(2) If $K_X \cdot C_t = -3$ and $\dim T = 3$, then either $X \simeq Q_3$, the threedimensional quadric, or X is a $\mathbf{P}_2-$bundle over a smooth curve.

(3) Assume $K_X \cdot C_t = -2$ and $\dim T = 2$.

(3.1) If X is non-algebraic and the (C_t) fill up a surface $S \subset X$, then $S \simeq \mathbf{P}_2$ with normal bundle $N_{S|X} = \mathcal{O}(-1)$ (the same holds for X projective if S is normal)

(3.2) If X is coveredby the C_t, then one of the following holds.

(3.2.1) X is Fano with $b_2(X) = 1$ and index 2.

(3.2.2) X is a quadric bundle over a smooth curve with C_t contained in fibers

(3.2.3) X is a $\mathbf{P}_1-$bundle over a surface, the C_t being the fibers.

(3.2.4) X is the blow-up of a $\mathbf{P}_2-$bundle over a curve along a section. Here the C_t are the strict transforms of the lines in the $\mathbf{P}_2$'s meeting the section

(4) Let $K_X \cdot C_t = -1$ and $\dim T = 1$. Then the C_t fill up a surface S. Assume X non-algebraic.

(4.1) If S is normal, then one of the following holds.

(4.1.1) $S = \mathbf{P}_2$ with $N_S = \mathcal{O}(-2)$.

(4.1.2) $S = \mathbf{P}_1 \times \mathbf{P}_1$ with $N_S = \mathcal{O}(-1,-1)$.

(4.1.3) $S = Q_0$ (a quadric cone) with $N_S = \mathcal{O}(-1)$.

(4.1.4) S is a ruled surface over a smooth curve and X is the blow-up of a smooth threefold along C.

(4.2) Let S be non-normal. Then $\kappa(X) = -\infty$. If moreover X can be approximated by algebraic threefolds, then we have $a(X) = 1$ and X has the structure of a "generic conic bundle" over a surface Y with $a(Y) = 1$. The general fiber of

the algebraic reduction f which can be taken holomorphic in this case is an almost homogeneous $\mathbf{P}_1-$bundle over an elliptic curve. The surface S consists of reducible conics and is contracted by f to a point.

The essential content of the theorem can be rephrased as follows. Assume that C is a rational curve with $K_X \cdot C = k, -1 \geq k \geq -4$. If no deformation of C splits, then the conclusions of the theorem hold. The point is that automatically the dimension of the deformation is at least k.

We will have a more careful look at the case (4.2) in sect.5.

The paper [CP94] will be refered to as Part I.

3. Limits of extremal rays

(3.1) For most of this section we fix a family $\pi : \mathcal{X} \longrightarrow \Delta$ of compact Kähler threefolds $X_t = \pi^{-1}(t)$ over the unit ball $\Delta \subset \mathbf{C}^m$. Let t_ν be a sequence converging to 0. Assume that all X_{t_ν} are projective, whereas X_0 is **not**. In other words, $\mathcal{X}$ is an algebraic approximation of $X = X_0$. Furthermore we assume that all $K_{X_{t_\nu}}$ are not nef. Fix some ν_0 and consider an extremal ray R_{ν_0} on $X_{t_{\nu_0}}$ generated by the extremal rational curve C_{ν_0}. Let $\varphi_{\nu_0} : X_{t_{\nu_0}} \longrightarrow Y_{t_{\nu_0}}$ be the contraction associated with R_{ν_0}. We are going to construct from R_{ν_0} a sequence of extremal rays R_ν living on X_ν. In order to do this we first examine the structure of R_{ν_0}. Let m denote the length of R_{ν_0}, i.e.

$$m = \min\{-K_{X_{t_{\nu_0}}} \cdot C | [C] \in R_{\nu_0}\}.$$

Then we have :

3.2 Lemma *Either $m = 1$ or $m = 2$.*

Proof. Assume $m \geq 3$. Because of the a priori bound $m \leq 4$ we have only to exclude the cases $m = 4$ and $m = 3$. In the first case we have $X_{t_{\nu_0}} \simeq \mathbf{P}_3$, in the second $X_{t_{\nu_0}} \simeq \mathbf{Q}_3$, the threedimensional quadric or $X_{t_{\nu_0}}$ is a $\mathbf{P}_2-$ bundle over a curve [Wi89]. Hence always $H^2(X_{t_{\nu_0}}, \mathcal{O}_{X_{t_{\nu_0}}}) = 0$. Therefore $H^2(X_0, \mathcal{O}_{X_0}) = 0$ and X_0 is projective by Kodaira's well-known theorem, contradiction.

The same argument shows

3.3 Lemma *The contraction φ_{ν_0} cannot be a del Pezzo fibration (i.e.* $\dim Y_{t_{\nu_0}} \neq 1$*) nor a $\mathbf{P}_1-$ or a conic bundle over a surface of Kodaira dimension $-\infty$.*

In fact, otherwise we would have $H^2(X_0, \mathcal{O}_{X_0}) = 0$ by virtue of

$$R^i \varphi_{\nu_0 *}(\mathcal{O}_{X_{t_{\nu_0}}}) = 0, i \geq 1, \text{ and } H^2(Y_{t_{\nu_0}}, \mathcal{O}_{Y_{t_{\nu_0}}}) = 0.$$

(3.4) From [Mo82] we obtain the following structure of φ_{ν_0} :
(a) a $\mathbf{P}_1-$bundle over a surface with nonnegative Kodaira dimension
(b) a conic bundle over a surface with nonnegative Kodaira dimension
(c) a birational contraction contracting an irreducible divisor E such that
(c.1) $E = \mathbf{P}_2$ with normal bundle $N = \mathcal{O}(-1)$,
(c.2) $E = \mathbf{P}_2, N = \mathcal{O}(-2)$,
(c.3) $E = \mathbf{P}_1 \times \mathbf{P}_1, N = \mathcal{O}(-1,-1)$,
(c.4) $E = Q_0$, the quadric cone, with $N = \mathcal{O}(-1)$.
(d) a birational contraction which contracts an irreducible divisor E such that $\dim\varphi_{\nu_0} = 1$ and φ_{ν_0} is the blow-up of the smooth curve $\varphi_{\nu_0}(E)$ in the manifold $Y_{t_{\nu_0}}$.
We will call the different contractions of type (a), (b), (c.i) and (d), respectively.

Now consider the extremal rational curve $C_{\nu_0} \subset X_{t_{\nu_0}}$. We assume that $-K_{X_{t_{\nu_0}}} \cdot C_{\nu_0}$ is minimal, C_{ν_0} is then smooth. Then the normal bundle N of C_{ν_0} in $X_{t_{\nu_0}}$ is of the form

$$N = \mathcal{O}(a) \oplus \mathcal{O}(b)$$

with $(a,b) \in \{(0,0),(0,-1),(1,-1),(1,-2)\}$.
If $a, b \geq -1$, the deformations of C_{ν_0} in $X_{t_{\nu_0}}$ as well as in $\mathcal{X}$ are unobstructed, since

$$N_{C_{\nu_0}|\mathcal{X}} \simeq \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}.$$

Let $\mathcal{C} = (C_s)_{s\in S}$ be the family of deformations of C_{ν_0} in $\mathcal{X}$, in particular we obtain a limit family (C_s^0) in X_0. However it is not clear whether e.g. a $\mathbf{P}_1-$bundle structure on $X_{t_{\nu_0}}$ converges to a $\mathbf{P}_1-$bundle structure on X_0. More precisely, the subfamily $(C_s)_{s\in S_{\nu_0}}$ of deformations of C_{ν_0} inside $X_{t_{\nu_0}}$ forms a non-splitting family of rational curves in the sense of Part 1; however the limit family may split. The simplest example (in the case of projective families) is the specialisation of $\mathbf{P}_1 \times \mathbf{P}_1$ into the Hirzebruch surface $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ where one of the two rulings converges to a splitting family of rational curves.

We also want to have a limit family in case $(a,b) = (1,-2)$. This case occurs if either φ_{ν_0} is a conic bundle, here C_{ν_0} must be the reduction of a non-reduced conic, or $E = \mathbf{P}_2$ with $N = \mathcal{O}(-2)$. In the first case we just take C_{ν_0} not to be the reduction of a non-reduced conic, but an irreducible component of a reducible reduced conic; then $N = \mathcal{O} \oplus \mathcal{O}(-1)$ and we can conclude. If $E = \mathbf{P}_2$ with $N = \mathcal{O}(-2)$ then by deformation theory the deformations of E in $\mathcal{X}$ are unobstructed, so we can deform E in $\mathcal{X}$ in a 1-dimensional family (note $N_{E|\mathcal{X}} = \mathcal{O}(-2) \oplus \mathcal{O}$). So our family $\mathcal{C}$ exists also here.

3.5 Theorem *There exists a surjective holomorphic map*

$$\psi_0 : X_0 \longrightarrow Z_0$$

to a normal complex space Z_0 *in class* $\mathcal{C}$ *of type (a),(b), (c.1) or (c.2) , as described in (3.4), such that* $\rho(X) = \rho(Y)+1$*. Moreover there exists a morphism* $\psi_t : X_t \longrightarrow Z_t$ *for general* t *of the same type fitting into a family* $\psi : \mathcal{X} \longrightarrow \mathcal{Z}$ *outside a proper analytic set* $A \subset \Delta$ *not containing* 0*. The* Z_{t_ν} *are projective except possibly in case (c.2).*

Recall that $\rho(X)$ is the Picard number of X and that a normal compact complex space is in class $\mathcal{C}$ if it is bimeromorphically equivalent to a Kähler manifold.

Proof. We consider the family $\mathcal{C}$ constructed in (3.4). In particular we have the family (C_s^0) in X_0. For simplicity of notations we skip the upper index 0.

(1) Assume first that the family (C_s) does not split. Then we apply the Main Theorem of Part 1 (0.4) and obtain the following.

If $-K_{X_0} \cdot C_s = -K_{X_{t_{\nu_0}}} \cdot C_{\nu_0} = 2$, then, X_0 being non-algebraic, the C_s define either a $\mathbf{P}_1$-bundle structure or a birational contraction of type (c.1) with $E = \mathbf{P}_2$ and $N = \mathcal{O}(-1)$. This defines our morphism $\psi_0 : X_0 \longrightarrow Z_0$. It is clear that the corresponding families in the nearby fibers defines maps ψ_t of the same type and that all maps fit together. Moreover $\psi_{t_{\nu_0}} = \varphi_{t_{\nu_0}}$.

Now let $-K_{X_0} \cdot C_s = 1$. Let $E_0 = \bigcup C_s \subset X_0$. If E_0 is normal, then we are in case (c.i) ($2 \leq i \leq 4$) or (d) and everything holds in the same way as above. So let E_0 be non-normal. Then X_0 has the structure of a so-called generic conic bundle in the sense of Part 1. If the C_s define an extremal ray in $\overline{NE}(X_0)$, or, equivalently, if the associated full family of "conics" splits only in the standard way in the sense of (5.1), then we conclude by (5.2) resp. (5.3). Otherwise we consider the general smooth conic and the associated family $(\tilde{C}_t)$. Then we have a non-standard splitting

$$\tilde{C}_{t_0} = \sum_{i=1}^{p} C_i'$$

with $p \geq 3$ or with $p = 2$ and, say, $-K_{X_0} \cdot C_1' \geq 2$. In any case we have a new C_i' with $-K_{X_0} \cdot C_i' \geq 1$ which deforms in X_0. Then we proceed as in (2) below.

(2) Assume now that (C_s) splits :

$$C_{s_0} = \sum C_i'.$$

Say that $-K_{X_0} \cdot C_1' > 0$. Then C_1' deforms in an at least 1-dimensional family, say (C_t'), in X_0. If this family does not split, then we argue as in (1); note that the C_t'

deform also to the nearby fibers since in the normal bundle we add a trivial factor (see (3.4) for details). If the family splits again, we take a splitting part C_1'' with $-K_{X_0} \cdot C_1'' > 0$ etc. The only thing left is to prove that this procedure terminates.

(a) First assume that no generic conic bundle comes up in our procedure. Then we consider the cone $\overline{NE}(X_0) \subset N_1(X_0)$; note that every effective curve represents an integer point in $\overline{NE}(X_0)$, denote this set of integer points by $\overline{NE}(X_0)_{\mathbf{Z}}$. Now introduce a norm $\|\cdot\|$ in $N_1(X_0)$ such that $\|x\|^2 \in \mathbf{N}$ for $x \in \overline{NE}(X_0)_{\mathbf{Z}}$. Let x_1 be the class of a general element of our first family (C_s), x_2 the one of the second etc. Then $x_1 = x_2 + x_2'$ with $0 \neq x_2 \in \overline{NE}(X_0)_{\mathbf{Z}}$. Therefore

$$\|x_2\| < \|x_1\|$$

(note that $\overline{NE}(X_0) \cap -\overline{NE}(X_0) = \{0\}$ since X_0 is Kähler). After finitely many steps we reach $\|x_n\| = 0$, so $x_n = 0$, which means that the family in step $n-1$ does not split.

(b) If generic conic bundles come into to the game, it seems at first glance that they cause difficulties because we then pass from x to $2x$. However this difficulty disappears if X_0 carries only finitely many generic conic bundle structures. Indeed, X_0 can carry at most one generic conic bundle structure $X_0 \longrightarrow Z_0$, because a second one would produce another covering family of rational curves on Z_0 so that Z_0 would be algebraic, hence X_0 would be algebraic.

The claim on the Picard number is clear, since by construction a curve C is contracted by ψ_0 iff $C \equiv aC_s$.

Finally, ψ_0 extends to ψ_t for t near 0, because our non-splitting family of rational giving rise to ψ_0 extends to non-splitting families on the X_t, therefore the claim follows rather easily from (0.4).

3.6 Addendum *In cases (a), (b),(c.1) and (d) the space Z_0 is smooth. In particular Z_0 is a Kähler surface in (a) and (b), and the curves contracted by ψ_0 form an extremal ray in $\overline{NE}(X_0)$. In the case of (c.1) it is classical that Z_0 is Kähler, too, hence we have the same conclusion.*

It is clearly important to know whether Z_0 is Kähler also in the remaining cases.

3.7 Theorem *Let X be a (smooth) compact Kähler threefold, $\varphi : X \longrightarrow Y$ be a birational morphism contracting the irreducible divisor E to a point p. Let N_E be its normal bundle. Assume that (E, N_E) is one of the following :*

$(\mathbf{P}_2, \mathcal{O}(-2)), (\mathbf{P}_1 \times \mathbf{P}_1, \mathcal{O}(-1,-1)), (Q_0, \mathcal{O}(-1))$. *In case $E = \mathbf{P}_1 \times \mathbf{P}_1$, assume also that $s \times \mathbf{P}_1 \equiv \mathbf{P}_1 \times t$ in X_0 for all s, t.*

Then Y is a normal Kähler space.

3.8 Explanations (1) A reduced complex space X is Kähler (cp. Grauert [Gr62]) if there is a Kähler form ω on the smooth part reg$X$ such that the following holds. For every singular point $x \in X$ there exists an open neighborhood $U \subset X$, an open set $V \subset \mathbf{C}^N$, a biholomorphic map $f : U \longrightarrow A \subset V$ onto a closed subspace A and a Kähler form ω' on V with

$$f^*(\omega')|\mathrm{reg}X \cap U = \omega.$$

(2) Let us explain (3.7) in case where X is projective. Then Y is projective, too. In fact, let L be an ample line bundle on X. Then we can write $L|E = \mathcal{O}_E(-kE)$ with some $k > 0$ (in case $E = \mathbf{P}_2$ we eventually substitute L by $L^{\otimes 2}$). Note that k exists in case $E = \mathbf{P}_1 \times \mathbf{P}_1$ because of our assumption $s \times \mathbf{P}_1 \equiv \mathbf{P}_1 \times t$. Now there exists a line bundle L' on Y with

$$L \otimes \mathcal{O}_X(kE) \simeq \varphi^*(L'),$$

and it is immediately clear that L' is ample .

(3) Assume X projective and $E = \mathbf{P}_1 \times \mathbf{P}_1$ with $s \times \mathbf{P}_1 \not\equiv \mathbf{P}_1 \times t$. We want to describe the situation; this is of course well known. Consider the blow-down $\varphi_i : X \longrightarrow Y_i$ contracting E along the two projections to curves $C_i \subset Y_i$. Then $C_i \simeq \mathbf{P}_1$ with normal bundle $N_{C_i} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. So we obtain birational maps $\psi_i : Y_i \longrightarrow Y$ contracting exactly the C_i. Then Y is projective if and only if φ is the contraction of an extremal face which just means that the face generated by the fibers of $\varphi_i, i = 1, 2$ is extremal.

On the other hand there are examples where Y is not projective.

Proof of 3.7 (I) In a first step we construct a semi-positive closed (1,1)-form ω' on X which is positive on $X \setminus E$ such that $\omega'|E = 0$.

(I.1) Fix a Kähler form ω on X. As in the projective case (3.8(2)) we can choose $\lambda > 0$ such that

$$[\omega] + \lambda c_1(\mathcal{O}_X(E))|E = 0 \qquad (*)$$

in $H^2(E, \mathbf{R})$.

Fix a representative η of $c_1(\mathcal{O}_X(E))$. We note that if η' is another representative of $c_1(\mathcal{O}_X(E))$, then we have $\eta - \eta' = \overline{\partial}\rho$, but we have even more, namely

$$\eta - \eta' = \partial\overline{\partial}h \qquad (+)$$

with a $C^\infty-$function h on a neighborhood U of E. This is completely standard: since $\overline{\partial}\partial\rho = 0$, $\partial\rho$ is a holomorphic 2-form, therefore by the holomorphic $d-$Poincaré lemma (note that $H^1(U, \mathbf{C}) = H^1(E, \mathbf{C}) = 0$ choosing U such that E

is a retract of U) we can write $\partial\rho = d\varphi$ with a holomorphic 1-form φ. Now put $\tilde\rho = \rho - \varphi$; then $\overline{\partial}\tilde\rho = \eta - \eta'$ and $\partial\tilde\rho = 0$. Hence $\tilde\rho = \partial h$ by Dolbeault (note $H^1(U, \mathcal{O}_U) = 0$ if U is strongly pseudo-convex) and (+) follows.

(I.2) Let $\mathcal{D}$ denote the space of bidimension (1,1)-currents on X. Let $W \subset \mathcal{D}$ be the subspace of those *closed* with supp$T \subset E$. Obviously W is closed. Following [HL83] we let $\mathcal{P} \subset \mathcal{D}$ be the closure of the cone of the positive (1,1)-currents T. Let $\mathcal{B} \subset \mathcal{D}$ be the subspace given by the $(1,1)$-parts of the currents of the form dS. Letting $\kappa : \mathcal{D} \longrightarrow \mathcal{D}/W = \tilde{\mathcal{D}}$ be the projection, we set $\tilde{\mathcal{P}} = \kappa(\mathcal{P}), \tilde{\mathcal{B}} = \kappa(\mathcal{B})$. Then we claim :

$$\tilde{\mathcal{P}} \cap \tilde{\mathcal{B}} = \{0\}. \tag{$**$}$$

In fact, let $0 \neq T \in \mathcal{D}$ with $\kappa(T) \in \tilde{\mathcal{P}} \cap \tilde{\mathcal{B}}$. We may assume that $T \in \mathcal{P}$. Then there are currents T' and S with supp$T' \subset E, dT' = 0$ such that

$$T = T' + (dS)_{(1,1)}.$$

It follows that $\partial\overline{\partial}T = 0$.

Decompose $T = \chi_E T + \chi_{X\setminus E}T$. A difficulty arising is that T is in general only $\partial\overline{\partial}-$closed and not $d-$closed. Let us first prove (**) in the simpler case $dT = 0$. Then by [Sk82] we have $d(\chi_E T) = 0$. Now observe that $T'(\omega + \lambda\eta) = 0$ by (+) because we can choose η in such a way that $\omega + \lambda\eta = 0$ in a neighborhood of E (see I.3). Hence we get

$$T(\omega + \lambda\eta) = 0.$$

This is still independent on theclosedness assumption. By the same reason as for T', we also have $\chi_E T(\omega + \lambda\eta) = 0$. Here we have used $d\chi_E T = 0$, so that we can take η as we want. Hence $\chi_{X\setminus E}T(\omega + \lambda\eta) = 0$. On the other hand, $\chi_{X\setminus E}T(\omega) \geq 0$ and by (3.7.a), also $\chi_{X\setminus E}T(\eta) \geq 0$, because this does not depend on the choice of η ! Therefore $\chi_{X\setminus E}T = 0$ which was to be proved.

The difficulty in the general case is to prove

$$\partial\overline{\partial}\chi_E T = 0. \tag{A}$$

Once we know (A) we can conclude as before; since (A) implies that $\chi_E T(\eta)$ (and hence $\chi_{X\setminus E}T(\eta)$) does not depend on the choice of η. Clearly (A) will follow from

$$\partial\overline{\partial}\chi_E T \geq 0, \tag{B}$$

where $\geq$ is to be understood in the sense of currents (just apply (B) to a constant non-zero function).

In order to prove (B) we make use of the following theorem on currents.

(C) *Let $\Omega \subset \mathbf{C}^n$ be an open set, $E \subset \Omega$ be a complex submanifold and T a positive current with $\partial\overline{\partial}T \geq 0$ (called plurisubharmonic in the literature). Then* $\partial\overline{\partial}\chi_E T \geq 0$.

This follows from Bassanelli's paper, [Ba94,1.24,3.5], as was pointed to me by L.Alessandrini [Al96]. Now (C) implies our claim (B) in case E is smooth. In case E is singular, i.e. a quadric cone with vertex x_0, we need an additional argument. By (B) we certainly know that

$$\partial\overline{\partial}\chi_E T | X \setminus \{x_0\}$$

is positive, so does its trivial extension $(\partial\overline{\partial}\chi_E T)^0$. Since

$$\chi_E T = (\chi_E T | X \setminus \{x_0\})^0$$

(note $\chi_{\{x_0\}}T = 0$, [Ba94,1.13]), we obtain from [AB93,5.11]

$$\partial\overline{\partial}\chi_E T = (\partial\overline{\partial}\chi_E T)^0 + \lambda T_{x_0},$$

where $\lambda \leq 0$ and T_{x_0} is the Dirac distribution of x_0. We want to prove $\lambda = 0$. Take a smooth closed (2,2)-form $u \leq 0$ such that the current $T + u$ is negative:

$$T + u \leq 0.$$

E.g. consider the Kähler form ω and let $u = -k\omega \wedge \omega$ for some large k. Having in mind $\chi_E(T + u) = \chi_E T$ and

$$\partial\overline{\partial}\chi_E(T + u) \geq 0$$

on $X \setminus \{x_0\}$, we can apply [AB93] and obtain

$$\partial\overline{\partial}\chi_E(T + u) = (\partial\partial\chi_E(T + u)|X \setminus \{x_0\})^0 + \mu T_{x_0}$$

with $\mu \geq 0$. Therefore we conclude $\lambda = \mu = 0$ and we get $\partial\overline{\partial}\chi_E T = 0$ also in the case of the quadric cone.

(I.3) To finish the proof of (**), we finally show that we can choose a representative of $\omega + \lambda\eta$, which is 0 on U. For this one needs to solve the equation

$$\omega + \lambda\eta = -\overline{\partial}\rho$$

on U which leads to ask for the injectivity of

$$H^1(U, \Omega^1_U) \longrightarrow H^1(E, \Omega^1_E).$$

This follows immediately from

$$H^1(E, N_E^*) = H^1(E, N_E^{*\mu} \otimes \Omega_U^1) = 0, \mu \geq 1.$$

Note that these arguments also work in the case E is the quadric cone, we leave the details to the reader.

(I.4)) Having verified (**) we find by the Hahn-Banach theorem a linear functional $\tilde{\Phi} : \tilde{\mathcal{D}} \longrightarrow \mathbf{R}$ which is strictly positive on $\tilde{\mathcal{P}} \setminus 0$ and 0 on $\tilde{\mathcal{B}}$. Lift $\tilde{\Phi}$ to a functional $\Phi : \mathcal{D} \longrightarrow \mathbf{R}$. Then Φ is given by a $\mathcal{C}^\infty(1,1)-$form ω' with the following properties.
(a) $d\omega' = 0$
(b) $\omega' | X \setminus E$ is positive, so ω' itself is semipositive
(c) $[\omega' | E] = 0$ in cohomology.

(b) and (c) together give $\omega' | E = 0$.

(II) We next claim that ω' is induced by a $(1,1)-$form ω_0 on Y and which is almost a Kähler metric on Y. Of course, ω_0 exists on $Y \setminus p$, where $p = \varphi(E)$. We show that there is an open neighborhood V of p and a plurisubharmonic function g on V such that $\omega_0 = \partial\overline{\partial} g$ on $V \setminus p$. Then ω_0 exists as form on Y. However it might not quite be a Kähler form on Y since g is possible not strictly plurisubharmonic at p. But then take a closed $(1,1)-$form λ on Y which is positive at p and let $\tilde{\omega} = \omega_0 + \epsilon\lambda$. For sufficiently small $\epsilon, \tilde{\omega}$ will be a Kähler form on Y. So it remains to prove the existence of g. For this we need to construct a $\mathcal{C}^\infty-$function f on a suitable neighborhood U of E such that $f | U \setminus E$ is strictly plurisubharmonic and $\omega' = \partial\overline{\partial} f$ on $U \setminus E$. Then automatically $f = \varphi^*(g)$.

Let $\mathcal{H}$ denote the sheaf of plurisubharmonic functions on X. Taking real parts of holomorphic functions, we have an exact sequence

$$0 \longrightarrow \mathbf{R} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{H} \longrightarrow 0 \qquad (S).$$

Let U be a strongly pseudo-convex neighborhood of E, such that E is a deformation retract of U. By [HL83] we have

$$H^1(U, \mathcal{H}) \simeq \frac{\{\psi \in \mathcal{A}_{\mathbf{R}}^{1,1}(U) | d\psi = 0\}}{\partial\overline{\partial} \mathcal{A}_{\mathbf{R}}^{0,0}(U)},$$

where $\mathcal{A}_{\mathbf{R}}^{p,q}$ is the sheaf of real $(p,q)-$forms. Therefore $\omega' | U$ defines a class $[\omega'] = 0 \in H^1(U, \mathcal{H}_U)$ and we must show $[\omega'] = 0$. Since $\omega' | E = 0$, we have $[\omega' | E] = 0$ in $H^1(E, \mathcal{H}_E)$, and so it is sufficient to verify that the restriction map

$$H^1(U, \mathcal{H}_U) \longrightarrow H^1(E, \mathcal{H}_E) \qquad (R)$$

is an isomorphism. By the exact sequence (S) on U resp. E and the obvious facts

$$H^i(U, \mathcal{O}_U) \simeq H^i(E, \mathcal{O}_E) = 0, i = 1, 2$$

we have

$$H^1(U, \mathcal{H}_U) \simeq H^2(U, \mathbf{R}),$$

$$H^1(E, \mathcal{H}_E) \simeq H^2(E, \mathbf{R}).$$

Hence (R) follows from the fact that

$$\text{rest} : H^2(U, \mathbf{R}) \longrightarrow H^2(E, \mathbf{R})$$

is an isomorphism, E being a deformation retract of U. So $[\omega'|U] = 0$ in $H^1(U, \mathcal{H}_U)$, and $\omega'|U = \partial\overline{\partial}f$ with $f \in \mathcal{A}^{0,0}(U)$. Since ω' is positive on $X \setminus E$, the function f is strictly plurisubharmonic on $U \setminus E$.

3.7.a Sub-Lemma *Let X be a compact Kähler manifold, D an irreducible divisor on X. Let T be a positive closed current on X of bidegree $(n-1, n-1)$, where $n = \dim X$. Assume $\chi_D T = 0$. Then the intersection number $T \cdot D \geq 0$.*

Proof. I am indepted to Jean-Pierre Demailly for communicating to me the following short proof using his approximation theorem for positive closed curents [De92]. We can write D (the current of integration over D) as a weak limit of smooth closed forms Θ_ε in the same cohomology class as $[D]$ such that

$$\Theta_\varepsilon \geq -\lambda_\varepsilon u - O(\varepsilon)\omega,$$

where ω is a positive (1,1)-form, u a suitable semi-positive (1,1)-form on X depending on the global structure of X and (λ_ϵ) a decreasing family of non-negative smooth functions $(0 < \varepsilon < 1)$ converging pointwise to 0 on $X \setminus D$ and to the multiplicity $m(D, x)$ on D. It follows

$$D \cdot T = \Theta_\varepsilon \cdot T \geq -\int_X \lambda_\varepsilon u \wedge T - O(\varepsilon).$$

Now the monotone convergence theorem gives convergence to 0, because $\chi_D T = 0$.

3.7.b Remark $T \cdot D$ can be computed either by $[T] \cdot [D]$, where $[T]$ and $[D]$ are the classes in $H^2(X, \mathbf{R})$ resp. $H^{2n-2}(X, \mathbf{R})$ or by representing D by a positive closed $(1,1)-$form (the curvature form of a metric on the line bundle associated to D), say η, and computing $T(\eta)$.

If merely $\partial\bar{\partial}T = 0$, then we still can define a class $[T]$, as explained detailed in the proof of (3.11), and everything in (3.7.a) remains true (the regularisation is applied to the current D and not to T!)

3.7.c Remark Theorem 3.7 should be true in a more general context : Let X be a compact Kähler manifold, $f : X \longrightarrow Y$ a birational morphism to a normal complex space. Assume that the exceptional set of f is an irreducible divisor E and that $\rho(X) = \rho(Y) + 1$. Then Y is Kähler.

The proof should be the same as before except that one needs a singular version of Bassanelli's theorem [Ba94,3.5] which must be true.

3.9 Corollary *Let $\varphi : X \longrightarrow Y$ be a contraction of type (c). If $\rho(X) = \rho(Y) + 1$, then Y is Kähler and $R = \mathbf{R}_+[l] \subset \overline{NE}(X)$ is an extremal ray, where l is any curve contracted by φ.*

If conversely R is an extremal ray, then obviously $\rho(X) = \rho(Y) + 1$ and Y is Kähler.

Proof. (a) If $\rho(X) = \rho(Y) + 1$, then the case $E = \mathbf{P}_1 \times \mathbf{P}_1, \dim\varphi(E) = 0, s \times \mathbf{P}_1 \not\equiv \mathbf{P}_1 \times t$ is excluded, hence 3.7 applies and it is immediately checked that R is extremal.

(b) If R is an extremal ray then the same arguments apply.

3.10 Remarks (1) The case $E = \mathbf{P}_1 \times \mathbf{P}_1, \dim\varphi(E) = 0, s \times \mathbf{P}_1 \not\equiv \mathbf{P}_1 \times t$ is not relevant for us, since then the associated contraction is not defined by an extremal ray and not given by one non-splitting family of rational curves, cp. (3.8).

(2) Clearly the proof of (3.7) also works at least in the case where X is a compact Kähler manifold of dimension n and where $\varphi : X \longrightarrow Y$ is an extremal contraction contracting a smooth divisor to a point.

It remains to treat birational contractions of type (d), i.e. blow-ups of smooth curves. Already in case X_0 is projective, Y_0 will not be projective "in general". This happens exactly if the defining ray R is extremal. In order to generalise this to the Kähler case we consider the closed dual cone $\overline{NA}(X)$ to the Kähler cone. So $\overline{NA}(X) \subset H^4(X, \mathbf{R})$ resp. $H^{2,2}(X)$. In general, $\overline{NE}(X)$ is a proper subcone of $\overline{NA}(X)$ and we can view $\overline{NA}(X)$ as the cone generated by the classes of positive closed currents of bidegree $(2, 2)$.

3.11 Proposition *Let X be a compact Kähler threefold, $\varphi : X \longrightarrow Y$ a birational map of type (d), i.e. X is the blow-up of a smooth curve C in the manifold Y. Let $E = \varphi^{-1}(C)$ and $l \subset E$ a ruling line. Then Y is Kähler if and only if $R = \mathbf{R}_+[l]$ is extremal in $\overline{NA}(X)$.*

Proof. (a) Assume that Y is Kähler. In order to show that l is extremal, we take positive closed currents T_1, T_2 with $[l] = [T_1] + [T_2]$. Then $0 = \varphi_*(T_1) + \varphi_*(T_2)$, and since Y is Kähler, we have for a Kähler form ω on Y that $\varphi_*(T_i)(\omega) = 0$, hence

$\varphi_*(T_i) = 0$. Therefore the T_i are supported on fibers of φ, i.e. the T_i are linear combinations of currents "integration over a fiber" and therefore $[T_i] \in R$.

(b) Conversely, assume that R is extremal in $\overline{NA}(X)$. By [HL83] it is sufficient to show the following:

(*) if T is a positive current of bidegree (2,2) with $T = \partial\overline{S} + \overline{\partial}S$, then $T = 0$.

For the proof of (*) write

$$T = \chi_C T + \chi_{Y \setminus C} T.$$

If $dT = 0$, then by [Si74] $\chi_C T = \lambda T_C$ where T_C is the current "integration over C." In the general case this is due to Bassanelli [Ba94]. Putting $\tilde{T} = \chi_{Y \setminus C} T$, we can write $\lambda[T_C] + [\tilde{T}] = 0$.

Let us first assume $dT = 0$. Then $\tilde{T}$ defines canonically a current T' on X by taking the trivial extension of $T|X \setminus E$. By [Sk82], T' is closed. Letting $C_0 \subset E = \mathbf{P}(N^*_{C|Y})$ be a section with minimal self-intersection, we obtain :

$$\lambda[T_{C_0}] + [T'] = \mu[l].$$

Obviously $\mu > 0$, since X is Kähler. Since l is extremal, we must have $T' = 0$, i.e. $\tilde{T} = 0$, and $\lambda = 0$, hence $T = 0$.

Now we treat the general case. We define $\tilde{T}$ and T' as before. We prove that $\partial\overline{\partial}T' = 0$, the analogous statement for $\tilde{T}$ being completely parallel. By [Ba94,3.5], we have

$$\partial\overline{\partial}T' = (\partial\overline{\partial}T|X \setminus E)^0 + R,$$

where $(\ )^0$ denotes again the trivial extension and R is a negative current supported on E. Choosing a negative closed (2,2)-form u on X as in the proof of (3.7) so that $T' + u \leq 0$ we have furthermore by loc.cit.

$$\partial\overline{\partial}(T' + u) = (\partial\overline{\partial}(T + u)|X \setminus E)^0 + R' = (\partial\overline{\partial}T|X \setminus E)^0 + R',$$

where R' is a positive current supported on E. In total we get $R = R' = 0$, hence $\partial\overline{\partial}T' = 0$.

It is clear that we still have the equation

$$\lambda[C] + [\tilde{T}] = 0.$$

Next we define a class $[T'] \in H^{2,2}_{\mathbf{R}}(X)$ ad hoc in the following way. By duality we define instead a linear form

$$[T'] : H^{1,1}_{\mathbf{R}}(X) \longrightarrow \mathbf{R},$$

by setting $[T'](\alpha) = T'(\eta)$, where η is any $\partial-$ ($= \overline{\partial}-$ closed representative, since the class is real) of α. Then there is still an equation

$$\lambda[T_{C_0}] + [T'] = \mu[l]$$

as before. Cupping with $[\omega]$ gives $\mu \geq 0$ and in fact $\mu > 0$, if $T \neq 0$. Noting $[T'] \in \overline{NA}(X)$, the extremality of l yields $\lambda = 0, T' = 0$, so $T = 0$.

Remark Of course one would expect that a stronger version of 3.11 holds, namely that Y is Kähler if and only if l is extremal in $\overline{NE}(X)$.

We go back to our special situation and assume that Z_0 is not Kähler in our contraction $\psi : X_0 \longrightarrow Z_0$, which blows up the curve $C_0 \subset Z_0$. Then the fibers of ψ_0 deform into X_t and we obtain a family $\psi_t : X_t \longrightarrow Z_t$ for small t. Every ψ_t is a blow-down of a divisor E_t which fit together to a divisor E. In principle we would like to show that by chosing φ_{t_0} at the beginning carefully, Y_0 must be Kähler. However we do not know how to do this at the moment. Therefore we examine a simpler situation, namely that X_{t_ν} has only one extremal ray, so that there is no choice. In that case the extremal rays should converge to an extremal ray on X_0, i.e. Y_0 should be Kähler. Let us still simplify the situation, namely that all X_t are projective and hence that all Y_t are projective, $t \neq 0$. Then still we cannot conclude that Y_0 is Kähler because the Kähler cone on X_0 might be smaller than the Kähler cone of the nearby fibers. At least we can state the following proposition which seems to be of independent interest.

3.12 Proposition *Let $\pi : \mathcal{X} \longrightarrow \Delta$ be a family of compact Kähler threefolds, $E \subset \mathcal{X}$ a family of ruled surfaces over Δ and $\varphi = (\varphi_t) : \mathcal{X} \longrightarrow \mathcal{Y}$ be the simultaneous blow-down of E to the manifold $\mathcal{Y} \longrightarrow \Delta$. Let $h^{1,1}(X_0) = 2$. Assume that there is a sequence t_ν converging to 0 such that the Y_{t_ν} are Kähler. Then Y_0 is Kähler (hence all Y_{t_ν} are Kähler for small t.)*

Proof. Choose a family (ω_t) of Kähler metrics on (X_t). Assuming Y_0 to be non-Kähler and letting $C_t = \varphi_t(E_t)$, we find a positive $\partial\overline{\partial}-$closed current $\tilde{T}$ such that

$$C_0 + \tilde{T} = \partial\overline{S} + \overline{\partial}S,$$

compare (3.11). We define its class $[\tilde{T}] \in H^{2,2}_{\mathbf{R}}(Y_0)$ as in (3.11). Then $[C_0] + [\tilde{T}] = 0$. Now consider the current $\varphi_{0*}(\omega_0)$; it defines a class in $H^{1,1}_{\mathbf{R}}(Y_0)$, and we obtain

$$([C_0] + [\tilde{T}]) \cdot [\varphi_{0*}(\omega_0)] = 0.$$

Let T' be the trivial extension of $\tilde{T}|X_0 \setminus E_0$ to X_0. By the same arguments as in (3.11), we have $\partial\overline{\partial}T' = 0$ and define its class $[T'] \in H^{2,2}_{\mathbf{R}}(X_0)$. Let $l \subset E$ be a fiber of φ_0. Then

$$[T'] = (\varphi_0)^*[\tilde{T}] + \lambda[l],$$

with $\lambda \leq 0$, by application of (3.7.a,b). Therefore from

$$[T'] \cdot [\omega_0] = [\tilde{T}] \cdot [\varphi_{0*}(\omega_0)] + \lambda[l] \cdot [\omega_0]$$

we obtain $[\tilde{T}] \cdot [\varphi_{0*}(\omega_0)] \geq 0$, hence

$$[C_0] \cdot [\varphi_{0*}(\omega_0)] \leq 0. \qquad (*)$$

Of course the same considerations hold on Y_t. But now $[\varphi_{t_\nu}(\omega_{t_\nu})]$ is represented by a Kähler form, since $h^{1,1}(Y_{t_\nu}) = 1$. Therefore

$$[C_{t_\nu}] \cdot [\varphi_{t_\nu}(\omega_{t_\nu})] > 0.$$

By continuity we obtain from $(*)$ that

$$[C_0] \cdot [\varphi_{0*}(\omega_0)] = 0.$$

Since $h^{1,1}(Y_0) = 1$, this means $[C_0] = 0$ in $H^4(Y_0, \mathbf{R})$. Since $[C_t]$ is an integer point, we conclude that $[C_t] = 0$ for all t, contradiction.

3.13 Remark If in (3.12) we have $h^{1,1}(Y_t) \geq 2$, then we cannot conclude because we don't necessarily have

$$[C_t] \cdot [\varphi_{t*}(\omega_t)] > 0.$$

Certainly we can choose some Kähler form ω' on X_t with this property, but this ω' might not fit into a family (ω_t) of Kähler metrics on all of $\mathcal{X}$. So it seems not impossible that (3.12) does not hold in general.

Unfortunately (3.12) does not give anything in our special situation, because

$$h^{1,1}(X_0) = 2$$

forces X_0 to be projective :

3.14 Proposition *Assume that in (3.12) all the X_{t_ν} are projective. Then all Y_{t_ν} are projective and X_0 is projective.*

Proof. Assume X_0 non projective, equivalently Y_0 non projective (3.12).

Since $h^{1,1}(Y_{t_\nu}) = 1$, we have

$$\mathrm{Pic}(Y_{t_\nu})/\mathrm{torsion} \simeq \mathbf{Z}.$$

Fix a big generator $\mathcal{O}_{Y_{t_\nu}}(1)$ of $\mathrm{Pic}(Y_{t_\nu})/\equiv$. Then

$$K_{Y_{t_\nu}} \equiv \mathcal{O}_{Y_{t_\nu}}(\alpha)$$

for some $\alpha \in \mathbf{Z}$. If $\alpha > 0$, then Y_{t_ν} is of general type and so does Y_0. Hence X_0 is projective. If $\alpha < 0$, then $\kappa(-K_{Y_{t_\nu}}) = 3$ and so does $\kappa(-K_{Y_{t_{\nu_0}}})$. The conclusion is as before. So we are left with $\alpha = 0$. Then $\chi(\mathcal{O}_{Y_{t_\nu}}) = 0$ by Riemann-Roch. X_0 being non-algebraic by assumption, we have $H^2(\mathcal{O}_{X_0}) \neq 0$. Since $h^3(\mathcal{O}_{Y_{t_\nu}}) = h^0(K_{Y_{t_\nu}}) \leq 1$, we conclude that the irregularity $q(Y_{t_\nu}) = h^1(\mathcal{O}_{Y_{t_\nu}}) \geq 1$. Hence we have a non-trivial Albanese map

$$\alpha : Y_{t_\nu} \longrightarrow \mathrm{Alb}(Y_{t_\nu}).$$

Since $\kappa(Y_{t_\nu}) = 0, \alpha$ is surjective [Be83]. This immediately contradicts $h^{1,1}(Y_{t_\nu}) = 1$.

But now an examination of the proof of (3.14) shows that it demonstrates at the same time

3.15 Proposition *Assume the situation of (3.14) but instead of an assumption on $h^{1,1}$ assume $\rho(X_{t_\nu}) = 2$ for some ν and moreover that X_0 is not projective. Then all Y_t are tori, in particular Y_0 is Kähler.*

In fact, with the arguments in the proof of (3.14) we arrive at the Albanese map $\alpha : Y_{t_\nu} \longrightarrow \mathrm{Alb}(Y_{t_\nu})$ which is onto and has connected fibers. Since $\rho(Y_{t_\nu}) = 1$, dim $\mathrm{Alb}(Y_{t_\nu}) = 3$ and α does not contract any divisor. But since $\mathrm{Alb}(Y_{t_\nu})$ is smooth, α cannot contract any curve either. So $Y_{t_\nu} = \mathrm{Alb}(Y_{t_\nu})$ (alternatively apply directly the decomposition theorem [Be83]). Hence our claim follows easily.

Some parts of the arguments of this section do not need the condition on algebraic approximability. Indeed we have

3.16 Theorem *Let X be a compact Kähler threefold with $\kappa(X) \geq 0$. Assume that there exists a rational curve $C \subset X$ such that $K_X \cdot C < 0$. Then there exists a contraction $\varphi : X \longrightarrow Y$ as described in (3.4)*

Proof. As already explained, we construct from the curve C a non-splitting family (C_t) of rational curves. Then we can apply again the main theorem of Part 1 and are through. It was necessary to assume that $\kappa(X) \geq 0$ in order to exclude the case $-K_X \cdot C_t = 1$ and the C_t fill up a non-normal surface (in which case we only have informations if X can be approximated algebraically or if $\kappa(X) = -\infty$).

4. An Openness Theorem

In this section we shall prove

4.1 Theorem *Let $\pi : \mathcal{X} \longrightarrow \Delta$ be an algebraic approximation of the non-algebraic compact Kähler threefold X_0. Let $(t_\nu) \to 0$ be a sequence such that X_{t_ν} is projective for every ν. If K_{X_0} is not nef, then all $K_{X_{t_\nu}}$ are not nef for $\nu \gg 0$.*

We will also prove that on threefolds with $\kappa(X) = 2$ and K_X not nef, there exists a rational curve C with $K_X \cdot C < 0$ (theorem 4.10). If $\kappa(X) = 1$, then we can at least prove that there is *some* curve C with $K_X \cdot C < 0$ (theorem 4.15).

4.2 Remark If $\mathcal{X} = (X_t)$ is a family of smooth projective threefolds such that K_{X_0} is not nef, then all K_{X_t} are not nef. This follows e.g. by deforming extremal rational curves in X_0 to X_t. For details and the higherdimensional context see [Pe95a].

The proof of (4.1) will be given in several steps according to the possible values of the Kodaira dimension. We fix a sequence (t_ν) as above and assume that $K_{X_{t_\nu}}$ is nef for all ν.

4.3 Proposition *We have $\kappa(X_t) \geq 0$ for all $t \in \Delta$.*

Proof. Assume that $\kappa(X_{t_0}) = -\infty$ for some $t_0 \in \Delta$. If X_{t_0} is projective, then X_{t_0} is uniruled [Mo88],[Mi88],[Ka92], hence all X_t are uniruled, contradiction.

So let X_{t_0} be non-projective; again we want to show that X_{t_0} is uniruled which ends the proof (this proof works also in the projective case, making the argument independent from the deep papers cited above). Since $K_{X_{t_\nu}}$ is nef, we have

$$\chi(X_{t_\nu}, \mathcal{O}_{X_{t_\nu}}) \leq 0$$

by [Mi87]. Therefore $\chi(\mathcal{O}_{X_{t_0}}) \leq 0$. Since $h^3(\mathcal{O}_{t_0}) = h^0(K_{X_{t_0}}) = 0$ by our assumption, we conclude that the irregularity

$$q(X_{t_0}) = h^1(\mathcal{O}_{X_{t_0}}) \geq 1.$$

So we have a non-trivial Albanese map

$$\alpha : X_{t_0} \longrightarrow \mathrm{Alb}(X_{t_0}).$$

If dim $\alpha(X_{t_0}) = 1$, then by $C_{3,1}$ (see [Ue87]) the general fiber F has $\kappa(F) = -\infty$, hence X_{t_0} is uniruled. If dim $\alpha(X_{t_0})$, then by $C_{3,2}$ [Ue87], the general fiber of α is $\mathbf{P}_1$ and again X_{t_0} is uniruled. If finally dim $\alpha(X_{t_0}) = 3$, then $\kappa(X_{t_0}) \geq 0$, a contradiction.

Next we deal with two easy cases. Observe first that $\kappa(X_{t_{\nu_0}})$ does not depend on ν_0 since by [Ka92]

$$\kappa(X_{t_{\nu_0}}) = \max \{m \in \mathbf{N} \cup \{0\} | K^m_{X_{t_\nu}} \not\equiv 0\}.$$

4.4 Proposition *(1) If $K_{X_{t_\nu}} \equiv 0$ for some ν, then $K_{X_0} \equiv 0$ and X_0 is up to finite etale cover either a torus or a product of an elliptic curve with a K3 surface.*

(2) If $K^3_{X_{t_\nu}} > 0$ (i.e. $K_{X_{t_\nu}}$ is big) for some ν, then $\kappa(X_0) = 3$ and X_0 is projective.

Proof. (1) Just notice that $c_1(X_{t_\nu}) = 0$ in $H^2(X_{t_\nu}, \mathbf{Q})$ implies that $c_1(X_0) = 0$ in $H^2(X_0, \mathbf{Q})$. Hence we conclude by [Be83].

(2) By Grauert-Riemenschneider or Kawamata-Viehweg vanishing, we have for $q \geq 1$ that

$$H^q(X_{t_\nu}, mK_{X_{t_\nu}}) = 0 \qquad (*)$$

for all ν and all $m \geq 2$. By the coherence of $R^q\pi_*(mK_{\mathcal{X}})$, equation (*) holds for all $t \in \Delta \setminus S$, with $S \subset \Delta$ a proper analytic subset. Passing to a local 1-dimensional submanifold of Δ through 0, we may assume that (*) holds for all $t \neq 0$. By Riemann-Roch we conclude

$$\chi(X_t, mK_{X_t}) = h^0(X_t, mK_{X_t}) \sim m^3$$

for $t \neq 0$, and this function in t is constant. By semi-continuity of cohomology it follows

$$h^0(X_0, mK_{X_0}) \sim m^3,$$

so that X_0 is Moishezon. Since X_0 is Kähler, it must be projective, contradiction.

We are therefore reduced to the intermediate cases $\kappa(t_\nu) = 1$ or 2. These cases are much more complicated and we shall need some preparations. The first lemma is probably well-known but apparently never stated explicitly.

4.5 Lemma *Let X be a compact Kähler manifold and L a line bundle on X. Then L is nef if and only if $c_1(L)$ is in the closure of the Kähler cone.*

Proof. (a) First assume that L is nef. Take an arbitrary Kähler metric ω on X. Then we must show that $c_1(L) + [\omega]$ is represented by a Kähler metric. For $0 < \varepsilon < 1$ choose a metric h_ε on L with curvature

$$\Theta_{L,h_\varepsilon} \geq -\epsilon\omega.$$

Then $\Theta_{L,h_\varepsilon} + \omega \geq (1-\varepsilon)\omega$, hence $\Theta_{L,h_\varepsilon} + \omega$ is a Kähler form representing $c_1(L) + [\omega]$.

(b) Now suppose that $c_1(L)$ is in the closure of the Kähler cone. Fix a Kähler metric ω on X. Let $\varepsilon > 0$. By [Su88] we find a Kähler metric ω_ε such that

$$[\omega_\varepsilon - \varepsilon\omega] = c_1(L).$$

Choose a metric h_ε on L with curvature $\Theta_{L,h_\varepsilon} = \omega_\varepsilon - \varepsilon\omega$. Then $\Theta_{L,h_\varepsilon} \geq -\varepsilon\omega$. Since $\varepsilon > 0$ was arbitrary, L is nef.

4.6 Lemma *Let X be a smooth compact Kähler surface or threefold and $A \subset X$ a compact submanifold. Let $\pi : \hat{X} \longrightarrow X$ be the blow-up of A. Let L be a line bundle on X. Then L is nef if and only if $\pi^*(L)$ is nef.*

Proof. For simplicity of notations we only treat the threefold case. One direction being obvious, we assume that $\pi^*(L)$ is nef. Let $KC(X)$ and $KC(\hat{X})$ denotes the Kähler cones on X resp. $\hat{X}$. By the easy part of (4.5) we have

$$c_1(\pi^*(L)) \in KC(\hat{X}).$$

We claim that this implies

$$c_1(L) \in KC(X). \tag{1}$$

Then the proof is finished by applying the difficult part of (4.5).
For the proof of (1) we need to show that

$$T \cdot c_1(L) \geq 0 \tag{2}$$

for every positive closed current of bidegree $(2,2)$. We know that $\hat{T} \cdot c_1(\pi^*(L)) \geq 0$ for all closed positive currents $\hat{T}$ on $\hat{X}$ of bidegree $(2,2)$. Therefore the proof of (2) is reduced to the following.
(3) For every closed positive current T on X of bidegree $(2,2)$ there exists a closed positive current $\hat{T}$ on $\hat{X}$ such that

$$\pi_*(\hat{T}) = T.$$

(Note that the dual cone to the Kähler cone is the cone of positive closed (1,1)-currents).
Write

$$T = \chi_A T + \chi_{X \setminus A} T.$$

If $\dim A = 0$, then $\chi_A T = 0$, otherwise $\chi_A T = \lambda T_A$ with $\lambda \geq 0$.
Now let $T' = T|X \setminus A$. Identifying $X \setminus A$ with $\hat{X} \setminus \hat{A}$, we can consider the trivial extension T_1 of T' on $\hat{X}$. By [Sk82] this is a closed positive current and obviously

$$\pi_*(T_1) = \chi_{X \setminus A} T.$$

We are thus reduced to dealing with $\chi_A T$ and may assume $\dim A = 1$. But then we take a section

$$C \subset \hat{A} = \mathbf{P}(N^*_{A|X})$$

and let $T_2 = \lambda T_C$. Hence $\pi_*(T_2) = \chi_A T$ and we are done, putting $\hat{T} = T_1 + T_2$.

Remark. Of course (4.6) should be true in any dimension.

Lemma 4.6 enables us to define nefness on any reduced compact complex space of dimension at most three

4.7 Definition Let X be a reduced compact complex space, $\dim X \leq 3$. Let L be a line bundle on X. Then L is *nef*, if there exists a desingularisation $\pi : \hat{X} \longrightarrow X$ such that $\pi^*(L)$ is nef.

By (4.6) the definition does not depend on the choice of the resolution.

4.8 Problem There is another natural way to define nefness for a line bundle L on a reduced compact complex space X : we require that for any $\varepsilon > 0$ there exists a metric h_ε on L such that the curvature satisfies $\Theta_{L,h_\varepsilon} \geq -\varepsilon\omega$ on the smooth part of X, where ω is a fixed positive (1,1)-form on X.

These both notions of nefness should coincide.

The next lemma will be very important for the main results of this section; it is of course trivial in the projective case.

4.9 Lemma *Let X be a compact Kähler manifold, L a line bundle on X. Assume that there is an effective divisor D on X such that $L = \mathcal{O}_X(D)$. If $L|D$ is nef, then L is nef.*

Proof. Note that the Kähler cone is the dual cone of the cone of of classes of positive closed currents of bidegree $(n-1, n-1)$ on X. So by (4.5) we only need to show

$$\int_X c_1(L) \wedge T \geq 0 \tag{$*$}$$

for every positive closed $(n-1, n-1)-$ current T. Of course we may assume that D is a reduced prime divisor. By Skoda [Sk82], the current $\chi_D T$ is closed, hence we obtain from (*) :

$$\int_X c_1(L) \wedge T = \int_X c_1(L) \wedge \chi_D T + \int_X c_1(L) \wedge \chi_{X \setminus D} T. \tag{$**$}$$

Appyling (4.6) to an embedded resolution of singularities for D, we may assume D to be smooth from the beginning. Then it is an easy exercise to conclude

$$\int_X c_1(L) \wedge \chi_D T \geq 0$$

(extend metrics on $L|D$ to a neighborhood).

Now the second term in (**) is also non-negative by (3.7.a), hence (*) follows and the claim is proved.

4.10 Theorem *Let X be a compact Kähler threefold with $\kappa(X) = 2$. If K_X is not nef, there exists a rational curve C with $K_X \cdot C < 0$.*

Proof. Let $m \gg 0$ so that $V = |mK_X|$ defines a rational map

$$f : X \rightharpoonup Y$$

to a projective surface Y. We choose a sequence of blow-ups $\pi : \hat{X} \longrightarrow X$ such that the induced map $\hat{f} : \hat{X} \longrightarrow Y$ is holomorphic. We note that f is almost holomorphic, i.e. there is a non-empty Zariski open set $U \subset X$ such that $f|U$ is holomorphic and proper. In fact, otherwise the exceptional set of π would contain a component lying surjectively over S and therefore X would be projective by [Ca81].

Let $D_0 \in |mK_X|$ be a general element. By (4.9) $K_X|D_0$ is not nef. Observe that V must have a positive dimensional base locus, otherwise K_X would be nef. We claim that V must even have fixed components:

otherwise the general D_0 is irreducible. It follows from (4.14) below (applied to a desingularisation) that there is a curve $C \subset D_0$ with $K_X \cdot C < 0$. This is impossible since then C would deform in X_0 in a 1-dimensional family [Ko91] to fill up D_0 which is clearly absurd.

Hence V has fixed components, D_0 is reducible and we can write

$$D_0 = B + \sum_i \lambda_i A_i$$

where B is the movable part of D_0, hence irreducible and with multiplicity 1, and the A_i are the fixed components.

(4.10.1) Our first aim is to show that there must be an irreducible curve $C \subset X$ (not necessarily rational) with $K_X \cdot C < 0$. By the arguments proving the existence of the fixed components it follows also that we can find some i_0 such that $K_X|A_{i_0}$ is not nef (apply the above argument for B instead of D_0). Moreover it suffices in order to get a contradiction to show that A_{i_0} has a non-constant meromorphic function and that the case (4.13(2)) does not occur. Let $A = A_{i_0}$ for simplicity of notations, and let $\mu = \lambda_{i_0}$. We can choose $\pi : \hat{X} \longrightarrow X$ such that the strict transform $\hat{A}$ of A is smooth. We investigate the structure of $\hat{A}$ by distinguishing cases according to $\dim \hat{f}(\hat{A})$.

(a) $\dim \hat{f}(\hat{A}) = 0$.

Let $\hat{F}$ denote the fiber of $\hat{f}$ containing $\hat{A}$, equipped with the natural structure. Write $\hat{F} = \lambda\hat{A} + R$. Then the conormal bundle $N^*_{\hat{F}|\hat{X}}|\hat{A}$ is generated by global sections, but is clearly not trivial, hence

$$\kappa(N^*_{\hat{F}|\hat{X}}|\hat{A}) \geq 1,$$

in particular $a(A) \geq 1$. Therefore we only need to exclude the case (4.13(2)) to conclude. So let $\hat{h} : \hat{A} \longrightarrow \hat{C}$ be the algebraic reduction of $\hat{A}$ and assume

$$\pi^*(K_X|A) \equiv \hat{h}^*(\hat{G})$$

with $\hat{G}^*$ nef and not numerically trivial, hence ample. Clearly $\hat{h}$ descends to a map $h : A \longrightarrow C$ with $g : \hat{C} \longrightarrow C$ being the normalisation. Then $K_X|A \equiv h^*(G)$ with G^* ample on C. Write

$$mK_X = D_0 = \mu A + M.$$

Then by "adjunction" we have

$$K_{\mu A}|A = mK_X|A - M',$$

where M' is effective on A and supported on $M \cap A$. On the other hand by ordinary adjunction

$$K_{\mu A}|A = K_X|A + N_{\mu A|X}|A,$$

hence in total $N^*_{\mu A|X}|A = N_A^{*\mu}$ is effective by the ampleness of G and effectivity of M'. Therefore from $K_A = K_X|A + N_A$ we see that $\kappa(K_A) = -\infty$. Since $K_{\hat{A}}$ is a subsheaf of $\pi^*(K_A)$, it follows $\kappa(\hat{A}) = -\infty$ and $\hat{A}$ is algebraic, contradiction.

(b) $\dim \hat{f}(\hat{A}) = 1$.

Now it is obvious that $a(\hat{A}) \geq 1$. The remaining case (4.13(2)) is excluded as in (a).

(c) $\dim \hat{f}(\hat{A}) = 2$.

Then $\hat{f}|\hat{A}$ is onto Y. However Y is algebraic and so does A. Then our claim is obvious.

(4.10.1) is thus completely proved.

(4.10.2) We next claim that there is a **rational** curve $C \subset X$ such that $K_X \cdot C < 0$. If such a rational C does not exist then we find an irrational curve $C \subset X$ with $K_X \cdot C < 0$, hence C deforms (with irreducible parameter space) in an at least 1-dimensional family $(C_t)_{t \in T}$ [Ko91] with $C = C_0$. Moreover we may assume that no deformation of C splits. We choose T maximal (but of course irreducible, as always). Then $\bigcup_{t\in T} C_t$ is a fixed component of V, say A. Write a general element of $|mK_X|$ as

$$D_0 = m_1 A + R$$

so that R does not contain A. Then by adjunction

$$m_1 K_A = m_1 K_X|A + m_1 A|A \subset m_1 K_X|A + D_0|A = (m_1+m)K_X|A.$$

Therefore we obtain the following basic inequality

$$K_A \cdot B \leq (1+\frac{m}{m_1})K_X \cdot B < K_X \cdot B \tag{I}$$

for all but finitely many curves $B \subset A$ with $K_X \cdot B < 0$. Hence

$$K_A \cdot C_t \leq -2. \tag{$*$}$$

Let $\nu : \tilde{A} \longrightarrow A$ be the normalisation of A and $\pi : \hat{A} \longrightarrow A$ the minimal desingularisation. Then we have

$$K_{\tilde{A}} = \nu^*(K_A) - E$$

with E an effective Weil divisor supported exactly on the preimage of the non-normal locus of A, cp. [Mo82]. Moreover

$$K_{\hat{A}} = \pi^*(K_{\tilde{A}}) - E'$$

with an effective divisor E' which is 0 if and only if $\tilde{A}$ has only rational double points.

Let $t \in T$ be general and $\hat{C}_t$ be the strict transform of C_t in $\hat{A}$. Then (*) yields

$$K_{\hat{A}} \cdot \hat{C}_t \leq -2. \tag{$**$}$$

The general fiber $\hat{F}$ of $\hat{f}$ (which is an elliptic curve) induces a unique maximal family $(\hat{F}_t)_{t\in\hat{T}}$ with graph

$$q : \mathcal{C} \longrightarrow \hat{T}, p : \mathcal{C} \longrightarrow \hat{X}$$

(taking closure in the cycle space). The map p is clearly bimeromorphic, f being almost holomorphic, and q is an elliptic fibration. Of course we may assume $\hat{T}$ smooth. Let $\overline{A}$ denote the strict transform of $\hat{A}$ in $\mathcal{C}$; we may assume $\overline{A}$ smooth (by possibly choosing an embedded resolution of singularities and then flattening q). Let $g = q|\overline{A} : \overline{A} \longrightarrow D \subset \hat{T}$. By (**) we have

$$\kappa(\hat{A}) = -\infty.$$

Note also that D is a rational or elliptic curve.

(a) Let us first assume that the genus $g(C_t) \geq 2$ (= genus of the normalisation). Our aim is to construct a new family of elliptic or rational curves (C'_t) such that $K_X \cdot C'_t < 0$.

Let $\hat{G}_t = \hat{F}_t \cap \overline{A}$ for $t \in D$. Since f is almost holomorphic, we have

$$p^*\pi^* K_X \cdot \hat{F}_t = 0.$$

Now take a component $\hat{C} \subset \hat{F}_t$ with dim $C = \dim \pi \circ p(\hat{C}) = 0$. If

$$p^*\pi^* K_X \cdot \hat{C} > 0,$$

then we find another $\hat{C}' \subset \hat{F}_t$ with dim $p \circ \pi(\hat{C}') = 1$ such that

$$p^*\pi^*(K_X \cdot \hat{C}') < 0.$$

Let $C' = \pi \circ p(\hat{C}')$. Then $K_X \cdot C' < 0$ and we have found a family of elliptic or rational curves (C'_t) with $K_X \cdot C'_t < 0$.

Therefore we may assume $K_X \cdot C = 0$ for anychoice of $\hat{C} \subset \hat{F}_t, t \in D$ and $C = \pi \circ p(\hat{C})$. We shall assume now that the general G_t is elliptic, the case that G_t is rational being even easier (and therefore omitted). We consider the possibly not relatively minimal fibration

$$g : \overline{A} \longrightarrow D \simeq \mathbf{P}_1.$$

Furthermore we have a "ruling" $h : \overline{A} \longrightarrow E$, so that the general fiber of h is a smooth rational curve and every fiber is a tree of $\mathbf{P}_1$. Let

$$L = p^*\pi^*(K_X).$$

Then $L \cdot C = 0$ for every component of every fiber of g. Since f is almost holomorphic, we even have

$$\pi^* p^*(K_X) \cdot \hat{F}_t = \mathcal{O}_{\hat{F}_t}$$

for general t. From semi-continuity it follows then easily that $L|C = \mathcal{O}_C$ for all C, possibly up to a torsion line bundle (in case C appears with multiplicity in the fiber). After eventually passing from L to L^m we get

$$L \equiv g^*(\mathcal{O}_D(a))$$

for some integer a. Now let C_t be a general element of our family (C_t) and let $\overline{C}_t$ be its strict transform in $\overline{A}$. Then

$$L \cdot \overline{C}_t < 0.$$

Since $\overline{C}_t$ is a multi-section of g, we conclude $a < 0$. Now consider a general fiber l of h. Then we conclude $L \cdot l < 0$, and therefore

$$K_X \cdot \pi p(l) < 0.$$

Therefore the images of l define the family C'_t of rational curves.

(b) We are now reduced to the case that $(C_t)_{t \in T}$ is a family of elliptic curves and moreover we may assume that the family is non-splitting in the sense that no component of any member is a rational curve (but multiples of elliptic curves are allowed). We take T of course maximal and assume T normal. Denote by $\mathcal{D}$ the normalisation of the graph of the family with projections $p_1 : \mathcal{D} \longrightarrow A$ and $q_1 : \mathcal{D} \longrightarrow T$.

First assume $\dim T \geq 2$. Note for the following that T is algebraic (= Moishezon) since A is algebraic and $A = \bigcup C_t$. Choose a general point $x \in A$ and introduce

$$T(x) = \{t \in T | x \in C_t\}.$$

Then $\dim T(x) \geq 1$. Choose an irreducible curve $\Delta \subset T(x)$. Let $p_\Delta : \mathcal{D}_\Delta \longrightarrow A$ be the normalisation of the induced graph with projection $q_\Delta : \mathcal{D}_\Delta \longrightarrow \Delta$. Then q_Δ admits a multi-section Z with $\dim p_\Delta(Z) = 0$, so that $Z \subset \mathcal{D}_\Delta$ is exceptional. This contradicts Sublemma (4.10.a).

We are therefore left with the case $\dim T = 1$. Since K_A is a subsheaf of $K_X|A$, the inequality

$$K_X \cdot C_t \leq -1$$

implies

$$K_A \cdot C_t \leq -1.$$

For general t, let $\hat{C}_t$ be the strict transform of C_t in $\hat{A}$, as before. Then

$$K_{\hat{A}} \cdot \hat{C}_t \leq -1.$$

If $K_{\hat{A}} \cdot \hat{C}_t \leq -2$, then $\hat{C}_t$ moves in an at least 2-dimensional family; hence we are reduced to

$$K_{\hat{A}} \cdot \hat{C}_t = -1.$$

Therefore we conclude that the general C_t will not meet the non-normal locus of A. Since not all C_t pass through a fixed point, we see that (4.10.a) that

$$C_t \cap \mathrm{Sing} A = \emptyset$$

for general t. Therefore C_t is a Cartier divisor in A and we can consider $\mathcal{L} = \mathcal{O}_A(C_t)$. Since dim $T = 1$, we have $h^0(A, \mathcal{L}) = 2$. Since $|\mathcal{L}|$ is base point free, it defines a holomorphic map $\alpha : A \longrightarrow \mathbf{P}_1$ with C_t being a fiber. This contradicts $C_t^2 = 1$.

The proof of (4.10) is now complete modulo the following

4.10.a Sublemma *Let S be a smooth projective surface and $f : S \longrightarrow C$ a holomorphic elliptic submersion onto a smooth curve C. Then there is no irreducible curve $A \subset S$ with $A^2 < 0$.*

Proof. Let g be the genus of C. By adjunction

$$\deg K_A = K_S \cdot A + A^2.$$

The canonical bundle formula for elliptic fibrations (see e.g. [BPV84,chap.5, 12.1 and chap.3, 18.3]) gives $K_S \equiv f^*(K_C)$. Let $d = \deg f|A$. Then

$$K_S \cdot A = d(2g-2).$$

Let $\nu : \tilde{A} \longrightarrow A$ be the normalisation of A; then $\deg K_{\tilde{A}} \leq \deg K_A$. Now the formula of Riemann-Hurwitz yields

$$\deg K_{\tilde{A}} = d(2g-2) + \deg R,$$

where R is the ramification divisor of $\tilde{A} \longrightarrow C$. Putting things together we obtain

$$\deg K_A = d(2g-2) + A^2 \geq d(2g-2) + \deg R,$$

hence $\deg R \leq A^2 < 0$, which is absurd.

4.11 Corollary *Let X be a compact Kähler threefold with $\kappa(X) = 2$. Assume that K_X is not nef. Then there exists a birational extremal contraction $f : X \longrightarrow Y$ (i.e. of type (c) or (d) in (3.4)).*

Proof. By (4.10) exists a rational curve $C \subset X_0$ with $K_{X_0} \cdot C < 0$; then apply (3.16).

(4.12) In particular we have proved (4.1) in case $K_{X_0}^3 = 0, K_{X_0}^2 \neq 0$. In fact, if K_{X_0} would not be nef, then we find a rational curve $C \subset X_0$ with $K_{X_0} \cdot C < 0$ and which can be deformed to X_t. In fact, taking over the notations of the last proof, we conclude that in all cases but the case of $\mathbf{P}_2$ with $N_S = \mathcal{O}(-2)$ we conclude that

$$N_{C_t|\mathcal{X}} \simeq \bigoplus \mathcal{O}(a_i)$$

with $a_i \geq -1$, so that the deformations of C_t in $\mathcal{X}$ are unobstructed. Moreover

$$h^0(\mathcal{X}, N_{C_t|\mathcal{X}}) > h^0(X_0, N_{C_t|X_0}),$$

so that the C_t can be deformed to the neighbouring fibers X_s, so that K_{X_s} cannot be nef. In the remaining case we see by a similar argument that we can deform directly the $\mathbf{P}_2$ out of X_0 and are also done.

In the proof (4.10) we made use of the following

4.13 Proposition *Let S be a smooth compact Kähler surface with algebraic dimension $a(S) = 1$. Let $f : S \longrightarrow A$ be the algebraic reduction to the smooth curve A. Let L be a line bundle on S. Let $\equiv$ denote topological equivalence. Then*
(1) L is nef if and only if $L \equiv f^*(G)$ with G nef
(2) L is not nef if and only if either
(a) there is a curve $C \subset S$ with $\mathcal{L} \cdot C < 0$
(b) $L \equiv f^*(G'), G'^*$ ample.

Proof. (1) One direction being obvious, we let L be nef. First we show that $L \cdot C = 0$ for every curve $C \subset S$. In fact, assume $L \cdot C > 0$ for some C. Then

$$(kL + C)^2 = k^2 L^2 + 2kL \cdot C + C^2.$$

Since $L^2 \geq 0$, we obtain by [DPS94] $(kL + C)^2 > 0$ for $k \gg 0$, hence S would be algebraic.

Note that if $L \equiv f^*(G)$, it is clear that G must be nef (otherwise we obtain a nef line bundle whose dual has a section with zeroes which is impossible by [DPS94]).

So assume now $L \cdot C = 0$ for all curves C. By (4.6) we may assume S minimal.

(a) First let $\kappa(S) = 1$. Since $L \cdot K_S = 0$, Riemann-Roch gives

$$\chi(S, mL) = \chi(S, \mathcal{O}_S) > 0.$$

Hence $h^0(mL) > 0$ or $h^0(mL^* + K_S) > 0$. In the first case we conclude that $mL|F = \mathcal{O}_F$ for every fiber of the algebraic reduction f, hence $mL = f^*(G)$. In the second case we can write

$$K_S = mL + D$$

with D effective. Since $D \cdot F = 0$, we get $D|F = \mathcal{O}_F$, hence

$$\lambda L = f^*(G).$$

(b) Now let $\kappa(S) = 0$. Then either S is a K3 surface or S is a torus. If S is a K3 surface, then $\chi(S, \mathcal{O}_S) = 2$ and we can conclude as in (a). So let S be a torus.

Then f is an elliptic fiber bundle over an elliptic curve. The line bundle L defines a section

$$s \in H^0(C, R^1 f_*(\mathcal{O}_S^*)).$$

Fix a point $y_0 \in C$. Then we find a topologically trivial line bundle H on S such that

$$L|f^{-1}(y_0) \otimes H|f^{-1}(y_0) \simeq \mathcal{O}.$$

Since S is not algebraic, we automatically have

$$L|f^{-1}(y) \otimes H|f^{-1}(y) \simeq \mathcal{O}$$

for all $y \in C$; otherwise we would get a multisection of f. This implies our claim.

(2) Again one direction is clear. So let L be not nef and assume $L \not\equiv f^*(G')$. Then by the proof of (1), there is a curve C_0 with $L \cdot C_0 \neq 0$. If our claim would be false, then $L \cdot C_0 > 0$ and $L \cdot C \geq 0$ for every curve $C \subset S$. Let F bethe general fiber of f. Then

$$L \cdot F = 0,$$

otherwise $(L + kF)^2 > 0$ for large k. Since $\dim f(C_0) = 0$, we therefore find $C_1 \subset f^{-1}f(C_0)$ with $L \cdot C_1 < 0$, contradiction.

4.14 Corollary *Let X be a non-algebraic compact Kähler surface. Let L be a line bundle on X such that $L \cdot C \geq 0$ for all curves $C \subset X$. Assume moreover that some power L^m has a section. Then L is nef.*

Proof. (a) First we assume that $a(X) = 1$ and let $f : X \longrightarrow C$ be the algebraic reduction which is an elliptic fibration. Assume that L is not nef. By (4.13) we find (posssibly after passing to a multiple of L) a topologically trivial line bundle G on X such that

$$L = f^*(H) \otimes G$$

with some negative line bundle H on C. We may assume that already L has a section. We conclude that

$$H^0(C, H \otimes f_*(G)) \neq 0. \qquad (*)$$

Note that $f_*(G)$ is a torsion free sheaf, hence locally free. Since G is topologically trivial, we have $f_*(G) \neq 0$ if and only if $G|F = \mathcal{O}_F$ for the general fiber F of f. On the other hand $f_*(G)$ must be non-zero by (*). Moreover (*) proves that $f_*(G)$ and hence G has a section (we may a priori assume that H^* is effective). Now this section cannot have zeroes and therefore $G = \mathcal{O}_X$. But then (*) gets absurd.

(b) Assume now that $a(X) = 0$. Let $\pi : X \longrightarrow X'$ be the map to the minimal model. We can write, L being effective,

$$L = \sum \lambda_i E_i + \pi^*(L')$$

where the E_i are the exceptional components of π and $\lambda_i \geq 0$. Since L is algebraically nef, it is clear that all $\lambda_i = 0$. So we may assume X minimal and X is either a torus or K3. On the torus we conclude that $L = \mathcal{O}$ and in the K3 case we write $L = \sum a_i C_i$. But the C_i are all exceptional as well as their union so that again we have $a_i = 0$. (The only curves in a K3 surface without meromorphic functions are $(-2)-$curves).

4.15 Theorem *Let X be a smooth compact Kähler threefold with $\kappa(X) = 1$. Assume K_X is not nef. Then there exists an irreducible curve $C \subset X$ such that $K_X \cdot C < 0$.*

Proof. We follow the lines of (4.10.1). We assume that K_X is algebraically nef and show that K_X is nef. Let

$$f : X \rightharpoonup C$$

be the meromorphic map defined by $|mK_X|$. Let $\pi : \hat{X} \longrightarrow X$ be a sequence of blow-ups such that the induced map $\hat{f} : \hat{X} \longrightarrow C$ is a morphism. Write again

$$mK_X = B + \sum \lambda_i A_i,$$

where B is the movable part. Then $mK_X|B$ is effective. Hence by (4.14) we conclude that $K_X|B$ is nef. It only remains to show that $K_X|A_i$ is nef for all i. Fix some i and let $A = A_i$. If $a(A) \geq 1$, the arguments of (4.10.1) work. So assume $a(A) = 0$. Let $\hat{A}$ be the strict transform of A in $\hat{X}$; we may assume $\hat{A}$ smooth. Let $\hat{F}$ denote the fiber of $\hat{f}$ containing $\hat{A}$ and write

$$\hat{F} = \lambda \hat{A} + R.$$

It follows from the exact sequence

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} = N^*_{\hat{F}|\hat{X}}|\hat{A} \longrightarrow N^*_{\lambda\hat{A}|\hat{X}}|\hat{A} \longrightarrow N^*_{\lambda\hat{A}|\hat{F}}|\hat{A} \longrightarrow 0$$

that

$$H^0(\hat{A}, \det N^{*\lambda}_{\hat{A}}) \neq 0.$$

Since $\kappa(\hat{A}) = 0$ ($\hat{A}$ being bimeromorphic to a torus or a K3-surface), it follows from the adjunction formula that

$$H^0(\hat{A}, \lambda K_{\hat{X}}|\hat{A}) \neq 0.$$

Let $g : \tilde{A} \longrightarrow A$ be the normalisation of A, then $\pi|\hat{A} = g \circ h$ with $h : \hat{A} \longrightarrow \tilde{A}$ the induced map. Hence

$$H^0(\hat{A}, g^*K_X - \lambda \sum \mu_i C_i) \neq 0,$$

where

$$K_{\hat{X}} = \pi^*(K_X) + \sum \mu_i E_i$$

and $C_i = \hat{A} \cap C_i$. Then we obtain by applying h_* that

$$H^0(\tilde{A}, g^*(\lambda K_X)) \neq 0.$$

Now apply (4.14)!

4.16 Theorem *Let X be a compact Kähler threefold with $\kappa(X) = 1$. Assume that K_X is not nef and X is algebraically approximable. Then there exists a rational curve $C \subset X$ with $K_X \cdot C < 0$.*

Proof. If $K_{X_{t_\nu}}$ is not nef for some ν, then the assertion is clear (sect. 3). So assume $K_{X_{t_\nu}}$ nef. We have already seen that this implies $K_X^2 = 0$. This is the only conclusion we draw from the algebraic approximability. Write as in (4.15)

$$mK_X = B + \sum_{i=1}^{k} \lambda_i A_i.$$

Let $\mathcal{L} = \mathcal{O}_X(B)$ and let $f : X \rightharpoonup S$ be the meromorphic map defined by $H^0(X, \mathcal{L})$ to the curve S.

(1) First let us assume that f is holomorphic, i.e. $H^0(X, \mathcal{L})$ is base point free. Then

$$\dim f(A_i) = 0$$

for all i. In fact, consider the general fiber F of f. Then $K_F = K_X|F$ and from $K_X^2 = 0$ we get $K_F^2 = 0$. Since $\kappa(F) = 0$, the fiber F must be minimal, hence $K_F \equiv 0$. Now if $\dim f(A_i) = 1$ for some i, then $mK_X|F$ would be non-zero effective, contradiction.

By $K_X^2 = 0$ we obtain

$$(\sum \lambda_i A_i)^2 = 0.$$

All A_i being contained in fibers of f, this implies that $\sum \lambda_i A_i \equiv \rho F$ which is impossible (i.e. kK_X is generated by global sections).

(2) Now assume that f is not holomorphic. Let $\pi : \hat{X} \longrightarrow X$ be a sequence of blow-ups such that $\hat{f} : \hat{X} \longrightarrow S$ is holomorphic. Therefore we obtain some exceptional

divisor E for $\hat{f}$ such that $\dim \hat{f}(E) = 1$. It follows that $a(\hat{F}) \leq 1$ for the general fiber $\hat{F}$ of $\hat{f}$, otherwise $\hat{X}$ would be algebraically connected, hence projective [Ca 81].

Assume first that $a(\hat{F}) = 0$ and observe $B = \pi(\hat{F})$. Taking another section in $H^0(X, mK_X)$, we obtain $D \in |mK_X|B|$ with $D^2 = 0$. Hence $\pi^*(D)^2 = 0$. But on a (blown up) torus or K3 surface without meromorphic functions there is no such effective non-zero divisor.

If however $a(\hat{F}) = 1$, then, arguing in the same way, we deduce that $(\pi|\hat{F})^*(D)$ consists of multiples of fibers of the algebraic reduction

$$h : \hat{F} \longrightarrow C.$$

Since h has no multi-sections, there is a map $h' : B \longrightarrow C'$ to a possibly non-normal curve C' (with normalisation C) such that D consists of multiples of fibers of h'. Hence $kK_X|B$ is generated by $H^0(X, kK_X)$ for $k \gg 0$. This gives a contradiction to $\dim f(B) = 0$.

(4.17) Finally we prove Theorem 4.1 in the only remaining case $K_{X_0}^2 \equiv 0$ but $K_{X_0} \not\equiv 0$ by the same reasoning as in (4.12).

Putting together (3.16) and (4.1) we now obtain the Main Theorem for *algebraically approximable* X as stated in the Introduction.

Combining (3.16) with (4.10) and (4.16) we obtain as another part of the Main Theorem:

4.18 Theorem *Let X be a smooth compact Kähler threefold with K_X not nef. Assume that $\kappa(X) = 1$ or 2 and in case $\kappa(X) = 1$ assume furthermore that X can be approximated algebraically. Then X carries an extremal contraction.*

5. Generic conic bundles

(5.1) In Part 1, (2.11) and (2.12) we had proved the following. Let X be a non-projective compact Kähler threefold which can be approximated algebraically or assume that X is uniruled. Assume that X admits a non-splitting 1-dimensional family (C_t) of rational curves with $-K_{X_t} \cdot C_t = 1$, filling up a non-normal surface S. Then the algebraic dimension $a(X) = 1$, and X has a "generic conic bundle structure" in the following sense.

The algebraic reduction is a holomorphic map $f : X \longrightarrow C$ to a smooth curve C whose general fiber is an almost homogeneous $\mathbf{P}_1$−bundle over an elliptic curve. The surface S is a fiber of f. Moreover there exists an almost holomorphic meromorphic map $g : X \rightharpoonup Y$ to a normal surface Y such that f factorises over g and such that the following holds:

(a) if $U = f^{-1}(\{c \in C | f^{-1}(c)$ is smooth or $S = f^{-1}(c)\}$, then $g|U$ is holomorphic and proper

(b) $g|U$ is a conic bundle

(c) g is a rational quotient of X.

The general smooth fiber of g is a $\mathbf{P}_1$ with normal bundle $\mathcal{O} \oplus \mathcal{O}$ and therefore we have a 2-dimensional family $(\tilde{C}_t)_{t \in \tilde{T}}$ such that for every $t_1 \in T$ there is a $t_2 \in T$ and a $t \in T$ with $C_{t_1} + C_{t_2} = \tilde{C}_t$.

In general of course, the map g will not be holomorphic: just perform some birational transformation on a conic bundle. However we shall prove a criterion when g is actually holomorphic.

We say that a 2-dimensional family $(\tilde{C}_t)$ of rational curves splits only in the **standard way** if the following holds.

If $\tilde{C}_{t_0}$ is a reducible member of the 2-dimensional family, then $\tilde{C}_{t_0} = C'_{t_1} + C'_{t_2}$ with smooth rational curves C'_{t_i} meeting transversally at one point and with $K_X \cdot C'_{t_i} = -1$.

So *all* $\tilde{C}_t$ are conics.

In this terminology we have:

5.2 Theorem *Let things be as in the setting (5.1) and assume that the induced family $(\tilde{C}_t)$ splits only in the standard way. Then Y can be taken smooth so that g is holomorphic and is a conic bundle over Y whose fibers are just the $\tilde{C}_t$.*

Proof. Let $p : \mathcal{C} \longrightarrow X$ be the graph of the family $(\tilde{C}_t)_{t \in \tilde{T}}$ with projection $q : \mathcal{C} \longrightarrow \tilde{T}$. Here we consider $\tilde{T}$ reduced. By (5.1) there is a Zariski open set $U \subset \tilde{T}$ such that $p|q^{-1}(U)$ is an isomorphism. Hence p is bimeromorphic, and, by considering the normalisation of $\tilde{T}$, it follows that p has connected fibers. Hence $\mathcal{C}$ is normal and so does $\tilde{T}$. It now suffices to prove that p is finite, then p is biholomorphic and our claim follows.

Assume there exists $x \in X$ such that there is a curve $B \subset \tilde{T}$ with $x \in \tilde{C}_t$ for all $t \in B$. Then we form the surface

$$A = \bigcup_{t \in B} \tilde{C}_t = p(q^{-1}(B)).$$

(a) First assume that no $\tilde{C}_t, t \in B$ splits. Then, A being normal by Part1, (2.3), similar arguments (even easier) as in Part1, (2.1), lead to a contradiction (by the fact that the $\tilde{C}_t$ can be moved out of A).

(b) So there exists $t_0 \in B$ such that $\tilde{C}_{t_0}$ splits. By our assumptions the components of the splitting deform in a non-splitting family. Hence it follows from what we have said in (5.1) (i.e. (2.12)) that there is no common point of the $\tilde{C}_t, t \in B$, since the map g is holomorphic near the reducible conics (the maps g attached to the various 1-dimensional non-splitting families attached to the full family of conics are of course the same).

Hence p is finite, hence biholomorphic. The smoothness of Y comes from deformation theory and the smoothness of X.

5.3 Corollary *Assume in (5.1) that the C_t define a geometrically extremal ray in $\overline{NE}(X)$. Then g is a conic bundle over a smooth surface Y with $a(Y) = 1$.*

Proof. We have to prove that the family $(\tilde{C}_t)$ splits only in the standard way. Assume to the contrary that $\tilde{C}_{t_0} = \sum_i^p C_i'$ with $p \geq 2$. Since $[C_t]$ is geometrically extremal and since $[\tilde{C}_{t_0}] = 2[C_t] \in \overline{NE}(X)$, we have $[C_i'] \in \overline{NE}(X)$, hence $K_X \cdot C_i' < 0$. Hence the claim is clear.

It is interesting to have a general look at conic bundles over non-algebraic surfaces. The next proposition follows of course from (5.1) and (5.3) but it is instructive to see a direct argument.

5.4 Proposition *Let $\varphi : X \longrightarrow S$ be a conic bundle over the Kähler surface S with $a(S) = 0$ such that $\rho(X) = \rho(S) + 1$. Then the discriminant locus $\Delta = \emptyset$, so that φ is an analytic $\mathbf{P}_1-$bundle.*

Proof. Assume $\Delta \neq \emptyset$. By the Kodaira classification S is birationally equivalent to a torus or a K3 surface. In particular all curves in S are smooth rational curves. Now it is a basic fact on conic bundles with $\rho(X) = \rho(S) + 1$ that every smooth rational component $C \subset \Delta$ has to meet $\overline{\Delta \setminus C}$ in at least two points (see e.g. [Mi83], the arguments remaining true in the non-algebraic case). This already rules out the torus case. In the K3 case first notice that the above condition implies $\Delta^2 = 0$. The union of all rational curves, and in particular Δ, is however contractible, therefore $\Delta^2 < 0$, contradiction.

The same type of argument together with Kodaira's classification of singular fibers of elliptic surfaces proves

5.5 Proposition *Let $\varphi : X \longrightarrow S$ be a conic bundle over a Kähler surface S with $a(S) = 1$, satisfying $\rho(X) = \rho(S) + 1$. Let Δ be its discrminant locus. Then $\Delta = \bigcup F_i$, where the F_i are smooth fibers of the algebraic reduction $f : S \longrightarrow C$ (which is an elliptic fibration) or reduction of multiple smooth fibers.*

5.6 Proposition *Let us assume the situation of (5.5). Then $a(X) = 1$ and $f \circ \varphi$ is an algebraic reduction of X.*

Proof. If $a(X) \neq 1$, then $a(X) = 2$. This case is already ruled out in Part 1. The argument is as follows. Let $g : X \rightharpoonup Z$ be an algebraic reduction and F a general fiber of φ. Since g is an "elliptic fibration", we have $\dim g(F) = 1$. It follows easily that any two points in X can be joined by a chain of curves, i.e. X is algebraically connected. Hence X is Moishezon by [Ca81], contradiction.

6. Minimal Models

(6.1) The weak minimal model conjecture (WMMC) in the Kähler case predicts that every compact Kähler manifold X is either uniruled or birationally equivalent to a Kähler $n-$fold X' with at most terminal singularities such that $K_{X'}$ is nef; such an X' is called minimal model.

The abundance conjecture (AC) says that every minimal model is semi-ample, i.e. some multiple $mK_{X'}$ is generated by global sections. A minimal model with semi-ample $K_{X'}$ is also called good minimal model.

The strong minimal model conjecture (SMMC) states that starting from a compact Kähler manifold X one get derive either a $\mathbf{Q}-$Fano fibration or a minimal model by a sequence of birational divisorial contractions or flips.

See [KMM87] for the background in the algebraic case. SMMC holds for algebraic threefolds by [Mo88] and AC by [Mi88] and [Ka92].

(6.2) Both WMMC and AC have been proved by Nakayama [Na88] in case the compact Kähler threefold carries an elliptic fibration, in particular if $a(X) = 2$.

(6.3) We next describe the structure theorem of Fujiki [Fu83] for non-algebraic non-uniruled compact Kähler threefolds with $a(X) \leq 1$.

(a) If $a(X) = 1$, then either we have a holomorphic algebraic reduction $f : X \longrightarrow C$ with the general smooth fiber being a torus or X is birationally $(C \times S)/G$ with C a compact Riemann surface, S a torus or a K3-surface with $a(S) = 0$ and G a finite group acting on both C and S and on $C \times S$ by $g(x, y) = (gx, gy)$.

(b) If $a(X) = 0$, then either X is a Kummer manifold, i.e. X is birationally T/G, where T is a torus and G a finite group or X is simple, i.e. X does not carry a covering family of compact subvarieties and does not admit a meromorphic map to a Kummer manifold.

6.4 Theorem *Kummer threefolds with $a(X) = 0$ have good minimal models.*

Proof. Let X be a Kummer threefold with $a(X) = 0$. So X is birationally $Y = T/G$ with a torus T of algebraic dimension 0. Since T does not carry positive-dimensional subvarieties, Y can have only isolated singularities and these are quotient singularities, since $g : T \longrightarrow T/G$ is unramified outside a finite set. In particular Y has only canonical singularities (see e.g. [KMM87]). By Reid [Re83] there is a partial resolution

$$f : X' \longrightarrow Y$$

such that X' has terminal singularities and f is crepant, i.e. $K_{X'} = f^*(K_Y)$. Since obviously $K_Y \equiv 0, X'$ is a minimal model. Since $K_T = g^*(K_Y)$, we even have $mK_Y = \mathcal{O}_Y$, hence X' is good.

(6.5) Remark (1) It might also be possible to construct directly a minimal model in case $X \sim (C \times S)/G$, but the case of the holomorphic torus fibration is certainly harder. Possibly one has to prove SMMC in that case to proceed to a minimal model. See (6.7) for more comments on SMMC.

(2) A consequence of the existence of good minimal models is the non-existence of simple threefolds as in (6.3) defined; see the Motiviation following the Introduction.

We finally show that once we know the existence of good minimal models, then the notions of algebraic nefness and nefness coincide (for threefolds)

6.6 Theorem *Let X be a smooth compact Kähler threefold. Assume that X has a good minimal model. Then K_X is nef if and only if $K_X \cdot C \geq 0$ for all curves $C \subset X$.*

Proof. Due to (4.10) and (4.15) we could restrict ourselves to the case $\kappa(X) = 0$. However we will give a simultaneous proof in all cases making the arguments independent of section 4.

One direction being obvious we assume that $K_X \cdot C \geq 0$ for all curves $C \subset X$. Let

$$h : X \rightharpoonup X'$$

be a bimeromorphic map to a good minimal model. Choose a sequence of blow-ups $f : \hat{X} \longrightarrow X$ such that the induced map $g : \hat{X} \longrightarrow X'$ is a morphism. Let

$$D' \in |mK_{X'}|$$

be a general smooth member and $D \subset X$ be its strict transform (in case $\kappa(X) = 0$, we have $D' = 0$). Then we can write

$$mK_X = D + \sum \mu_i A_i, \qquad (*)$$

with $mu_i > 0$. To determine the structure of the A_i, we write

$$K_{\hat{X}} = g^*(K_{X'}) + \sum a_j F_j, a_j > 0.$$

Let $\hat{D}$ be the strict transform of D in $\hat{X}$. Then

$$g^*(D') + \sum ma_j F_j = \hat{D} + \sum c_j F_j \in |mK_{\hat{X}}|$$

with $c_j \geq 0$. Therefore

$$D + \sum c_j f(F_j)_0 \in |mK_X|,$$

where the index 0 indicates that $f(F_j)$ is omitted if $\dim f(E_j) \leq 1$. So $A_i = f(E_j)_0$ and in particular all A_i are algebraic.

By (4.9) it is sufficient to show that $K_X|D$ is nef and that $K_X|A_i$ is nef for all i. Since the A_i are algebriac, the second statement is clear.

(1) If $\kappa(X) = 0$, then $D' = 0$, hence $D = 0$ and there is nothing to prove.

(2) If $\kappa(X) \geq 1$, we note that $mK_X|D$ is effective, hence we conclude by (4.14).

6.7 Corollary *Assume that WMMC and AC hold in dimension 3. Let X be a compact Kähler threefold with at most terminal singularities. Then K_X is nef if and only $K_X \cdot C \geq 0$ for all curve $C \subset X$.*

6.8 Remark The most difficult step in the construction of minimal models for algebraic threefolds is of course the existence of flips. However this is reduced in [Ka88] to a local analytic problem around the rational curves which have to be flipped. [Ka88] shows also that in the analytic this local reduction works. Since Mori proves in [Mo88] the local analytic existence of flips, we have already the existence of flips in the analytic category. The proof of termination is just the same as in the algebraic case.

References

[AB93] Alessandrini, L; Bassanelli, G.: Plurisubharmonic currents and their extensions across analytic subsets. Forum Math. 5, 577–602 (1993)

[Al96] Alessandrini, L.: Letter to the author, february 1996

[Ba94] Bassanelli, G.: A cut-off theorem for plurisubharmonic currents. Forum Math. 6, 567–595 (1994)

[Be83] Beauville, A.: Variétés kähleriennes dont la premiere classe de Chern est nulle. J. Diff. Geom. 18, 755–782 (1983)

[BPV84] Barth, W.; Peters, C.;van de Ven, A.: Compact complex surfaces. Erg. d. Math., 3. Folge, Band 4. Springer 1984

[Ca81] Campana, F.: Coréduction algébrique d'un espace analytique faiblement kählérien compact. Inv, math. 63, 187–223 (1981)

[CP94] Campana, F; Peternell, T.: Towards a Mori theory on compact Kähler threefolds, I. To appear in Math. Nachr. 1996

[De92] Demailly, J.P.: Regularisation of closed positive currents and intersection theory. J. Alg. Geom. 1, 361–410 (1992)

[DPS94] Demailly, J.P.; Peternell, T.; Schneider, M.: Compact complex manifolds with numerically effective tangent bundles. J. Alg. Geom. 3, 295–345 (1994)

[Fu83] Fujiki, A.: On the strucutre of compact complex manifolds in class $\mathcal{C}$. Adv. Stud. Pure Math. 1, 231–302 (1983)

[HL83] Harvey, R.; Lawson, B.: An intrinsic characterisation of Kähler manifolds. Inv. math. 74, 169–198 (1983)

[Ka85] Kawamata, Y.: Minimal models and the Kodaira dimension of algebraic fiber spaces. J. reine u. angew. Math. 363, 1–46 (1985)

[Ka86] Kawamata, Y.: On the plurigenera of minimal algebraic threefolds with $K \equiv 0$. Math. Ann. 275, 539–546 (1986)

[Ka88] Kawamata, Y.: The crepant blowing-ups of 3-dimensional canonical singularities and its application to degeneration of surface. Ann. Math. 127, 93–163 (1988)

[Ka92] Kawamata, Y.: Abundance theorem for minimal threefolds. Inv. math. 108, 229–246 (1992)

[KMM87] Kawamata, Y.; Matsuda, K.; Matsuki, K.: Introduction to the minimal

model problem. Adv. Stud. Pure Math. 10, 283–360 (1987)

[Mi87] Miyanishi, M.: Algebraic threefolds. Adv. Stud. Pure Math. 1, 69–99 (1983)

[Mi87] Miyaoka, Y.: The Chern classes and Kodaira dimension of a minimal variety. Adv. Stud. Pire Math. 10, 449–476 (1987) Adv. Stud. Pure Math. 10,

[Mi88] Miyaoka, Y.: Abundance conjecture for threefolds: $\nu = 1$ case. Comp. math. 68, 203–220 (1988)

[Mo82] Mori, S.: Threefolds whose canonical bundles are not numerically effective. Ann. Math. 116, 133–176 (1982)

[Mo88] Mori, S.: Flip theorem and the existence of minimal models for 3-folds. J. Amer. Math. Soc. 1, 117–253 (1988)

[Na88] Nakayama, N.: On Weierstrass models. Alg.Geom. and Comm. Algebra; vol. in honour of Nagata, vol. 2, 405–431. Kinokuniya, Tokyo 1988

[Pe95] Peternell, T.: Minimal varieties with trivial canonical class, I. Math. Z. 217, 377–407 (1994)

[Pe95a] Peternell, T.: On the limits of manifolds with nef canonical bundles. Preprint 1995

[Re83] Reid, M: Minimal models of canonical threefolds. Adv. Stud. Pure Math. 1, 131–180 (1983)

[Sk82] Skoda, H.: Prolongement des courants, positifs, fermés, de masse finie. Inv. math. 66, 361–376 (1982)

[Su88] Sugiyama, Y.: A geometry of Kähler cones. Math. Ann. 281, 135–144 (1988)

[Ue87] Ueno, K.: On compact analytic threefolds with non-trivial Albanese torus. Math. Ann. 278, 41–70 (1987)

[Wi89] Wisniewski, J.: Length of extremal rays and generalised adjunction. Math. Z. 200, 409–427 (1989)

Thomas Peternell, Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany (peternel@btm8x1.mat.uni-bayreuth.de)